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On J-Noetherian modules and J-coherent modules

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Abstract. This paper continues the study of the recently introduced notion of J-ideals by Khashan and Bani-Ata, extending it to a module-theoretic and homological framework. First, we introduce and investigate J-injective modules, providing a Baer-type characterization and studying their relation to J-torsion-free and J-divisible modules. We then define and examine J-Noetherian modules and rings, showing that many classical properties such as the ascending chain condition and the Cartan–Eilenberg–Bass Theorem admit J-analogues. Finally, we develop the theory of J-flat and J-coherent modules, characterizing J-coherent rings through homological conditions and examples. Our results reveal that many homological behaviors persist in this broader setting, shedding light on the structural richness of the J-ideal framework.

Key Words: J-submodule, J-injective module, J-Noetherian module, J-Noetherian ring, J-flat module, J-coherent module, J-coherent ring.
2010 MSC: 13A15, 13A18, 13F05, 13G05, 13C20.

1 Introduction

Let R be commutative ring with identity element, $J(R)$ its Jacobson radical, $Z(R)$ the set of all zero-divisors of R, and $\mathrm{Nil}(R)$ the set of all nilpotent elements of R.

Recently, Khashan and Bani-Ata [14] introduced the notion of J-ideals as follows: a proper ideal I of a ring R is called a J-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then $b \in I$. The authors study various properties and provide examples of this class of ideals. Furthermore, they explore its relationship with several well-known types of ideals, including r-ideals, prime ideals, primary ideals, and maximal ideals. They also extend the notion to J-submodules of an R-module M and investigate their properties, particularly in the case of multiplication modules.

In [18], Mimouni further investigates this concept in various contexts of commutative rings, specifically in trivial ring extensions, amalgamated constructions of rings, and pullback diagrams. Examples are provided to illustrate the scope and limitations of the results.

This notion has also been studied in other frameworks, such as the S-version, the graded version, and within multiplicative lattices [1, 2, 7, 11, 16, 22].

In contrast to the above works, the present paper approaches the theory of J-ideals from a module-theoretic and homological perspective. Our goal is to investigate this notion not only within the structure of modules but also through the lens of homological algebra, thereby revealing new insights and extending the applicability of J-ideals beyond the classical ideal-theoretic framework.

This paper aims to extend the classical theory of modules and rings by incorporating the concept of J-ideals, recently introduced by Khashan and Bani-Ata, into a broader module-theoretic and

* Corresponding author.

homological framework.

In Section 2, we introduce and investigate the notion of J-injective modules, which generalizes the classical concept of injectivity. We establish a version of Baer's Criterion in this setting and explore the relationship between J-injectivity and J-torsion-free modules via Ext-vanishing conditions. Particular attention is given to ZJ-rings, in which the zero-divisors are contained in the Jacobson radical, and under which many simplifications occur.

Section 3 is devoted to the study of J-Noetherian modules and rings. We show that J-Noetherianity satisfies analogues of the ascending chain condition and the maximal condition for J-submodules. We also present a J-version of the Cartan–Eilenberg–Bass Theorem, characterizing J-Noetherian rings through the behavior of direct sums and direct limits of J-injective modules.

In Section 4, we develop the theory of J-flat and J-coherent modules. We define J-flatness via the exactness of tensor sequences involving finitely generated J-ideals and characterize J-coherent modules as those for which every finitely generated J-submodule is finitely presented. Several equivalent conditions for a ring to be J-coherent are provided, involving vanishing Tor functors, behavior of direct products, and the existence of J-flat preenvelopes.

These developments demonstrate that many familiar homological properties can be meaningfully extended to the J-setting, offering a rich generalization of classical module theory.

2 On J-injective modules

In this section, we initiate the development of a new theory through the study of certain submodules.

Definition 2.1. Let R be a ring and M an R-module. The module M is said to be *J-torsion free* if, for every $s \in R \setminus J(R)$ and every $m \in M$, the condition $sm = 0$ implies $m = 0$.

An R-submodule N of M is called a *J-submodule* of M if either $N = 0$ or the quotient module M/N is J-torsion free; that is, for every $s \in R \setminus J(R)$ and every $m \in M$, the condition $sm \in N$ implies $m \in N$.

Remark 2.2.

1. An ideal I of R is a J-submodule if and only if either $I = 0$ or I is a J-ideal in the sense of [14].
2. When R is a local ring, it is easy to see that every submodule of an R-module is a J-submodule. In particular, every ideal of R is a J-submodule of R.

Definition 2.3. A proper submodule $N \subsetneq M$ is called a *J-submodule in the sense of [14]* if, for all $a \in R$ and $m \in M$,

$$am \in N \quad \text{and} \quad a \notin \big(J(R)M : M\big) \quad \Longrightarrow \quad m \in N.$$

It is important to compare our definition of J-submodules with the one given by the authors of [14]. The following Proposition 2.4 clarifies this comparison.

Proposition 2.4. *For every R-module M and every submodule $N \subsetneq M$, if N is a J-submodule in the sense of Definition 2.1, then N is a J-submodule in the sense of Definition 2.3.*

Proof. Recall that

$$(J(R)M : M) = \{a \in R \mid aM \subseteq J(R)M\}, \quad J(R) \subseteq (J(R)M : M).$$

Hence,

$$R \setminus (J(R)M : M) \subseteq R \setminus J(R). \tag{$*$}$$

Assume N satisfies Definition 2.1. Take $a \in R \setminus (J(R)M : M)$ and $m \in M$ such that $am \in N$. By $(*)$, we have $a \in R \setminus J(R)$, so the defining property of Definition 2.1 forces $m \in N$. Thus, N also satisfies Definition 2.3. □

Remark 2.5. The converse of Proposition 2.4 is *false* in general as we will show in the following Example 2.6.

Example 2.6. Let p be a prime, set $R = \mathbb{Z}$ (so $J(R) = 0$), and take

$$M = \mathbb{Z}/p\mathbb{Z}, \qquad N := 0 \subsetneq M.$$

- N **satisfies Definition 2.3.** Here $J(R)M = 0$ and $(J(R)M : M) = p\mathbb{Z}$. For $a \in \mathbb{Z} \setminus p\mathbb{Z}$ and $m \in M$, $am = 0$ implies $m = 0$ because a is invertible modulo p. Hence N is a J-submodule in the sense of Definition 2.3.
- N **fails Definition 2.1.** The quotient $M/N = M$ is not J-torsion free: choose $s = p \in R \setminus J(R)$ and $m = \overline{1} \in M$, then $sm = 0 \in N$ but $m \notin N$. Thus N is *not* a J-submodule in the sense of Definition 2.1.

Hence Definition 2.3 does not imply Definition 2.1.

As in the classical case, we introduce a new generalization of injective modules as follows.

Definition 2.7. Let R be a ring and E an R-module. We say that E is *J-injective* if, for every extension of R-modules $M \subset N$, where M is a J-submodule of N, any R-homomorphism from M to E can be extended to a homomorphism from N to E.

Remark 2.8.

1. Clearly, every injective module is J-injective. From Remark 2.2, every J-injective module over a local ring is injective.
2. Let R be a ring. Since every R-module is an epimorphic image of a free R-module, a standard verification shows that an R-module E is J-injective if and only if $\mathrm{Ext}^1_R(M,E) = 0$ for every J-torsion-free R-module M.

We now establish an analogue of the well-known Baer's Criterion to characterize J-injectivity in a manner similar to the classical case.

Theorem 2.9. Let R be a ring and E an R-module. Then E is J-injective if and only if every R-homomorphism from a J-ideal of R to E can be extended to a homomorphism from R to E.

Proof. The necessity follows directly from Remark 2.2 (1).

Conversely, let M be a J-submodule of an R-module N, and let $f : M \to E$ be an R-homomorphism. Consider the collection $\mathscr{C}$ of all pairs (C, g) such that $M \subseteq C \subseteq N$, C is a J-submodule of N, and $g : C \to E$ is an R-homomorphism satisfying $g|_M = f$. Clearly, $\mathscr{C} \neq \emptyset$ since $(M, f) \in \mathscr{C}$. Partially order $\mathscr{C}$ by declaring $(C, g) \leq (C', g')$ if $C \subseteq C'$ and $g'|_C = g$. Then $\mathscr{C}$ is inductive, and by Zorn's Lemma, it has a maximal element, say (C_0, g_0).

Suppose $C_0 \neq N$. Choose $x \in N \setminus C_0$ and define $I = \{r \in R : rx \in C_0\}$. We claim that I is a J-ideal of R: indeed, for any $a, s \in R$ with $s \notin J(R)$ and $as \in I$, it follows that $s(ax) \in C_0$, and since C_0 is a J-submodule of N, we conclude $ax \in C_0$, hence $a \in I$.

Now define $h : I \to E$ by $h(r) = g_0(rx)$. Then h is an R-homomorphism and extends to $h' : R \to E$ by assumption. Define $\bar{g} : C_0 + Rx \to E$ by $\bar{g}(c_0 + rx) = g_0(c_0) + h'(r)$.

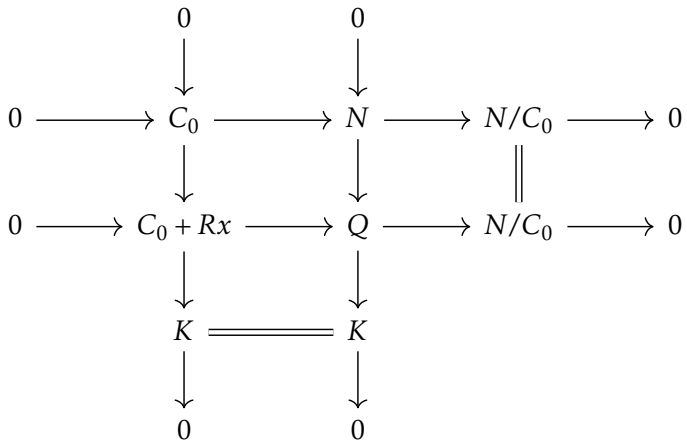
If $c_0 + rx = c_0' + r'x$, then $(c_0 - c_0') = (r' - r)x \in C_0$ and hence $r' - r \in I$. It follows that

$$g_0(c_0 - c_0') = g_0((r' - r)x) = h(r' - r) = h'(r' - r),$$

so

$$g_0(c_0) + h'(r) = g_0(c_0') + h'(r').$$

Therefore, $\bar{g}$ is well-defined and agrees with f on M. Consider the following commutative diagram with exact rows and columns:



Since N/C_0 is a J-torsion free R-module and C_0 is a J-submodule of N, we conclude that C_0+Rx is a J-submodule of Q (and hence of N). Thus, $(C_0+Rx,\bar{g})\in\mathscr{C}$, contradicting the maximality of (C_0,g_0). Therefore, $C_0=N$, completing the proof. □

Corollary 2.10. *Let R be a ring and E an R-module. Then E is J-injective if and only if $\mathrm{Ext}^1_R(R/I,E)=0$ for every J-ideal I of R.*

Proof. This follows immediately from Theorem 2.9. □

Corollary 2.11. *The class of J-injective modules is closed under direct products.*

Proof. This follows directly from the fact that Ext commutes with products in the second variable. □

In the remainder of this paper, we focus on rings R satisfying the condition $Z(R)\subseteq J(R)$. We refer to such rings as ZJ-rings. In particular, every local ring is a ZJ-ring.

Corollary 2.12. *Let R be a ZJ-ring. The following statements are equivalent for an R-module E:*

1. *E is J-injective;*
2. *$\mathrm{Ext}^k_R(M,E)=0$ for every J-torsion-free R-module M and every $k\in\mathbb{N}^*$;*
3. *$\mathrm{Ext}^1_R(M,E)=0$ for every J-torsion-free R-module M;*
4. *$\mathrm{Ext}^k_R(R/I,E)=0$ for every J-ideal I of R and every $k\in\mathbb{N}^*$;*
5. *$\mathrm{Ext}^1_R(R/I,E)=0$ for every J-ideal I of R.*

Proof. The implications (2) ⇒ (4), (4) ⇒ (5), (2) ⇒ (3), and (3) ⇒ (5) are all immediate.

(5) ⇒ (1) follows directly from Corollary 2.10.

(1) ⇒ (2): The case $k=1$ follows from Remark 2.8 (2). Assume now that $k>1$, and let M be a J-torsion-free R-module. Consider a projective resolution of M:

$$0\to K\to F_{k-2}\to\cdots\to F_0\to M\to 0,$$

where each F_i is a free R-module. Then K is a torsion-free R-module. Since R is a ZJ-ring, every torsion-free R-module is J-torsion-free. In particular, K is J-torsion-free, and thus

$$\mathrm{Ext}_R^k(M,E) \cong \mathrm{Ext}_R^1(K,E) = 0$$

by dimension shifting. Therefore, statement (2) holds for all $k \in \mathbb{N}^*$. □

It is well known that in the classical setting, injective modules are divisible. We now introduce an analogous concept for J-injectivity.

Definition 2.13. Let R be a ring, $s \in R$, and M an R-module. We say that s is a *J-regular* element if s is regular in R and the ideal (s) generated by s is a J-ideal.

An R-module M is said to be *J-divisible* if $aM = M$ for every J-regular element $a \in R$.

