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Strong S-valuation power series rings

Mohamed Khalifa[1] and Abdelamir Dabbabi[2]
[1]University of Sousse, Higher institute of transports and logistics, 4002 Sousse, Tunisia.
e-mail: *khalifa.mohamed_cr@yahoo.fr*
[2]Mathematics Department, Faculty of Sciences of Monastir, Monastir 5000 Tunisia.
e-mail: *amir.dababi.25@gmail.com*

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Abstract. Let R be an integral domain and S a multiplicative subset of R. We prove that $R[[X]]$ is a strong S-valuation domain if and only if $R[[X]]$ is an S-valuation domain if and only if the localization ring R_S is a field and R is anti-Archimedean. We study when the $D+M$ ring construction is a strong S-valuation domain (respectively an S-valuation domain). We deduce that if D is an integral domain with quotient field K, then $D+XK[[X]]$ is a strong S-valuation domain (respectively an S-valuation domain) if and only if the D-module K is strong S-uniserial (respectively S-uniserial). We prove that if R is Dedekind domain, then $R((X))$ is a strong S-valuation domain if and only if $R((X))$ is an S-valuation domain if and only if R is a strong S-valuation domain.
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1 Introduction

Throughout this paper, all rings considered are commutative integral domains with identity. Let R be a ring and $S \subseteq R$. Recall that S is said to be a multiplicative subset (or multiplicatively closed subset) of R if $1 \in S$, $0 \notin S$ and whenever $s,t \in S$, $st \in S$. Since Anderson and Dumitrescu ([3]) defined the S-Noetherian rings in order to generalize Noetherian rings, many authors generalized numerous notions in commutative ring theory using the S-concept (see for example [7, 11, 13, 15]). D. K. Kim and J. W. Lim generalized valuation domains as follows, an integral domain R is called an S-valuation domain if for each nonzero $a,b \in R$, there exists $s \in S$ such that a divides sb or b divides sa [14]. N. Guennach and N. Mahdou generalized S-valuation domains to rings with zero divisors as follows, a ring R is said to be an S-valuation ring if for all $a,b \in R$, there exists $s \in S$ such that $sa \in bR$ or $sb \in aR$ [12]. A. Benhissi and A. Dabbabi called an integral domain R to be an S-valuation domain if for every two ideals I,J of R, there exists $s \in S$ such that $sI \subseteq J$ or $sJ \subseteq I$ [9]. To avoid the confusion between the definition of D. K. Kim and J. W. Lim and the definition of A. Benhissi and A. Dabbabi, we follow the following definitions.

Definition 1.1. Let I,J be two ideals of an integral domain R and S a multiplicative subset of R.

1. We say that I and J are S-comparable if there exists $s \in S$ such that $sI \subseteq J$ or $sJ \subseteq I$.
2. We say that R is an S-valuation domain if every two principal ideals of R are S-comparable.
3. We say that R is a strong S-valuation domain if every two ideals of R are S-comparable.

