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S-Classical Prime Submodules

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Abstract. We introduce the concept of S-classical prime submodules as a simultaneous generalization of classical prime and S-prime submodules. Several equivalent characterizations of these objects are derived, and we elucidate the interplay between S-prime and S-classical prime submodules. Additionally, we determine which commutative rings possess the property that arbitrary intersections of S-prime submodules yield S-classical prime submodules. As a principal application, we establish an S-analogue of Cohen's theorem in the module-theoretic setting.

Key Words: Prime submodules, S-Prime submodule, Classical prime submodule.
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1 Introduction

Throughout this paper, we assume all rings to be commutative with unity and all modules to be unitary. Unless stated otherwise, let R be a commutative ring, M an R-module, and S a multiplicatively closed subset (m.c.s. for short) of R. For any ideal I of R, a nonempty set $X \subseteq R$, and a submodule $N \leq M$, we use the following notation: $(N : X) = \{r \in R \mid rX \subseteq N\}$, $\mathrm{Zd}_R(M) = \{r \in R \mid rm = 0 \text{ for some nonzero } m \in M\}$, $(N :_M I) = \{m \in M \mid Im \subseteq N\}$. For simplicity, we denote $(N : \{m\})$ by $(N : m)$ and $(N :_M Rs)$ by $(N :_M s)$.

Prime ideals and prime submodules are among the most important concepts in commutative algebra and module theory. The significance of these structures has motivated the introduction of several generalizations aimed at understanding various algebraic phenomena. One such generalization is the concept of an S-prime ideal, which was first introduced by Hamed and Malek [9]. Specifically, an ideal $\mathcal{P}$ of R with $\mathcal{P} \cap S = \emptyset$ is termed an S-prime ideal provided there is an element $s \in S$ such that whenever $ab \in \mathcal{P}$, we have either $sa \in \mathcal{P}$ or $sb \in \mathcal{P}$. This notion has been generalized, and several related notions have also been studied; see, for example, [8, 10, 13].

The analogous notion for modules was introduced in [11]. A proper submodule P of an R-module M satisfying $(P : M) \cap S = \emptyset$ is called an S-prime submodule if there exists $s \in S$ such that for any $a \in R$ and $m \in M$ with $am \in P$, we have either $sm \in P$ or $saM \subseteq P$. The collection of all S-prime submodules of M is denoted by $\mathrm{Spec}_S(M)$. When $S = \{1\}$, this definition reduces to the standard definition of a prime submodule.

A second important generalization of prime submodules is given by the notion of a classical prime submodule. This concept appeared initially as a weakly prime submodule in [7] and was subsequently studied under its present name (see, for example, [6]). A proper submodule P of M is said to be classical prime if for every $a, b \in R$ and $m \in M$, the condition $abm \in P$ implies $am \in P$ or $bm \in P$.

While the theories of S-prime and classical prime submodules have evolved in parallel, their mutual relationship has received limited investigation. The theory of S-prime submodules is intimately connected to localization at multiplicatively closed sets, whereas classical prime submodules are characterized by a condition involving the product of two elements from the ring and one element from the module. This observation suggests the possibility of a unified framework that encompasses both classes of submodules. With this motivation in mind, we introduce the concept of an S-classical prime submodule as a natural extension that simultaneously generalizes S-prime and classical prime submodules.

The main goal of this paper is to establish a foundational theory for S-classical prime submodules. We derive multiple equivalent characterizations, examine their properties under localization, module homomorphisms, tensor products, and finite direct sums, and identify when S-prime and S-classical prime submodules coincide. Additionally, we characterize commutative rings having the property that all finite intersections of S-prime submodules are S-classical prime. As an application of our results, we provide an S-analogue of Cohen's classical theorem on the characterization of Noetherian modules.

2 S-Classical prime submodules

Definition 2.1. Let P be a proper submodule of R-module M satisfying $(P:M)\cap S=\emptyset$. We say that P is an *S-classical prime submodule* if there is $s\in S$ such that $abm\in P$ implies $sam\in P$ or $sbm\in P$ for all $a,b\in R$ and $m\in M$. Such an element s is said to be *associated with* P.

We denote the collection of all S-classical prime submodules of M by $\mathrm{cl.Spec}_S({}_RM)$. Whenever the ambient module is clear from the context, we simply write $\mathrm{cl.Spec}_S(M)$. If $S=\{1\}$, we write $\mathrm{cl.Spec}(M)$.

Every classical prime submodule, as well as every S-prime submodule, satisfies the above condition and is therefore S-classical prime. The following example demonstrates that the class of S-classical prime submodules properly contains both the class of classical prime submodules and the class of S-prime submodules.

Example 2.2.

(1)Let $R=\mathbb{Z}$, and let $s_1,\ldots,s_n$ be pairwise relatively prime positive numbers. Put $S=\{s_1^{m_1}\cdots s_n^{m_n}\mid m_i\in\mathbb{N}_0,\ 1\le i\le n\}$. Consider the submodule

$$P=s_1^{k_1}\mathbb{Z}\oplus s_2^{k_2}\mathbb{Z}\oplus\cdots\oplus s_n^{k_n}\mathbb{Z}\oplus p\mathbb{Z}\oplus(0)\le M=\mathbb{Z}\oplus\mathbb{Z}\oplus\cdots\oplus\mathbb{Z},$$

where p is a prime number satisfying $s_i\notin p\mathbb{Z}$ for every $1\le i\le n$, and $k_i\ge 2$.

One easily checks that $(P:M)=(0)$. Moreover, $p(0,0,\ldots,0,1,0)\in P$, while for every $s\in S$, $s(0,0,\ldots,0,1,0)\notin P$ and $spM\not\subseteq P$. On the other hand, $s_1^{k_1}s_2^{k_2}(1,1,0,\ldots,0)\in P$, whereas neither $s_1^{k_1}(1,1,0,\ldots,0)$ nor $s_2^{k_2}(1,1,0,\ldots,0)$ belongs to P. Then, P is neither an S-prime submodule nor a classical prime submodule.

We claim that $P\in\mathrm{cl.Spec}_S(M)$. Let $ab(c_1,\ldots,c_{n+2})\in P$, where $c_1,\ldots,c_{n+2}\in\mathbb{Z}$. Replacing a or b by zero, if necessary, we may suppose that $a,b\neq 0$. It then follows that $c_{n+2}=0$. Set $s=s_1^{k_1}\cdots s_n^{k_n}$. Since $abc_{n+1}\in p\mathbb{Z}$ and p is prime, either $a\in p\mathbb{Z}$ or $bc_{n+1}\in p\mathbb{Z}$. Hence $sa(c_1,\ldots,c_{n+2})\in P$ or $sb(c_1,\ldots,c_{n+2})\in P$. This proves that $P\in\mathrm{cl.Spec}_S(M)$.