Remark 2.14. Clearly, when R is a local ring, J-regularity and J-divisibility coincide with classical regularity and divisibility, respectively. In particular, by [24, Theorem 2.4.5 (3)], over an integral domain, a (J)-torsion-free R-module E is J-injective if and only if E is injective.

Corollary 2.15. *Let R be a ring and E an R-module. If E is J-injective, then E is J-divisible.*

Proof. This follows immediately from Corollary 2.10 and [24, Exercise 3.3 (3)]. □

3 J-Noetherian Modules

Having introduced the notion of J-injective modules as a generalization of injectivity, we now follow the classical approach to define a new class of modules and rings that extends the concept of Noetherianity.

Definition 3.1. Let R be a ring and M an R-module. We say that M is *J-Noetherian* if M is finitely generated and every J-submodule of M is also finitely generated.

The ring R is called a *J-Noetherian ring* if it is a J-Noetherian module over itself.

Remark 3.2. Clearly, every Noetherian R-module is J-Noetherian. In particular, for a local ring R, Noetherianity and J-Noetherianity coincide by Remark 2.2 (2).

Let A be a commutative ring and let E be an A-module. The *trivial ring extension* of A by E, also called the *idealization* of E, is the set

$$A \propto E := \{(a,e) \mid a \in A, e \in E\},$$

with addition defined componentwise:

$$(a,e) + (a',e') := (a+a', e+e'),$$

and multiplication defined by:

$$(a,e)(a',e') := (aa', ae' + a'e).$$

It is easy to verify that $A \propto E$ is a commutative ring with unity $(1,0)$, where 0 denotes the zero of E.

The ring $A \propto E$ contains a natural ideal $0 \propto E := \{(0,e) \mid e \in E\}$, which is isomorphic to E as an A-module. Also, $A \propto E/(0 \propto E) \cong A$, so the projection map onto the first component gives a ring homomorphism $A \propto E \to A$.

This construction, introduced by D. D. Anderson and M. Winders [3], is often used to transfer or study ring-theoretic and module-theoretic properties in a new context (for instance, [13]), particularly for investigating properties like coherence, clean rings, or Gorenstein dimensions.

It is important to illustrate that the J-Noetherian property does not imply classical Noetherianity. The following example provides such a case.

Example 3.3. Let $\pi : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ be the projection map defined by $\pi(a,b) = a$. It is easy to verify that π is a ring homomorphism. Let $A = \mathbb{Z}\times\mathbb{Z}$. Then every abelian group can be viewed as an A-module via the structure induced by π. Let p be a prime integer, and set $R = A \propto \mathbb{Z}_{p^\infty}$. Then R is a J-Noetherian ring that is not Noetherian.

Proof. First, observe that $\mathbb{Z}_{p^\infty}$ is not a finitely generated abelian group. Consequently, it is not a finitely generated A-module, and thus R is not a Noetherian ring.

On the other hand, as noted in [4], the group $\mathbb{Z}_{p^\infty}$ is *almost Noetherian*, meaning that every proper subgroup is finitely generated. Therefore, every proper abelian subgroup of $\mathbb{Z}_{p^\infty}$ is a finitely generated R-module.

Next, observe that $J(R) = \mathrm{Nil}(R) = 0 \propto \mathbb{Z}_{p^\infty}$, which is not a J-ideal of R. Indeed, we have:

$$((1,0),0)\cdot((0,1),0) = ((0,0),0) \in J(R),$$

yet neither $((1,0),0)$ nor $((0,1),0)$ belongs to $J(R)$.

Suppose, for contradiction, that there exists a J-ideal $I \subsetneq J(R)$ that is not finitely generated. Then I has an infinite generating set of the form $\{(0,0),e_i\}_i$. It is easy to see that $R((0,0),e_i) = (0,0) \propto Ae_i$ for each i, and so

$$I = (0,0) \propto \sum_i Ae_i \subsetneq J(R).$$

Hence, $H = \sum_i Ae_i$ is a proper A-submodule of $\mathbb{Z}_{p^\infty}$. Since H is a finitely generated abelian subgroup of $\mathbb{Z}_{p^\infty}$, it is also a finitely generated A-module — a contradiction.

Therefore, every J-ideal of R contained in $J(R)$ must be finitely generated. By [14, Proposition 2.2], we conclude that R is a J-Noetherian ring. □

We now show that J-Noetherian modules satisfy the ascending chain condition and the maximal condition, in a form analogous to the classical Noetherian case.

Theorem 3.4. Let R be a ring. The following statements are equivalent for an R-module M:

1. M is a J-Noetherian module;
2. M satisfies the ascending chain condition (ACC) on J-submodules, that is, for any ascending chain
$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots,$$
where each M_i is a J-submodule of M, there exists a positive integer m such that $M_n = M_m$ for all $n \geq m$;
3. M satisfies the J-maximal condition, that is, every nonempty set of J-submodules of M contains a maximal element.

Proof. (1) ⇒ (2): Let $\{M_n\}$ be an ascending chain of J-submodules of M, and define $N := \bigcup_n M_n$. By assumption, N is a J-submodule of M, and hence finitely generated. Write $N = Rx_1 + \cdots + Rx_k$. Then, for some m, all x_i belong to M_m, so $N \subseteq M_m$. Since $M_m \subseteq N$, we have $N = M_m$. Therefore, the chain stabilizes at M_m.

(2) ⇒ (3): Let Γ be a nonempty set of J-submodules of M with no maximal element. Choose any $M_1 \in \Gamma$. Since M_1 is not maximal, there exists $M_2 \in \Gamma$ such that $M_1 \subset M_2$. Repeating this process, we obtain an infinite strictly ascending chain

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots,$$

contradicting the assumption that every ascending chain of J-submodules stabilizes.

(3) $\Rightarrow$ (1): Let N be a J-submodule of M, and consider the set

$$\Gamma := \{A \subseteq N \mid A \text{ is a finitely generated } J\text{-submodule of } N\}.$$

Clearly, Γ is nonempty since $0 \in \Gamma$. By assumption, Γ has a maximal element, say A. If $A \neq N$, then there exists $x \in N \setminus A$, and $A_1 := A + Rx$ is a finitely generated submodule of N strictly containing A. As in the proof of Theorem 2.9, one can verify that A_1 is a J-submodule of M, contradicting the maximality of A. Hence, $N = A$ is finitely generated, and M is J-Noetherian. □

Corollary 3.5. *Let R be a ring. The following statements are equivalent:*

1. *R is a J-Noetherian ring;*
2. *R satisfies the ascending chain condition (ACC) on J-ideals, that is, for any ascending chain*

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots,$$

   there exists a positive integer m such that $I_n = I_m$ for all $n \geq m$;
3. *R satisfies the J-maximal condition, that is, every nonempty set of J-ideals of R has a maximal element.*

Proof. This follows immediately from Theorem 3.4. □

We now establish the relationship between exact sequences and J-Noetherianity, in analogy with the classical theory.

Theorem 3.6. Let R be a ring, and let

$$0 \to M' \to M \to M'' \xrightarrow{\pi} 0$$

be a short exact sequence of R-modules, where M' is a J-submodule of M. Then M is a J-Noetherian module if and only if both M' and M'' are J-Noetherian modules.

Proof. Assume that M is J-Noetherian. Let X be a J-submodule of M'. To show that X is finitely generated, it suffices to prove that X is a J-submodule of M. Let $x \in M$ and $s \in R \setminus J(R)$ such that $sx \in X$. Then $s\pi(x) = 0$ in M''. Since M'' is J-torsion free, it follows that $\pi(x) = 0$, and hence $x \in M'$. So $x \in X$ as $sx \in X$ and X is a J-submodule of M'. Thus, X is a J-submodule of M, and therefore finitely generated. Hence M' is J-Noetherian.

Now let Y be a J-submodule of M'', and consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & P & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

Since $M''/Y \cong P/M$, and M''/Y is J-torsion free, we conclude that P is a J-submodule of M. By assumption, M is J-Noetherian, so P is finitely generated. It follows that Y is finitely generated, and hence M'' is J-Noetherian.

Conversely, suppose M' and M'' are J-Noetherian. Let

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$$

be an ascending chain of J-submodules of M. Then

$$M' \cap M_1 \subseteq M' \cap M_2 \subseteq \cdots$$

is an ascending chain of J-submodules of M', and

$$\frac{M_1 + M'}{M'} \subseteq \frac{M_2 + M'}{M'} \subseteq \cdots$$

is an ascending chain of J-submodules of M/M': for each i, we have

$$\frac{M_i + M'}{M'} \cong \frac{M_i}{M_i \cap M'}.$$

Let $s \in R \setminus J(R)$ and suppose $\bar{x} \in M/M'$ satisfies $s\bar{x} \in M_i/(M_i \cap M')$. Then $sx - m_i \in M_i \cap M'$ for some $m_i \in M_i$, so $sx \in M_i$ and since M_i is a J-submodule, it follows that $x \in M_i$, hence $\bar{x} \in M_i/(M_i \cap M')$. Thus, $M_i/(M_i \cap M')$ is a J-submodule of M/M'.

Since M' and M/M' are J-Noetherian, the above chains stabilize at some $m \in \mathbb{N}$. That is,

$$M' \cap M_n = M' \cap M_m \quad \text{and} \quad M' + M_n = M' + M_m \quad \text{for all } n \geq m.$$

It follows that $M_n = M_m$ for all $n \geq m$, so M is J-Noetherian by Theorem 3.4. □

Corollary 3.7. *Let R be a ring and $\{M_i\}_{i \in I}$ a finite collection of J-torsion-free R-modules. Then $\bigoplus_{i \in I} M_i$ is a J-Noetherian R-module if and only if each M_i is J-Noetherian.*

Proof. We proceed by induction on the number of summands, applying Theorem 3.6. □

Corollary 3.8. *Let R be a ring and I a J-ideal of R. Then R is a J-Noetherian ring if and only if both I and R/I are J-Noetherian R-modules.*

Proof. This follows immediately from Theorem 3.6. □

Corollary 3.9. *Let R be a ZJ-ring that is J-Noetherian, and let M be a J-torsion-free R-module. Then M is J-Noetherian if and only if it is a finitely generated R-module.*

Proof. The forward direction is immediate, since J-Noetherian modules are, in particular, finitely generated. Conversely, suppose M is finitely generated. Then M is a quotient of a finite direct sum $R^{(n)}$. Since $Z(R) \subseteq J(R)$, R is J-torsion-free, and by Corollary 3.7, $R^{(n)}$ is J-Noetherian. Therefore, M is J-Noetherian by Theorem 3.6. □

Theorem 3.10 (J-version of Cartan–Eilenberg–Bass)**.** The following statements are equivalent for a ring R:

1. R is J-Noetherian;
2. every direct limit of J-injective R-modules is J-injective;
3. every direct sum of J-injective R-modules is J-injective.

Proof. (1) $\Rightarrow$ (2): Let $E = \varinjlim E_j$, where each E_j is a J-injective R-module, and let I be a J-ideal of R. Then

$$\operatorname{Ext}^1_R(R/I, E) = \varinjlim \operatorname{Ext}^1_R(R/I, E_j)$$

by [24, Theorem 3.9.4 (1)]. Since each E_j is J-injective, each term vanishes, so $\operatorname{Ext}^1_R(R/I, E) = 0$. Thus, E is J-injective by Corollary 2.10.

(2) $\Rightarrow$ (3): This follows from [24, Example 2.5.30 (3)], since direct sums are special cases of direct limits.

(3) $\Rightarrow$ (1): Suppose R is not J-Noetherian. Then, by Theorem 3.5, there exists a strictly ascending chain $I_1 \subset I_2 \subset \cdots$ of J-ideals that does not stabilize. Let $I = \bigcup_{i=1}^{\infty} I_i$. Then $I/I_i \neq 0$ for all i.

Define f_i as the composition of the natural map $\tau_i : I \to I/I_i$ with the inclusion into its injective envelope $E(I/I_i)$. Define $f : I \to \bigoplus_{i=1}^{\infty} E(I/I_i)$ by $f(a) = (f_i(a))_i$. Since each $a \in I$ belongs to only finitely many I_i, $f(I)$ is well-defined.

If $\bigoplus_{i=1}^{\infty} E(I/I_i)$ were J-injective, then f would extend to an R-homomorphism $g : R \to \bigoplus_{i=1}^{\infty} E(I/I_i)$ by Theorem 2.9. Let π_j be the projection onto the j-th component. Then, for sufficiently large i, we have $\pi_i \circ g(1) = 0$, so for $a \in I$,

$$f_i(a) = \pi_i \circ f(a) = \pi_i \circ g(a) = a \cdot \pi_i(g(1)) = 0.$$

This implies that each f_i is eventually zero, contradicting the assumption that $I/I_i \neq 0$ for all i. Hence, the direct sum is not J-injective, and the result follows. □

Remark 3.11. Let $R = A \propto \mathbb{Z}_{p^\infty}$, where $A = \mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Z}_{p^\infty}$ is the Prüfer p-group viewed as an A-module via the projection map $\pi((a, b)) = a$. Over this ring (see Example 3.3), there exists a J-injective R-module which is not injective. Consider the R-module

$$E := \frac{R}{J(R)} \cong A = \mathbb{Z} \times \mathbb{Z}.$$

(i) E is J-injective. By Theorem 2.9, an R-module E is J-injective if every R-homomorphism $\varphi : I \to E$, where I is a J-ideal of R, extends to a homomorphism $R \to E$. In this case, the only proper J-ideal of R is $J(R) = 0 \propto \mathbb{Z}_{p^\infty}$. Since $J(R) \cdot E = 0$, any such homomorphism must be the zero map, which clearly extends to R. Thus, E is J-injective.