Assume that R is an integral domain with quotient field K and S a multiplicative subset of R. It is well known that the power series ring $R[[X]]$ is a valuation domain if and only if $R = K$ is a field. Also the ring $R + XK[[X]]$ is a valuation domain if and only if R is a valuation domain. The goal of this paper is to study when each of the rings $R[[X]]$, $R + XK[[X]]$ and $R((X))$ is a strong S-valuation domain (respectively an S-valuation domain).
In the second section, we show that the polynomial ring $R[X]$ is never an S-valuation domain (Proposition 2.2). We prove that $R[[X]]$ is a strong S-valuation domain if and only if $R[[X]]$ is an S-valuation domain if and only if the localization ring R_S is a field and R is anti-Archimedean (Theorem 2.3). We deduce that $R[[X]]$ is a strong R^*-valuation domain (where $R^* = R\backslash 0$) if and only if R is anti-Archimedean (Corollary 2.4).
In the third section, we study when the $D + M$ ring construction is a strong S-valuation domain. Let M be an R-module. We say that M is strong S-uniserial if for every two submodules N, N' of M, there exists $s \in S$ such that $sN \subseteq N'$ or $sN' \subseteq N$. Let T be a quasi-local domain of the form $T = K + M$ (where K is a field and M is the maximal ideal of T), D a subring of K and $R = D + M$. We prove that R is a strong S-valuation domain if and only if T is a valuation domain and the D-module K is strong S-uniserial (Theorem 3.1). We deduce that if D is a subring of a field K, then $D + XK[[X]]$ is a strong S-valuation domain if and only if the D-module K is strong S-uniserial (Corollary 3.2). Let A be a subring of an integral domain B, S a multiplicative subset of A and $\mathcal{R} = A + XB[[X]]$. We prove that the ring $\mathcal{R}$ is a strong S-valuation domain if and only if the $\mathcal{R}$-module $B[[X]]$ is strong S-uniserial. Moreover in this case, we prove that B is anti-Archimedean, the localization ring B_S is a field and the A-module B is strong S-uniserial (Proposition 3.5).
In the fourth section, we study when the $D + M$ ring construction is an S-valuation domain. Let M be an R-module. We prove that R is an S-valuation domain if and only if T is a valuation domain and the D-module K is S-uniserial (Theorem 4.1). We deduce that if D is a subring of a field K, then $D + XK[[X]]$ is an S-valuation domain if and only if the D-module K is S-uniserial (Corollary 4.2). We prove that if $\mathcal{R}$ is an S-valuation domain, then B is anti-Archimedean, the localization ring B_S is a field and the A-module B is S-uniserial (Proposition 4.6). We deduce that if A_S is a field and B is an overring of A, then $\mathcal{R}$ is an S-valuation domain if and only if B is anti-Archimedean and the localization ring B_S is a field (Corollary 4.7).
In the fifth section, we study when each of the rings $R(X) = \{f/g \mid f, g \in R[X] \text{ and } C(f) = R\}$ and $R((X)) = \{f/g \mid f, g \in R[[X]] \text{ and } C(f) = R\}$ (where $C(f)$ is the ideal of R generated by all coefficient of f) is a strong S-valuation domain. First, we prove that if A is a subring of an integral domain B and S a multiplicative subset of A such that each ideal of B is extended from A and $IB \cap A = I$ for each ideal I of A, then B is a strong S-valuation domain if and only if A is a strong S-valuation domain (Theorem 5.1). We deduce that if R is a Dedekind domain, then $R((X))$ is a strong S-valuation domain if and only if R is a strong S-valuation domain if and only if $R((X))$ is an S-valuation domain (Corollary 5.2). Also, we deduce that if R is a Prüfer domain, then $R(X)$ is a strong S-valuation domain if and only if R is a strong S-valuation domain (Corollary 5.3). Finally, we deduce that if R is a Dedekind domain, then $R(X)$ is an S-valuation domain if and only if R is a strong S-valuation domain (Corollary 5.4).

2 Strong S-valuation power series rings

Throughout this paper, let R be an integral domain and S a multiplicative subset of R. Clearly, we have the following.

valuation domain $\Longrightarrow$ strong S-valuation domain $\Longrightarrow$ S-valuation domain.

None of these implications is reversible.

Example 2.1. We denote $R^* = R\backslash\{0\}$.
1) Note that R is a strong R^*-valuation domain. Let $\mathbb{Z}$ be the ring of integers. So $\mathbb{Z}$ is a strong $\mathbb{Z}^*$-valuation domain but it is not a valuation domain.
2) [9, Example 2.3-(4)]. Let $\mathbb{Q}$ be the field of rational numbers. A. Benhissi and A. Dabbabi showed that the ring $\mathbb{Z} + X\mathbb{Q}[[X]]$ is a $\mathbb{Z}^*$-valuation domain but it is not a strong $\mathbb{Z}^*$-valuation domain.

Note that S will be a multiplicative subset of R even when we consider the rings $R[X]$ or $R[[X]]$. It is well known that a polynomial ring is never a valuation domain.

Proposition 2.2. *The polynomial ring $R[X]$ is never an S-valuation domain (so is not a strong S-valuation domain).*

Proof. If not, then there exists $s \in S$ such that $sX \in (1-X)$ or $s(1-X) \in (X)$. In the second case, $s = 0$, a contradiction. Let $h \in R[X]$ such that $sX = (1-X)h$. Since $h^{(0)} = 0$, $h = Xg$ for some $g \in R[X]$ and so $s = (1-X)g = g - Xg$. If $g \neq 0$ and c is the leading coefficient of g, then $c = 0$, a contradiction. It follows that $g = 0$ and so $s = 0$, a contradiction. □

It is well known that a power series ring $R[[X]]$ is a valuation domain if and only if R is a field. Recall that an integral domain R is called anti-Archimedean if $\cap_{n\geq 1} r^n R \neq 0$ for each $0 \neq r \in R$ [5]. It was shown that R is anti-Archimedean if and only if the localization ring $R[[X]]_{R^*}$ is quasi-local (of course with maximal ideal $(X)_{R^*}$) [5, Lemma 3.1]. For each power series $f \in R[[X]]$ and each integer $i \geq 0$, $f^{(i)}$ denotes the coefficient of X^i in f.

Theorem 2.3. The following are equivalent.