(2) Let p and q be distinct prime numbers, let $R=\mathbb{Z}$, and let $S=\{p^n\mid n\in\mathbb{N}_0\}$. Consider the zero submodule $P=(0)$ of $M=\mathbb{Z}\oplus\mathbb{Z}_q\oplus\mathbb{Z}_p\oplus\mathbb{Z}_{p^2}\oplus\cdots\oplus\mathbb{Z}_{p^k}$, where $k\ge 2$. Again, $(P:M)=(0)$. Moreover, $q(0,\bar{1},\bar{0},\ldots,\bar{0})\in P$, whereas for every $s\in S$, $s(0,\bar{1},\bar{0},\ldots,\bar{0})\notin P$ and $sqM\not\subseteq P$. Further, $p^kq(0,\bar{1},\bar{1},\ldots,\bar{1})\in P$, but neither $p^k(0,\bar{1},\bar{1},\ldots,\bar{1})$ nor $q(0,\bar{1},\bar{1},\ldots,\bar{1})$ is contained in P. Thus P is neither an S-prime submodule nor a classical prime submodule.

It remains to verify that $P\in\mathrm{cl.Spec}_S(M)$. Let $ab(c_1,\bar{c}_2,\ldots,\bar{c}_{k+2})\in P$, where $a,b,c_1\in\mathbb{Z}$, $\bar{c}_2\in\mathbb{Z}_q$, and $\bar{c}_i\in\mathbb{Z}_{p^i}$ for every $3\le i\le k+2$. Choose $s=p^k$. Assume that $a,b\neq 0$. Then $c_1=0$. If

$sa(c_1, \bar{c}_2, \ldots, \bar{c}_{k+2}) \notin P$, then $sac_2 \notin q\mathbb{Z}$. Since $abc_2 \in q\mathbb{Z}$, we must have $b \in q\mathbb{Z}$. Consequently, $sb(c_1, \bar{c}_2, \ldots, \bar{c}_{k+2}) \in P$. Therefore, $P \in \mathrm{cl.Spec}_S(M)$.

Proposition 2.3. *Let $P \leq M$ with $(P : M) \cap S = \emptyset$. The following conditions are equivalent.*

1. $P \in \mathrm{cl.Spec}_S(M)$.
2. *For some $s \in S$, $abK \subseteq P$ implies $saK \subseteq P$ or $sbK \subseteq P$ for all $a, b \in R$ and all $K \leq M$.*
3. *For some $s \in S$, $IJm \subseteq P$ implies $sIm \subseteq P$ or $sJm \subseteq P$ for all ideals I, J of R and all $m \in M$.*
4. *For some $s \in S$, $IJK \subseteq P$ implies $sIK \subseteq P$ or $sJK \subseteq P$ for all ideals I, J of R and all $K \leq M$.*

Proof. The implications (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) require no proof.

(1) $\Rightarrow$ (2). Assume that P is an S-classical prime submodule associated with some fixed element $s \in S$. Let $a, b \in R$, and suppose that $K \leq M$ satisfies $abK \subseteq P$. Define $L = \{x \in K \mid sax \in P\}$ and $N = \{x \in K \mid sbx \in P\}$. Clearly, both L and N are submodules of K. Since P is S-classical prime, every element of K belongs to at least one of these submodules. Thus $K = L \cup N$. Since a module cannot be expressed as the union of two proper submodules, either $L = K$ or $N = K$. Hence either $saK \subseteq P$ or $sbK \subseteq P$.

(2) $\Rightarrow$ (4). Choose $s \in S$ satisfying (2). Let I and J be ideals of R, and let $K \leq M$ satisfy $IJK \subseteq P$. Suppose that $sIK \nsubseteq P$. Then there exist $i \in I$ and $x \in K$ such that $six \notin P$. Since $ijK \subseteq P$ for every $j \in J$, applying (2) to the pair i, j yields $sjK \subseteq P$ for each $j \in J$. Therefore $sJK \subseteq P$, establishing (4). □

For the remainder of this section, $S^{-1}M$ will denote the localization of M with respect to S. Although $S^{-1}M$ may be regarded as either an R-module or an $S^{-1}R$-module, throughout this section we consider it only as an $S^{-1}R$-module.

Let $U(R)$ denote the group of units of R. We set $S^* = \{x \in R \mid \frac{x}{1} \in U(S^{-1}R)\}$ and $\bar{S} = \left\{\frac{s}{1} \mid s \in S\right\}$. It is immediate that $S \subseteq S^*$, and that both S^* and $\bar{S}$ are multiplicatively closed subsets of R and $S^{-1}R$, respectively.

Proposition 2.4. *Let $P \leq M$. Then the following statements hold.*

1. *If $P \in \mathrm{cl.Spec}(M)$ and $(P : M) \cap S = \emptyset$, then $P \in \mathrm{cl.Spec}_S(M)$.*
2. *If $P \in \mathrm{cl.Spec}_S(M)$ and $S \subseteq U(R)$, then $P \in \mathrm{cl.Spec}(M)$.*
3. *Let T be a m.c.s. of R with $T \subseteq S$. If $P \in \mathrm{cl.Spec}_T(M)$ and $(P : M) \cap S = \emptyset$, then $P \in \mathrm{cl.Spec}_S(M)$.*
4. *$P \in \mathrm{cl.Spec}_S(M)$ if and only if $P \in \mathrm{cl.Spec}_{S^*}(M)$.*
5. *$P \in \mathrm{cl.Spec}_S(M)$ if and only if $(P :_M s) \in \mathrm{cl.Spec}_S(M)$ for every $s \in S$.*
6. *If J is an ideal of R satisfying $J \cap S \neq \emptyset$ and $P \in \mathrm{cl.Spec}_S(M)$, then $JP \in \mathrm{cl.Spec}_S(M)$.*

Proof. Parts (1)–(3) follow immediately from the definitions.

(4). Assume first that $P \in \mathrm{cl.Spec}_S(M)$. We claim that $(P : M) \cap S^* = \emptyset$. Indeed, if $x \in (P : M) \cap S^*$, then $\frac{x}{1}$ is invertible in $S^{-1}R$. Hence $uxy \in S$ for some $u \in S$ and $y \in R$. Because $x \in (P : M)$, we obtain $uxy \in (P : M) \cap S$, contrary to the assumption. Thus $(P : M) \cap S^* = \emptyset$. Since $S \subseteq S^*$, part (3) yields $P \in \mathrm{cl.Spec}_{S^*}(M)$.