(ii) E is not injective. Consider the short exact sequence

$$0 \longrightarrow J(R) \xrightarrow{\iota} R \xrightarrow{\pi} E \longrightarrow 0.$$

If E were injective, this sequence would split, implying that $R \cong E \oplus J(R)$ as R-modules. Hence, $J(R)$ would be isomorphic to eR for some idempotent $e \in R$. However, since $J(R) \subseteq \mathrm{Nil}(R)$, and the Jacobson radical contains no nontrivial idempotents, we must have $e = 0$, contradicting the assumption $J(R) \cong eR \neq 0$. Therefore, the sequence does not split and E is not injective.

4 On J-flat Modules and J-coherent Modules

We now generalize the concept of flat modules in the context of the J-setting.

Definition 4.1. Let R be a ring and F an R-module. We say that F is *J-flat* if, for every finitely generated J-ideal I of R, the sequence

$$0 \to I \otimes_R F \to R \otimes_R F$$

is exact.

Remark 4.2. Clearly, every flat R-module is J-flat. Moreover, by Remark 2.2, J-flatness coincides with classical flatness in certain cases.

It is straightforward to verify that the class of J-flat modules is closed under direct sums.

Proposition 4.3. *Let R be a ring and F an R-module. Then F is J-flat if and only if $\mathrm{Tor}_1^R(R/I,F) = 0$ for every finitely generated J-ideal I of R. Moreover, if R is a ZJ-ring, then F is J-flat if and only if $\mathrm{Tor}_k^R(R/I,F) = 0$ for every $k \in \mathbb{N}^*$ and every finitely generated J-ideal I of R.*

Proof. The first assertion follows by adapting the classical flatness criterion via Tor. The second assertion follows similarly to the argument used in the proof of Corollary 2.12. □

In what follows, we denote by $\mathscr{F}^J$ the class of all finitely presented cyclic J-torsion-free R-modules.

Definition 4.4. Let R be a ring and M an R-module. We say that M is *J-coherent* if M is finitely generated and every finitely generated J-submodule of M is finitely presented.

We say that R is a *J-coherent ring* if R is a J-coherent module over itself.

Remark 4.5. A ring R is J-coherent if and only if it is $\mathscr{F}^J$-coherent in the sense of [27], where $\mathscr{C} = \mathscr{F}^J$.

Example 4.6. Following [6], a ring R is said to be *left J-coherent* if its Jacobson radical $J(R)$ is a coherent left R-module.

It is easy to see that J-coherence in the sense of [6] implies our notion of J-coherence. In particular, by [6, Example 2.8], let R be a non-coherent commutative domain and let G be a free abelian group of infinite rank. Then the group ring RG is not coherent, yet it is J-coherent in our sense.

Theorem 4.7. Let R be a ZJ-ring. If R is a J-Noetherian ring, then R is a J-coherent ring.

Proof. Let I be a J-ideal of R. Since R is J-Noetherian, I is finitely generated. Consider the following pullback diagram of R-modules:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^{(n-1)} & \longrightarrow & P & \longrightarrow & I & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^{(n-1)} & \longrightarrow & R^{(n)} & \longrightarrow & R & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & R/I & = & R/I & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since R is a ZJ-ring, Corollary 3.7 implies that $R^{(n)}$ is J-Noetherian. Because P is a J-submodule of $R^{(n)}$, Theorem 3.6 shows that P is also J-Noetherian. Therefore, as a J-submodule of itself, P is finitely presented. By [10, Theorem 2.1.2 (2)], it follows that I is a finitely presented ideal. Hence, R is J-coherent. □

Here is another example of a J-coherent ring in our sense, which is not J-coherent in the sense of [6], constructed via a trivial ring extension.

Example 4.8. Let $R = \mathbb{Z} \propto \mathbb{Q}$, the trivial ring extension of $\mathbb{Z}$ by $\mathbb{Q}$. Then R is a J-coherent ring in our sense, since the only finitely generated J-ideal of R is the zero ideal.

Indeed, by [3, Corollary 3.4] and [14, Proposition 2.2], any nonzero J-ideal I of R is of the form $I = 0 \propto G$ where G is a subgroup of $\mathbb{Q}$.

If $G = \mathbb{Q}$, then I is a prime ideal but not finitely generated. Suppose instead that G is a proper subgroup of $\mathbb{Q}$. Then there exists $\frac{p}{q} \in \mathbb{Q} \setminus G$.

Assume $\mathbb{Z} \subset G$. Then we have:

$$(q,0)(0,\tfrac{p}{q}) = (0,p) \in 0 \propto G,$$

but $(0,\frac{p}{q}) \notin 0 \propto G$, contradicting the definition of a J-ideal. Thus, $\mathbb{Z} \not\subset G$, and there exists $\alpha \in \mathbb{Z} \setminus G$.

Now choose $\beta \in \mathbb{Z}^* \cap G$. Such β exists: if $\frac{x}{s} \in G \setminus \{0\}$, then $x \in sG \subset G$ (since G is a group), and we may take $\beta = x$. Then $\frac{\alpha}{\beta} \notin G$, but:

$$(\beta^2,0)(0,\tfrac{\alpha}{\beta}) = (0,\beta\alpha) \in 0 \propto G.$$

Hence, $0 \propto G$ is not a J-ideal of R, a contradiction. It follows that the only J-ideal of R is zero, which is trivially finitely generated. Thus, R is J-coherent in our sense, but not J-coherent in the sense of [6], as it is not coherent.

The following result is an analogue of the well-known behavior described in [10, Theorem 2.2.1].

Theorem 4.9. Let R be a ring and consider a short exact sequence of R-modules

$$0 \to P \xrightarrow{f} N \xrightarrow{g} M \to 0$$

such that P is a J-submodule of N. Then:

1. If N is a J-coherent R-module and P is finitely generated, then M is a J-coherent R-module.
2. If M is a coherent R-module and P is a J-coherent R-module, then N is a J-coherent R-module.
3. If both N and M are J-coherent R-modules, then so is P.

Proof. (1) Since N is finitely generated, so is M. Let F be a finitely generated J-submodule of M. Then $F \cong A/P$, where A is a finitely generated J-submodule of N and P is a submodule of A. Since N is J-coherent, A is finitely presented. Therefore, by [15, (4.54) Lemma], the quotient F is finitely presented. Thus, M is a J-coherent module.

(2) Since M is coherent and P is finitely generated, N is finitely generated by [10, Theorem 2.1.2 (1)]. Let X be a finitely generated J-submodule of N, and consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(g_{|X}) & \longrightarrow & X & \xrightarrow{g} & g(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \xrightarrow{f} & N & \xrightarrow{g} & M & \longrightarrow & 0
\end{array}$$

Since X is finitely generated, $g(X)$ is also finitely generated. As M is coherent, $g(X)$ is finitely presented. Moreover, $\ker(g_{|X}) \subseteq P$ is a J-submodule and hence finitely presented by the J-coherence of P. It follows that X is finitely presented, and by [10, Theorem 2.1.2], N is J-coherent.

(3) By [10, Theorem 2.1.2 (3)], P is finitely generated. Let G be a finitely generated J-submodule of P. Then G is also a J-submodule of N. Since N is J-coherent, G is finitely presented. Hence P is J-coherent. □

Theorem 4.10. Let R be a ring and let I be a finitely generated ideal contained in $J(R)$. Then an R/I-module M is J-coherent if and only if it is J-coherent as an R-module. In particular:

1. If R is a J-coherent ring and $I \subseteq J(R)$ is a finitely generated ideal, then R/I is a J-coherent ring.

2. If R/I is a coherent ring and I is a J-coherent R-module, then R is a J-coherent ring.

Proof. We use the result from [10, Theorem 2.1.8], which states that an R-module M is finitely presented if and only if M/IM is a finitely presented R/I-module.

Now assume that M is a J-coherent R-module. We claim that M is also a J-coherent R/I-module. Let N be a finitely generated J-submodule of M, considered as an R/I-module. Then N is also a finitely generated J-submodule of M as an R-module. Since M is assumed J-coherent over R, it follows that N is finitely presented as an R-module. Hence, by the above result, N is finitely presented over R/I, and thus M is a J-coherent R/I-module.

Conversely, suppose M is a J-coherent R/I-module. Let N be a finitely generated J-submodule of M as an R-module. Then N is also a finitely generated J-submodule of M considered as an R/I-module. Since M is J-coherent over R/I, N is finitely presented over R/I, and therefore, by [10, Theorem 2.1.8], N is also finitely presented over R. Thus, M is J-coherent over R.

For part (2), assume that R/I is a coherent ring and I is a J-coherent R-module. Then by part (1) and Theorem 4.9 (2), it follows that R is a J-coherent ring. □

To characterize J-coherent rings, we introduce a J-analogue of the well-known concept of FP-injective modules.

Definition 4.11. An R-module M is said to be *J-FP-injective* if $\operatorname{Ext}^1_R(R/I, M) = 0$ for every finitely generated J-ideal I of R.

Theorem 4.12. The following statements are equivalent for a ring R:

1. R is J-coherent;
2. For any finitely generated J-ideal I of R and any direct system $(M_\alpha)_{\alpha\in A}$ of R-modules, we have
$$\varinjlim \operatorname{Ext}^1_R(R/I, M_\alpha) \cong \operatorname{Ext}^1_R(R/I, \varinjlim M_\alpha);$$
3. For any family $\{N_\alpha\}$ of R-modules and any finitely generated J-ideal I of R, we have
$$\operatorname{Tor}^R_1\Big(\prod N_\alpha, R/I\Big) \cong \prod \operatorname{Tor}^R_1(N_\alpha, R/I);$$
4. Any direct product of copies of R is J-flat;
5. Any direct product of J-flat R-modules is J-flat;
6. Any direct limit of J-FP-injective R-modules is J-FP-injective;
7. Any direct limit of injective R-modules is J-FP-injective;
8. An R-module M is J-FP-injective if and only if M^+ is J-flat;
9. An R-module M is J-FP-injective if and only if M^{++} is J-FP-injective;
10. An R-module M is J-flat if and only if M^{++} is J-flat;
11. For any ring S, finitely generated J-ideal I of R, (R,S)-bimodule B, and injective S-module E, we have
$$\operatorname{Tor}^R_1(\operatorname{Hom}_S(B,E), R/I) \cong \operatorname{Hom}_S\Big(\operatorname{Ext}^1_R(R/I, B), E\Big);$$
12. Every R-module has a J-flat preenvelope.

Proof. This follows immediately from Remark 4.5 and [27, Theorem 3.3]. □

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A going-down theoretic proof that straight domains are locally divided

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Abstract. A result on going-down domains is used to give a new proof of the recent result of Spencer Secord that all straight domains are locally divided domains. Corollaries collect characterizations of straight domains (equivalently, of locally divided domains) and of prime ideals of a base domain R that are straight with respect to all overring extensions of R. For integrally closed (but not necessarily quasi-local) base domains R, it is shown how to use the "going-down to P" property and the behavior of a relatively small class of singly generated overrings of R in order to characterize the locally divided prime ideals of R, while avoiding any mention of "straight"-related concepts.
Key Words: Integral domain, prime ideal, going-down, going-down domain, divided domain, locally divided domain, torsion-free module, quotient field, overring, straight domain
2010 MSC: Primary 13B21, 13F05; Secondary 13G05, 13A15, 13C13.

1 Introduction

In this note, all rings are associative and commutative, and they are usually (integral) domains; all inclusion of rings denote subrings; and all rings, ring homomorphims, and modules are unital. If R is a domain with quotient field K, then by an *overring* of R, we mean any ring T such that $R \subseteq T \subseteq K$. For any ring A, we let $\mathrm{Spec}(A)$ (resp., $\mathrm{Max}(A)$) denote the set of all prime (resp., all maximal) ideals of A. In the spirit of [12], we let GD denote the going-down property of ring homomorphisms (for instance, of ring extensions). For the sake of completeness, this Introduction next gives two paragraphs of background material; then a short paragraph stating the recent result of Spencer Secord that will be reproved in Theorem 2.1; then a paragraph explaining why we believe that it would be natural to use GD-theoretic methods while seeking a new proof of Secord's result; and, finally, a short paragraph announcing some new characterizations of straight domains and locally divided prime ideals that will be proved in Corollary 2.2 and Proposition 2.4, as well as the simpler characterizations in Section 3 when the base domain is integrally closed.