1. The power series ring $R[[X]]$ is a strong S-valuation domain.
2. The power series ring $R[[X]]$ is an S-valuation domain.
3. The localization ring R_S is a field and R is anti-Archimedean.

Proof. 1) $\Rightarrow$ 2) Trivial.
2) $\Rightarrow$ 3) Let $r \in R$ a nonzero element. There exists $s \in S$ such that $sX \in (r-X)$ or $s(r-X) \in (X)$. In the second case $sr = 0$, a contradiction. Let $h \in R[[X]]$ such that $sX = (r-X)h$. Since $h^{(0)} = 0$, $h = Xg$ for some $g \in R[[X]]$ and so $s = (r-X)g = rg - Xg$. Thus $s = rg^{(0)} \in rR$ and so $\frac{1}{r} \in R_S$. Then R_S is a field (which is of course the quotient field of R). For each positive integer i, $rg^{(i+1)} = g^{(i)}$. Therefore, $s = r^2 g^{(1)} = r^3 g^{(2)} = ... = r^n g^{(n-1)} \in r^n R$ for all positive integer n. So $\cap_{n\geq 1} r^n R \neq 0$. Hence R is anti-Archimedean.
3) $\Rightarrow$ 1) Let $\mathcal{I}, \mathcal{J}$ be two nonzero ideals of $R[[X]]$. If there exists a nonzero $r \in \mathcal{I} \cap R$, then $r\mathcal{J} \subseteq \mathcal{I}$. Since R_S is a field, there exists $s \in S \cap rR$. Thus $s\mathcal{J} \subseteq \mathcal{I}$. So we can assume that $\mathcal{I} \cap R = 0$ and $\mathcal{J} \cap R = 0$. Since R is an anti-Archimedean domain, the localization ring $R[[X]]_{R^*}$ is quasi-local with maximal ideal $(X)_{R^*}$ and so $\mathcal{I} \subseteq (X)_{R^*}$. Thus $\mathcal{I} \subseteq (X)$. Let n be the maximal positive integer such that $\mathcal{I} \subseteq (X^n)$ and m the maximal positive integer such that $\mathcal{J} \subseteq (X^m)$. We claim that $\mathcal{I} = X^n\mathcal{I}'$ for some ideal $\mathcal{I}'$ of $R[[X]]$ such that $\mathcal{I}' \cap A \neq 0$. Let $\mathcal{I}' = \{f \in R[[X]] \text{ such that } X^n f \in \mathcal{I}\}$. It is clear that $\mathcal{I}'$ is an ideal of $R[[X]]$ and $X^n\mathcal{I}' = \mathcal{I}$. If $\mathcal{I}' \cap R = 0$, then $\mathcal{I}' \subseteq (X)$ and so $\mathcal{I} \subseteq (X^{n+1})$, which conradicts the maximality of n. Thus $\mathcal{I}' \cap R \neq 0$. Similarly, $\mathcal{J} = X^m\mathcal{J}'$ for some ideal $\mathcal{J}'$ of $R[[X]]$ such that $\mathcal{J}' \cap R \neq 0$. Assume that $n \geq m$. Let $0 \neq r \in R$ such that $r \in \mathcal{J}'$. Since R_S is a field, there exists $s \in S \cap rR$. Then $s\mathcal{I}' \subseteq \mathcal{J}'$ and hence $s\mathcal{I} = sX^n\mathcal{I}' \subseteq X^n\mathcal{J}' \subseteq X^{n-m}\mathcal{J} \subseteq \mathcal{J}$. Similarly, if $n < m$, then there exists $s \in S$ such that $s\mathcal{J} \subseteq \mathcal{I}$. □

Corollary 2.4. *The following are equivalent.*

1. *The power series ring $R[[X]]$ is a strong R^*-valuation domain.*
2. *The power series ring $R[[X]]$ is an R^*-valuation domain.*
3. *R is anti-Archimedean.*

Proof. Because R_{R^*} is a field (the quotient field of R) and by Theorem 2.3. □

Another example of a strong S-valuation domain R that is not a valuation domain and $S \neq R^*$.

Example 2.5. R. Gilmer showed that there exists an integral domain D with quotient field K such $D \neq K$ and $K((X))$ is the quotient field of $D[[X]]$ [10, Example]. By [10, Theorem 1-(3)], $\cap_{i\geq 1} d_i D \neq 0$ for all $d_1, d_2, ... \in D^*$. In particular, D is an anti-Archimedean domain. By Corollary 2.4, $D[[X]]$ is a strong D^*-valuation domain. Since D is not a field, $D[[X]]$ is not a valuation domain.