Conversely, suppose that $P \in \mathrm{cl.Spec}_{S^*}(M)$. If $a, b \in R$ and $m \in M$ satisfy $abm \in P$, then $xam \in P$ or $xbm \in P$ for some $x \in S^*$. Since $\frac{x}{1}$ is invertible in $S^{-1}R$, we can choose $u, s \in S$ and $y \in R$ so that $us = uxy$. It follows that $usam = uxyam \in P$ or $usbm = uxybm \in P$. Hence $P \in \mathrm{cl.Spec}_S(M)$.

(5). One readily checks that $(P : M) \cap S = \emptyset$ if and only if $((P :_M s) : M) \cap S = \emptyset$.

Assume that $(P :_M s) \in \mathrm{cl.Spec}_S(M)$ for every $s \in S$. Fix $s \in S$ and let $t \in S$ be an element associated with $(P :_M s)$. If $abm \in P$, then certainly $abm \in (P :_M s)$. Therefore, $tam \in (P :_M s)$ or $tbm \in (P :_M s)$. Equivalently, $stam \in P$ or $stbm \in P$. This proves that $P \in \mathrm{cl.Spec}_S(M)$.

Conversely, suppose that $P \in \mathrm{cl.Spec}_S(M)$ and let $t \in S$ be an associated element. If $abm \in (P :_M s)$, then $sabm \in P$. By the defining property, $tsam \in P$ or $tsbm \in P$. Hence $tam \in (P :_M s)$ or $tbm \in (P :_M s)$, which shows that $(P :_M s) \in \mathrm{cl.Spec}_S(M)$.

(6). Choose $t \in J \cap S$, and let $s \in S$ be associated with P. Since $(JP : M) \subseteq (P : M)$, we have $(JP : M) \cap S = \emptyset$. Now assume that $a, b \in R$ and $m \in M$ satisfy $abm \in JP$. As $JP \subseteq P$, we also have $abm \in P$. Consequently, $sam \in P$ or $sbm \in P$. Multiplying by t gives $tsam \in JP$ or $tsbm \in JP$. Therefore $JP \in \mathrm{cl.Spec}_S(M)$. □

Let M be an R-module and let P be a submodule of M. A classical characterization asserts that P is prime if and only if, for every $m \in M \setminus P$, the residual ideal $(P : m)$ is prime and, moreover, $(P : m) = (P : M)$.

An analogous description is available for classical prime submodules. More precisely, P is classical prime exactly when $(P : m)$ is a prime ideal of R for every $m \in M \setminus P$, and the family $\{(P : m) \mid m \in M \setminus P\}$ is linearly ordered by inclusion (see, for example, [6, Proposition 1.1]).

The next two propositions provide S-versions of these well-known characterizations.

Proposition 2.5. *Let $P \leq M$ satisfy $(P : M) \cap S = \emptyset$. The following statements are equivalent.*

1. $P \in \mathrm{Spec}_S(M)$.
2. *For some $s \in S$, whenever $K \leq M$ satisfies $s \notin (P : K)$, one has $s(P : K) \subseteq (P : M)$.*
3. *For some $s \in S$, whenever $m \in M$ satisfies $s \notin (P : m)$, one has $s(P : m) \subseteq (P : M)$.*

Proof. (1) ⇒ (2). Assume that P is an S-prime submodule associated with some $s \in S$, and let K be a submodule of M for which $s \notin (P : K)$. Then there exists $k \in K$ satisfying $sk \notin P$. Now let $a \in (P : K)$. Since $ak \in P$, while $sk \notin P$, the defining property of S-prime submodules implies that $saM \subseteq P$. Thus $sa \in (P : M)$, which is equivalent to $s(P : K) \subseteq (P : M)$.

(2) ⇒ (3). This is obtained by taking $K = Rm$.

(3) ⇒ (1). Let $am \in P$ for some $a \in R$ and $m \in M$. If $sm \in P$, then the required conclusion already holds. Otherwise, $s \notin (P : m)$. By (3), $s(P : m) \subseteq (P : M)$. Since $a \in (P : m)$, we obtain $sa \in (P : M)$, and therefore $saM \subseteq P$. This verifies that P is an S-prime submodule. □

For a submodule N of M, its S-saturation is defined by $N_S = \{m \in M \mid (N : m) \cap S \neq \emptyset\}$ (see, for instance, [3]). Equivalently, $N_S = \bigcup_{s \in S} (N :_M s)$.

Corollary 2.6. *Let $P \leq M$ satisfy $P_S = P$. Then*

$$P \in \mathrm{Spec}(M) \iff P \in \mathrm{Spec}_S(M) \text{ and } (P : M) \in \mathrm{Spec}(R).$$

Proof. (⇒). Suppose that P is a prime submodule. If $(P : M) \cap S \neq \emptyset$, choose $s \in (P : M) \cap S$. Then $sM \subseteq P$, which implies $P_S = M$, contradicting the assumption $P_S = P$. Hence $(P : M) \cap S = \emptyset$. It now follows from [11, Proposition 2.2(i)] that $P \in \mathrm{Spec}_S(M)$. Moreover, the annihilator of a prime submodule is a prime ideal, so $(P : M) \in \mathrm{Spec}(R)$.

(⇐). Assume that $P \in \mathrm{Spec}_S(M)$ and $(P : M)$ is prime. Let $s \in S$ be associated with P, and $am \in P$ for some $a \in R$ and $m \in M$.

If $m \in P$, there is nothing to prove. Because $P_S = P$, the assumption $m \notin P$ would imply $sm \notin P$. Hence the defining property of S-prime submodules gives $sa \in (P : M)$. Since $s \notin (P : M)$ and $(P : M)$ is prime, we conclude that $a \in (P : M)$. Therefore $aM \subseteq P$, showing that P is a prime submodule. □

The next result may be regarded as an S-analogue of [6, Proposition 1.1]. We begin by introducing the corresponding notion of S-comparability.

Definition 2.7. Let $\mathcal{A}$ be a collection of submodules of an R-module M. We say that $\mathcal{A}$ is *S-comparable* if, for some $s \in S$, every pair of submodules $N, K \in \mathcal{A}$ satisfies either $sN \subseteq K$ or $sK \subseteq N$. In this situation, $\mathcal{A}$ is said to be *associated with* s.

Likewise, a family of ideals of R is called *S-comparable* whenever it is S-comparable as a family of submodules of the regular module R.

Every totally ordered family of submodules is clearly S-comparable. The converse need not hold. Indeed, let $\mathcal{A} = \{2^i\mathbb{Z} \mid i \in \mathbb{N}\} \cup \{3^j\mathbb{Z} \mid 0 \leq j \leq n\}$, where n is a fixed positive integer, and put $S = \{3^k \mid k \in \mathbb{N}_0\}$. Then $\mathcal{A}$ is S-comparable, associated with the element 3^n, although its members are not totally ordered by inclusion.