Recall that if $f : A \to B$ is a (unital) ring homomorphism (for instance, a ring extension $A \hookrightarrow B$), then B can be canonically viewed as an A-module, via $a \cdot b := f(a)b$ for all $a \in A$ and $b \in B$. This fact is particularly useful when one is also given an ideal I of A, for f then induces a ring homomorphism $A/I \to B/IB$ that allows B/IB to be viewed as a module over A/I. Some of the rest of this paragraph results from editing the first and second paragraphs of the Introduction in [9]. As in [7], a ring homomorphism $R \to T$ is said to be a *prime morphism* if T/PT is torsion-free over (the domain) R/P for each $P \in \mathrm{Spec}(R)$ (when the induced ring homomorphism $R/P \to T/PT$ is used to view T/PT as a module over R/P). It was shown, *inter alia*, in Proposition 2.3 of [9] that a ring homomorphism $R \to T$ is a prime morphism if and only if the induced map $R_M \to T_M$ ($:= T_{R\setminus M}$) is a prime morphism for each $M \in \mathrm{Max}(R)$. As in [9], a domain R is said to be a *straight domain* if $R \hookrightarrow T$ is a prime morphism for each overring T of R. The most natural examples of straight domains are Pruefer domains (the two-fold point being that Pruefer domains are the domains all of whose overrings are flat; and any

flat ring homomorphism is a prime morphism). Moreover, it follows from the proof of Proposition 3.5 (b) of [9] that if R is a domain, then: R is a straight domain $\Leftrightarrow$ R_P is a straight domain for each $P \in \mathrm{Spec}(R) \Leftrightarrow R_M$ is a straight domain for each $M \in \mathrm{Max}(R)$.

Recall from [5] and [8] that a domain R, with quotient field K, is said to be a *going-down domain* if $R \subseteq T$ satisfies GD for each overring T of R (equivalently, if $R \subseteq V$ satisfies GD for each valuation overring V of R; equivalently, if $R \subseteq T$ satisfies GD for each domain T having R as a subring; equivalently, if $R \subseteq R[u]$ satisfies GD for each $u \in K$). The most natural examples of going-down domains are domains of Krull dimension 1 and Pruefer domains. It was noted in Remark 2.7 (b) of [6] that a domain R is a going-down domain if and only if R_M is a going-down domain for all $M \in \mathrm{Max}(R)$ (equivalently, if R_Q is a going-down domain for all $Q \in \mathrm{Spec}(R)$). Next, recall from [6] that a domain R is said to be a *divided domain* if $PR_P = P$ for all $P \in \mathrm{Spec}(R)$. (Such domains were introduced by T. Akiba in [1], where they were called AV-domains.) The most natural example of a divided domain is a valuation domain. It was shown in Proposition 2.1 of [6] that each divided domain is a quasi-local going-down domain. (Example 2.9 of [6] showed that if $2 \leq n \leq \infty$, then there exists a quasi-local going-down domain of Krull dimension n which is not a divided domain.) Recall from Remark 2.7 (b) of [6] that a domain R is said to be a *locally divided domain* if R_M is a divided domain for all $M \in \mathrm{Max}(R)$ (equivalently, if R_Q is a divided domain for all $Q \in \mathrm{Spec}(R)$); and that any localization of a locally divided domain is itself locally divided (hence, divided domains are the same as the quasi-local locally divided domains).

In Corollary 3.1.1 of [14], Secord recently proved that a domain is a straight domain if and only if it is a locally divided domain. As noted in [14], the "if" assertion was already known: see Corollary 3.8 of [9]. The next paragraph explains why/how Secord's result settled a natural open question; and why we have decided to devote Theorem 2.1 to giving a new proof of (the "only if" assertion in) the theorem of Secord.

A noteworthy difference between our proof in Theorem 2.1 and the exposition in [14] is that our proof features roles for the GD property of ring extensions and for the class of going-down domains. This seems appropriate to us for two reasons: all the relevant kinds of domains considered here (and many of those in [14]) are examples of going-down domains; and a technical result on quasi-local going-down domains from [6] (namely, its Corollary 2.6) will justify the only nontrivial step (apart from a displayed calculation) in our proof of Theorem 2.1. For historical and motivational purposes, we next recall the first sentence of a paragraph on page 760 of [9]: "The following observations constitute much of the motivation for the present article." That paragraph goes on to discuss the hint that was given by Kaplansky for Exercise 37 on pages 44-45 of [12]. That material details a torsion-theoretic characterization of the (commutative) ring homomorphisms that satisfy GD. The paragraphs ends as follows: "As noted in [7], it follows easily that each prime morphism satisfies GD." Since straight domains are defined as the domains R such that $R \hookrightarrow T$ is a prime morphism for each overring T of R, each straight domain is a going-down domain. As it was shown in Remark 2.7 (b) of [6] that each locally divided domain is a going-down domain, it is natural to ask what connections, if any, exist between the class of straight domains and the class of locally divided domains. In that regard, Corollary 3.8 of [9] established that each locally divided domain is a straight domain; and, after investigating a number of special contexts, the authors of [9] concluded its Remark 3.15 by stating that "we do not know an example of a straight domain that is not locally divided." The question that was implicit in that quotation was answered by Secord in Corollary 3.1.1 of [14], approximately 15 years after the publication of [9]. Frankly, one purpose of this note is to show that appropriate use of some material which was available at the time that [9] was written would have been sufficient (when conjoined with a suitable calculation) to provide a quick answer to the question that had been left open in Remark 3.15 of [9].

Corollary 2.2 collects some characterizations of straight domains. Arguably, its condition (7) goes beyond what can be found in [14]. Proposition 2.4 collects some characterizations of locally divided

prime ideals of a domain. Its conditions (6) and (7) mark the first explicit appearances of sets of the form $\Omega(R,P)$ in such characterizations. (The definition of $\Omega(R,P)$ is given in Section 2.) The main point of Section 3 is that the locally divided prime ideals of an integrally closed base domain R can be characterized, without any mention of the "straight" or "prime morphism" concepts, by suitably using the "going-down to P" concept while restricting attention to overrings generated as R-algebras by single elements of $\Omega(R,P)$.

2 A new proof of a result of Secord and an application

Theorem 2.1. (Corollary 3.1.1 of [14]) Let R be a domain. Then R is a straight domain if and only if R is a locally divided domain.

Proof. The "if" assertion was proven in Corollary 3.8 of [9]. We turn to the "only if" assertion. By the material that was recalled above, we can assume, without loss of generality, that R is quasi-local, say with (unique) maximal ideal M. Suppose that the assertion fails. Then R is a quasi-local straight domain but not a divided domain. As we recalled above that any straight domain is a going-down domain, an application of Corollary 2.6 of [6] yields $w \in K$ and $P \in \operatorname{Spec}(R)$ such that $w^n \in P$ for all integers $n \geq 2$ and $w \in PR_P \setminus P$ $(= PR_P \setminus R)$. Observe that $P \neq M$ (since $MR_M = MR = M$). As $w \in PR_P$, we can write $w = p/z$ for some nonzero elements $p \in P$ and $z \in R \setminus P$. In fact, $z \in M \setminus P$, since the facts that (R,M) is quasi-local, $w \notin R$, and $p \in P$ combine to entail that z is a nonunit of R.

Now, consider the overring $T := R[w]$ of R. Since $w^2 \in R$, it follows that $T = R + Rw$. As R is a straight domain, T/PT is a torsion-free module over R/P. Consider the elements $\rho := z + P \in R/P$ and $\tau := w + PT \in T/PT$. We have $\rho \neq 0$ since $z \notin P$; and $\rho \cdot \tau = 0$, since

$$z \cdot w = z(p/z) = p \in P \subseteq PT.$$

As T/PT is a torsion-free module over R/P, we get $\tau = 0$; that is, $w \in PT = P(R + Rw) = P + Pw$. Thus, $w = p_0 + p_1 w$ for some elements $p_0, p_1 \in P$. Then

$$(1 - p_1)w = w - p_1 w = p_0 \in P.$$

Since $p_1 \in P \subseteq M$ (and M is the Jacobson radical of R), $1 - p_1$ is a unit of R. Hence, $w = (1 - p_1)^{-1} p_0 \in Rp_0 \subseteq P$, the desired contradiction. □

In the spirit of the (equivalent) characterizations of going-down domains given in Theorem 1 of [8], it seems reasonable to ask if one can harmlessly focus on some naturally occurring (and possibly proper) subset of "test overrings" T in applying Theorem 2.1. In other words, is there a naturally occurring class of overrings T of a given domain R such that $R \hookrightarrow T$ being a prime morphism for all T in *that* class would ensure that R is a locally divided domain? For the special case where R is a conducive domain, it was shown in Theorem 3.19 of [9] that it is enough to use the fractional overrings of R as the test overrings T. For an arbitrary domain R with quotient field K, it is enough to use the finite-type R-subalgebras of K as the test overrings T (since it was shown in Proposition 2.4 of [9] that the direct limit of prime morphisms is a prime morphism). One can actually do better, in general. Indeed, Corollary 2.2 will use the proof of Theorem 2.1 and some globalization reasoning to achieve an analogue of the characterization of going-down domains in condition (a) of Theorem 1 of [8], by showing that if R is an arbitrary domain with quotient field K, it is enough to use the singly generated overrings of R (that is, the rings of the form $R[u]$, with $u \in K$) as the test overrings T.

The following notation will be useful in this section and in Section 3. Let R be a domain with quotient field K and let $P \in \operatorname{Spec}(R)$. Put

$$\Omega(R,P) := \{u \in K \mid u \in PR_P \setminus PR_Q \text{ for some } Q \in \operatorname{Spec}(R) \text{ with } P \subseteq Q\}$$

and $\Lambda(R,P) := K \setminus (R \cup \Omega(R,P))$. Note that in the above definition of $\Omega(R,P)$, it is harmless to replace the condition "$Q \in \mathrm{Spec}(R)$" with "$Q \in \mathrm{Max}(R)$".

Corollary 2.2 collects some characterizations of straight domains (equivalently, of locally divided domains). One can make the case that the equivalence of conditions (1) and (5) in Corollary 2.2 is readily obtainable from what is explicitly in [14]. We do believe, however, that the equivalence of conditions (1) and (7) in Corollary 2.2 is new here, since the possible localization/globalization behavior of sets of the form $\Omega(R,P)$ was not addressed in [14].

Corollary 2.2. *Let R be a domain with quotient field K. Then the following conditions are equivalent:*

(1) R is a locally divided domain;

(2) R is a straight domain;

(3) $R \hookrightarrow T$ is a prime morphism for all overrings T of R;

(4) T/PT is a torsion-free module over R/P for all overrings T of R and all $P \in \mathrm{Spec}(R)$;

(5) $R \hookrightarrow R[u]$ is a prime morphism for all $u \in K$;

(6) $R[u]/PR[u]$ is a torsion-free module over R/P for all $P \in \mathrm{Spec}(R)$ and all $u \in K$;

(7) For all $P \in \mathrm{Spec}(R)$, $R \hookrightarrow R[u]$ is a prime morphism for all $u \in \Omega(R,P)$ and $R \hookrightarrow R[u]$ satisfies GD *for all $u \in \Lambda(R,P)$.*

Proof. (1) $\Leftrightarrow$ (2) by Corollary 3.1.1 of [14] (or by the above Theorem 2.1); (2) $\Leftrightarrow$ (3) by the definition of a straight domain; each of the equivalences (3) $\Leftrightarrow$ (4) and (5) $\Leftrightarrow$ (6) follows from the definition of a prime morphism; and each of the implications (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) is trivial. Moreover, (5) $\Rightarrow$ (7) since any prime morphism satisfies GD (cf. Exercise 37, pages 44-45 of [12]). Therefore, it will suffice to prove that (7) $\Rightarrow$ (1).

(7) $\Rightarrow$ (1): Assume (7). Fix an arbitrary $Q \in \mathrm{Spec}(R)$. Our task is to prove that R_Q is a divided domain. As it was shown in Proposition 2.3 of [9] that the "prime morphism" property is preserved by the formation of rings of fractions, (7) yields that $R_Q \hookrightarrow R[u]_{R\setminus Q}$ $(= R_Q[u])$ is a prime morphism (and hence satisfies GD) for all $u \in \Omega(R,P)$ for all $P \in \mathrm{Spec}(R)$ such that $P \subseteq Q$. On the other hand, if $u \in \Lambda(R,P)$ for some $P \in \mathrm{Spec}(R)$ such that $P \subseteq Q$, then $R \hookrightarrow R[u]$ satisfies GD by (7), and so $R_Q \hookrightarrow R[u]_{R\setminus Q}$ $(= R_Q[u])$ also satisfies GD. Thus, $R_Q \subseteq R_Q[u]$ satisfies GD for all $u \in K$. As K is (also) the quotient field of R_Q, an application of Theorem 1 of [8] shows that (the quasi-local domain) R_Q is a going-down domain. Bearing in mind that a quasi-local locally divided domain is the same thing as a divided domain, we can now revisit the proof of Theorem 2.1, beginning with its fourth sentence, with the current R_Q playing the role that R had played from the fourth sentence onward in the proof of Theorem 2.1.

