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S-Artinian and finite S-cogeneration properties of amalgamated algebras along an ideal

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Abstract. Let $f : A \to B$ be a ring homomorphism and J an ideal of B. We study the S-Artinian property on the amalgamation $A \bowtie^f J$ for a multiplicative subset S of $A \bowtie^f J$. We characterize when $A \bowtie^f J$ is S-Artinian, treat the cases where J is annihilated by an element of S or $J \subseteq f(A)$, and give an example separating S-Artinian from Artinian. We also give necessary conditions for pullbacks and amalgamations to be finitely S-cogenerated, and introduce strongly S-Artinian modules, which satisfy the extension property and yield full characterizations for pullbacks and amalgamations.
Key Words: S-Artinian, finitely S-cogenerated, strongly S-Artinian, Artinian ring, Amalgamation.
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1 Introduction

Throughout this paper all rings are commutative with unity and all modules are unitary. Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A,\ j \in J\},$$

called the amalgamation of A and B along J with respect to f (see [11]). In particular, the authors of [11] studied amalgamation in the frame of pullbacks, which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [10, 12, 13, 14]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealization) (cf. [21]). Moreover, other classical constructions (such as $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of amalgamations ([11], Examples 2.5 and 2.6), and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [9]), are strictly related to it ([11], Example 2.7 and Remark 2.8). In [11] the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Motivations and additional applications of the amalgamations are discussed in detail in [3, 5, 15].

In [1], D. D. Anderson and T. Dumitrescu generalized the concept of Noetherian rings to S-Noetherian rings. A ring R is called S-Noetherian, where S is a multiplicative set of R, if each ideal I of R is S-finite, i.e., there exist an $s \in S$ and a finitely generated ideal F such that $sI \subseteq F \subseteq I$. In [19], J. W. Lim and D. Y. Oh transferred the notion of S-Noetherian rings to amalgamated algebras and characterized when $A \bowtie^f J$ is an S-Noetherian ring. Several works have investigated S-versions of classical ring-theoretic properties; see, for instance, [2, 4, 5, 6, 7, 16, 17, 20]. Since the descending chain condition is the natural dual of the ascending one, it is natural to ask whether a similar transfer is possible for the Artinian property; this is the question addressed here.

Let A be a ring, M an A-module and S a multiplicative subset of A. In [22, 23], the authors introduced the notions of S-Artinian rings and S-Artinian modules, which extend the Artinian ones. They call A *S-Artinian* if each decreasing sequence $I_1 \supseteq I_2 \supseteq \cdots$ of ideals of A is *S-stationary*, i.e., there exist $s \in S$ and $k \in \mathbb{N}^*$ such that $sI_k \subseteq I_n$ for each integer $n \geq k$. Also an A-module M is called S-Artinian if each decreasing sequence $M_1 \supseteq M_2 \supseteq \cdots$ of submodules of M is S-stationary, i.e., there exist $s \in S$ and $k \in \mathbb{N}^*$ such that $sM_k \subseteq M_n$ for each integer $n \geq k$. The motivation is that many rings which are far from Artinian satisfy a chain condition "up to a multiplier"; for instance, if e is an idempotent, then the ring $K \times \mathbb{Z}$ is $\{1, e\}$-Artinian for $e = (1, 0)$ although $\mathbb{Z}$ is not Artinian (see Example 3.7). Since amalgamations unify pullbacks, idealizations and the $D + M$ constructions, characterizing when $A \bowtie^f J$ is S-Artinian settles the question for all these constructions at once.

In Section 2 we collect the basic properties of S-Artinian modules (including a self-contained proof of the extension property) and give a characterization of S-Artinian pullbacks. In Section 3 we characterize S-Artinian amalgamations for an arbitrary multiplicative subset (Theorem 3.1), deduce the case of sets of the form $\{(s, f(s))\}$ (Corollary 3.3), treat two special cases (Propositions 3.5 and 3.6), and give an example (Example 3.7). In Section 4 we study finitely S-cogenerated modules and give necessary conditions for pullbacks and amalgamations to be finitely S-cogenerated (Theorems 4.4 and 4.5). In Section 5 we introduce strongly S-Artinian modules, which imply both S-Artinian and finitely S-cogenerated modules, prove that they satisfy the extension property, and obtain full characterizations for pullbacks and amalgamations (Theorems 5.10 and 5.11).

