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Convergence of the arithmetic means for numerical sequences having a countable adherence values set

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Abstract. This paper presents a convergence of arithmetic means for a large class of numerical sequences having a countable adherence values set in $\overline{\mathbb{R}}$ the real straight line completed. Using the adherence values and their corresponding ratios for a numerical sequence having a countable set of adherence values, we prove that the arithmetic means of this sequence is transformed into a sum of the arithmetic means of all its convergent subsequences multiplied by their ratios in this numerical sequence to the order of the sum of its arithmetic means. Then we show that the limit of the arithmetic means of these numerical sequence exists and that its value can be expressed in terms of the adherence values of this subsequences and their corresponding ratios. The convergence discussions in $\mathbb{R}$ of arithmetic means with some illustrative examples for a large class of numerical sequences will be investigated from different cases. Some conclusions about the Reciprocal of Cezàro Theorem and the Relationships with probability theory will be presented.
Key Words: Sequence, Adherence value, Countable set, arithmetic mean, Ratios, Cezàro Theorem.
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1 Introduction

Elements in the form of ordered numbers, such as decimals, fractions, or integers, constitute an important class of mathematical concepts. They often model natural phenomena like plant growth, physics, or finance and can exhibit regularities such as the Fibonacci sequence, demonstrating a "countable stickiness" in its structure within large-scale systems. These sequences are fundamental to understand how things grow, shrink, or repeat in the physical world, and reveal regularities in behaviors that are both predictable and complex.
Cezàro's theory generally refers to Cezàro's convergence Theorem in analysis. It gives that for any convergent numerical sequence in the real straight line completed $\overline{\mathbb{R}}$, its Cezàro mean converges to the same limit [6, 8, 10, 11, 12]. The reciprocal result is not generally true except for monotonic numerical sequences or under the two conditions : strong and weak ones presented in Tauberian Theorems [4, 7].
This work presents a more convergence of arithmetic means in the real straight line completed $\overline{\mathbb{R}}$. Using the adherence values and their corresponding ratios for a class of numerical sequences having a countable adherence values set, the arithmetic means of any numerical sequence is transformed into a summation of arithmetic means of all its convergent subsequences multiplied by their ratios in this numerical sequence to the order of the sum of its arithmetic means.
Based on the properties of the strictly increasing applications defined from $\mathbb{N}$ towards $\mathbb{N}$ related

to the numerical subsequences, the limit of the arithmetic means for a large class of numerical sequences always exists in the real straight line completed as a finite or infinite summation of the multiplication between the adherence values and their corresponding ratios. The presence of indeterminate forms to the product or the summation can be solved by introducing the limit in the calculated formulates.
Some illustrative examples about the convergence of numerical sequences related to different cases will be presented.

2 Preliminaries

The set of adherence points of any sequence $(x_n)_n$ is, generally, not equal to the set of limits of its extracted subsequences. This is only true in sequential topological spaces such as countable basis spaces.
In this previous case, there is no difference between adherence points and adherence values and moreover the set of adherence values of any sequence is a countable set.
The space of the real numbers $\mathbb{R}$ is a sequential topological space (like first-countable ones).
Let $(u_n)_n$ be a numerical sequence. Suppose that its adherence values is a countable set.
Consider the real sequence $(v_n)_n$ as its arithmetic means defined by

$$v_n = \frac{\sum_{k=0}^{n} u_k}{n+1}. \tag{1}$$

The convergence of $(v_n)_n$ will be study in the real straight line completed $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. The following definitions and notations will be used throughout this paper.
Let $(u_n)_n$ be a sequence. Let us consider $\{l_i \,|\, i \in I = I_1 \cup I_2\}$ the set of all its adherence values where $\{l_i \,|\, i \in I_1\}$ is the set of all its finite adherence values ($l_i < +\infty$) and $\{l_i \,|\, i \in I_2\}$ is that of all its infinite adherence values ($l_i = +\infty$) or ($l_i = -\infty$) .
We know that every adhence value of a numerical sequence is defined as a limit to the one of their subsequences in a metric space, more generally first-countable.
Note that I is associated to a subset of the space of integer numbers $\mathbb{N}$ because I is a countable set.
For all $i \in I$ and $n \in \mathbb{N}$, let us consider the subsequence $(u_{\varphi_i(n)})_n$ of the sequence (u_n) which is convergent to l_i where $\varphi_i : \mathbb{N} \longmapsto \mathbb{N}$ is defined as a strictly increasing application and one can easily show that $\varphi_i(n) \geq n$ for every integer number n. Then φ_i has a rate of change greater than 1 because for all $m > n$ which n and m are two integer numbers, one has

$$\frac{\varphi_i(m) - \varphi_i(n)}{m-n} = \frac{\sum_{k=n}^{m-1} (\varphi_i(k+1) - \varphi_i(k))}{m-n} \geq \frac{\sum_{k=n}^{m-1} 1}{m-n} = \frac{m-n}{m-n} = 1. \tag{2}$$