The following words of caution pertain to that revisiting. We will not need to use any assertions in the proof to the effect that R_Q is known to be a straight domain; we can apply Corollary 2.6 of [6] to the base ring R_Q and a supposed nondivided prime ideal $\mathfrak{P} = PR_Q$ of R_Q (with P a suitable prime ideal of R contained in Q) to find a suitable element $w \in PR_P \setminus PR_Q$, with $w = p/z$ for some $p \in P$ and $z \in Q \setminus P$, such that $w^2, w^3 \in \mathfrak{P}$; as before, we do consider $T := R_Q[w]$; most importantly, even though we do not (yet) know that R_Q is a straight domain, we *do* get from (7), since $w \in \Omega(R,P)$ (and the formation of rings of fractions preserves prime morphisms), that $R_Q \hookrightarrow T$ is a prime morphism, whence T/PT $(= T/\mathfrak{P}T)$ is a torsion-free module over $R_Q/\mathfrak{P}$: and then (with R_Q still playing the earlier role of R) the second paragraph of the proof of Theorem 2.1 carries over uneventfully, thus (obtaining the desired contradiction and) completing the proof. □

Although Corollary 2.2 showed that the singly generated overrings of a domain R suffice as a family of "test overrings" (relative to appropriate torsion-free behavior) in determining whether R is a straight domain (equivalently, a locally divided domain), one may plausibly assert that such a conclusion would also be deducible from the methods of Secord in [14]. Proposition 2.4 will begin the process of sharpening Corollary 2.2 by recording that the "straight domain" (or "locally divided

domain") conclusion for a given quasi-local domain R can be obtained by using a significantly smaller class of "test overrings" of a domain R. To ease the way to Proposition 2.4, we devote two paragraphs to recalling some definitions and facts about certain local/global behavior and then give Lemma 2.3, which collects some elementary but useful facts concerning such behavior. Proposition 2.4 sharpens the characterization of "sharp prime ideals" that was given in the main result of [14]. We wish to recognize that the proof of Proposition 2.4 makes explicit use of use of an argument from [14]. We wish to reiterate that for the case of integrally closed (but not necessarily quasi-local) domains, Section 3 will present an alternate GD-theoretic approach to the results and themes in the present section that will make no appeal to material from [14] (apart from using its definition of a sharp prime ideal of a domain).

By definition, a prime ideal P of a domain R is said to be a *locally divided prime ideal of* R if PR_M is a divided prime ideal of R_M for each maximal ideal M of R that contains P; equivalently, that $PR_P = PR_M$ for any such M, since

$$(PR_M)(R_M)_{PR_M} = (PR_M)R_P = P(R_M R_P) = PR_P.$$

In particular, if R is a quasi-local domain, a locally divided prime ideal of R is the same as a divided prime ideal of R.

Recall from [9] that if $R \to T$ is a ring homomorphism (for instance, an inclusion of rings) and $P \in \operatorname{Spec}(R)$, then we say that f is a *prime morphism at* P if T/PT is a torsion-free module over R/P (with respect to the module-theoretic structure induced by $R \to T$). In case f is an inclusion of domains $R \hookrightarrow T$ and $P \in \operatorname{Spec}(R)$, it will be convenient to use the terminology "P is straight with respect to $R \subseteq T$" to describe this condition. If a prime ideal P of a domain R is straight with respect to $R \subseteq T$ for all overrings T of R (contained in some preassigned quotient field of R), we will say that "P is a straight prime ideal of R". This replaces the terser (but possibly ambiguous) usage "P is straight" which occurred notably in Theorem 3.1 of [14], a result that has Corollary 3.1.1 of [14] (that is, the above Theorem 2.1) as a consequence. Note that a domain R is a straight domain if and only if each prime ideal of R is straight with respect to $R \subseteq T$ for each overring T of R, that is, if and only if each prime ideal of R is a straight prime ideal of R.

Lemma 2.3. *Let R be a domain.*

(a) Let E be an R-module. Then E is a torsion-free R-module if and only if $E_{R\setminus M}$ is a torsion-free module over R_M for all $M \in \operatorname{Max}(R)$.

(b) Let $P \in \operatorname{Spec}(R)$ and let T be a domain containing R as a subring. Then P is straight with respect to $R \subseteq T$ if and only if PR_M is straight with respect to $R_M \subseteq T_{R\setminus M}$ for all $M \in \operatorname{Max}(R)$ that contain P.

(c) Let R be a domain with quotient field K, and let $P \in \operatorname{Spec}(R)$. Let $\mathcal{C}$ be a nonempty collection of overrings of R that are contained in K. Let $M \in \operatorname{Max}(R)$ such that $P \subseteq M$; let $\mathcal{C}_M := \{T_{R\setminus M} \mid T \in \mathcal{C}\}$. Then $\mathcal{C}_M$ is a nonempty collection of overrings of R_M that are contained in K and, moreover, each overring S of R_M that is contained in K satisfies $S = SR_M = S_{R\setminus M}$ and is an overring of R.

(d) Let R, K, P and $\mathcal{C}$ be as in (c). Then the following conditions are equivalent:

(1) P is straight with respect to $R \subseteq T$ for all $T \in \mathcal{C}$;

(2) For each $M \in \operatorname{Max}(R)$ that contains P, PR_M is straight with respect to $R_M \subseteq S$ for all $S \in \mathcal{C}_M$.

(e) Let R, K and $\mathcal{C}$ be as in (c). Then the following conditions are equivalent:

(1) Each prime ideal of R is straight with respect to $R \subseteq T$ for all $T \in \mathcal{C}$;

(2) For each $P \in \operatorname{Spec}(R)$ and each $M \in \operatorname{Max}(R)$ that contains P, PR_M is straight with respect to $R_M \subseteq S$ for all $S \in \mathcal{C}_M$.

Proof. (a) This assertion follows easily via globalization.

(b) Apply (a) to $E := T/PT$, noting the canonical isomorphisms

$$(R/P)_{M/P} \cong R_M/PR_M \text{ and } E_{R/P\setminus M/P} \cong T_{R\setminus M}/PR_M T_{R\setminus M}$$

for all $M \in \mathrm{Max}(R)$ that contain P.

(c) The assertion is easily verified.

(d) It suffices to apply (b) to each $T \in \mathcal{C}$.

(e) It suffices to apply (d) to each prime ideal P of R. □

Recall that one purpose of this paper is to show that some material which was available at the time that [9] was written would have been sufficient to provide a quick (and self-contained) answer to the question that had been left open in Remark 3.15 of [9]. The proof of Theorem 2.1 has done just that. However, by focusing on one prime ideal at a time in the statement of Theorem 3.1 of [14], Secord has produced a stronger result. Proposition 2.4 will sharpen that stronger result of Secord. We believe that an objective reading of the rest of this section will show that the only part of [14] that we will be using there (apart from some material which was available at the time that [9] was written) is Lemma 2.2 of [14]. For that reason, we have concluded that Lemma 2.2 of [14] is the most important contribution in [14], and we wish to express the hope that the method of proof of that result will find additional applications in the future, even though, prior to the statement of Lemma 2.2 of [14], Secord expressed the view that his Theorem 3.1 "will make this Lemma [his Lemma 2.2] redundant".

Next, in the spirit of Corollary 2.2, the $\Omega(R,P)$ concept appears in two entries in a list of characterizations of locally divided prime ideals.

Proposition 2.4. *Let R be a domain, with quotient field K. Let $P \in \mathrm{Spec}(R)$. Then:*

(a) The following five conditions are equivalent:

(1) P is a locally divided prime ideal of R;

(2) P is straight with respect to $R \subseteq T$ for each overring T of R;

(3) T/PT is a torsion-free module over R/P for all overrings T of R;

(4) P is straight with respect to $R \subseteq R[w]$ for each $w \in K$;

(5) $R[w]/PR[w]$ is a torsion-free module over R/P for each $w \in K$.

(b) Assume also that R is quasi-local, with maximal ideal M. Then the conditions (1)-(5) in (a) are also equivalent to the following two conditions:

(6) P is straight with respect to $R \subseteq R[w]$ for each $w \in \Omega(R,P)$ and $R \subseteq R[u]$ satisfies "GD to P" for all $u \in \Lambda(R,P)$;

(7) P is straight with respect to $R \subseteq R[w]$ for each $w \in \Omega(R,P)$.

Proof. (a) Both of the equivalences (2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) follow from the above definition of a prime ideal of R being straight with respect to a given domain extension of R. On the other hand, each of the implications (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) is trivial. Therefore, it will suffice to prove the two implications (1) $\Rightarrow$ (3) and (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3): Assume (1). As explained above, this assumption means that $PR_P = PR_M$ for each maximal ideal M of R that contains P. On the other hand, it follows from Lemma 2.3 (e), with $\mathcal{C}$ defined as the set of all overrings of R contained in K, that (3) is equivalent to PR_M being straight with respect to $R_M \subseteq T_{R\setminus M}$ for all overrings T of R (contained in K) and all maximal ideals M of R that contain P. Hence, we can assume, without loss of generality, that R is quasi-local, say with maximal ideal M. Thus $PR_P = P$, and our task is to prove that T/PT is a torsion-free module over R/P for each overring T of R. This, in turn, can be done by repeating the second paragraph of the proof of Proposition 3.7 (a) of [9] (with $A := R$ and $B := T$).

(5) $\Rightarrow$ (1): Consider the analyses, in the above proof of the implication (1) $\Rightarrow$ (3), of the condition (1) and of the special case of the condition (3) where the rings T are taken to be of the form $T = R[w]$, as w varies over K. The upshot is that here, in the proof of the implication (5) $\Rightarrow$ (1), we can assume, without loss of generality, that R is quasi-local, say with maximal ideal M. Our task is to prove that $PR_P = P$, given that (5) holds. It will be useful to note that (4) also holds.

Suppose that the assertion fails. Choose $\xi \in PR_P \setminus P$ $(= PR_P \setminus R)$. Consider the ring $T := R[\xi^{-1}]$. By Lemma 2.3 (b) and (5), T/PT is torsion-free over R/P. Also, as noted in the proof of Lemma 2.2 (i) of [14], it follows that $\xi \in PR[\xi]$. (For the sake of completeness, we recall its proof: as $\xi = p/z$ for some $p \in P$ and $z \in R \setminus P$, one need only combine the fact that $z\xi = p \in PR[\xi]$ with the hypothesis that P is straight with respect to $R \subseteq R[\xi]$.) Next, as (4) gives that P is straight with respect to $R \subseteq T$, it follows from the hint given by Kaplansky for Exercise 37 on pages 44-45 of [12] that $R \subseteq T$ satisfies "GD to P". That fragment of GD-like behavior is enough to allow us to adapt the first paragraph of the proof of Lemma 2.4 (a) of [6] in order to conclude that ξ is integral over R. In particular, $S := R[\xi]$ is a finitely generated R-module. To get the desired contradiction (that $S = R$), it will suffice (by Nakayama's Lemma) to show that $M(S/R) = S/R$, that is, to show that $S \subseteq R + MS$. As S is the subring of K generated by $R \cup \{\xi\}$, we need only note that $R \subseteq R + MS$ and $\xi \in PR[\xi] = PS \subseteq MS \subseteq R + MS$.

(b) (4) $\Rightarrow$ (6): As $\Omega(R,P) \subseteq K$, it suffices to appeal once again to Exercise 37 on pages 44-45 of [12].

(6) $\Rightarrow$ (1): Since R is assumed quasi-local, it suffices to rework the second paragraph of the above proof that (5) $\Rightarrow$ (1). Observe that the element ξ in that paragraph is assumed to be in $\Omega(R,P)$. In reworking the earlier argument, it is also necessary to observe that $\xi^{-1} \in \Lambda(R,P)$. Since $\xi \notin R$, this comes down to showing that $\xi^{-1} \notin \Omega(R,P)$. This, in turn, holds, since the proper ideal PR_P of R_P would otherwise contain both ξ and ξ^{-1}, whence $\xi\xi^{-1} \in PR_P$, a patent contradiction.

(6) $\Rightarrow$ (7): Trivial.

(7) $\Rightarrow$ (1): In our opinion, this is the main application in [14]. For the sake of completeness, we summarize Secord's argument here. As in the above proof that (5) $\Rightarrow$ (1), one uses Nakayama's Lemma to show that R is integrally closed in $R + PR_P$; and as noted in the proof of Lemma 2.2 (i) of [14], one shows that each $\xi \in \Omega(R,P)$ satisfies $\xi \in PR[\xi]$. Thus, since R is quasi-local, we get that ξ is the root of a polynomial in $R[X]$ having a unit coefficient. So, since $\xi \notin R$, it follows from a well known generalization of Seidenberg's "u, u^{-1} Lemma" (cf. Lemma 3.8 of [10]) that $\xi^{-1} \in R$. This would give a contradiction, since ξ would be an element of the proper ideal PR_P of the larger ring $R + PR_P$. Hence, no such ξ can exist. This completes the proof. □

We close this section with a multi-faceted remark/*mélange*.

Remark 2.5. (a) The equivalent conditions (4) and (5) in the statement of Proposition 2.4 gave the sharpest "straight-theoretic" or "torsion-theoretic" characterizations of a locally divided prime ideal of a domain that I was able to develop by using mostly only "old" material involving GD and going-down domains. About "mostly": I humbly acknowledge that the final two sentences of the proof that (5) $\Rightarrow$ (1) in the proof of Proposition 2.4 were a mild adaptation of an innovative argument using Nakayama's Lemma that was part of Secord's proof of Lemma 2.2 (ii) in [14]. Thus, apart from that appeal to Nakayama's Lemma (and the use of the definition of a prime ideal being straight with respect to a ring extension), our GD-theoretic proof that (4) and (5) each characterize locally divided prime ideals of a domain is, in effect, independent of [14].

As the property of being a locally divided prime ideal of a domain is a local property, it seemed reasonable to include a part (b) of Proposition 2.4 to address the context of a quasi-local base domain. For that context, the proof that condition (6) characterizes a divided prime ideal (of a quasi-local domain) featured roles (that we found pleasant) for the "$\Omega(R,P)$" and "GD to P" concepts, with proofs that were also, in effect, independent of [14] (apart from Secord's above-mentioned innovative use of Nakayama's Lemma). *That* explains our interest in including condition (6) in the statement of Proposition 2.4. I am glad to acknowledge and appreciate the superiority of the condition (7) (*vis-à-vis* condition (6)) in Proposition 2.4 as a characterization of a divided prime ideal of a quasi-local domain. In that regard, we note again that the above proof that (7) $\Rightarrow$ (1) in Proposition 2.4 is merely a reworking of some of Secord's arguments.