2 S-Artinian pullbacks

A multiplicative subset of a ring is a subset containing 1, closed under products, and not containing 0. When a ring homomorphism g is applied to a multiplicative set S, we assume $S \cap \operatorname{Ker} g = \emptyset$, so that $g(S)$ is again a multiplicative set not containing 0. (Without this assumption the statements involving $g(S)$ remain true but degenerate: if $0 \in g(S)$, every ring or module is trivially $g(S)$-Artinian.) For the pullback $D = \alpha \times_C \beta$ we have $\operatorname{Ker} p_A = \{0\} \times \operatorname{Ker} \beta$ and $\operatorname{Ker} p_B = \operatorname{Ker} \alpha \times \{0\}$, and for $A \bowtie^f J$ we have $\operatorname{Ker} p_A = \{0\} \times J$ and $\operatorname{Ker} p_B = f^{-1}(J) \times \{0\}$. For $S \subseteq A$ with $0 \notin S$ the set $S' = \{(s, f(s)) \mid s \in S\}$ does not contain 0, never meets $\operatorname{Ker} p_A$, and meets $\operatorname{Ker} p_B$ if and only if $S \cap \operatorname{Ker} f \neq \emptyset$; this is why Corollary 3.3 assumes $S \cap \operatorname{Ker} f = \emptyset$.

Let $f : A \to B$ be a homomorphism of rings. Then B is an A-module for the multiplication $a \cdot b = f(a)b$, and f is a homomorphism of A-modules. If f is surjective, the ideals of B are exactly the A-submodules of B.

Our first goal is to compare the S-Artinian property of a ring with that of its homomorphic images. The next lemma shows that, for a surjection f, passing from A to B only replaces S by $f(S)$.

Lemma 2.1. *Let $f : A \to B$ be a surjective homomorphism of rings and S a multiplicative set of A with $S \cap \operatorname{Ker} f = \emptyset$. A decreasing sequence $I_1 \supseteq I_2 \supseteq \cdots$ of ideals of B (i.e., of A-submodules of B) is $f(S)$-stationary as a sequence of ideals if and only if it is S-stationary as a sequence of A-submodules. In particular, B is $f(S)$-Artinian if and only if the A-module B is S-Artinian.*

Proof. The sequence is $f(S)$-stationary if and only if there exist $s \in S$ and $k \in \mathbb{N}^*$ with $f(s)I_k \subseteq I_n$ for all $n \geq k$, if and only if $s \cdot I_k \subseteq I_n$ for all $n \geq k$, if and only if it is S-stationary as a sequence of A-submodules. The last assertion is clear. □

The proof of our pullback theorem requires the extension property of S-Artinian modules. This is known [8], but we include the short proof to keep the paper self-contained.

Lemma 2.2. *Let N be a submodule of an A-module M and S a multiplicative subset of A.*

1. *If M is S-Artinian, then N and M/N are S-Artinian.*
2. *If N and M/N are S-Artinian, then M is S-Artinian.*

Proof. (1) A chain of submodules of N is a chain of submodules of M. A chain of submodules of M/N has the form $P_1/N \supseteq P_2/N \supseteq \cdots$ with $P_n \supseteq N$ submodules of M; if $sP_k \subseteq P_n$ for $n \geq k$, then $s(P_k/N) \subseteq P_n/N$.

(2) Let $M_1 \supseteq M_2 \supseteq \cdots$ be submodules of M. Since N is S-Artinian there are $s_1 \in S$ and k_1 with $s_1(N \cap M_k) \subseteq N \cap M_n$ for $n \geq k \geq k_1$. Since M/N is S-Artinian there are $s_2 \in S$ and k_2 with $s_2((M_{k_2} + N)/N) \subseteq (M_n + N)/N$ for $n \geq k_2$. Let $k = \max(k_1, k_2)$, $n \geq k$ and $x \in M_k$. As $M_k + N \subseteq M_{k_2} + N$, we get $s_2 x = m + y$ with $m \in M_n$ and $y \in N$. Then $y = s_2 x - m \in M_k \cap N$, so $s_1 y \in N \cap M_n$, and $s_1 s_2 x = s_1 m + s_1 y \in M_n$. Thus $s_1 s_2 M_k \subseteq M_n$ for all $n \geq k$. □

Corollary 2.3. *Let $g : R \to R'$ be a surjective ring homomorphism and S a multiplicative subset of R with $S \cap \operatorname{Ker} g = \emptyset$. If R is S-Artinian, then R' is $g(S)$-Artinian.*

Proof. $R' \simeq R/\operatorname{Ker} g$ is a quotient of the S-Artinian R-module R, hence an S-Artinian R-module by Lemma 2.2(1), and Lemma 2.1 applies. □

We now turn to pullbacks, which is the framework in which amalgamations are best understood.