For all $i \in I$, let us denote by $N_{i,n}$ the Cardinal of the set

$$\{k \in \mathbb{N} \,|\, k \leq \varphi_i(k) \leq n\} \tag{3}$$

and $\dfrac{N_{i,n}}{n+1}$ is the ratio in this numerical sequence to the order n of the sum of its arithmetic means.
The sets $\{\varphi_i(k) \,|\, k \in \mathbb{N}\}$ depend on $i \in I$ and they can be chosen to verifies

$$\bigcup_{i \in I} \Big\{\varphi_i(k) \,|\, k \in \mathbb{N}\Big\} = \mathbb{N} \quad \text{and} \quad \Big\{\varphi_i(k) \,|\, k \in \mathbb{N}\Big\} \bigcap \Big\{\varphi_j(k) \,|\, k \in \mathbb{N}\Big\} = \emptyset \quad \forall i \neq j, \tag{4}$$

because the set of the adherence values is supposed to be countable, moreover separable and we always have the possibility of separating the subsequences.

3 Convergence theorem for the arithmetic means

The following Lemma will be needed for our convergence of arithmetic means.

Lemma 3.1. *For all $i \in I$, the limit of $\dfrac{N_{i,n}}{n+1}$ when the integer number n tends to $+\infty$, exists and is a finite real value.*

Proof. It is clearly that $0 \leq \dfrac{N_{i,n}}{n+1} \leq 1$ for all $i \in I$ because

$$\{k \in \mathbb{N} \,|\, k \leq \varphi_i(k) \leq n\} \subseteq \{k \in \mathbb{N} \,|\, k \leq n\}.$$

Moreover $\varphi_i : \mathbb{N} \longmapsto \mathbb{N}$ is a strictly increasing application with a rate of change greater than 1 for all $i \in I$ (2), so the variation in its increasing can only be vary between polynomial function (in particular, an affine function with slope greater than 1) or functions with accelerated growth (puissance function) or functions with irregular growth (composed of polynomial and puissance functions) then φ_i can be a polynomial function or puissance function or composed of polynomial and puissance functions or a function that from a certain rank varies in discrete steps solely on an ongoing basis as a one of the three functions already mentioned :

First case : $\varphi_i(k) = \displaystyle\sum_{s=0}^{m} a_i^s k^s$ where $a_i^s \in \mathbb{Z}$ and $a_i^m \in \mathbb{N}^*$ with $a_i^m > \displaystyle\sum_{s=0}^{m-1} |a_i^s|$ if at least it exists a_i^s strictly negative numbers. Then if $m = 1$ one has

$$\varphi_i(k) \leq n \Longleftrightarrow a_i^1 k + a_i^0 \leq n \Longleftrightarrow k \leq \left[\frac{n - a_i^0}{a_i^1}\right],$$

where $[.]$ defined the integral part of real numbers and we deduce that

$$N_{i,n} = \left[\frac{n - a_i^0}{a_i^1}\right] + 1.$$

Thus $\dfrac{n - a_i^0}{(n+1)a_i^1} < \dfrac{N_{i,n}}{n+1} \leq \dfrac{n - a_i^0}{(n+1)a_i^1} + \dfrac{1}{n+1}$ and

$$\lim_{n \longrightarrow +\infty} \frac{N_{i,n}}{n+1} = \frac{1}{a_i^1} = r_i.$$

For $m > 1$ one has $\varphi_i(k) \leq n \Longleftrightarrow \displaystyle\sum_{s=0}^{m} a_i^s k^s \leq n$. On the one hand if all a_i^s are positive numbers, one has

$$a_i^m k^m \leq n \Longleftrightarrow k^m \leq \frac{n}{a_i^m} \Longleftrightarrow k \leq \sqrt[m]{\frac{n}{a_i^m}} \Longleftrightarrow k \leq \left[\sqrt[m]{\frac{n}{a_i^m}}\right]$$

and we deduce that

$$N_{i,n} = \left[\sqrt[m]{\frac{n}{a_i^m}}\right] + 1.$$

Then

$$\sqrt[m]{\frac{n}{a_i^m}}\frac{1}{n+1} < \frac{N_{i,n}}{n+1} \leq \sqrt[m]{\frac{n}{a_i^m}}\frac{1}{n+1} + \frac{1}{n+1} \quad \text{and} \quad \lim_{n \longrightarrow +\infty} \frac{N_{i,n}}{n+1} = 0 = r_i.$$