Proposition 2.8. *Let $P \leq M$ satisfy $(P : M) \cap S = \emptyset$. Consider the following statements:*

1. *$P \in \mathrm{cl.Spec}_S(M)$.*
2. *There is an element $s \in S$ such that, if $K \leq M$ satisfies $(P : K) \cap S = \emptyset$, then $(P : K)$ is an S-prime ideal associated with s.*
3. *There is an element $s \in S$ such that $\{(P : K) \mid K \leq M,\ (P : K) \cap S = \emptyset\}$ is an S-comparable family of S-prime ideals associated with s.*
4. *There is an element $s \in S$ such that $\{(P : m) \mid m \in M,\ (P : m) \cap S = \emptyset\}$ is an S-comparable family of S-prime ideals associated with s.*
5. *There is an element $s \in S$ such that, if $m \in M$ satisfies $(P : m) \cap S = \emptyset$, then $(P : m)$ is an S-prime ideal associated with s.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$. Moreover, if $(P : P_S) \cap S \neq \emptyset$, then $(5) \Rightarrow (1)$.

Proof. The implication $(1) \Rightarrow (2)$ follows directly from Proposition 2.3. Likewise, $(3) \Rightarrow (4)$ and $(4) \Rightarrow (5)$ follow directly.

$(2) \Rightarrow (3)$. Let $K, K' \leq M$ and put $I = (P : K)$ and $J = (P : K')$. Since both I and J are disjoint from S, the same is true for $(P : K+K')$. Hence $(P : K+K')$ is an S-prime ideal. Moreover, $IJ \subseteq I \cap J = (P : K+K')$. Applying the defining property of S-prime ideals, we obtain $sI \subseteq (P : K+K') \subseteq J$ or $sJ \subseteq (P : K+K') \subseteq I$ for some $s \in S$. Thus the family $\{(P : K) \mid K \leq M,\ (P : K) \cap S = \emptyset\}$ is S-comparable.

$(5) \Rightarrow (1)$. Assume that $(P : P_S) \cap S \neq \emptyset$, and choose $t \in (P : P_S) \cap S$. Let $a, b \in R$ and $m \in M$ satisfy $abm \in P$. Equivalently, $ab \in (P : m)$.

If $(P : m) \cap S = \emptyset$, then $(P : m)$ is an S-prime ideal associated with some $s \in S$. Hence $sa \in (P : m)$ or $sb \in (P : m)$, which yields $sam \in P$ or $sbm \in P$.

Now suppose that $(P : m) \cap S \neq \emptyset$. Then $m \in P_S$, so $tm \in P$. Consequently, $tam \in P$ and $tbm \in P$.

Therefore, in either case, $stam \in P$ or $stbm \in P$ for the element $st \in S$. This proves that $P \in \mathrm{cl.Spec}_S(M)$. □

Proposition 2.8 shows that if P is an S-classical prime submodule associated with some $s \in S$, then $(P : M)$ is an S-prime ideal associated with the same element. The following example shows that the converse does not hold. In addition, it demonstrates that the additional assumption $(P : P_S) \cap S \neq \emptyset$ in Proposition 2.8 cannot be omitted.

Example 2.9. Let $R = \mathbb{Z}$, let p and q be distinct prime numbers, and let $S = \{p^n \mid n \in \mathbb{N}_0\}$. Consider the R-module $M = \mathbb{Z}(p^\infty) \times \mathbb{Z}_q$, where $\mathbb{Z}(p^\infty) = \left\{\frac{a}{p^k} + \mathbb{Z} \mid a \in \mathbb{Z},\ k \in \mathbb{N}\right\}$, and let $P = (0)$. Clearly, $(P : M) = 0$ is a prime ideal (and therefore an S-prime ideal) of R.

Take $m = \left(\frac{a}{p^l} + \mathbb{Z}, c\right) \in M$, where $a \in \mathbb{Z}$, $c \in \mathbb{Z}_q$, and $l \in \mathbb{N}_0$. Assume that $a \notin p\mathbb{Z}$. Then $(P : m) = p^l\mathbb{Z}$ whenever $c = 0$, whereas $(P : m) = p^l q\mathbb{Z}$ whenever $c \neq 0$. Accordingly, $(P : m)$ is an S-prime ideal whenever $(P : m) \cap S = \emptyset$.

Nevertheless, P fails to be an S-classical prime submodule. Indeed, for any $s = p^n \in S$, we have $p^{n+1}p^{n+1}\left(\frac{1}{p^{2n+2}} + \mathbb{Z}, 0\right) \in P$, while $sp^{n+1}\left(\frac{1}{p^{2n+2}} + \mathbb{Z}, 0\right) \notin P$. Hence $P \notin \mathrm{cl.Spec}_S(M)$.

Finally, $P_S = \bigcup_{k \in \mathbb{N}_0}(P :_M p^k) = \bigcup_{k \in \mathbb{N}_0}(M_k \times (0)) = \mathbb{Z}(p^\infty) \times (0)$, and therefore $(P : P_S) \cap S = \emptyset$.

Recall that an R-module M is called *S-finite* provided there exist an element $s \in S$ and a finitely generated submodule F of M such that $sM \subseteq F$. The module M is called *S-Noetherian* if each of its submodules is S-finite (see [2]).

Following [1], we say that a multiplicatively closed subset S satisfies the *maximal multiple condition* (abbreviated m.m.c.) whenever there exists an element $s \in S$ with the property that $s \in Rt$ for every $t \in S$. This requirement is fulfilled, for instance, whenever S is finite or every element of S is invertible in R.

Whenever the m.m.c. holds, one may fix an element $t \in S$ that serves simultaneously as an associated element for every S-prime ideal of R and every S-prime submodule of any R-module.

Lemma 2.10. *Let $N \leq M$. Then $(N : N_S) \cap S \neq \emptyset$ in each of the following situations:*

1. *N_S is S-finite (and hence whenever M is S-Noetherian).*
2. *S satisfies the maximal multiple condition.*

Proof. (1). Assume that N_S is S-finite. Then there exist $t \in S$ and finitely many elements $n_1, \ldots, n_k \in N_S$ such that $tN_S \subseteq Rn_1 + \cdots + Rn_k$. Since $n_i \in N_S$ for each i, we may choose $s_i \in S$ satisfying $s_i n_i \in N$. Consequently, $ts_1 \cdots s_k N_S \subseteq N$, which implies that $ts_1 \cdots s_k \in (N : N_S) \cap S$. Hence $(N : N_S) \cap S \neq \emptyset$.

(2). Suppose now that S satisfies the maximal multiple condition, and let $s \in S$ be chosen so that $s \in Rt$ for every $t \in S$. Take any $m \in N_S$. Then $tm \in N$ for some $t \in S$. Since $s = rt$ for a suitable $r \in R$, it follows that $sm = rtm \in N$. Therefore, $sN_S \subseteq N$, showing that $s \in (N : N_S) \cap S$. □

Combining Lemma 2.10 with Proposition 2.8 immediately yields the following characterization.