Definition 2.4. Let $\alpha : A \to C$ and $\beta : B \to C$ be two homomorphisms of rings. The subring $D = \alpha \times_C \beta = \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the pullback of α and β. The canonical projections $p_A : D \to A$ and $p_B : D \to B$ are defined by $p_A(a, b) = a$ and $p_B(a, b) = b$. We obtain the commutative diagram

$$D := \alpha \times_C \beta @ > p_A >> A @ V p_B V V @ V V \alpha V B @ > \beta >> C$$

If β is surjective, then p_A is surjective. Moreover, $\operatorname{Ker}(\beta)$ and A are D-modules via p_B and p_A respectively:

$$d \cdot b := p_B(d)b \quad \text{and} \quad d \cdot a := p_A(d)a \qquad (d \in D,\ a \in A,\ b \in \operatorname{Ker} \beta).$$

Lemma 2.5. *With the notations of Definition 2.4, assume β is surjective and let S be a multiplicative set of D with $S \cap \operatorname{Ker} p_A = \emptyset$. The ring A is $p_A(S)$-Artinian if and only if the D-module A is S-Artinian.*

Proof. Since β is surjective, so is p_A, and the result is Lemma 2.1 applied to $p_A : D \to A$. □

The following theorem is the basic tool of the paper: the S-Artinian property of a pullback splits into a condition on one factor and a condition on a kernel, exactly as in the Noetherian case.

Theorem 2.6. Let $D = \alpha \times_C \beta$ be a pullback with β surjective, and let S be a multiplicative set of D with $S \cap (\{0\} \times \operatorname{Ker} \beta) = \emptyset$. The following are equivalent:

1. D is S-Artinian.
2. A is $p_A(S)$-Artinian and the D-module $\operatorname{Ker} \beta$ is S-Artinian (for the structure induced by p_B).

Proof. Since β is surjective, $p_A : D \to A$ is a surjective homomorphism of D-modules, so $A \simeq D/\operatorname{Ker} p_A$ as D-modules, and $\operatorname{Ker} p_A = \{0\} \times \operatorname{Ker} \beta \simeq \operatorname{Ker} \beta$ as D-modules. By Lemma 2.2, D is S-Artinian if and only if the D-modules A and $\operatorname{Ker} p_A$ are S-Artinian. By Lemma 2.5, A is S-Artinian as a D-module if and only if the ring A is $p_A(S)$-Artinian. Hence (1) and (2) are equivalent. □

3 S-Artinian amalgamations

We now specialize to amalgamations. Let $\pi : B \to B/J$ be the canonical map and $\widehat{f} = \pi \circ f$. Then $A \bowtie^f J = \widehat{f} \times_{B/J} \pi$, with $\operatorname{Ker} \widehat{f} = f^{-1}(J)$ and $\operatorname{Ker} \pi = J$. The projection p_B makes B an $A \bowtie^f J$-module via $(a, f(a)+j)\cdot b = (f(a)+j)b$, and J is a submodule of B. Since $p_B(A \bowtie^f J) = f(A)+J$, the $A \bowtie^f J$-submodules of $f(A)+J$ are exactly the ideals of the ring $f(A)+J$.

The following theorem reduces the S-Artinian property of $A \bowtie^f J$ to properties of A and of the subring $f(A)+J$ of B. It holds for an arbitrary multiplicative subset, which is the natural level of generality since multiplicative sets of $A \bowtie^f J$ need not come from A.

Theorem 3.1. Let $f : A \to B$ be a ring homomorphism, J an ideal of B and T a multiplicative subset of $A \bowtie^f J$ with $T \cap (\{0\} \times J) = \emptyset$ and $T \cap (f^{-1}(J) \times \{0\}) = \emptyset$ (so $p_A(T)$ and $p_B(T)$ are multiplicative sets not containing 0). The following statements are equivalent:

1. $A \bowtie^f J$ is T-Artinian.
2. A is $p_A(T)$-Artinian and J is a T-Artinian $A \bowtie^f J$-module.
3. A is $p_A(T)$-Artinian and $f(A)+J$ is $p_B(T)$-Artinian.

Proof. (1) $\Leftrightarrow$ (2). It suffices to apply Theorem 2.6 to $\widehat{f} \times_{B/J} \pi$.

(1) $\Rightarrow$ (3). The rings A and $f(A)+J$ are the images of $A \bowtie^f J$ under the surjections p_A and p_B, so Corollary 2.3 applies.

(3) $\Rightarrow$ (2). The $A \bowtie^f J$-submodules of J are exactly the ideals of $f(A)+J$ contained in J, and $t = (a, f(a)+j) \in T$ acts on J as multiplication by $p_B(t) = f(a)+j$. Hence a decreasing chain of submodules of J is T-stationary if and only if it is $p_B(T)$-stationary as a chain of ideals of $f(A)+J$, and the hypothesis gives (2). □

Remark 3.2. Assume that A and B are nonzero and take $T = \{(1,1)\}$ in Theorem 3.1. The two conditions on T hold, and the theorem reads: $A \bowtie^f J$ is Artinian if and only if A and $f(A)+J$ are Artinian (and then J is an Artinian $A \bowtie^f J$-module).

The most important multiplicative sets are those induced by a multiplicative set S of A. For these the theorem takes a simpler form.