In the other hand if it exists a_i^s strictly negative numbers, one has

$$\sum_{s=0}^{m} a_i^s k^s = \sum_{a_i^s \geq 0} a_i^s k^s + \sum_{a_i^s < 0} a_i^s k^s \leq n.$$

Since $\left|\sum_{a_i^s<0} a_i^s k^s\right| \leq \sum_{a_i^s<0} |a_i^s| k^m$ then $\sum_{a_i^s<0} a_i^s k^s \geq -\sum_{a_i^s<0} |a_i^s| k^m$ and

$$\sum_{a_i^s \geq 0} a_i^s k^s - \sum_{a_i^s<0} |a_i^s| k^m \leq \sum_{s=0}^{m} a_i^s k^s \leq n \implies \sum_{a_i^s \geq 0, s \neq m} a_i^s k^s + \left[a_i^m - \sum_{a_i^s<0} |a_i^s|\right] k^m \leq n$$

So $k^m \leq \dfrac{n}{a_i^m - \sum_{a_i^s<0} |a_i^s|} \implies k \leq \sqrt[m]{\dfrac{n}{a_i^m - \sum_{a_i^s<0} |a_i^s|}}$ and

$$N_{i,n} = \left[\sqrt[m]{\frac{n}{a_i^m - \sum_{a_i^s<0} |a_i^s|}}\right] + 1.$$

We deduce that

$$\sqrt[m]{\frac{n}{a_i^m - \sum_{a_i^s<0} |a_i^s|}} \frac{1}{n+1} < \frac{N_{i,n}}{n+1} \leq \sqrt[m]{\frac{n}{a_i^m - \sum_{a_i^s<0} |a_i^s|}} \frac{1}{n+1} + \frac{1}{n+1}$$

and

$$\lim_{n \longrightarrow +\infty} \frac{N_{i,n}}{n+1} = 0 = r_i.$$

Second case : $\varphi_i(k) = a^k$ where a is an integer greather than or equal to 2. One has

$a^k \leq n \iff \exp(k \ln a) \leq n \iff k \ln a \leq \ln n \iff k \leq \dfrac{\ln n}{\ln a}$

and $N_{i,n} = \left[\dfrac{\ln n}{\ln a}\right] + 1$. Then

$\dfrac{\ln n}{(n+1)\ln a} < \dfrac{N_{i,n}}{n+1} \leq \dfrac{\ln n}{(n+1)\ln a} + \dfrac{1}{n+1}$ and $\lim_{n \longrightarrow +\infty} \dfrac{N_{i,n}}{n+1} = 0 = r_i.$

Third case : $\varphi_i(k) = a^{\sum_{s=0}^{m} a_i^s k^s}$ where $a_i^s \in \mathbb{Z}$ and $a_i^m \in \mathbb{N}^*$ with $a_i^m > \sum_{s=0}^{m-1} |a_i^s|$ if at least it exists a_i^s strictly negative numbers and a is an integer greather than or equal to 2. We repeat the same reasoning as the first case then one has :

For $m = 1$ $N_{i,n} = \left[\dfrac{\ln n}{\ln a} - a_i^0}{a_i^1}\right] + 1$ and $\lim_{n \longrightarrow +\infty} \dfrac{N_{i,n}}{n+1} = 0 = r_i.$

For $m > 1$ and all a_i^s are positive numbers one has $N_{i,n} = \left[\sqrt[m]{\dfrac{\ln n}{a_i^m \ln a}}\right] + 1$ so $\lim_{n \longrightarrow +\infty} \dfrac{N_{i,n}}{n+1} = 0 = r_i.$

For $m > 1$ and it exists a_i^s strictly negative numbers one has

$$N_{i,n} = \left[\sqrt[m]{\frac{\frac{\ln n}{\ln a}}{a_i^m - \sum\limits_{a_i^s<0} |a_i^s|}} \right] + 1 \text{ and } \lim_{n\longrightarrow+\infty} \frac{N_{i,n}}{n+1} = 0 = r_i. \qquad \square$$

Remark 3.2. In general, the ratio r_i is equal to 0, except in cases where the φ_i is affine or varies in discrete steps solely on an ongoing basis as affine functions from a certain rank onward, in which cases the ratio is equal to the reciprocal of the leading coefficient or the sum of the reciprocals of several leading coefficients multiplied by their ratios depending on the discrete steps corresponding respectively.