3 Strong S-valuation domain of the form D+M

Recall that an R-module E is called uniserial if whenever N, N' submodules of E, $N \subseteq N'$ or $N' \subseteq N$. N. Guennach and N. Mahdou [12] generalized uniserial module as follows: an R-module E is called S-uniserial if for all $x, y \in E$, there exists $s \in S$ such that $sx \in Ry$ or $sy \in Rx$. The goal of this section is to study when the ring construction $D + M$ is a strong S-valuation domain. We need first the following definition.

Definition 3.1. We say that an R-module E is strong S-uniserial if whenever N, N' submodules of E, there exists $s \in S$ such that $sN \subseteq N'$ or $sN' \subseteq N$.

Theorem 3.2. Let T be a quasi-local domain of the form $T = K + M$ where M is the nonzero maximal ideal of T and K is a field. Let D be a subring of K, S a multiplicative subset of D and $R = D + M$. The following statements are equivalent:

1. R is a strong S-valuation domain.
2. T is a valuation domain and the D-module K is strong S-uniserial.

Proof. 1) ⇒ 2) If I, J are two proper ideals of T, then $I, J \subseteq M \subset R \subseteq T$ and so I, J are ideals of R. Thus there exists $s \in S$ such that $sI \subseteq J$ or $sJ \subseteq I$. Since $S \subseteq D^* \subseteq K^*$ which elements are invertible in T, $I \subseteq J$ or $J \subseteq I$. Then T is a valuation domain. Let H, L be two nonzero D-submodules of K and $0 \neq r \in M$. Note that $Hr+Mr$ (also $Lr+Mr$) is an ideal of R and so there exists $s \in S$ such that $s(Hr+Mr) \subseteq Lr+Mr$ or $s(Lr + Mr) \subseteq Hr + Mr$. Thus $s(H + M) \subseteq L + M$ or $s(L + M) \subseteq H + M$. Since $K \cap M = 0$, $sH \subseteq L$ or $sL \subseteq H$.
2) ⇒ 1) Let I, J be two nonzero proper ideals of R. Note that M is a divided prime ideal of R (by [2, Lemma 2.1-(c)]) and so each ideal of R is comparable (under inclusion) to M. If $I \subseteq M \subseteq J$ or $J \subseteq M \subseteq I$, then choose $s = 1$. Assume that $M \subset I$ and $M \subset J$. By [6, Theorem 2.1-(d)], there exist two ideals I', J' of D such that $I = I'+M$ and $J = J'+M$. There exists $s \in S$ such that $sI' \subseteq J'$ or $sJ' \subseteq I'$. Then $sI \subseteq J$ or $sJ \subseteq I$. We can assume that $I \subset M$ and $J \subset M$. By [6, Theorem 2.1-(n)], if $C \subset M$ is an ideal of R, then either C is an ideal of T or CT is a principal ideal of T. Moreover, if C is not an ideal of T and $CT = aT$ for some $a \in C$, then $C = Wa + Ma$ where $W = \{u \in K | ua \in C\}$ which is a D-submodule of K.

Case 1: if I, J are both ideals of T, then I and J are comparable and so we can choose $s = 1$.

Case 2: if I is an ideal of T and J is not an ideal of T. Thus $J = Wa + Ma$ for some $a \in J$ such that $JT = aT$ and $W = \{u \in K | ua \in J\}$. If $JT \subseteq I$, then $J \subseteq I$ and so we can choose $s = 1$. If not, then $I \subset JT = aT$ and so $a \notin I$. For every $x \in I$, there exists $t \in T$ such that $x = at$. Since $a \notin I$ and $a = t^{-1}x$, $t^{-1} \notin T$ and so $t \in M$. Therefore, $x \in aM$. Thus $I \subseteq aM \subseteq J$ and so we can choose $s = 1$.

Case 3: if J is an ideal of T and I is not an ideal of T. Similar argument as given in the case 2.

Case 4: if I, J are not ideals of T and so $I = Wa + Ma$, $J = \Omega b + Mb$ where $a \in I$, $b \in J$ such that $IT = aV$, $JT = bT$, $W = \{u \in K | ua \in I\}$ and $\Omega = \{u \in K | ub \in J\}$. For example, $a \in bT$ and so let $c \in T$ such that $a = bc$.
Case 4.1: Assume that $c \in M$. Therefore, $Wc + Mc \subseteq c(K + M) = cT \subseteq M$ and so $I = Wbc + Mbc \subseteq Mb \subseteq J$. we can choose $s = 1$.
Case 4.2: Assume that $c \notin M$ (so c is invertible in T and $Mc = M$). Let so $u \in K^*$ and $r \in M$ such that $c = u + r$. There exists $s \in S$ such that $sWu \subseteq \Omega$ or $s\Omega \subseteq Wu$.