Corollary 3.3. *Let $f : A \to B$ be a ring homomorphism, J an ideal of B, and S a multiplicative set of A with $S \cap \operatorname{Ker} f = \emptyset$. Put $S' = \{(s, f(s)) \mid s \in S\}$. The following are equivalent:*

1. *$A \bowtie^f J$ is S'-Artinian.*
2. *A is S-Artinian and J is an S'-Artinian $A \bowtie^f J$-module.*
3. *A is S-Artinian and $f(A)+J$ is $f(S)$-Artinian.*

Proof. Apply Theorem 3.1 with $T = S'$: here $p_A(S') = S$, $p_B(S') = f(S)$, $S' \cap (\{0\} \times J) = \emptyset$ since $0 \notin S$, and $S' \cap (f^{-1}(J) \times \{0\}) = \emptyset$ since $S \cap \operatorname{Ker} f = \emptyset$. □

The case $f = \mathrm{id}_A$ gives the amalgamated duplication $A \bowtie I := A \bowtie^{\mathrm{id}_A} I$ (see [10]); here the answer is particularly clean: $A \bowtie I$ inherits the S-Artinian property from A and no condition on I is needed.

Corollary 3.4. *Let A be a ring, I an ideal of A and S a multiplicative set of A. Then $A \bowtie I$ is S'-Artinian, with $S' = \{(s,s) \mid s \in S\}$, if and only if A is S-Artinian.*

Proof. Take $f = \mathrm{id}_A$ in Corollary 3.3: then $f(A)+I = A$ and $f(S) = S$, so (3) reads "A is S-Artinian". □

We next ask when the condition on J in Corollary 3.3 is automatic. This happens in two natural situations: when J is killed by an element of S (it is then "S-torsion"), and when J is already contained in $f(A)$.

Proposition 3.5. *Let S be a multiplicative subset of A with $S \cap \operatorname{Ker} f = \emptyset$ such that $f(s)J = 0$ for some $s \in S$. Then $A \bowtie^f J$ is S'-Artinian if and only if A is S-Artinian.*

Proof. ($\Rightarrow$) is Corollary 3.3. ($\Leftarrow$) By Corollary 3.3(2) it suffices to show that J is an S'-Artinian module. Let $N_1 \supseteq N_2 \supseteq \cdots$ be submodules of J. Then $(s, f(s)) \cdot N_1 = f(s)N_1 \subseteq f(s)J = 0 \subseteq N_n$ for all $n \geq 1$, so the chain is S'-stationary with $k = 1$. □

Proposition 3.6. *Let S be a multiplicative subset of A with $S \cap \operatorname{Ker} f = \emptyset$ and assume $J \subseteq f(A)$ (this holds, e.g., if f is surjective or $f = \mathrm{id}_A$). Then $A \bowtie^f J$ is S'-Artinian if and only if A is S-Artinian.*

Proof. Here $f(A) + J = f(A)$. If A is S-Artinian, then $f(A)$ is $f(S)$-Artinian by Corollary 2.3, and Corollary 3.3(3) applies. The converse is Corollary 3.3. □

Finally, we show that the S-Artinian property is strictly more general than the Artinian one, even for amalgamations.

Example 3.7. Let K be a field, $A = B = K \times \mathbb{Z}$, $f = \mathrm{id}_A$, $e = (1, 0)$ and $S = \{(1, 1), e\}$ (a multiplicative set, since $e^2 = e$ and $0 \notin S$). Then A is S-Artinian: the ideals of A are of the form $I^K \times I^{\mathbb{Z}}$ with $I^K \in \{0, K\}$, so for a decreasing chain $I_n = I_n^K \times I_n^{\mathbb{Z}}$ the K-components stabilize from some k on, hence $eI_k = I_k^K \times 0 \subseteq I_n$ for all $n \geq k$.

Take $J = 0 \times \mathbb{Z}$. Then $f(e)J = eJ = 0$, so by Proposition 3.5 the ring $A \bowtie J$ is S'-Artinian. It is not Artinian: otherwise its image $A = p_A(A \bowtie J)$ would be Artinian by Corollary 2.3 (with $S = \{1\}$), which is false since $\mathbb{Z}$ is not Artinian.

4 Finitely S-cogenerated modules and pullbacks

Besides chains, a second classical description of the Artinian property uses finitely cogenerated modules. In this section we study the corresponding S-notion for pullbacks and amalgamations; only necessary conditions are obtained here, and the reason why a converse is hard to prove motivates the stronger notion of Section 5.

Recall from [23] that an R-module M is *finitely S-cogenerated* (where S is a multiplicative subset of R) if for every family $(N_i)_{i \in \Lambda}$ of submodules of M with $\bigcap_{i \in \Lambda} N_i = 0$ there exist a finite subset $F \subseteq \Lambda$ and $s \in S$ with $s \bigcap_{i \in F} N_i = 0$ (an empty intersection being M). A ring R is finitely S-cogenerated if it is so as an R-module.

We first record three elementary facts, needed to transfer the property along the maps p_A and p_B.