Definition 3.3. For any real sequence $(u_n)_n$ and for all $i \in I$, the ratios $(r_i)_{i\in I}$ of all its numerical subsequences are defined by

$$r_i = \lim_{n\longrightarrow+\infty} \frac{N_{i,n}}{n+1}. \tag{5}$$

Corollary 3.4. *For any numerical sequence $(u_n)_n$. The ratios $(r_i)_{i\in I}$ of all its numerical subsequences verify*

$$\sum_{i\in I} r_i = 1. \tag{6}$$

Proof. One has

$$\begin{aligned}
1 &= \lim_{n\longrightarrow+\infty} \frac{Card\{u_k \quad | \quad k \in \mathbb{N} \quad \text{and} \quad k \leq n\}}{n+1} \\
&= \lim_{n\longrightarrow+\infty} \frac{Card\{u_{\varphi_i(k)} \quad | \quad i \in I, \quad k \in \mathbb{N} \quad \text{and} \quad k \leq \varphi_i(k) \leq n\}}{n+1} \\
&= \lim_{n\longrightarrow+\infty} \left(\frac{Card\left(\bigcup\limits_{i\in I} \left\{ u_{\varphi_i(k)} \quad | \quad k \in \mathbb{N} \quad \text{and} \quad k \leq \varphi_i(k) \leq n \right\} \right)}{n+1} \right) \\
&= \lim_{n\longrightarrow+\infty} \left(\frac{Card\left(\bigcup\limits_{i\in I} \{ k \in \mathbb{N} \quad | \quad k \leq \varphi_i(k) \leq n \} \right)}{n+1} \right)
\end{aligned}$$

From (3) and (4) one has

$$\begin{aligned}
1 &= \lim_{n\longrightarrow+\infty} \sum_{i\in I} \left(\frac{Card\{k \in \mathbb{N} \quad | \quad k \leq \varphi_i(k) \leq n\}}{n+1} \right) \\
&= \lim_{n\longrightarrow+\infty} \sum_{i\in I} \left(\frac{N_{i,n}}{n+1} \right),
\end{aligned}$$

and we have the possibility to permute the limit and sum because there is an independence between the integer number n and the indice i. Then

$$1 = \sum_{i\in I} \lim_{n\longrightarrow+\infty}\left(\frac{N_{i,n}}{n+1}\right) = \sum_{i\in I} r_i. \qquad \square$$

Throughout the rest of this article, we will refer to $(r_i)_{i\in I}$ and $(l_i)_{i\in I}$ as the ratios and the limits, respectively, of all numerical subsequences $(u_{\varphi_i(n)})_n$ of the sequence $(u_n)_n$ having a countable adherence values set. Our main convergence theorems of the present paper reads as follows.

Theorem 3.5. The arithmetic means $(v_n)_n$ of any sequence $(u_n)_n$ converge in the real straight line completed $\overline{\mathbb{R}}$ when all $r_i l_i$ as follows as their sum are a well determinate forms, towards

$$\sum_{i\in I_1\cup I_2} r_i l_i. \tag{7}$$

Proof. One has

$$\begin{aligned}\lim_{n\longrightarrow+\infty}\frac{\sum_{k=0}^{n} u_k}{n+1} &= \lim_{n\longrightarrow+\infty}\sum_{i\in I}\left(\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{n+1}\right)\\ &= \lim_{n\longrightarrow+\infty}\sum_{i\in I}\left(\frac{N_{i,n}}{n+1}\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{N_{i,n}}\right),\end{aligned}$$

and we have the possibility of permuting the limit and the sum because there is an independence between the integer number n and the indice i. Then

$$\begin{aligned}\lim_{n\longrightarrow+\infty}\frac{\sum_{k=0}^{n} u_k}{n+1} &= \sum_{i\in I}\lim_{n\longrightarrow+\infty}\left(\frac{N_{i,n}}{n+1}\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{N_{i,n}}\right)\\ &= \sum_{i\in I}\left(\lim_{n\longrightarrow+\infty}\left(\frac{N_{i,n}}{n+1}\right).\lim_{n\longrightarrow+\infty}\left(\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{N_{i,n}}\right)\right)\\ &= \sum_{i\in I} r_i l_i\end{aligned}$$

if this sum is a well determinate form. $\square$

In the following result, we show that the same result can be obtained if indeterminate forms exist for a certain indices of the set I_2, in the product of the adherence values by their ratios $(r_i l_i)_i$ or sum of the product $\sum_j r_j l_j$.