If $sWu \subseteq \Omega$, then:

$$sI = s(Wbc + Mbc) = b(sWc + sMc) \subseteq b(sWu + M) \subseteq b(\Omega + M) = J$$

If $s\Omega \subseteq Wu$, then $s\Omega \subseteq W(c - r) \subseteq Wc + M$. Thus:

$$sJ = s(\Omega b + Mb) \subseteq s\Omega b + sMb \subseteq (Wc + M)b + Mb \subseteq Wbc + Mb = Wbc + Mbc = I$$

□

Immediately, we deduce that

Corollary 3.3. *Let D be a subring of a field K and S a multiplicative subset of D. The ring $D + XK[[X]]$ is a strong S-valuation domain if and only if the D-module K is strong S-uniserial.*

Example 3.4. For each $n \in \mathbb{Z}^*$, $\mathbb{Z}[\frac{1}{n}]$ is the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\frac{1}{n}$. Let $p \neq q$ two prime numbers. Assume that there exists $s \in \mathbb{Z}^*$ such that $s\mathbb{Z}[\frac{1}{p}] \subseteq \mathbb{Z}[\frac{1}{q}]$. For each positive integer n, $\frac{s}{p^n} = \frac{k}{q^m}$ for some $k \in \mathbb{Z}$ and some integer $m \geq 0$. Thus $sq^m \in p^n\mathbb{Z}$ and so $s \in p^n\mathbb{Z}$. Therefore, $s \in \cap_{n \geq 1} p^n\mathbb{Z} = 0$, a contradiction. Then the $\mathbb{Z}$-module $\mathbb{Q}$ is not strong S-uniserial. Hence the ring $\mathbb{Z} + X\mathbb{Q}[[X]]$ is not a strong S-valuation domain for any multiplicative subset S of $\mathbb{Z}$.

Remark 3.5. Let D be an integral domain with quotient field K and S a multiplicative subset of D.

1. If K is a strong S-uniserial D-module, then the localization ring D_S is a valuation domain.

   Indeed, if $0 \neq x \in K$, then there exists $s \in S$ such that $sD \subseteq Dx$ or $sDx \subseteq D$. Thus $sx^{-1} \in D$ or $sx \in D$. Therefore, $x^{-1} \in D_S$ or $x \in D_S$.

2. The converse does not hold. Indeed, let p be a prime number and $S = \mathbb{Z} \setminus p\mathbb{Z}$ which is a multiplicative subset of $\mathbb{Z}$ because $p\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. It is well known that $\mathbb{Z}_S$ is a valuation domain. But the $\mathbb{Z}$-module $\mathbb{Q}$ is not strong S-uniserial by Example 3.4.

Example 3.6. Assume that there exists $s \in \mathbb{Z}^*$ such that $s\frac{1}{\pi} \in \mathbb{Z}$. Thus $s \in \pi\mathbb{Z}$ and so $s = 0$, a contradiction, because it is well known that $\pi \notin \mathbb{Q}$ is not a rational number. Assume that $s \in \mathbb{Z}\frac{1}{\pi}$. Thus $s\pi \in \mathbb{Z}$, a contradiction. Then the $\mathbb{Z}$-module $\mathbb{R}$ is not S-uniserial (so is not strong S-uniserial). Hence the ring $\mathbb{Z}+X\mathbb{R}[[X]]$ is not a strong S-valuation domain for any multiplicative subset S of $\mathbb{Z}$.

Proposition 3.7. *Let A be a subring of an integral domain B, S a multiplicative subset of A and $\mathcal{R} = A + XB[[X]]$. Consider the following statements.*

1. *$\mathcal{R}$ is a strong S-valuation domain.*
2. *The $\mathcal{R}$-module $B[[X]]$ is strong S-uniserial*
3. *The following hold*
   - *The localization ring B_S is a field.*
   - *B is anti-Archimedean.*
   - *The A-module B is strong S-uniserial.*

Then (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3).