Corollary 2.11. *Let $P \leq M$ satisfy $(P : M) \cap S = \emptyset$. If S satisfies the maximal multiple condition or P_S is S-finite, then the conditions (1)–(5) of Proposition 2.8 are equivalent.*

The next proposition together with its corollary follows directly from the definitions. Since the arguments are entirely routine, their proofs are omitted.

Proposition 2.12. *Let $\phi : M_1 \to M_2$ be an R-module homomorphism. Then:*

1. *If $P_1 \in \mathrm{cl.Spec}_S(M_1)$ and $\mathrm{Ker}(\phi) \subseteq P_1$, then $\phi(P_1) \in \mathrm{cl.Spec}_S(\phi(M_1))$.*
2. *If $P_2 \in \mathrm{cl.Spec}_S(M_2)$ and $(\phi^{-1}(P_2) : M_1) \cap S = \emptyset$, then $\phi^{-1}(P_2) \in \mathrm{cl.Spec}_S(M_1)$.*

Proof. (1) We first show that $(\phi(P_1) : \phi(M_1)) \cap S = \emptyset$. Suppose that $t \in (\phi(P_1) : \phi(M_1)) \cap S$. Then $t\phi(x) \in \phi(P_1)$ for every $x \in M_1$. Hence, for each $x \in M_1$, there exists $p \in P_1$ such that $\phi(tx) = \phi(p)$. Thus $tx - p \in \ker(\phi) \subseteq P_1$. Since $p \in P_1$, we obtain $tx \in P_1$, and so $t \in (P_1 : M_1)$, contradicting $(P_1 : M_1) \cap S = \emptyset$.

Now let $a, b \in R$ and $\phi(m) \in \phi(M_1)$ satisfy $ab\phi(m) \in \phi(P_1)$. Then $\phi(abm) \in \phi(P_1)$, so there exists $p \in P_1$ such that $\phi(abm) = \phi(p)$. Hence $abm - p \in \ker(\phi) \subseteq P_1$, and therefore $abm \in P_1$. Since P_1 is an

S-classical prime submodule of M_1, there exists $s \in S$ such that $sam \in P_1$ or $sbm \in P_1$. Applying ϕ, we obtain $sa\phi(m) \in \phi(P_1)$ or $sb\phi(m) \in \phi(P_1)$. Hence $\phi(P_1) \in \mathrm{cl.Spec}_S(\phi(M_1))$.

(2) Let $a, b \in R$ and $m \in M_1$ be such that $abm \in \phi^{-1}(P_2)$. Then $ab\phi(m) = \phi(abm) \in P_2$. Since P_2 is an S-classical prime submodule of M_2, there exists $s \in S$ such that $sa\phi(m) \in P_2$ or $sb\phi(m) \in P_2$. Hence $\phi(sam) \in P_2$ or $\phi(sbm) \in P_2$, which implies that $sam \in \phi^{-1}(P_2)$ or $sbm \in \phi^{-1}(P_2)$. Since $(\phi^{-1}(P_2) : M_1) \cap S = \emptyset$, it follows that $\phi^{-1}(P_2)$ is an S-classical prime submodule of M_1. □

Corollary 2.13. *Let $L \leq M$. Then:*

1. *If $P \in \mathrm{cl.Spec}_S(M)$ and $(P : L) \cap S = \emptyset$, then $L \cap P \in \mathrm{cl.Spec}_S(L)$.*
2. *Suppose $L \subseteq P \leq M$. Then $P \in \mathrm{cl.Spec}_S(M)$ if and only if $\frac{P}{L} \in \mathrm{cl.Spec}_S\left(\frac{M}{L}\right)$.*

As Example 2.2 indicates, an S-classical prime submodule is not necessarily classical prime. We note that similar investigations for S-prime ideals appear in [5]. The next result identifies a natural hypothesis under which these two concepts agree.

Proposition 2.14. *Let $P \leq M$ be such that $\mathrm{Zd}\left(\frac{M}{P}\right) \cap S = \emptyset$. Then the following statements are equivalent.*

1. *$P \in \mathrm{cl.Spec}(M)$.*
2. *$P \in \mathrm{cl.Spec}_S(M)$.*
3. *$S^{-1}P \in \mathrm{cl.Spec}(S^{-1}M)$.*
4. *$S^{-1}P \in \mathrm{cl.Spec}_{\bar{S}}(S^{-1}M)$.*

Proof. Since $(P : M) \subseteq \mathrm{Zd}\left(\frac{M}{P}\right)$, condition $\mathrm{Zd}\left(\frac{M}{P}\right) \cap S = \emptyset$ implies $(P : M) \cap S = \emptyset$. Thus (1) $\Rightarrow$ (2) is a direct consequence of Proposition 2.4(1). Moreover, $\mathrm{Zd}\left(\frac{M}{P}\right) \cap S = \emptyset$ implies $\mathrm{Zd}\left(\frac{S^{-1}M}{S^{-1}P}\right) \cap \bar{S} = \emptyset$, so (3) $\Rightarrow$ (4) again follows from Proposition 2.4(1).

(2) $\Rightarrow$ (3). Suppose that P is an S-classical prime submodule associated with $s \in S$. Take $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$ and $\frac{m}{t} \in S^{-1}M$ such that $\frac{r_1}{s_1}\frac{r_2}{s_2}\frac{m}{t} \in S^{-1}P$. Then $ur_1r_2m \in P$ for some $u \in S$. Since P is S-classical prime, either $sur_1m \in P$ or $usr_2m \in P$. Hence $\frac{r_1}{s_1}\frac{m}{t} = \frac{sur_1m}{sus_1t} \in S^{-1}P$ or $\frac{r_2}{s_2}\frac{m}{t} = \frac{usr_2m}{uss_2t} \in S^{-1}P$. Therefore, $S^{-1}P$ is a classical prime submodule of $S^{-1}M$.

(4) $\Rightarrow$ (1). Assume that $S^{-1}P$ is an $\bar{S}$-classical prime submodule associated with $\frac{s}{1} \in \bar{S}$. Let $a, b \in R$ and $m \in M$ satisfy $abm \in P$. Passing to the localization gives $\frac{a}{1}\frac{b}{1}\frac{m}{1} \in S^{-1}P$. Hence, $\frac{s}{1}\frac{a}{1}\frac{m}{1} \in S^{-1}P$ or $\frac{s}{1}\frac{b}{1}\frac{m}{1} \in S^{-1}P$. Accordingly, there exists $u \in S$ such that $usam \in P$ or $usbm \in P$. Since $us \notin \mathrm{Zd}\left(\frac{M}{P}\right)$, multiplication by us induces an injective endomorphism of M/P. Consequently, $am \in P$ or $bm \in P$, showing that P is a classical prime submodule. □

The following example shows that the hypothesis $\mathrm{Zd}\left(\frac{M}{P}\right) \cap S = \emptyset$ cannot be omitted from Proposition 2.14.