Corollary 3.6. *If there are indeterminate forms to the product $r_i l_i$ as example in the subset I_3 of I_2 or to the summation of the terms $\sum_{i\in I_4} r_i l_i$ as example in the subset I_4 of I_2 and the problem can be solved by introducing the limit of the product or the summation to obtain well-determined forms, then its arithmetic means $(v_n)_n$ converges towards*

$$\sum_{\{i\in I\setminus(I_3\cup I_4)\}} r_i l_i + \sum_{i\in I_3}\lim_{n\longrightarrow+\infty}(r_i l_i) + \lim_{n\longrightarrow+\infty}\left(\sum_{i\in I_4} r_i l_i\right) \tag{8}$$

when all the sums are of a well-defined form.
Once again when the last sum presents an indeterminate form, thus the problem can be solved by introducing the limit over this sum to obtain well-determined form and the arithmetic means $(v_n)_n$ converges towards

$$\lim_{n\longrightarrow+\infty}\left(\sum_{\{i\in I\setminus(I_3\cup I_4)\}} r_i l_i+\sum_{i\in I_3} r_i l_i+\sum_{i\in I_4} r_i l_i\right) \tag{9}$$

Proof. One has

$$\begin{aligned}\lim_{n\longrightarrow+\infty}\frac{\sum_{k=0}^{n} u_k}{n+1} &= \lim_{n\longrightarrow+\infty}\left(\sum_{\{i\in I\setminus(I_3\cup I_4)\}}\left(\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{n+1}\right)+\sum_{i\in I_3}\left(\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{n+1}\right)+\sum_{i\in I_4}\left(\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{n+1}\right)\right)\\ &= \sum_{m=3}^{5}\left(\lim_{n\longrightarrow+\infty}\left(\sum_{i\in I_m}\left(\frac{N_{i,n}}{n+1}\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{N_{i,n}}\right)\right)\right)\end{aligned}$$

where $I_5 = I\setminus(I_3\cup I_4)$ and we have the possibility to permute the limit and the sum because there is an independence between the integer number n and the indices i then

$$\begin{aligned}\lim_{n\longrightarrow+\infty}\frac{\sum_{k=0}^{n} u_k}{n+1} &= \sum_{m=3}^{5}\left(\lim_{n\longrightarrow+\infty}\left(\sum_{i\in I_m}\left(\frac{N_{i,n}}{n+1}\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{N_{i,n}}\right)\right)\right)\\ &= \sum_{m=3}^{5}\lim_{n\longrightarrow+\infty}\left(\sum_{i\in I_m}\lim_{n\longrightarrow+\infty}\left(\frac{N_{i,n}}{n+1}\right).\lim_{n\longrightarrow+\infty}\left(\sum_{k=1}^{N_{i,n}}\frac{u_{\varphi_i(k)}}{N_{i,n}}\right)\right)\\ &= \sum_{i\in I_3}\lim_{n\longrightarrow+\infty}(r_i l_i)+\lim_{n\longrightarrow+\infty}\left(\sum_{i\in I_4} r_i l_i\right)+\sum_{i\in I_5} r_i l_i\\ &= \sum_{\{i\in I\setminus(I_3\cup I_4)\}} r_i l_i+\sum_{i\in I_3}\lim_{n\longrightarrow+\infty}(r_i l_i)+\lim_{n\longrightarrow+\infty}\left(\sum_{i\in I_4} r_i l_i\right)\end{aligned}$$

if the previous three sums are of a well-defined form.
Once again, when these sums are of well-defined form but its sum has an indeterminate form, the problem can be solved by introducing the limit of this sum to obtain a well-determinate form and the arithmetic means $(v_n)_n$ converges towards

$$\lim_{n\longrightarrow+\infty}\left(\sum_{\{i\in I\setminus(I_3\cup I_4)\}} r_i l_i+\sum_{i\in I_3} r_i l_i+\sum_{i\in I_4} r_i l_i\right).$$ □

4 Convergence discussions in $\mathbb{R}$

According to results (7), (8) and (9), we deduce the convergence cases of the arithmetic means of any numerical sequence :
First case $I_2 = \emptyset$: If I_1 and I_2 are a finite sets then the corresponding arithmetic means converges. But if the cardinal of the set I_1 is infinite the corresponding arithmetic means perhaps converges perhaps no because the summation $\sum_{i\in I_1} r_i l_i$ can be a finite real when the sequence of $(l_i)_{i\in I_1}$ is bounded or

monotonic in modulus towards any real number as example.