Proof. 1) $\Rightarrow$ 2) Note that if Γ is an $\mathcal{R}$-submodule of $B[[X]]$, then $X\Gamma \subseteq \mathcal{R}$ and so is an ideal of $\mathcal{R}$. Let Γ, Δ be two $\mathcal{R}$-submodules of $B[[X]]$. There exists $s \in S$ such that $sX\Gamma \subseteq X\Delta$ or $sX\Delta \subseteq X\Gamma$. Then $s\Gamma \subseteq \Delta$ or $s\Delta \subseteq \Gamma$.
2) $\Rightarrow$ 1) Trivial since each ideal of $\mathcal{R}$ is an $\mathcal{R}$-submodule of $B[[X]]$.
1) $\Rightarrow$ 3) Note that if $\mathcal{I}$ is an ideal of $B[[X]]$, then $X\mathcal{I}$ is an ideal of $\mathcal{R}$. Let $\mathcal{I}, \mathcal{J}$ be two ideals of $B[[X]]$. There exists $s \in S$ such that $sX\mathcal{I} \subseteq X\mathcal{J}$ or $sX\mathcal{J} \subseteq X\mathcal{I}$. Thus $s\mathcal{I} \subseteq \mathcal{J}$ or $s\mathcal{J} \subseteq \mathcal{I}$. Then $B[[X]]$ is an S-valuation domain. By Theorem 2.3, the localization ring B_S is a field and B is anti-Archimedean. Note that if W is an A-submodule of B, then $W + XB[[X]]$ is an $\mathcal{R}$-submodule of $B[[X]]$. Let W, Ω be two A-submodules of B. There exists $s \in S$ such that $s(W + XB[[X]]) \subseteq \Omega + XB[[X]]$ or $s(\Omega + XB[[X]]) \subseteq W + XB[[X]]$. Thus $sW \subseteq \Omega$ or $s\Omega \subseteq W$. □

4 S-valuation domain of the form D+M

Theorem 4.1. Let T be a quasi-local domain of the form $T = K + M$ where M is the nonzero maximal ideal of T and K is a field. Let D be a subring of K, S a multiplicative subset of D and $R = D + M$. The following statements are equivalent:

1. R is an S-valuation domain.
2. T is a valuation domain and the D-module K is S-uniserial.

Proof. 1) $\Rightarrow$ 2) With a similar argument as given in the begining of the proof of Theorem 3.2, we show easily that T is a valuation domain. Let $x, y \in K$. We can assume that $x, y \neq 0$. Note that R and T have the same quotient field because they have a nonzero common ideal M. Since $xy^{-1} \in T$, there exists $s \in S$ such that $sxy^{-1} \in R$ or $sx^{-1}y \in R$. Since $K \cap R = D$, $sxy^{-1} \in D$ or $sx^{-1}y \in D$. Therefore, $sx \in Dy$ or $sy \in Dx$.
2) $\Rightarrow$ 1) Let $r_1, r_2 \in R$, $d_1, d_2 \in D$ and $c_1, c_2 \in M$ such that each $r_i = d_i + c_i$. There exists $s \in S$ such that $sd_1 \in Dd_2$ or $sd_2 \in Dd_1$.
Case 1: Assume that each $d_i \neq 0$. Thus $r_2^{-1} = d_2^{-1} + c_3$ for some $c_3 \in M$. So $r_1 r_2^{-1} = d_1 d_2^{-1} + c_4$ for some $c_4 \in M$. Similarly, $r_2 r_1^{-1} = d_2 d_1^{-1} + c_5$ for some $c_5 \in M$. Since $sd_1 d_2^{-1} \in D$ or $sd_2 d_1^{-1} \in D$, $sr_1 r_2^{-1} \in R$ or

$sr_1r_2^{-1} \in R$. Therefore, $sr_1 \in Rr_2$ or $sr_2 \in Rr_1$.
Case 2: Assume that each $d_1 = d_2 = 0$. Since T is a valuation domain, there exists $t \in T$ such that $c_1 = tc_2$ or $c_2 = tc_1$. We can assume that $c_1 = tc_2$. Let $u \in K$ and $c_6 \in M$ such that $t = u + c_6$. There exists $s \in S$ such that $su \in D$ or $s \in Du$.
Case 2.1: $su \in D$. Thus $st \in R$ and so $sc_1 = stc_2 \in Rc_2$.
Case 2.2: $s \in Du$. Thus $u \neq 0$ and so t is a unit of T. Since $t^{-1} = u^{-1} + c_7$ for some $c_7 \in M$, $sc_2 = st^{-1}c_1 \in Rc_1$ because $st^{-1} \in R$.
Case 3: Assume that $d_1 \neq 0$ but $d_2 = 0$. Since M is a divided prime ideal of R and $r_1 \notin M$, $M \subset r_1R$ and so $r_2 = c_2 \in Rr_1$. □

It is easy to see that the D-module K is S-uniserial if and only if D_S is a valuation domain with quotient field K. Also Kim and Lim showed that an integral domain D is an S-valuation domain if and only if the localization ring D_S is a valuation domain (see [14, Theorem 2.8]).

Corollary 4.2. *Let T be a quasi-local domain of the form $T = K + M$ where M is the nonzero maximal ideal of T and K is a field. Let D be a subring of K, S a multiplicative subset of D and $R = D + M$. The following statements are equivalent:*

1. *R_S is a valuation domain.*
2. *T is a valuation domain and D_S is a valuation domain with quotient field K.*

Immediately, we deduce that

Corollary 4.3. *Let D be a subring of a field K and S a multiplicative subset of D. The ring $D + XK[[X]]$ is an S-valuation domain if and only if D_S is a valuation domain with quotient field K.*

Remark 4.4. If $K = D_{D^*}$ is the quotient field of D, then the D-module K is D^*-uniserial. Hence $D + XK[[X]]$ is a D^*-valuation domain for every integral domain D with quotient field K by Corollary 4.3.