Example 2.15. Suppose that $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$, $M = \mathbb{Q} \oplus \mathbb{Q}$, and $P = \mathbb{Z} \oplus (0)$, where $\mathbb{Q} = S^{-1}\mathbb{Z}$. Then $(P : M) \cap S = \emptyset$, whereas $\mathrm{Zd}\left(\frac{M}{P}\right) \cap S = S$.

Fix $s \in S$. Choose a nonzero integer k with $k \nmid s$. Then $k^2\left(\frac{1}{k^2}, 0\right) \in P$, whereas $sk\left(\frac{1}{k^2}, 0\right) \notin P$. Hence $P \notin \mathrm{cl.Spec}_S(M)$. On the other hand, $S^{-1}M$ is a vector space over the field $S^{-1}R$. Since every prime submodule of a vector space is classical prime, $S^{-1}P \in \mathrm{Spec}(S^{-1}M) \subseteq \mathrm{cl.Spec}(S^{-1}M)$.

The proof of the next proposition is essentially identical to that of [11, Theorem 2.15]. Accordingly, we omit it.

Proposition 2.16. *Let $S = \prod_{i=1}^{n} S_i$, $R = \prod_{i=1}^{n} R_i$, $M = \prod_{i=1}^{n} M_i$, and $P = \prod_{i=1}^{n} P_i$, where, for each $1 \leq i \leq n$, S_i is a multiplicatively closed subset of R_i, M_i is an R_i-module, and P_i is a submodule of M_i. Then the following conditions are equivalent.*

1. *$P \in \mathrm{cl.Spec}_S(M)$.*
2. *For some $i \in \{1, \ldots, n\}$, one has $P_i \in \mathrm{cl.Spec}_{S_i}(M_i)$ and $(P_j : M_j) \cap S_j \neq \emptyset$ for every $j \neq i$.*

The next lemma provides several alternative characterizations of S-classical prime submodules that will be needed later in our study of tensor products.

Lemma 2.17. *Let $P \leq M$ with $(P : M) \cap S = \emptyset$. The following conditions are equivalent.*

1. *$P \in \mathrm{cl.Spec}_S(M)$.*
2. *For some $s \in S$, $s(P :_M ab) \subseteq (P :_M a) \cup (P :_M b)$ for all $a, b \in R$.*
3. *For some $s \in S$, $s(P :_M IJ) \subseteq (P :_M I) \cup (P :_M J)$ for all ideals I, J of R.*

Proof. (1) $\Rightarrow$ (2). Assume that P is an S-classical prime submodule associated with an element $s \in S$. Let $a, b \in R$, and choose $m \in (P :_M ab)$. Then $abm \in P$. By the defining property of P, $sam \in P$ or $sbm \in P$. Equivalently, $sm \in (P :_M a)$ or $sm \in (P :_M b)$. Hence $s(P :_M ab) \subseteq (P :_M a) \cup (P :_M b)$.

(2) $\Rightarrow$ (1). Let $a, b \in R$ and suppose that $abm \in P$. Since $m \in (P :_M ab)$, condition (2) implies that $sm \in (P :_M a) \cup (P :_M b)$. Consequently, $sam \in P$ or $sbm \in P$, so P is an S-classical prime submodule.

The equivalence of (1) and (3) follows by the same argument, replacing elements with ideals and applying Proposition 2.3(3). □

Proposition 2.18. *Let $P \in \mathrm{cl.Spec}_S(M)$. If F is a flat R-module satisfying $(F \otimes P : F \otimes M) \cap S = \emptyset$, then $F \otimes P \in \mathrm{cl.Spec}_S(F \otimes M)$.*

Proof. Let $s \in S$ be associated with P. By Lemma 2.17, $s(P :_M ab) \subseteq (P :_M a) \cup (P :_M b)$ for all $a, b \in R$.

Take $f \in F$ and $m \in (P :_M ab)$. Then $sm \in (P :_M a)$ or $sm \in (P :_M b)$. Therefore,

$$s(f \otimes m) = f \otimes sm \in F \otimes (P :_M a) \cup F \otimes (P :_M b).$$

Hence,

$$s\Big(F \otimes (P :_M ab)\Big) \subseteq F \otimes (P :_M a) \cup F \otimes (P :_M b).$$

On the other hand, [4, Lemma 3.2] yields $F \otimes (P :_M ab) = (F \otimes P :_{F \otimes M} ab)$. Consequently,

$$s(F \otimes P :_{F \otimes M} ab) \subseteq F \otimes (P :_M a) \cup F \otimes (P :_M b).$$

Another application of Lemma 2.17 now shows that $F \otimes P$ is an S-classical prime submodule of $F \otimes M$. □

Recall that an R-module M is called an *S-multiplication module* if, for every submodule N of M, one can find an ideal I of R together with an element $s \in S$ such that $sN \subseteq IM \subseteq N$. It is proved in [1, Proposition 4] that, for such modules, a submodule is S-prime exactly when its annihilator ideal is S-prime. Combining this characterization with Proposition 2.8, we conclude that every S-classical prime submodule of an S-multiplication module is S-prime, and hence the two notions agree on this class of modules.

Proposition 2.19. *Let M be an S-multiplication R-module and let $P \leq M$. The following conditions are equivalent.*

1. $P \in \mathrm{Spec}_S(M)$.

2. $P \in \mathrm{cl.Spec}_S(M)$.

3. $(P : M) \in \mathrm{Spec}_S(R)$.

Following [14], an ideal $\mathcal{P}$ of R satisfying $\mathcal{P} \cap S = \emptyset$ is called *S-maximal* provided that there exists an element $s \in S$ such that, whenever $\mathcal{P} \subseteq I$, one has either $sI \subseteq \mathcal{P}$ or $I \cap S \neq \emptyset$. The collection of all S-maximal ideals is denoted by $\mathrm{Max}_S(R)$. It is known from [14, Proposition 10] that every S-maximal ideal is necessarily S-prime.

By [7], an R-module is called *compatible* whenever every classical prime submodule is prime; equivalently, the classes of prime and classical prime submodules agree.

Motivated by this terminology, we call an R-module *S-prime compatible* if each S-classical prime submodule is S-prime. Equivalently, the families $\mathrm{Spec}_S(M)$ and $\mathrm{cl.Spec}_S(M)$ coincide. A ring R will be called *S-prime compatible* if every R-module has this property. By Proposition 2.19, every S-multiplication module is S-prime compatible. Our next result provides a characterization of S-prime compatible rings.