Lemma 4.1. *If M is a finitely S-cogenerated R-module, then every submodule of M is finitely S-cogenerated. In particular every ideal of a finitely S-cogenerated ring is a finitely S-cogenerated module.*

Proof. A family of submodules of $N \subseteq M$ with zero intersection is a family of submodules of M with zero intersection. □

Lemma 4.2. *Let $g : R \to R'$ be a surjective ring homomorphism, S a multiplicative subset of R with $S \cap \operatorname{Ker} g = \emptyset$, and M an R'-module, regarded as an R-module via g. Then M is a finitely S-cogenerated R-module if and only if it is a finitely $g(S)$-cogenerated R'-module.*

Proof. Since g is surjective, the R-submodules and the R'-submodules of M coincide, and $s \in S$ acts as $g(s)$. So the two defining conditions are identical. □

Lemma 4.3. *If $sM = 0$ for some $s \in S$, then M is a finitely S-cogenerated R-module.*

Proof. Let $(N_i)_{i\in\Lambda}$ be a family with zero intersection. If $\Lambda = \emptyset$, then $M = 0$ and there is nothing to prove. Otherwise take $F = \{i_0\}$; then $sN_{i_0} \subseteq sM = 0$. □

Our first main result of this section says that finite S-cogeneration of a pullback is inherited by both kernels.

Theorem 4.4. Let $D = \alpha \times_C \beta$ be a pullback with β surjective, and let S be a multiplicative subset of D with $S \cap \operatorname{Ker} p_A = S \cap \operatorname{Ker} p_B = \emptyset$. Put $B' = p_B(D) = \beta^{-1}(\alpha(A))$, a subring of B containing $\operatorname{Ker}\beta$. If D is finitely S-cogenerated, then

1. $\operatorname{Ker}\alpha$ is a finitely $p_A(S)$-cogenerated A-module;
2. $\operatorname{Ker}\beta$ is a finitely $p_B(S)$-cogenerated B'-module.

Proof. By Lemma 4.1, every ideal of D is a finitely S-cogenerated D-module. We have $\operatorname{Ker} p_B = \operatorname{Ker}\alpha \times \{0\}$, and D acts on it through p_A, which is surjective because β is. By Lemma 4.2 (with $g = p_A$), $\operatorname{Ker}\alpha$ is a finitely $p_A(S)$-cogenerated A-module. Similarly $\operatorname{Ker} p_A = \{0\} \times \operatorname{Ker}\beta$ is a D-module through $p_B : D \to B'$ (surjective by definition of B'), so Lemma 4.2 with $g = p_B$ shows that $\operatorname{Ker}\beta$ is a finitely $p_B(S)$-cogenerated B'-module. □

Applying this to $A \bowtie^f J = \widehat{f} \times_{B/J} \pi$ gives the analogous statement for amalgamations.

Theorem 4.5. Let $f : A \to B$ be a ring homomorphism, J an ideal of B and T a multiplicative subset of $A \bowtie^f J$ with $T \cap \operatorname{Ker} p_A = T \cap \operatorname{Ker} p_B = \emptyset$. If $A \bowtie^f J$ is finitely T-cogenerated, then

1. J is a finitely $p_B(T)$-cogenerated $(f(A)+J)$-module (equivalently, a finitely T-cogenerated $A \bowtie^f J$-module);
2. $f^{-1}(J)$ is a finitely $p_A(T)$-cogenerated A-module.

Moreover, condition (1) holds automatically whenever $p_B(t)J = 0$ for some $t \in T$.

Proof. Apply Theorem 4.4 to $A \bowtie^f J = \widehat{f} \times_{B/J} \pi$, where $\operatorname{Ker}\widehat{f} = f^{-1}(J)$, $\operatorname{Ker}\pi = J$ and $p_B(A \bowtie^f J) = f(A) + J$. The parenthetical equivalence in (1) is Lemma 4.2 (with $g = p_B$), and the last assertion is Lemma 4.3. □

For the amalgamated duplication the conditions simplify, since $f^{-1}(I) = I$ and $f(A) + I = A$.

Corollary 4.6. *Let A be a ring, I an ideal of A and T a multiplicative subset of $A \bowtie I$ with $T \cap (I \times \{0\}) = T \cap (\{0\} \times I) = \emptyset$. If $A \bowtie I$ is finitely T-cogenerated, then I is a finitely $p_A(T)$-cogenerated A-module and a finitely $p_B(T)$-cogenerated A-module.*

The next example shows that the S-conditions above are genuinely weaker than the classical ones.

Example 4.7. Let A, J, S be as in Example 3.7. Since $f(e)J = 0$, the module $J = 0 \times \mathbb{Z}$ is finitely S'-cogenerated by Lemma 4.3. It is not finitely cogenerated: its submodules are $0 \times n\mathbb{Z}$, the family $(0 \times 2^k\mathbb{Z})_{k\geq 1}$ has zero intersection, and no finite subfamily has zero intersection.