(b) By way of partial summary of (b), I am pleased to acknowledge that Secord has given a sharper characterization of a divided prime ideal of a quasi-local domain than I could while using only GD-

theoretic methods. One may ask whether the characterization of (not necessarily quasi-local) straight domains (equivalently, locally divided domains) given by condition (7) in Corollary 2.2 surpasses anything that could reasonably be attributed to [14]. Because of the content of the next paragraph, I would caution against rushing to such a rash conclusion.

By combining Lemma 2.3 (b) and Proposition 2.4 (b), one gets the following result. Let R be a (not necessarily quasi-local) domain and let $P \in \mathrm{Spec}(R)$. Then P is a locally divided prime ideal of R if and only if, for each $M \in \mathrm{Max}(R)$ that contains P, PR_M is straight with respect to $R_M \subseteq R_M[w]$ for each $w \in \Omega(R,P)$. It seems reasonable to conclude that this result would be noticed by many readers of [14].

(c) If R is any domain, we can obtain another characterization of locally divided domains R (equivalently, straight domains R) by applying universal quantification over $P \in \mathrm{Spec}(R)$ to the result in the second paragraph of (b). That characterization deserves to be added to the list of equivalent conditions in the statement of Corollary 2.2.

(d) In view of the roles played by sets of the form $\Omega(R,P)$ and the prevalence of globalization considerations in this section, it is tempting to ask about the relationship, given an arbitrary domain R and prime ideals $P \subseteq M$ of R, between $\Omega(R_M, PR_M)$ and $\Omega(R,P)_{R\setminus M}$. In our opinion, this question merits further scrutiny.

(e) Although the concept of a prime ideal being straight with respect to a ring extension was innovative and has been fruitful, we believe that more can/should be learned about it. One possible source of questions is provided by known results on prime morphisms. The next four paragraphs discuss the use of one such source, namely, Proposition 2.6 of [9], which gave some general results on how the above-mentioned concept may be preserved or reflected under composition of (for our applications, injective) ring homomorphisms.

Suppose that $R \subset S$ are (distinct) domains, but not fields, such that $\mathrm{Spec}(R) = \mathrm{Spec}(S)$ (as sets). The first paper studying the "$\mathrm{Spec}(R) = \mathrm{Spec}(S)$" condition was [2]. The most familiar example of a ring extension satisfying this condition is given by an arbitrary pseudo-valuation domain R (in the sense of Hedstrom and Houston [11]) which is not a field and its canonically associated valuation overring S. By adapting the proof of Proposition 2.16 of [9], one can obtain the following conclusion (for $R \subset S$ distinct domains, but not fields, with the same prime ideals). Let $P \in \mathrm{Spec}(R)$ $(= \mathrm{Spec}(S))$ and let T be an overring of S (that is, an overring of R that contains S). Then: if P is straight with respect to $S \subseteq T$, then P is straight with respect to $R \subseteq T$.

The next paragraph will comment briefly on the proof of the above conclusion. Before that, we will make two points that help to explain the tractability of data satisfying the "$\mathrm{Spec}(R) = \mathrm{Spec}(S)$" condition. The first of these was noted in Proposition 2.1 (a) of [9]: because every vector space over a field K is a torsion-free K-module, if follows that if A is a (commutative) ring and $\mathcal{M} \in \mathrm{Max}(A)$, then $\mathcal{M}$ is a straight ideal of A (that is, $\mathcal{M}$ is straight with respect to $A \subseteq B$ for any commutative ring B containing A as a subring). Second, if $R \subset S$ are domains, but not fields, such that $\mathrm{Spec}(R) = \mathrm{Spec}(S)$, then R and S are each quasi-local, they have the same maximal ideal (say M), and S is an overring of R (this follows from Proposition 3.3 of [2]) and if in addition $P \in \mathrm{Spec}(R) \setminus \mathrm{Max}(R)$, then $R_P = S_{S\setminus P}$ (so, the stalks of the canonical structure sheaves of the affine schemes $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ differ *only* at the point M).

The proof of the above conclusion is an easy consequence of the proof of Proposition 2.6 (c) of [9], which uses the fact that the "$\mathrm{Spec}(R) = \mathrm{Spec}(S)$" hypothesis ensures that the inclusion map $R \hookrightarrow S$ is a prime-producing homomorphism that satisfies the quasi-lying-over property.

It seems natural to ask for additional hypotheses under which the converse of the above result would be valid. In other words: if $\mathrm{Spec}(R) = \mathrm{Spec}(S)$ (as sets) for domains $R \subset S$, T is an overring of S, and $P \in \mathrm{Spec}(R)$ is straight with respect to $R \subseteq T$, under what additional conditions (if any) must P be straight with respect to $S \subseteq T$? It seems that none of the *other* parts of Proposition 2.6 of [9] would be helpful in settling this question. Only some of the homological properties used in the

proof of the just-cited result, such as flatness, have relevance in the study of overrings of domains. Some other homological properties that were involved in that proof, such as faithfullyflatness, would be irrelevant for the kind of focused study that I am proposing. (After all, no domain has a proper faithfully flat overring.) I suggest that it may be necessary to develop a deeper study of some ideal-theoretic properties in order to obtain a satisfactory answer to this question. Such an answer will surely call for something that would go beyond a facile restatement involving conductors.

3 Integrally closed domains admit a simpler characterization of locally divided prime ideals

Many of the objects of study in this paper (certain kinds of domains or prime ideals) are defined, and often characterized, via formulations involving universal quantifications. Proposition 2.4 pursued Secord's approach of removing the universal quantification over the prime spectrum of a given domain R (which had been involved in the definition of a straight domain in [9]) by focusing on whether a given prime ideal P of R is a locally divided prime ideal of R (equivalently, a straight prime ideal of R) while still using universal quantification over the overrings of that domain R that can be generated over R by a single element of $\Omega(R,P)$. As Proposition 2.4 (b) recorded, that approach is successful when the base domain R is quasi-local. It is natural to ask if one can remove (or, at least, relax) that hypothesis of quasi-locality. An initial attempt along such lines was made in the second paragraph of Remark 2.5 (b), at the cost of replacing consideration of a triple (P,R,T) with consideration of the set of triples $\{(PR_M, R_M, T_{R\setminus M}) : M \in \mathrm{Max}(R), P \subseteq M\}$. In this brief section, we show that one *can*, for the special case where R is integrally closed, remove the "quasi-local" hypothesis *and* also avoid consideration of the above-mentioned maximal ideals M: see Corollary 3.5.

It seems natural to focus on something like the universe of integrally closed domains in studying the "straight" and "locally divided" concepts. Indeed, in Corollary 11 of [13], McAdam showed (using other terminology) that an integrally closed domain is a going-down domain if and only if it is a locally divided domain. (The statement of the just-cited result included the extra hypothesis that the given domain is quasi-local, but that stated result easily globalizes.) This was generalized by replacing "integrally closed" with "root closed" in Corollary 2.8 of [5], whose method of proof yields the further generalization in which "root closed" is replaced by "seminormal". In short, for a seminormal domain R: R is a straight domain $\Leftrightarrow$ R is a going-down domain $\Leftrightarrow$ R is a locally divided domain. Note, however, that a going-down domain need not be a straight domain (and hence need not be a locally divided domain). This was shown in Example 4.4 of [9] by further analyzing a certain two-dimensional quasi-local domain which had been constructed for other purposes by Boisen and Sheldon in [4] (and which had been shown to be a quasi-local going-down domain that is not a divided domain in Example 2.9 of [6]).

The first result in this section extracts part of what was actually shown in a result in [5]. This is then used in proving the main result of this section, Theorem 3.2 (b), which gives a companion for the results in the earlier sections by having a statement and proof featuring the "going-down to P" concept while ignoring the "P is straight with respect to $R \subseteq T$" concept. The subsequent corollaries specialize matters to integrally closed based domains and feature statements replacing the occurrences of "going-down to P" with their less general "straight" analogues.

Lemma 3.1. *Let R be a domain with quotient field K and let $u \in K \setminus \{0\}$, such that R is integrally closed in both $R[u]$ and $R[u^{-1}]$. Let $P \in \mathrm{Spec}(R)$ such that $R \subseteq R[u]$ satisfies* GD *to P. Then $R \subseteq R[u^{-1}]$ satisfies* GD *to P.*

Proof. The assertion can be extracted from the proof of Theorem 2.6 of [5]. □

Theorem 3.2. (a) Let R be a quasi-local domain, $P \in \mathrm{Spec}(R)$ and $u \in PR_P \setminus \{0\}$. Suppose that R is integrally closed in both $R[u]$ and $R[u^{-1}]$ and that $R \subseteq R[u]$ satisfies GD to P. Then $u \in P$.

(b) Let R be a domain, $P \in \mathrm{Spec}(R)$ and $u \in PR_P \setminus \{0\}$. Suppose that R is integrally closed in both $R[u]$ and $R[u^{-1}]$ and that $R \subseteq R[u]$ satisfies GD to P. Then $u \in PR_M$ for each maximal ideal M of R that contains P.

Proof. (a) By Lemma 3.1, $R \subseteq R[u^{-1}]$ also satisfies "GD to P". Therefore, it follows, as in the proof of Lemma 2.4 (a) of [5], that $1 \in PR[u^{-1}]$. Hence, $1 = p_0 + \sum_{i=1}^{n} p_i u^{-i}$ for some finite list $p_0, \ldots, p_n$ of elements of P (and some $n \geq 1$). Multiplying this equation through by u^n reveals that u is integral over R. As R is integrally closed in $R[u]$, we get $u \in R \cap PR_P = P$, as asserted.

(b) Fix a maximal ideal M of R that contains P. Then $u \in PR_P = (PR_M)(R_M)_{PR_M}$. Since $R \subseteq R[u]$ satisfies GD to P, it is easy to use the lore of localizations to check that $R_M \subseteq R[u]_{R\setminus M}$ $(= R_M[u])$ satisfies GD to PR_M. Also, the lore of integrality (cf. Proposition 5.12 of [3]) leads to R_M being integrally closed in both $R[u]_{R\setminus M}$ $(= R_M[u])$ and $R[u^{-1}]_{R\setminus M}$ $(= R_M[u^{-1}])$. Hence, by applying (a) to the base ring R_M and its prime ideal PR_M, we get $u \in PR_M$. □

Corollary 3.3. *Let R be a domain, $P \in \mathrm{Spec}(R)$ and $u \in PR_P \setminus \{0\}$. Suppose that R is integrally closed in both $R[u]$ and $R[u^{-1}]$ and that P is straight with respect to $R \subseteq R[u]$. Then $u \in PR_M$ for each maximal ideal M of R that contains P.*

Proof. Since P is straight with respect to $R \subseteq R[u]$, it follows from the oft-cited exercise of Kaplansky [12] that $R \subseteq R[u]$ satisfies "GD to P". So, the assertion follows from Theorem 3.2 (b). □

Corollary 3.4. *Let R be an integrally closed domain, $P \in \mathrm{Spec}(R)$ and $u \in PR_P \setminus \{0\}$ such that P is straight with respect to $R \subseteq R[u]$. Then $u \in PR_M$ for each maximal ideal M of R that contains P.*

Proof. Apply Corollary 3.3. □

We close by showing that if the base domain R is integrally closed (but not necessarily quasi-local), it suffices to use the overrings generated by a single element of $\Omega(R, P)$ to determine whether a prime ideal P of R is locally divided.

Corollary 3.5. *Let R be an integrally closed domain and $P \in \mathrm{Spec}(R)$, such that P is straight with respect to $R \subseteq R[u]$ for all $u \in \Omega(R, P)$. Then P is a locally divided prime ideal of R.*

Proof. It suffices to show that $PR_P = PR_M$ for each maximal ideal M of R that contains P. One has, on general principles, that $PR_M \subseteq PR_P$. If the assertion fails, there exists $u \in PR_P \setminus PR_M$ for some maximal ideal M of R that contains P. Then $u \in \Omega(R, P)$, and so an application of Corollary 3.4 gives the desired contradiction, thus completing the proof. □

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Canonical solution of universal problem for the prolongation of algebra by a local algebra

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Abstract. In this paper, we introduce the notion of prolongations of algebras by a local algebra. We propose a universal problem for the prolongations of algebras by a local algebra and we describe the existence and uniqueness of the solution of this problem. We provide several propositions which illustrate the properties of functors in the category of prolongations of algebra by a local algebra.

Key Words: Prolongation of algebra, symmetric algebra, local algebra, categories of algebras.
2010 MSC: Primary 13B02, 16S10; Secondary 16P10, 08C05.

1 Introduction

The universal property is a fundamental concept in mathematics that describes the unique and universal nature of certain mathematical constructions. Universal properties define objects uniquely up to a unique isomorphism [1]. Therefore, one strategy to prove that two objects are isomorphic is to show that they satisfy the same universal property. The universal property is a powerful tool in various areas of mathematics, such as category theory, algebra, topology, and beyond [2, 3, 6].