Two illustrative examples are presented below :

Example 4.1. Let us consider the numerical sequence $u_n = (-1)^n$ for all $n \in \mathbb{N}$. Then its arithmetic means (v_n) tends to $\left((\frac{1}{2}.1) + (\frac{1}{2}.(-1))\right) = 0$.

Here is an example where I_1 is an infinite set.

Example 4.2. For all $p \in \mathbb{N}$ and a numerical sequence $(u_n)_n$ defined by increasing applications from $\mathbb{N}$ to $\mathbb{N}$ and the corresponding real subsequences as follows

$$\begin{cases} \varphi_0(0) = 0 \quad \text{and} \quad \varphi_{p+1}(0) = \varphi_p(0) + p + 2 \\ \varphi_p(k+1) = \varphi_p(k) + p + k + 1 \quad \forall k \in \mathbb{N} \end{cases}$$

and $u_{\varphi_p(k)} = \dfrac{1}{(p+1)^s}$ where s is a non-zero integer number. Then its arithmetic means (v_n) tend to 0 because their adherence values are finite reals and their corresponding ratios are all zero numbers.

Second case $I_2 \neq \emptyset$: If I_1 is a finite set then the corresponding arithmetic means can be converges or not converges because the summation $\sum_{i \in I_1 \cup I_2} r_i l_i$ can be a finite real number when $r_i = 0$ and we can be solved the problem by introducing the limit of the product $r_i l_i$ to obtain well-determined forms for all $i \in I_2$. The same results can be obtained even if the cardinal of the set I_1 is infinite if the summation $\sum_{i \in I_1} r_i l_i$ converges following the conditions presented in the first case.

Two illustrative examples are presented below :

Example 4.3. Consider the numerical sequence $u_n = (-1)^n \sqrt[m]{n}$ for all $n \in \mathbb{N}^*$ and m is a real number strictly greater than 1. Then its arithmetic means (v_n) tend to 0.

Example 4.4. Let us consider the numerical sequence $u_n = \begin{cases} \sqrt{n} & \text{if} \quad n = k^2 \,|\, k \in \mathbb{N} \\ a & elseif \quad \text{Others cases} \end{cases}$

Then there are two subsequences of the sequence (u_n) where φ_0 is defined by $\varphi_0(k) = k^2$ and the other φ_1 is a strictly increasing function defined from $\mathbb{N}$ to $\{n \in \mathbb{N} \quad / \quad n \neq k^2 \quad \forall k \in \mathbb{N}\}$. There are two adherence values $l_0 = +\infty$ and $l_1 = a$ where $r_0 = 0$ and $r_1 = 1$. Thus its arithmetic means v_n tends to $\dfrac{1}{2} + a$ where a is a real number given.

Here is another example where I_2 is a finite set and I_1 is an infinite set.

Example 4.5. For all $p \in \mathbb{N}$ and $(u_n)_n$ a numerical sequence defined by increasing applications from $\mathbb{N}$ to $\mathbb{N}$ and the corresponding real subsequences as follows

$$\begin{cases} \varphi_0(0) = 0 \quad \text{and} \quad \varphi_{p+1}(0) = \varphi_p(0) + p + 2 \\ \varphi_p(k+1) = \varphi_p(k) + p + k + 1 \quad \forall k \in \mathbb{N} \end{cases}$$

and $u_{\varphi_0(k)} = k$ but for $p \neq 0 \quad u_{\varphi_p(k)} = \frac{1}{p^s}$ where s is a non-zero integer number.
Then its arithmetic means (v_n) tend to 1 because their adherence values are finite reals and their corresponding ratios are zero numbers for $p \neq 0$ but for $p = 0$ the limit of the multiplication between the infinite adherence value and the corresponding zero ratio equal to 1.

If I_1 and I_2 are infinite sets with $r_i = 0$ for all $i \in I_2$, the problem can be solved by introducing the limit of the product $r_i l_i$ to obtain well-determined forms, then the sum

$$\sum_{i\in I_1\cup I_2} r_i l_i$$

will be converges :

1- When the series defined by $\sum_{i\in I_2} \lim_{n\longrightarrow+\infty}(r_i l_i)$ converges and the sequence of $(l_i)_{i\in I_1}$ is bounded or monotonic in modulus towards any real number as an example.