Question 4.8. Let N be a submodule of an A-module M. If N and M/N are finitely S-cogenerated, is M finitely S-cogenerated ? (A positive answer would yield a converse to Theorems 4.4 and 4.5. The classical argument fails, since one only gets that the intersection of the relevant images is annihilated by some $s \in S$, not that it is zero. This is settled in Section 5 for a stronger notion.)

5 Strongly S-Artinian modules

Finite S-cogeneration only controls families with *zero* intersection, which is why the converse of Theorems 4.4 and 4.5 resists the methods above. We therefore introduce a notion that controls families with arbitrary intersection. It implies both S-Artinian and finitely S-cogenerated (Proposition 5.2) and satisfies the extension property, which yields full characterizations for pullbacks and amalgamations. For *rings* it coincides with S-Artinian by [18, Theorem 2.24]; for modules we prove the coincidence when S is countable (Theorem 5.7).

Definition 5.1. Let S be a multiplicative subset of A. An A-module M is called *strongly S-Artinian* if for every family $(N_i)_{i\in\Lambda}$ of submodules of M there exist a finite subset $F \subseteq \Lambda$ and $s \in S$ such that

$$s\bigcap_{i\in F} N_i \subseteq \bigcap_{i\in\Lambda} N_i,$$

where an empty intersection is M (so $F = \emptyset$ is allowed). The ring A is strongly S-Artinian if it is so as an A-module.

We first compare this notion with the two known ones.

Proposition 5.2. *Every strongly S-Artinian A-module M is S-Artinian and finitely S-cogenerated.*

Proof. Let $N_1 \supseteq N_2 \supseteq \cdots$ be a decreasing sequence and take F and s as in the definition. If $F = \emptyset$, then $sM \subseteq \bigcap_i N_i$, so $sN_1 \subseteq N_n$ for all n. Otherwise let $k = \max F$; then $\bigcap_{i\in F} N_i = N_k$, so $sN_k \subseteq \bigcap_i N_i \subseteq N_n$ for all $n \geq k$. Hence M is S-Artinian. If $\bigcap_i N_i = 0$, the defining inclusion reads $s\bigcap_{i\in F} N_i = 0$, so M is finitely S-cogenerated. □

The next theorem is the S-analogue of the classical fact that a module is Artinian if and only if all its quotients are finitely cogenerated.

Theorem 5.3. An A-module M is strongly S-Artinian if and only if every quotient M/N of M is finitely S-cogenerated.

Proof. ($\Rightarrow$) Let $N \subseteq M$ and let $(P_i/N)_{i\in\Lambda}$ be submodules of M/N ($N \subseteq P_i \subseteq M$) with zero intersection, i.e. $\bigcap_i P_i = N$. Take F, s with $s\bigcap_{i\in F} P_i \subseteq \bigcap_i P_i = N$. Then $s\bigcap_{i\in F}(P_i/N) = 0$.

($\Leftarrow$) Let $(N_i)_{i\in\Lambda}$ be a family and $K = \bigcap_i N_i$. In M/K the family (N_i/K) has zero intersection. Since M/K is finitely S-cogenerated, there are F and s with $s\bigcap_{i\in F}(N_i/K) = 0$, that is $s\bigcap_{i\in F} N_i \subseteq K$. □

To transfer the property through exact sequences and pullbacks we need the following stability properties.

Lemma 5.4. 1. *Submodules and quotients of a strongly S-Artinian module are strongly S-Artinian.*

2. *Let $g : A \to B$ be a surjective ring homomorphism, $S \subseteq A$ multiplicative with $S \cap \operatorname{Ker} g = \emptyset$, and M an B-module. Then M is strongly S-Artinian over A if and only if it is strongly $g(S)$-Artinian over B.*

3. *If $sM = 0$ for some $s \in S$, then M is strongly S-Artinian.*

Proof. (1) A family of submodules of $L \subseteq M$ is a family of submodules of M with the same intersections. For a quotient M/N, a family (P_i/N) has $\bigcap_i(P_i/N) = (\bigcap_i P_i)/N$, and $s\bigcap_{i\in F} P_i \subseteq \bigcap_i P_i$ gives $s\bigcap_{i\in F}(P_i/N) \subseteq \bigcap_i(P_i/N)$. (2) As in Lemma 4.2, the submodules and the action of s are the same in both settings. (3) For any family, $s\bigcap_{i\in F} N_i \subseteq sM = 0 \subseteq \bigcap_i N_i$. □

The key result of this section is the extension property, which is exactly what was missing in the finitely S-cogenerated case.

Theorem 5.5. Let N be a submodule of an A-module M. Then M is strongly S-Artinian if and only if N and M/N are strongly S-Artinian.

Proof. ($\Rightarrow$) By Lemma 5.4(1).