Let K be a commutative field. A local algebra on K is a finite-dimensional commutative algebra A with unit 1_A admitting a unique maximal ideal of codimension 1 on K (see [7, 8, 11, 12] for more details). The set $\mathbb{D} = \{a+\varepsilon b \,/\, \varepsilon^2 = 0, (a,b) \in \mathbb{R}\} = \mathbb{R}\oplus\varepsilon\mathbb{R}$ is a local algebra over $\mathbb{R}$ with maximal $\mathbf{m} = \varepsilon\mathbb{R}$. The local algebra $\mathbb{D}$ is called *algebra of dual numbers* the or *Study numbers*. It is also called *truncated polynomial algebra of degree* 1 (see [4, 7, 10]). We denote by $(1^*_{\mathbb{D}}, \varepsilon^*)$ the dual basis of the basis $(1_{\mathbb{D}}, \varepsilon)$ of $\mathbb{D}$. Let $\mathbf{m}$ be the unique maximal ideal of local algebra A. Then the set $A_0 = \{\lambda \cdot 1_A / \lambda \in K\}$ is a subalgebra of A of dimension 1. Moreover, $A_0 \cap \mathbf{m} = (0)$ et $A_0 + \mathbf{m} = A$. Thus, a local algebra A can be represented by a direct sum $A = A_0 \oplus \mathbf{m}$. The map $K \longrightarrow A_0, \lambda \longmapsto \lambda \cdot 1_A$ is an isomorphism of algebras, that is, $A_0 \simeq K$. So $A = K \cdot 1_A \oplus \mathbf{m}$. When A is a local algebra on K, then the surjection $\pi_A : A \longrightarrow A/\mathbf{m} \simeq K$ is a homomorphism of algebras called *augmentation* of A (see [4, 9, 13] for more details).

In this paper, we consider an algebra F and a local algebra A. The purpose of this paper is to find an algebra X and a homomorphism γ of algebras from F into $A \otimes X$ having the following universal property: For any algebra L and any homomorphism u from algebra F into algebra $A \otimes L$, there exists a unique homomorphism $\tilde{u}$ such that $(id_A \otimes \tilde{u}) \circ \gamma = u$. The pair (X, γ) is a *solution of this universal*

problem if the application

$$Hom_{Alg}(X,L) \longrightarrow Hom_{Alg}(F, A\otimes L), \varphi \longmapsto (id_A \otimes \varphi)\circ\gamma,$$

is a bijection.

The paper is organized as follows. In Section 2, we briefly recall some universal properties in commutative algebra. In Section 3, we define the prolongation of algebra by a local algebra and we construct the canonical solution to the universal problem corresponding to the algebras F and A in cases where A is a local algebra. Finally, in Section 4, we describe the covariant functor and the contravariant functor in the categories corresponding to the prolongations of algebras by a local algebra.

2 Preliminaries

In this section, we recall some universal properties in commutative algebra which will be necessary for the subsequent sections.

Throughout this section, A denotes a commutative ring with unit.

Theorem 2.1. [2] (Universal property of quotients). Let M and P be two A-modules, N a submodule of M and $\pi : M \longrightarrow M/N$ the quotient map. For every A-module homomorphism $f : M \longrightarrow P$ with $N \subseteq \ker f$, there exists a unique A-linear map $\tilde{f} : M/N \longrightarrow P$ such that $\tilde{f}\circ\pi = f$.

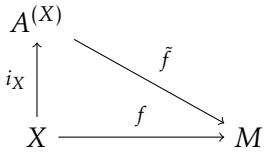
Let M and N be two A-modules. We assume that M is a free A-module and let $(e_i)_{i\in I}$ be its basis, where I is a set of finite indices. If $(y_i)_{i\in I}$ is a family of elements of N, then there exists a unique linear map $f : M \longrightarrow N$ such that $f(e_i) = y_i$, for all $i \in I$ (see [2, 6] for more details).

For any set X, we denote by $A^{(X)}$ the free A-module generated by X, and $(e_x)_{x\in X}$ its basis, where the map $e_x : X \longrightarrow A$ is defined by $e_x(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x. \end{cases}$

We denote by $i_X : X \longrightarrow A^{(X)}$ the map such that $i_X(x) = e_x$, for all $x \in X$.

Given any set X and an A-module M, we look an A-module $Y = A^{(X)}$ and a map i_X of X into $A^{(X)}$ having the following universal property

Theorem 2.2. [2] (Universal property of the pair $\left(A^{(X)}, i_X\right)$). Let X be any set and M be an A-module. If $f : X \longrightarrow M$ is a map, then there exists a unique A-linear map $\tilde{f} : A^{(X)} \longrightarrow M$ such that the following diagram



commutes, that is, $\tilde{f}\circ i_X = f$. Thus, $\tilde{f}(e_x) = f(x)$, for all $x \in X$.

Definition 2.3. The pair (Y, i_X) is called *solution to the universal problem* if the map $Hom_A(Y,M) \longrightarrow Ens(X,M), \varphi \longmapsto \varphi\circ i_X$ is a bijection.

Let M and N be two A-modules and $\mathcal{H}$ be the A-submodule of $A^{(M\times N)}$, generated by the elements of the form

$$\begin{cases} e_{(x_1+x_2,y)} - e_{(x_1,y)} - e_{(x_2,y)}, \\ e_{(x,y_1+y_2)} - e_{(x,y_1)} - e_{(x,y_2)}, \\ e_{(ax,y)} - ae_{(x,y)}, \\ e_{(x,ay)} - ae_{(x,y)}, \end{cases}$$

for all $x_1, x_2, x \in M$, $y_1, y_2, y \in N$ and $a \in A$.

A *tensor product* of M and N is an A-module quotient $M \bigotimes_A N = A^{(M\times N)}/\mathcal{H}$. The class of the element $e_{(x,y)} \in A^{(M\times N)}$ in $M \bigotimes_A N$ is denoted by $\overline{e_{(x,y)}} = x \otimes y$.

The map $i_{(M,N)} = \pi \circ i_{M\times N} : M \times N \xrightarrow{\pi} A^{(M\times N)} \xrightarrow{i_{M\times N}} A^{(M\times N)}/\mathcal{H} = M \bigotimes_A N$ defined by $i_{(M,N)}(x,y) = x \otimes y$ is A-bilinear. Every element $z \in M \bigotimes_A N$ can be written as a fini sum $z = \sum_{i\in I} x_i \otimes y_i$, with $x_i \in M$ and $y_i \in N$.

Theorem 2.4. [2] (Universal property of the pair $(M \bigotimes_A N, i_{(M,N)})$). Let M, N, P be three A-modules. For any bilinear map $f : M \times N \longrightarrow P$, there exists a unique linear map $\tilde{f} : M \bigotimes_A N \longrightarrow P$ such that $\tilde{f} \circ i_{(M,N)} = f$, that is, $\tilde{f}(x \otimes y) = f(x,y)$, for any $x \in M$ and for any $y \in N$.

In the category of modules over a commutative ring A, the universal property of a tensor product $M \bigotimes_A N$ is the possession of a bilinear mapping $M \times N \longrightarrow M \bigotimes_A N$, that is, $M \bigotimes_A N$ is a representing object for the covariant functor which sends a module P to the set of bilinear mappings $M \times N \longrightarrow P$.

An *algebra* on A or an *A-algebra* is an A-module E which is equipped with an A-bilinear map $\mu : E \times E \longrightarrow E, (x,y) \longmapsto x \cdot y$ such as:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for all $x, y, z \in E$.

Let A be a commutative ring with unit and let M be an A-module. For $n \geq 0$, we denote $T_A^0(M) = A$, $T_A^1(M) = M$ and $T_A^n(M) = M \otimes M \otimes ... \otimes M$, (n-times). A *tensor algebra* is an A-algebra $T_A(M) = \bigoplus_{n\geq 0} T_A^n(M$ which is equipped with a multiplication $\cdot$ defined by

$$(x_1 \otimes x_2 \otimes ... \otimes x_p) \cdot (x_{p+1} \otimes x_{p+2} \otimes ... \otimes x_{p+q}) = x_1 \otimes x_2 \otimes ... \otimes x_{p+q},$$

$$a \cdot (x_1 \otimes x_2 \otimes ... \otimes x_p) = a(x_1 \otimes x_2 \otimes ... \otimes x_p),$$

for all $p \geq 0, q \geq 0, a \in A$ and $x_i \in M$.

It is immediate that the multiplication $\cdot$ defined on $T_A(M)$ is associative and admits as unit element the unit element 1_A of $A = T_A^0(M)$. We denote by $i : T^1(M) = M \longrightarrow T(M)$, the canonical injection map. The tensor algebra $T_A(M)$ has the following universal property.

Theorem 2.5. [2, 6] (Universal property of tensor algebra). For any associative A-algebra E with a unital element 1_E, and any homomorphism $f : M \longrightarrow E$ of A-modules, there exists a unique homomorphism of A-algebras $\widetilde{f} : T_A(M) \longrightarrow E$, such that $\widetilde{f}(1_A) = 1_E$ and $\widetilde{f} \circ i = f$.

The *symmetric algebra* of an A-module M, denoted by $S(M)$, is the quotient algebra over A of the tensor algebra $T_A(M)$ by the two-sided ideal $\mathcal{I}$ generated by the elements $x \otimes y - y \otimes x$ of $T_A(M)$, where $x, y \in M$. The map $i : M \longrightarrow S(M)$ is called the canonical injection

Theorem 2.6. [2, 6] (Universal property of symmetric algebras). Let M be an A-module, E be an A-algebra with and $\varphi : M \longrightarrow E$ be a linear map such that $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$, for all $x, y \in M$. There exists a unique homomorphism of A-algebras $\tilde{\varphi} : S(M) \longrightarrow E$ such that $\tilde{\varphi} \circ i = \varphi$.

3 Canonical solution of universal problem for the prolongation of algebra by a local algebra

Given an algebra F and A a local algebra. Let I be a finite set, $(a_i)_{i\in I}$ a basis of the algebra A, $(a_i^*)_{i\in I}$ the dual basis of the basis $(a_i)_{i\in I}$ and $S(A^*\otimes F)$ the symmetric algebra of the vector space $A^*\otimes F$.

Definition 3.1. The *prolongation* of the algebra F by A, denoted by F^A, is the quotient algebra of $S(A^*\otimes F)$ by the ideal $\mathcal{J}$ of $S(A^*\otimes F)$ generated by the elements of the form $\varphi\otimes 1_F-\varphi(1_A)$ and $\varphi\otimes xy-\sum\limits_{j,k}\varphi(a_ja_k)(a_j^*\otimes x)(a_k^*\otimes y)$, for all $\varphi\in A^*$ and for all $x,y\in F$.

The class of the element $\varphi\otimes x\in A^*\otimes F$ in $F^A=S(A^*\otimes F)/\mathcal{J}$ is denoted by $\overline{\varphi\otimes x}$.

Proposition 3.2. *The map $\gamma_{(A,F)}:F\longrightarrow A\otimes F^A$ such that*

$$\gamma_{(A,F)}(x)=\sum_i a_i\otimes(\overline{a_i^*\otimes x}) \tag{1}$$

is a homomorphism of algebras, for all $x\in F$. Moreover, this writing does not depend on the chosen basis.

Proof. The map $\gamma_{(A,F)}$ is by construction well-defined and is linear. By definition of $\mathcal{J}$, for all $x,y\in F$ and for all $\varphi\in A^*$, we have

$$\overline{\varphi\otimes xy-\sum_{j,k}\varphi(a_ja_k)(a_j^*\otimes x)(a_k^*\otimes y)}=0.$$

This implies that,

$$\overline{\varphi\otimes xy}=\sum_{j,k}\varphi(a_ja_k)(\overline{a_j^*\otimes x})(\overline{a_k^*\otimes y}). \tag{2}$$

Thus, by (1) and (2), for all $x,y\in F$, we have

$$\begin{aligned}\gamma_{(A,F)}(xy) &= \sum_i a_i\otimes\left[\sum_{j,k}a_i^*(a_ja_k)(\overline{a_j^*\otimes x})(\overline{a_k^*\otimes y})\right]\\ &= \sum_{j,k}\left[\sum_i a_i^*(a_ja_k)a_i\otimes(\overline{a_j^*\otimes x})(\overline{a_k^*\otimes y})\right]\\ &= \sum_{j,k}a_ja_k\otimes(\overline{a_j^*\otimes x})(\overline{a_k^*\otimes y}).\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_{(A,F)}(xy) &= \sum_{j,k}\left[a_j\otimes(\overline{a_j^*\otimes x})\right]\left[a_k\otimes(\overline{a_k^*\otimes y}\right]\\ &= \left[\sum_j a_j\otimes(\overline{a_j^*\otimes x})\right]\left[\sum_k a_k\otimes(\overline{a_k^*\otimes y}\right]\\ &= \gamma_{(A,F)}(x)\gamma_{(A,F)}(y).\end{aligned}$$

Let $(b_j)_{j\in I}$ be an other basis of A. Let $(b_j^*)_{j\in I}$ be the dual basis of the basis $(b_j)_{j\in I}$. By (2), for all x in F, we have

$$\begin{aligned}\gamma_{(A,F)}(x) &= \sum_j b_j \otimes \overline{b_j^* \otimes x} \\ &= \sum_j b_j \otimes \left[\sum_i \overline{b_j^*(a_i)\, a_i^* \otimes x}\right] \\ &= \sum_i \left[\sum_j b_j^*(a_i)\, b_j \otimes \overline{\left(a_i^* \otimes x\right)}\right] \\ &= \sum_i a_i^*(a_i)\overline{\left(a_i^* \otimes x\right)}.\end{aligned}$$

Therefore, the homomorphism $\gamma_{(A,F)}$ does not depend on the chosen basis. □

The homomorphism $\gamma_{(A,F)}$ is called the *canonical homomorphism* of the algebra F into the algebra $A \otimes F^A$.