2- When the two series $\sum_{i\in I_2} \lim_{n\longrightarrow+\infty}(r_i l_i)$ and $\sum_{i\in I_1} r_i l_i$ diverge and its summation

$$\left(\sum_{i\in I_2} \lim_{n\longrightarrow+\infty}(r_i l_i) + \sum_{i\in I_1} r_i l_i\right)$$

is an indeterminate form but its limit

$$\lim\left(\sum_{i\in I_2} \lim_{n\longrightarrow+\infty}(r_i l_i) + \sum_{i\in I_1} r_i l_i\right)$$

exists and finite.

Example 4.6. For all $p \in \mathbb{N}$ and $(u_n)_n$ a numerical sequence defined by the increasing applications from $\mathbb{N}$ to $\mathbb{N}$ and the corresponding real subsequences as follows

$$\begin{cases} \varphi_0(0) = 0 \quad \text{and} \quad \varphi_{p+1}(0) = \varphi_p(0) + p + 2 \\ \varphi_p(k+1) = \varphi_p(k) + p + k + 1 \quad \forall k \in \mathbb{N} \end{cases}$$

with $u_{\varphi_p(k)} = \sqrt[p]{k+1}$ if p is an odd integer and $u_{\varphi_p(k)} = \dfrac{1}{(p+1)^s}$ for p is an even integer where s is a non-zero integer number. Then its arithmetic means (v_n) tends to 1 as the previous example because their adherence values are finite real numbers and their corresponding ratios are zero numbers for p an even integer and their adherence values are infinite real numbers and their corresponding ratios are zero numbers and, moreover, the limit of the multiplication between the infinite adherence value and the corresponding zero ratio equal to 0 for p an odd integer $\neq 1$ but for $p = 1$ the limit is equal to 1.

Example 4.7. For all $p \in \mathbb{N}$ and $(u_n)_n$ a numerical sequence defined by increasing applications from $\mathbb{N}$ to $\mathbb{N}$ and the corresponding real subsequences as follows

$$\begin{cases} \varphi_0(0) = 0 \quad \text{and} \quad \varphi_{p+1}(0) = \varphi_p(0) + p + 2 \\ \varphi_p(k+1) = \varphi_p(k) + p + k + 1 \quad \forall k \in \mathbb{N} \end{cases}$$

where for $p \neq 0 \quad u_{\varphi_p(k)} = (k+1)^{\frac{1}{p}}$ and $u_{\varphi_0(k)} = c$ for c any real number is given.
Then its arithmetic means (v_n) tend to 1 because their adherence values are infinite real numbers and their corresponding ratios are zero numbers and, moreover, the limit of the multiplication between the infinite adherence value and the corresponding zero ratio equal to 0 for any integer number $p \neq 1$ but for $p = 1$ the limit is equal to 1.

Another example where I_2 and I_1 are infinite sets.

If the cardinal of the set I_2 is infinite with $r_i = 0$ for all $i \in I_2$ and the cardinal of the set I_1 is finite, the problem can be solved by introducing the limit of the product $r_i l_i$ to obtain well-determined forms for all $i \in I_2$ and the summation $\sum_{i \in I_1 \cup I_2} r_i l_i$ will converge when the series defined by $\sum_{i \in I_2} \lim_{n \longrightarrow +\infty} (r_i l_i)$ converges.

Example 4.8. For all $p \in \mathbb{N}$ and $(u_n)_n$ a numerical sequence defined increasing applications from $\mathbb{N}$ to $\mathbb{N}$ and the corresponding real subsequences as follows

$$\begin{cases} \varphi_0(0) = 0 \quad \text{and} \quad \varphi_{p+1}(0) = \varphi_p(0) + p + 2 \\ \varphi_p(k+1) = \varphi_p(k) + p + k + 1 \quad \forall k \in \mathbb{N} \end{cases}$$

with $u_{\varphi_p(k)} = (k+1)^{\frac{1}{p}}$ if p is an odd integer and $u_{\varphi_p(k)} = c_p$ if p is an even integer where (c_p) are any real numbers given.
Then its arithmetic means (v_n) tend to 1 because their adherence values are infinite reals and their corresponding ratios are zero numbers and, moreover, the limit of their multiplications is equal to 0 for any odd integer number p but for $p = 1$ the limit is equal to 1. For any even p the limit of multiplication between the c_p adherence value and the corresponding zero ratio equal to 0.