($\Leftarrow$) Let $(M_i)_{i\in\Lambda}$ be a family of submodules of M (the case $\Lambda=\emptyset$ is trivial), and $K=\bigcap_i M_i$. Let $\mathcal{F}$ be the set of finite intersections of the M_i; it is closed under finite intersections and $\bigcap_{L\in\mathcal{F}} L = K$. It suffices to find $L_0\in\mathcal{F}$ and $s\in S$ with $sL_0\subseteq K$.

Apply the hypothesis on N to the family $(L\cap N)_{L\in\mathcal{F}}$ (its intersection is $K\cap N$): since $\mathcal{F}$ is closed under finite intersections, there are $L_1\in\mathcal{F}$ and $s_1\in S$ with $s_1(L_1\cap N)\subseteq K\cap N$. Apply the hypothesis on M/N to the family $((L+N)/N)_{L\in\mathcal{F}}$: there are a finite subfamily $F'\subseteq\mathcal{F}$ and $s_2\in S$ with $s_2\bigcap_{L\in F'}((L+N)/N)\subseteq\bigcap_{L\in\mathcal{F}}((L+N)/N)$. Put $L_2=\bigcap_{L\in F'}L\in\mathcal{F}$; since $L_2+N\subseteq\bigcap_{L\in F'}(L+N)$, we get

$$s_2(L_2+N)\subseteq\bigcap_{L\in\mathcal{F}}(L+N).$$

Let $L_0=L_1\cap L_2\in\mathcal{F}$ and $x\in L_0$. Let $L\in\mathcal{F}$ be arbitrary and $L'=L\cap L_0\in\mathcal{F}$. Since $s_2x\in L'+N$, write $s_2x=m+n$ with $m\in L'$ and $n\in N$. Then $n=s_2x-m\in L_0\cap N\subseteq L_1\cap N$, so $s_1n\in K\cap N\subseteq L$. Also $s_1m\in L'\subseteq L$. Hence $s_1s_2x=s_1m+s_1n\in L$. As $L\in\mathcal{F}$ was arbitrary, $s_1s_2x\in K$. Thus $s_1s_2L_0\subseteq K$ with $s_1s_2\in S$. □

Corollary 5.6. *A finite direct sum of strongly S-Artinian modules is strongly S-Artinian.*

A natural question is how the new notion relates to S-Artinian modules. For countable S they coincide.

Theorem 5.7. If S is countable (possibly finite), then an A-module M is strongly S-Artinian if and only if it is S-Artinian. Consequently, for countable S, a module is S-Artinian if and only if all its quotients are finitely S-cogenerated.

Proof. ($\Rightarrow$) By Proposition 5.2.

($\Leftarrow$) Let M be S-Artinian, (N_i) a family, $\mathcal{F}$ its set of finite intersections and $K=\bigcap_{L\in\mathcal{F}}L$. Suppose, for contradiction, that $sL\not\subseteq K$ for all $L\in\mathcal{F}$ and $s\in S$. Enumerate $S=\{s_1,s_2,\dots\}$ (with repetitions if S is finite) and choose $L_1\in\mathcal{F}$. Given $L_1,\dots,L_n$, for each pair (a,k) with $1\le a,k\le n$ pick $y_{a,k}\in s_aL_k\setminus K$ and $L^{(a,k)}\in\mathcal{F}$ with $y_{a,k}\notin L^{(a,k)}$ (possible since $K=\bigcap\mathcal{F}$), and put $L_{n+1}=L_n\cap\bigcap_{a,k\le n}L^{(a,k)}\in\mathcal{F}$. We get a decreasing sequence with $s_aL_k\not\subseteq L_{n+1}$ whenever $a,k\le n$. Hence for every $s=s_a\in S$ and every k there is $m\ge k$ with $sL_k\not\subseteq L_m$, so the sequence is not S-stationary, contradicting that M is S-Artinian. The last assertion follows from Theorem 5.3. □

Remark 5.8. For rings, [18, Theorem 2.24] shows that A is S-Artinian if and only if A is strongly S-Artinian, for an arbitrary multiplicative set S (no countability needed). We proved the module version only for countable S; whether strongly S-Artinian and S-Artinian modules coincide for uncountable S is left open.

Corollary 5.9. *A ring A is S-Artinian if and only if for every ideal I of A the A-module A/I is finitely S-cogenerated.*

Proof. By [18, Theorem 2.24], A is S-Artinian if and only if A is strongly S-Artinian, and by Theorem 5.3 this holds if and only if all quotients A/I are finitely S-cogenerated. □

We now apply the extension property to pullbacks, obtaining the converse of Theorem 4.4 in the strong setting.

Theorem 5.10. Let $D=\alpha\times_C\beta$ be a pullback with β surjective and S a multiplicative subset of D with $S\cap(\{0\}\times\operatorname{Ker}\beta)=\emptyset$. The following are equivalent:

1. D is strongly S-Artinian.
2. A is strongly $p_A(S)$-Artinian and $\operatorname{Ker}\beta$ is a strongly S-Artinian D-module.
3. Every quotient $D/\mathfrak{a}$ of the ring D is a finitely S-cogenerated D-module.