Example 4.5. Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ the field of rational numbers and $\mathbb{R}$ the field of real numbers. By Corollary 4.3, the ring $\mathbb{Z}+X\mathbb{Q}[[X]]$ is a $\mathbb{Z}^*$-valuation domain but the ring $\mathbb{Z}+X\mathbb{R}[[X]]$ is not a $\mathbb{Z}^*$-valuation domain.

Remark 4.6. Let A be a subring of an integral domain B, S a multiplicative subset of A and $\mathcal{R} = A + XB[[X]]$. The ring $\mathcal{R}$ is an S-valuation domain if and only if the $\mathcal{R}$-module $B[[X]]$ is S-uniserial.
Indeed: ($\Rightarrow$) Let $f, g \in B[[X]]$. There exists $s \in S$ such that $sXf \in Xg\mathcal{R}$ or $sXg \in Xf\mathcal{R}$. Thus $sf \in g\mathcal{R}$ or $sg \in f\mathcal{R}$.
($\Leftarrow$) Let $f, g \in \mathcal{R}$. Since $f, g \in B[[X]]$, there exists $s \in S$ such that $sf \in g\mathcal{R}$ or $sg \in f\mathcal{R}$.

Corollary 4.7. *Let A be a subring of an integral domain B, S a multiplicative subset of A such that A_S is a field and B is an overring of A. Denote $\mathcal{R} = A + XB[[X]]$. The following are equivalent.*

1. *The ring $\mathcal{R}$ is an S-valuation domain.*
2. *The localization ring B_S is a field and B is anti-Archimedean.*

Moreover, when these assertions hold, the A-module B is S-uniserial.

Proof. 1) $\Rightarrow$ 2) Let $f, g \in B[[X]]$. There exists $s \in S$ such that $sXf \in Xg\mathcal{R}$ or $sXg \in Xf\mathcal{R}$. Thus $sf \in g\mathcal{R}$ or $sg \in f\mathcal{R}$. So $sf \in gB[[X]]$ or $sg \in fB[[X]]$. Then $B[[X]]$ is an S-valuation domain. By Theorem 2.3, B_S is a field and B is anti-Archimedean.
2) $\Rightarrow$ 1) Let $f, g \in \mathcal{R}$. By Theorem 2.3, $B[[X]]$ is an S-valuation domain. There exists $s \in S$ such that $sf \in gB[[X]]$ or $sg \in fB[[X]]$. Assume that $sf = gh$ for some $h \in B[[X]]$. Since B is contained in the quotient field of A, $ah^{(0)} \in A$ for some $0 \neq a \in A$. Since A_S is a field, there exists $t \in S$ such that $t \in aA$. Therefore $st \in S$ and $stf = ga'ah$ where $a' \in A$, $t = a'a$ and $a'ah \in \mathcal{R}$. Then $stf \in g\mathcal{R}$.
To show that the A-module B is uniserial, let $\alpha, \beta \in B$. There exists $s \in S$ such that $sX\alpha \in X\beta\mathcal{R}$ or $sX\beta \in X\alpha\mathcal{R}$. Then $s\alpha \in A\beta$ or $s\beta \in A\alpha$. □

Remark 4.8. Let A be a subring of an integral domain B and S a multiplicative subset of A. The polynomial ring $A + XB[X]$ is never an S-valuation domain (so never a strong S-valuation domain). Indeed: If not, then $B[X]$ is an S-valuation domain which contradicts Proposition 2.2.

5 Strong S-valuation domain of the form R(X) and R((X))

Let R be an integral domain. For each $f \in R[[X]]$, let $C(f)$ be the ideal of R generated by the coefficient of f. It is well known that $N = \{f \in R[X] \text{ such that } C(f) = R\}$ is a multiplicative subset of R and the ring $R[X]_N$ of fractions of $R[X]$ by N is denoted $R(X)$. The power series analogs of the ring $R(X)$ is the ring $R((X)) = R[[X]]_{N'}$ where $N' = \{f \in R[[X]] \text{ such that } C(f) = R\}$.

Let A be a subring of a ring B. Recall that an ideal J of B is said to be extended from A if $J = IB$ for some ideal I of A (see [8, page 588]).

Theorem 5.1. Let A be a subring of an integral domain B and S is a multiplicative subset of A. If the following

- Each ideal of B is extended from A,
- $IB \cap A = I$ for each ideal I of A

holds, then the following are equivalent.