Proposition 2.20. *Consider the following assertions.*

1. *R is an S-prime compatible ring.*

2. *$R \oplus R$ is an S-prime compatible R-module.*

3. $\mathrm{Spec}_S(R) = \mathrm{Max}_S(R)$.

Then $(1) \Rightarrow (2) \Rightarrow (3)$. *Furthermore, if $(P : P_S) \cap S \neq \emptyset$ for every S-classical prime submodule P of every R-module, then* $(3) \Rightarrow (1)$.

Proof. $(1) \Rightarrow (2)$ is clear.

$(2) \Rightarrow (3)$. Choose $\mathcal{P}_1 \in \mathrm{Spec}_S(R)$, and let $s \in S$ be an element associated with $\mathcal{P}_1$. Suppose that I is an ideal satisfying $\mathcal{P}_1 \subseteq I$ and $I \cap S = \emptyset$. Consider the family $\mathcal{A} = \{J \triangleleft R \mid I \subseteq J,\ J \cap S = \emptyset\}$. An application of Zorn's lemma yields a maximal element $\mathcal{P}_2$ of $\mathcal{A}$. As is well known (see, for example, [12, Theorem 3.44]), $\mathcal{P}_2$ is a prime ideal.

Set $N = \mathcal{P}_1 \oplus \mathcal{P}_2$.

We claim that $N \in \mathrm{cl.Spec}_S(R \oplus R)$.

Indeed, let $a, b, x, y \in R$ satisfy $ab(x, y) \in N$. Then $abx \in \mathcal{P}_1$ and $aby \in \mathcal{P}_2$.

If $s(x, y) \in N$, there is nothing further to prove. Assume therefore that $s(x, y) \notin N$.

If $sx \notin \mathcal{P}_1$, the S-primality of $\mathcal{P}_1$ implies that $sa \in \mathcal{P}_1$ or $sb \in \mathcal{P}_1$. Hence $sa(x, y) \in N$ or $sb(x, y) \in N$.

Otherwise, $sx \in \mathcal{P}_1$, so $sy \notin \mathcal{P}_2$. Using the primality of $\mathcal{P}_2$ and the inclusion $aby \in \mathcal{P}_2$, we conclude that $a \in \mathcal{P}_2$ or $b \in \mathcal{P}_2$. Again, $sa(x, y) \in N$ or $sb(x, y) \in N$. This establishes the claim.

By hypothesis, N is in fact an S-prime submodule. Now let $r \in \mathcal{P}_2$. Since $r(0, 1) \in N$ while $s(0, 1) \notin N$, the defining property of S-prime submodules shows that $sr(R \oplus R) \subseteq N$. Consequently, $s\mathcal{P}_2 \subseteq \mathcal{P}_1$. Because $I \subseteq \mathcal{P}_2$, it follows that $sI \subseteq \mathcal{P}_1$. Hence $\mathcal{P}_1$ is an S-maximal ideal.

$(3) \Rightarrow (1)$. Let $P \in \mathrm{cl.Spec}_S(M)$. By assumption, $(P : P_S) \cap S \neq \emptyset$. Choose $s_1 \in (P : P_S) \cap S$. From Proposition 2.8, $(P : M) \in \mathrm{Spec}_S(R)$. Hypothesis (3) therefore implies that $(P : M)$ is S-maximal. Hence there exists $s_2 \in S$ such that, for every ideal $I \supseteq (P : M)$, one has $s_2 I \subseteq (P : M)$ or $I \cap S \neq \emptyset$.

Now let $a \in R$ and $m \in M$ satisfy $am \in P$. Since $(P : M) \subseteq (P : m)$, there are two cases.

If $(P : m) \cap S = \emptyset$, then $s_2(P : m) \subseteq (P : M)$. As $a \in (P : m)$, we obtain $s_2 a \in (P : M)$, whence $s_1 s_2 aM \subseteq P$.

If instead $(P : m) \cap S \neq \emptyset$, then $m \in P_S$, and therefore $s_1 m \in P$. It follows that $s_1 s_2 m \in P$.

In either case, P satisfies the defining condition of an S-prime submodule. Therefore $P \in \mathrm{Spec}_S(M)$. $\square$

Combining Lemma 2.10 with Proposition 2.20 immediately yields the following consequence.

Theorem 2.21. Let R be a ring. Suppose that the multiplicatively closed subset S satisfies the maximal multiple condition. Then the following assertions are equivalent.

1. R is an S-prime compatible ring.

2. $R \oplus R$ is an S-prime compatible R-module.

3. $\mathrm{Spec}_S(R) = \mathrm{Max}_S(R)$.

Recall that a ring R is called an *S-field* whenever (0) is an S-maximal ideal. In addition, R is said to be an *S-integral domain* provided there exists an element $s \in S$ such that, for any $a, b \in R$, the equality $ab = 0$ forces either $sa = 0$ or $sb = 0$ (see [14]). Equivalently, this means that the zero ideal of R is S-prime.

Corollary 2.22. *If R is an S-integral domain and an S-prime compatible ring, then R is an S-field.*

We proceed to identify those rings for which arbitrary intersections of S-prime submodules remain S-classical prime. The next proposition generalizes [9, Theorem 4].

Proposition 2.23. *Assume that S satisfies the maximal multiple condition, and let $\{\mathcal{P}_\alpha\}_{\alpha\in A}$ be a family of ideals of R. Write $A = B \cup C$, where $\{\mathcal{P}_\alpha\}_{\alpha\in B}$ is a nonempty S-comparable family of S-prime ideals and $\mathcal{P}_\alpha \cap S \neq \emptyset$ for every $\alpha \in C$. Then $\cap_{\alpha\in A}\mathcal{P}_\alpha$ is an S-prime ideal of R.*

Proof. Because S satisfies the m.m.c., there exists $t \in S$ with the property that $t \in Rs$ for every $s \in S$. Put $\mathcal{P} = \cap_{\alpha\in B}\mathcal{P}_\alpha$, $\mathcal{Q} = \cap_{\alpha\in C}\mathcal{P}_\alpha$, and $\overline{\mathcal{P}} = \mathcal{P} \cap \mathcal{Q}$. Take $a, b \in R$ such that $ab \in \overline{\mathcal{P}}$.

For every $\alpha \in C$, the assumption $\mathcal{P}_\alpha \cap S \neq \emptyset$ implies $t \in \mathcal{P}_\alpha$. Hence $ta, tb \in \mathcal{Q}$.

Suppose that $tb \notin \overline{\mathcal{P}}$. Then $tb \notin \mathcal{P}_{\alpha_0}$ for some $\alpha_0 \in B$. Since $ab \in \mathcal{P}_{\alpha_0}$ and $\mathcal{P}_{\alpha_0}$ is S-prime with associated element t, we obtain $ta \in \mathcal{P}_{\alpha_0}$.