Proof. Apply Theorem 5.5 to $0 \to \operatorname{Ker} p_A \to D \to A \to 0$, with $\operatorname{Ker} p_A = \{0\} \times \operatorname{Ker}\beta \simeq \operatorname{Ker}\beta$ as D-modules. Then D is strongly S-Artinian if and only if the D-modules A and $\operatorname{Ker}\beta$ are. By Lemma 5.4(2) applied to p_A, the D-module A is strongly S-Artinian if and only if the ring A is strongly $p_A(S)$-Artinian. This gives (1) $\Leftrightarrow$ (2). Statement (3) is equivalent to (1) by Theorem 5.3 applied to $M = D$. □

For amalgamations we obtain the complete analogue of Theorem 3.1, together with a description in terms of quotients.

Theorem 5.11. Let $f : A \to B$ be a ring homomorphism, J an ideal of B and T a multiplicative subset of $A \bowtie^f J$ with $T \cap (\{0\} \times J) = T \cap (f^{-1}(J) \times \{0\}) = \emptyset$. The following are equivalent:

1. $A \bowtie^f J$ is strongly T-Artinian.
2. A is strongly $p_A(T)$-Artinian and J is a strongly T-Artinian $A \bowtie^f J$-module.
3. A is strongly $p_A(T)$-Artinian and $f(A)+J$ is strongly $p_B(T)$-Artinian.
4. Every quotient of A is finitely $p_A(T)$-cogenerated and every quotient of $f(A)+J$ is finitely $p_B(T)$-cogenerated.

Proof. (1) $\Leftrightarrow$ (2). Apply Theorem 5.10 to $\widehat{f} \times_{B/J} \pi$.

(1) $\Rightarrow$ (3). A is strongly $p_A(T)$-Artinian by (2). Also $f(A)+J \simeq (A \bowtie^f J)/\operatorname{Ker} p_B$ is a quotient of $A \bowtie^f J$ through p_B, so it is strongly $p_B(T)$-Artinian by Lemma 5.4(1),(2).

(3) $\Rightarrow$ (2). J is an ideal, hence a submodule, of the strongly $p_B(T)$-Artinian ring $f(A)+J$, so J is strongly $p_B(T)$-Artinian over $f(A)+J$ (Lemma 5.4(1)), hence strongly T-Artinian over $A \bowtie^f J$ by Lemma 5.4(2).

(3) $\Leftrightarrow$ (4) is Theorem 5.3. □

Combining Theorems 3.1, 5.7 and 5.11 gives, for countable T, a description of the T-Artinian property by finite T-cogeneration of quotients.

Corollary 5.12. *With the notations of Theorem 5.11, if T is countable then the following are equivalent:*

1. *$A \bowtie^f J$ is T-Artinian.*
2. *Every quotient of $A \bowtie^f J$ is finitely T-cogenerated.*
3. *Every quotient of A is finitely $p_A(T)$-cogenerated and every quotient of $f(A)+J$ is finitely $p_B(T)$-cogenerated.*

In particular, if $A \bowtie^f J$ is T-Artinian with T countable, then $A \bowtie^f J$ is finitely T-cogenerated, and so are A, $f(A)+J$ (for $p_A(T)$ and $p_B(T)$), J and $f^{-1}(J)$ (for the sets of Theorem 4.5).

Proof. The sets $p_A(T)$ and $p_B(T)$ are countable when T is. (1) $\Leftrightarrow$ (2) is Theorem 5.7 for $M = A \bowtie^f J$. By Theorem 3.1, (1) is equivalent to A being $p_A(T)$-Artinian and $f(A)+J$ being $p_B(T)$-Artinian; by Theorem 5.7 this is equivalent to both being strongly Artinian for these sets, which is (3) by Theorem 5.11. The last assertion follows from Proposition 5.2 and Theorem 4.5. □

Remark 5.13. Since $A \bowtie^f J$, A and $f(A)+J$ are rings, Remark 5.8 shows that, for these rings, strongly T-Artinian and T-Artinian coincide for an arbitrary multiplicative set. Combining Theorems 3.1, 5.3 and 5.11 (and replacing Theorem 5.7 by Remark 5.8), the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) of the corollary above therefore hold without assuming that T is countable. The countability hypothesis is only needed for the module-level statement of Theorem 5.7.

Corollary 5.14. *Let A be a ring, I an ideal of A and T a multiplicative subset of $A \bowtie I$ with $T \cap (I \times \{0\}) = T \cap (\{0\} \times I) = \emptyset$. Then $A \bowtie I$ is strongly T-Artinian if and only if A is strongly $p_A(T)$-Artinian and strongly $p_B(T)$-Artinian.*

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