1. A is a strong S-valuation domain.
2. B is a strong S-valuation domain.

Proof. 1) $\Rightarrow$ 2) Let J_1, J_2 be two ideals of B. There exist two ideals I_1, I_2 of A such that each $J_i = I_iB$. There exists $s \in S$ such that $sI_1 \subseteq I_2$ or $sI_2 \subseteq I_1$. Then $sJ_1 \subseteq J_2$ or $sJ_2 \subseteq J_1$.
2) $\Rightarrow$ 1) Let I_1, I_2 be two ideals of A. There exists $s \in S$ such that $sI_1B \subseteq I_2B$ or $sI_2B \subseteq I_1B$. If $sI_1B \subseteq I_2B$, then $sI_1 = (sI_1)B \cap A = (s(I_1B)) \cap A \subseteq (I_2B) \cap A = I_2$. Similarly, if $sI_2B \subseteq I_1B$, then $sI_2 \subseteq I_1$. □

Corollary 5.2. *Let R be a Prüfer domain and S a multiplicative subset of R. The following are equivalent.*

1. *$R(X)$ is a strong S-valuation domain.*
2. *R is a strong S-valuation domain.*

Proof. By [4, Theorem 2.3], every ideal of $R(X)$ is extended from R. By [1, Proposition 1-(2)], $IR(X) \cap R = I$ for each ideal of R. By Theorem 5.1, 1) $\Leftrightarrow$ 2). □

If R is a Dedekind domain, then the ring extension $R \subset R((X))$ satisfies the two conditions in Theorem 5.1.

Corollary 5.3. *Let R be a Dedekind domain and S a multiplicative subset of R. The following are equivalent.*

1. *$R((X))$ is a strong S-valuation domain.*
2. *$R((X))$ is an S-valuation domain.*
3. *$R(X)$ is a strong S-valuation domain.*
4. *$R(X)$ is an S-valuation domain.*
5. *R is a strong S-valuation domain.*

Proof. By [4, Theorem 4.10], every ideal of $R((X))$ is extended from R. Let I be an ideal of R and $r \in IR((X)) \cap R$. There exists $h \in R[[X]]$ such that $C(h) = R$ and $rh \in I[[X]]$. Thus $rR = rC(h) = C(rh) \subseteq I$ and so $r \in I$. Then $IR((X)) \cap R = I$. By Theorem 5.1, 1) $\Leftrightarrow$ 5).
1) $\Rightarrow$ 2) and 3) $\Rightarrow$ 4) A strong S-valuation domain is an S-valuation domain.
2) $\Rightarrow$ 5) Let I, J be two nonzero ideals of R. Since R is Noetherian, there exist $f, g \in R[X]$ such that $I = C(f)$ and $J = C(g)$. There exists $s \in S$ such that $sf \in gR((X))$ or $sg \in fR((X))$. By [1, Theorem 7], $C(f)R(X) = fR(X)$. If $r \in C(f)$, then $rh \in fR[X]$ for some $h \in R[X]$ such that $C(h) = R$. Thus $rh \in fR[[X]]$ and so $r \in fR((X))$.

$$\text{Thus} \quad C(f)R((X)) = fR((X))$$

Similarly, $C(g)R((X)) = gR((X))$. Then $sIR((X)) \subseteq JR((X))$ or $sJR((X)) \subseteq IR((X))$. So $sI \subseteq J$ or $sJ \subseteq I$.
3) $\Leftrightarrow$ 5) By Corollary 5.2 and since every Dedekind domain is a Prüfer domain.
4) $\Rightarrow$ 5) Similar argument as given in the proof of " 2) $\Rightarrow$ 5) ". □

Example 5.4. Let R be a PID (i.e., principal ideal domain) and r an irreducible element of R. By [9, Example 2.9], R is a strong $(R \backslash rR)$-valuation domain. Hence each of the rings $R(X)$ and $R((X))$ is a strong $(R \backslash rR)$-valuation domain. In particular, each of the rings $\mathbb{Z}(X)$ and $\mathbb{Z}((X))$ is a strong $(\mathbb{Z} \setminus p\mathbb{Z})$-valuation domain for each prime number p.

Remark 5.5. If R is an anti-Archimedean domain and S a multiplicative subset of R such that R_S is a field, then $R((X))$ is a strong S-valuation domain. Indeed: By Theorem 2.3, $R[[X]]$ is a strong S-valuation domain. The result follows from the fact that the localization of a strong S-valuation domain is also a strong S-valuation domain. In particular, If R is an anti-Archimedean domain, then $R((X))$ is a strong R^*-valuation domain.

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