Now fix $\beta \in B$. As $\{\mathcal{P}_\alpha\}_{\alpha\in B}$ is S-comparable, either $t\mathcal{P}_\beta \subseteq \mathcal{P}_{\alpha_0}$ or $t\mathcal{P}_{\alpha_0} \subseteq \mathcal{P}_\beta$.

In the first case, $tb \notin \mathcal{P}_\beta$; otherwise, $t^2 b \in \mathcal{P}_{\alpha_0}$, which would imply $tb \in \mathcal{P}_{\alpha_0}$ because $\mathcal{P}_{\alpha_0}$ is S-prime, a contradiction. Hence $ta \in \mathcal{P}_\beta$.

In the second case, $t^2 a \in \mathcal{P}_\beta$. Using the S-primality of $\mathcal{P}_\beta$, we again conclude that $ta \in \mathcal{P}_\beta$.

Therefore $ta \in \mathcal{P}$. Since $ta \in \mathcal{Q}$ as well, we have $ta \in \overline{\mathcal{P}}$. Consequently, $\overline{\mathcal{P}}$ is an S-prime ideal of R. □

Theorem 2.24. If S satisfies the m.m.c., then the following are equivalent.

1. Whenever $\{P_\alpha\}_{\alpha\in\Lambda}$ is a family of S-prime submodules of an R-module M, the intersection $\cap_{\alpha\in\Lambda}P_\alpha$ belongs to $\mathrm{cl.Spec}_S(M)$.

2. Whenever $\{P_\alpha\}_{\alpha\in\Lambda}$ is a family of S-prime submodules of an R-module $R \oplus R$, the intersection $\cap_{\alpha\in\Lambda}P_\alpha$ belongs to $\mathrm{cl.Spec}_S(M)$.

3. $\mathrm{Spec}_S(R)$ is an S-comparable family.

Proof. $(1) \Rightarrow (2)$ is clear.

$(2) \Rightarrow (3)$. Choose $\mathcal{P}_1, \mathcal{P}_2 \in \mathrm{Spec}_S(R)$, and define $M = R \oplus R$, $N_1 = \mathcal{P}_1 \oplus R$, and $N_2 = R \oplus \mathcal{P}_2$. Clearly, N_1 and N_2 are S-prime submodules of M. Hence $N_1 \cap N_2 = \mathcal{P}_1 \oplus \mathcal{P}_2 \in \mathrm{cl.Spec}_S(M)$.

Let $s \in S$ be associated with $N_1 \cap N_2$. Assume that $s\mathcal{P}_1 \not\subseteq \mathcal{P}_2$, and choose $a \in \mathcal{P}_1$ with $sa \notin \mathcal{P}_2$. For any $b \in \mathcal{P}_2$, we have $ab(1,1) \in N_1 \cap N_2$. Since $sa(1,1) \notin N_1 \cap N_2$, the S-classical primality of $N_1 \cap N_2$ forces $sb(1,1) \in N_1 \cap N_2$. Thus $s\mathcal{P}_2 \subseteq \mathcal{P}_1$, establishing that $\mathrm{Spec}_S(R)$ is S-comparable.

(3) $\Rightarrow$ (1). Let $\{P_\alpha\}_{\alpha\in\Lambda}$ be a family of S-prime submodules of M, and put $P = \cap_{\alpha\in\Lambda} P_\alpha$. Choose $m \in M$ with $(P : m) \cap S = \emptyset$. According to Corollary 2.11, it is enough to verify that $(P : m)$ is an S-prime ideal.

Observe that $(P : m) = \cap_{\alpha\in\Lambda}(P_\alpha : m)$. At least one ideal $(P_\alpha : m)$ has empty intersection with S. Moreover, every such ideal belongs to $\mathrm{Spec}_S(R)$. Therefore, the desired conclusion follows directly from Proposition 2.23. □

By Cohen's theorem, a commutative ring is Noetherian if and only if all of its prime ideals are finitely generated (see, for example, [12]). Several module-theoretic extensions of this result have also been established.

The following theorem provides an S-analogue of Cohen's theorem for modules.

Theorem 2.25. Let M be an S-finite R-module. Then M is S-Noetherian if and only if every S-classical prime submodule of M is S-finite.

Proof. $(\Rightarrow)$ is immediate.

$(\Leftarrow)$. Assume that M is an S-finite R-module and that every S-classical prime submodule of M is S-finite. Let $\mathcal{A} = \{N \leq M \mid N \text{ is not } S\text{-finite}\}$. Suppose that $\mathcal{A} \neq \emptyset$, and choose a maximal element N of $\mathcal{A}$ by Zorn's lemma.

We first show that $(N : M) \cap S = \emptyset$. Indeed, if $s \in (N : M) \cap S$, then $sM \subseteq N$. As M is S-finite, there are $t \in S$ and a finitely generated submodule $F \leq M$ with $tM \subseteq F$. Hence $stM \subseteq sF \subseteq sM \subseteq N$. As sF is finitely generated, this shows that N is S-finite, contradicting the choice of N. Therefore $(N : M) \cap S = \emptyset$.

We claim that $N \in \mathrm{cl.Spec}_S(M)$. Suppose otherwise. Then for every $s \in S$ there exist $a, b \in R$ and $m \in M$ such that $abm \in N$, $sam \notin N$, and $sbm \notin N$.

Set $K = N + Ram$ and $L = (N :_M a)$. Since $am \notin N$, we have $N \subsetneq K$, and since $m \notin L$, we have $N \subsetneq L$. By the maximality of N, both K and L are S-finite.

Thus there exist $s_1, s_2 \in S$, elements $n_1, \ldots, n_p \in N$, $l_1, \ldots, l_q \in L$, and $z_1, \ldots, z_p \in R$ such that $s_1K \subseteq \sum_{i=1}^{p} R(n_i + z_i am)$ and $s_2L \subseteq \sum_{j=1}^{q} Rl_j$.

Let $x \in N$. Since $x \in K$, we may write $s_1x = \sum_{i=1}^{p} r_i(n_i + z_i am)$ for suitable $r_i \in R$. Hence $\sum_{i=1}^{p} r_i z_i m \in L$, and therefore $s_2 \sum_{i=1}^{p} r_i z_i m = \sum_{j=1}^{q} r'_j l_j$ for suitable $r'_j \in R$. Consequently, $s_1 s_2 x = s_2 \sum_{i=1}^{p} r_i n_i + a \sum_{j=1}^{q} r'_j l_j \in N$. It follows that N is S-finite, contradicting the choice of N.

Hence $N \in \mathrm{cl.Spec}_S(M)$. By hypothesis, every S-classical prime submodule is S-finite, contradicting $N \in \mathcal{A}$. Therefore $\mathcal{A} = \emptyset$, and consequently every submodule of M is S-finite. Thus M is S-Noetherian. □

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