5 Some results related to Reciprocal Cezàro theorems and probability theory

From all cases in which the Cezàro means converge, the convergence of the numerical sequence is not ensured except if this sequence has one adherence value which is true if one has the following properties :
Monotonicity of numerical sequences or the conditions strong and weak presented by Tauberian theory and Hardy result's [2], [4], [5], [7], [8] and [12].

Our contributions are introduced for developing the Cezàro and Tauber's theorems on the one hand and on the other for establishing connection with probability theory.

5.1 Reciprocal of Cezàro theorems

It is clearly and known that the monotonicity of $(u_n)_n$ proves his convergence in $\mathbb{R}$ and $\overline{\mathbb{R}}$ when its Cezàro mean converges in these spaces.

First case $I_2 = \emptyset$**:** We have shown that the limit to the Cezàro mean of any numerical sequence $(u_n)_n$ is

$$\sum_{i \in I_1} r_i l_i.$$

Then we have

1 - If $\lim_{n \longrightarrow +\infty} (u_{n+1} - u_n) = 0$, then the sequence $(u_n)_n$ cannot have more than one adherence value in the separate space $\mathbb{R}$ and will converge when its mean Cezàro converges.

2 - The following condition $\lim_{n \longrightarrow +\infty} (u_{n+1} - u_n) = 0$ is more strong when compared with the strong condition presented in the Tauber and Hardy's results.

Second case $I_2 \neq \emptyset$: We have shown that the limit to the Cezàro mean of any numerical sequence $(u_n)_n$ is

$$\sum_{i\in I_1\cup I_2} r_i l_i \quad \text{or} \quad \sum_{\{i\in I\setminus(I_3\cup I_4)\}} r_i l_i + \sum_{i\in I_3} \lim_{n\longrightarrow+\infty}(r_i l_i) + \lim_{n\longrightarrow+\infty}\left(\sum_{i\in I_4} r_i l_i\right)$$

$$\text{or} \quad \lim_{n\longrightarrow+\infty}\left(\sum_{\{i\in I\setminus(I_3\cup I_4)\}} r_i l_i + \sum_{i\in I_3} r_i l_i + \sum_{i\in I_4} r_i l_i\right).$$

Then we have

1 - If $I_1 \neq \emptyset$ there are no conditions for that the sequence $(u_n)_n$ converges even if its mean Cezàro converges.

2 - If $I_1 = \emptyset$ and $\lim\limits_{n\longrightarrow+\infty}(u_{n+1} - u_n) = 0$, then the sequence $(u_n)_n$ converges when its Cezàro mean converges. Furthermore, if the set $I = \{+\infty\}$ then the sequence converges to $+\infty$ and its Cezaro mean necessarily converges to $+\infty$. The same result is obtained if $I = \{-\infty\}$.

5.2 Relationships with probability theory

From the famous formulas (6) and (7)

$$\sum_{i\in I} r_i = 1 \qquad \text{and} \qquad \sum_{i\in I} r_i l_i. \tag{10}$$

If $I = I_1$ one can consider the limit of the Cezàro mean for any numerical sequence $(u_n)_n$ when I_1 is a finite set as a random variable X following the following discrete probability law :

$$X \sim P(X) \tag{11}$$

where $P(X = l_i) = p_i = r_i$ for all $i \in I_1$. Then we deduce the following approaches :

For the numerical sequences to time-dependent to different physicochemical parameters, as example, if the limit of their arithmetic means is $\sum\limits_{i\in I_1} r_i l_i$, then it is represented by the expectation of his associated random variable X (8) and all properties of this limit can be obtained from its random variable as the different order moments, the variance, the standard deviations. Furthermore, the covariance, the correlation and the correlation matrix between two or more numerical sequences can be study.

Conclusion

The convergence of arithmetic means for a significant class of numerical sequences and different examples in the completed real straight $\overline{\mathbb{R}}$ are presented. The transformation of the arithmetic means for any numerical sequence in this significant class into a sum of the arithmetic means of all its convergent subsequences multiplied by their corresponding ratios is obtained. The limit of the arithmetic means of a such numerical sequence exists and its value is expressed in terms of the adherence values of this subsequences and their corresponding ratios, sometimes introducing the limit in the presence of indeterminate forms arising from the product or the sum. This main result will open up many avenues of research concerning discussions on the convergence of arithmetic

means in $\mathbb{R}$ and the convergence of certain types of numerical series associated with the corresponding limits based on the strong condition presented in this paper, as well as many questions and suggestions concerning its various applications physicochemical moreover other disciplines and its different relationships with probability theory and mathematical statistics. The study of convergence of the arithmetic means for numerical sequences not having a countable adherence values set will be treated in the future work.

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