Proposition 3.3. *Let L be an algebra and A be a finite-dimensional algebra over K. Then, $(a_i \otimes 1_L)_{i\in I}$ is a basis of the L-module $A \otimes L$ defined by the map*

$$L \times A \otimes L \longrightarrow A \otimes L, (l, \sum_i a_i \otimes l_i) \longmapsto \sum_i a_i \otimes l l_i.$$

Proof. For $x \in A \otimes L$, there exists a family $(l_i)_{i\in I} \in L$ such that
$x = \sum_i a_i \otimes l_i = \sum_i l_i(a_i \otimes 1_L)$. Since x is arbitrary in $A \otimes L$, then the family $(a_i \otimes 1_L)_{i\in I}$ generates the L-module $A \otimes L$.

On the other hand, let $(l_i)_{i\in I}$ be a family of elements of L such that

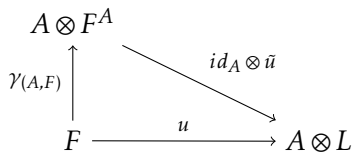
$$\sum_i l_i(a_i \otimes 1_L) = \sum_i a_i \otimes l_i = 0.$$

Then, for all $j \in I$, we have $(a_j^* \otimes id_L)(\sum_i a_i \otimes l_i) = 0$. Thus,

$$\sum_i a_i^*(a_i) l_i = \sum_i \delta_{ij} l_i = l_j = 0.$$

Hence, the family $(a_i \otimes 1_L)_{i\in L}$ is linearly independent. Therefore, $(a_i \otimes 1_L)_{i\in I}$ is a basis of the L-module $A \otimes L$. □

Theorem 3.4 (Universal property of the pair $(A \otimes F^A, \gamma_{(A,F)})$)**.** Let F be an algebra, A a finite-dimensional algebra over K and $\gamma_{(A,F)}$ the canonical homomorphism of F into $A \otimes F^A$. For any algebra L and any homomorphism u from algebra F into algebra $A \otimes L$, there exists a unique homomorphism $\tilde{u}$ such that the following diagram



commutes, that is,

$$(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)} = u. \tag{3}$$

Proof. The uniqueness of $\tilde{u}$: Let $v : F^A \longrightarrow L$ be an other algebra homomorphism such that $(id_A \otimes v) \circ \gamma_{(A,F)} = u$. For all $x \in F$, we have

$$\sum_i a_i \otimes v(\overline{a_i^* \otimes x}) = \sum_i a_i \otimes \tilde{u}(\overline{a_i^* \otimes x}).$$

This implies that,

$$\begin{aligned} 0 &= \sum_i a_i \otimes v(\overline{a_i^* \otimes x}) - \sum_i a_i \otimes \tilde{u}(\overline{a_i^* \otimes x}) \\ &= \sum_i \left[v(\overline{a_i^* \otimes x}) - \tilde{u}(\overline{a_i^* \otimes x}) \right] (a_i \otimes 1_L). \end{aligned}$$

Hence, by the Proposition 3.3, $v(\overline{a_i^* \otimes x}) = \tilde{u}(\overline{a_i^* \otimes x})$, for all $i \in I$. Since the elements $\overline{a_i^* \otimes x}$ generate F^A, then $v = \tilde{u}$.

For the existence of $\tilde{u}$: For all $x \in F$, we have $u(x) = \sum_i a_i \otimes (a_i^* \otimes id_L)[u(x)]$.

Consider the map $f : A^* \times F \longrightarrow L$ such that, for all $(\varphi, x) \in A^* \times F$, $f(\varphi, x) = (\varphi \otimes id_L)[u(x)]$. This map is well-defined and bilinear by construction. According to the universal property 2.4 applied to the pair $(A^* \otimes F, i_{(A^*,F)})$, where $i_{(A^*,F)}$ is the canonical map of $A^* \times F$ into $A^* \otimes F$, there exists a unique linear map $\tilde{f} : A^* \otimes F \longrightarrow L$ such that $\tilde{f} \circ i_{(A^*,F)} = f$. Let us denote by $i_{A^* \otimes F}$ the canonical injection of the module $A^* \otimes F$ in symmetric algebra $S(A^* \otimes F)$ of $A^* \otimes F$. By virtue of the universal property of symmetric algebra (Proposition 2.6) applied to the pair $(S(A^* \otimes F), i_{A^* \otimes F})$, there is a unique homomorphism $g : S(A^* \otimes F) \longrightarrow L$ such that $g \circ i_{A^* \otimes F} = \tilde{f}$, where $i_{A^* \otimes F} : A^* \otimes F \longrightarrow S(A^* \otimes F)$. Thus, for all $\varphi \in A^*$ and $x \in F$, $(g \circ i_{A^* \otimes F})(\varphi \otimes x) = \tilde{f}(\varphi \otimes x)$, that is,

$$g(\varphi \otimes x) = \tilde{f}(\varphi \otimes x) = f(\varphi, x) = (\varphi \otimes id_L)[u(x)].$$

For all $\varphi \in A$ and for all $x, y \in F$, we have

$$g\left((\varphi \otimes xy) - \sum_{jk} \varphi(a_j a_k)(a_j^* \otimes x)(a_k^* \otimes y) \right) = 0 \tag{4}$$

Indeed, for $\varphi \in A$ and for all $x, y \in F$, we have

$$\begin{aligned} & g\left[(\varphi \otimes xy) - \sum_{jk} \varphi(a_j a_k)(a_j^* \otimes x)(a_k^* \otimes y) \right] \\ = \; & g(\varphi \otimes xy) - g[\sum_{jk} \varphi(a_j a_k)(a_j^* \otimes x)(a_k^* \otimes y)] \\ = \; & \tilde{f}(\varphi \otimes xy) - \sum_{jk} \varphi(a_j a_k) g(a_j^* \otimes x) g(a_k^* \otimes y) \\ = \; & f(\varphi, xy) - \sum_{jk} \varphi(a_j a_k) \tilde{f}(a_j^* \otimes x) \tilde{f}(a_k^* \otimes y) \\ = \; & (\varphi \otimes id_L)[u(xy)] - \sum_{jk} \varphi(a_j a_k) f(a_j^*, x) f(a_k^*, y) \\ = \; & (\varphi \otimes id_L)[u(x)u(y)] \\ & - \sum_{jk} \varphi(a_j a_k)[(a_j^* \otimes id_L)[u(x)]][(a_k^* \otimes id_L)[u(y)]]. \end{aligned}$$

This implies that,

$$\begin{aligned}
& g[(\varphi\otimes xy)-\sum_{jk}\varphi(a_ja_k)(a_j^*\otimes x)(a_k^*\otimes y)] \\
= \;& (\varphi\otimes id_L)(\sum_{jk}a_i\otimes(a_j^*\otimes id_L)[u(x)]\sum_{jk}a_l\otimes(a_l^*\otimes id_L)[u(y)]) \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& (\varphi\otimes id_L)(\sum_{jk}a_i\otimes(a_j^*\otimes id_L)[u(x)]a_l\otimes(a_l^*\otimes id_L)[u(y)]) \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& (\varphi\otimes id_L)(\sum_{jk}a_ia_l\otimes[(a_j^*\otimes id_L)[u(x)]][(a_l^*\otimes id_L)[u(y)]]) \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& (\varphi\otimes id_L)(\sum_{jk}a_ia_l\otimes[(a_j^*\otimes id_L)[u(x)]][(a_l^*\otimes id_L)[u(y)]]) \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& \sum_{il}\varphi(a_ia_l)[(a_i^*\otimes id_L)[u(x)]][(a_l^*\otimes id_L)[u(y)]] \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& 0.
\end{aligned}$$

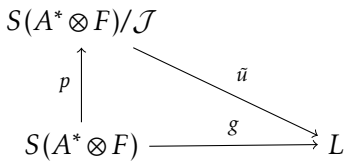
For all $\varphi\in A^*$, we have

$$g(\varphi\otimes 1_F-\varphi(1_A))=0. \tag{5}$$

Indeed, for all $\varphi\in A^*$, we have

$$\begin{aligned}
g(\varphi\otimes 1_F-\varphi(1_A)) &= \tilde{f}(\varphi\otimes 1_F)-\varphi(1_A)g(1_K) \\
&= f(\varphi,1_F)-\varphi(1_A)\cdot 1_L \\
&= (\varphi\otimes id_L)[u(1_F)]-\varphi(1_A)\cdot 1_L \\
&= (\varphi\otimes id_L)(1_A\otimes 1_L)-\varphi(1_A)\cdot 1_L \\
&= \varphi(1_A)\cdot id_L(1_L)-\varphi(1_A)\cdot 1_L \\
&= 0.
\end{aligned}$$

By (4) and (5), the map g vanishes on $\mathcal{J}$. Therefore, according to Theorem 2.1, there exists a unique linear map $\tilde{u}: F^A\longrightarrow L$ such that the following diagram



commutes, that is, $\tilde{u} \circ p = g$, where $p : S(A^* \otimes F) \longrightarrow F^A = S(A^* \otimes F)/\mathcal{J}$ is the canonical surjection. Furthermore, for all $\varphi \in A^*$and $x \in F$,

$$(\tilde{u} \circ p)(\varphi \otimes x) = g(\varphi \otimes x),$$

that is, $\tilde{u}(\overline{\varphi \otimes x}) = g(\varphi \otimes x)$. Hence, for all $\varphi_1, \varphi_2 \in A^*$ and for all $x_1, x_2 \in F$,

$$\begin{aligned} \tilde{u}[\overline{(\varphi_1 \otimes x_1)(\varphi_2 \otimes x_2)}] &= g[(\varphi_1 \otimes x_1)(\varphi_2 \otimes x_2)] \\ &= g(\varphi_1 \otimes x_1) g(\varphi_2 \otimes x_2) \\ &= \tilde{u}(\overline{\varphi_1 \otimes x_1})\tilde{u}(\overline{\varphi_2 \otimes x_2}). \end{aligned}$$

Since the elements $\overline{\varphi \otimes x}$ generate F^A, then $\tilde{u}$ is a homomorphism of algebras.

On the other hand, for all $x \in F$,

$$\begin{aligned} \Big[(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}\Big](x) &= (id_A \otimes \tilde{u})[\sum_i a_i \otimes (\overline{a_i^* \otimes x})] \\ &= \sum_i a_i \otimes \tilde{u}(\overline{a_i^* \otimes x}) \\ &= \sum_i a_i \otimes (a_i^* \otimes id_L)[u(x)] \\ &= u(x). \end{aligned}$$

Since the element x is arbitrary in F, we conclude $(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)} = u$. □

The pair $(F^A, \gamma_{(A,F)})$ is called *solution of the universal problem* for homomorphisms of an algebra F into an algebra $A \otimes F^A$.

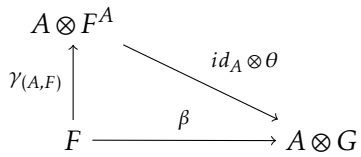
Remark 3.5. The existence and uniqueness of homomorphism $\tilde{u} : F^A \longrightarrow L$ such that $(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)} = u$ for every homomorphism $u : F \longrightarrow A \otimes L$ proves that the map

$$Hom(F^A, L) \longrightarrow Hom(F, A \otimes L), \tilde{u} \longmapsto (id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}$$

is a bijection.

Example 3.6. For $L = K$, there exists a canonical bijection of $Hom(F^A, K)$ into $Hom(F, A)$. In particular, if M is a paracompact smooth manifold and if A is a local algebra on $\mathbb{R}$, then the manifold $M^A = Hom_{Alg}(C^\infty(M), A)$ of infinitely near points is canonically in bijection with $Hom_{Alg}(C^\infty(M)^A, \mathbb{R})$ (see [7, 9, 13] for more details).

Proposition 3.7. *Let G be a commutative algebra with unit over K and β a homomorphism of the algebra G into the algebra $A \otimes L$. If the pair (G, β) is another solution of the same universal problem, then there exists a unique isomorphism θ of the algebra F^A into the algebra G such that the following diagram*



commutes, that is,

$$(id_A \otimes \theta) \circ \gamma_{(A,F)} = \beta. \tag{6}$$

Proof. The existence and uniqueness of homomorphism θ follows from Theorem 3.4 applied to algebra G and to homomorphism β of algebra F into the algebra $A \otimes G$. Moreover, let θ' be the unique homomorphism of G into F^A such that

$$(id_A \otimes \theta') \circ \gamma_{(A,F)} = \beta. \tag{7}$$

We deduce from relations (6) and (7) that

$$[id_A \otimes (\theta' \circ \theta)] \circ \gamma_{(A,F)} = \gamma_{(A,F)} \tag{8}$$

and

$$[id_A \otimes (\theta \circ \theta')] \circ \beta = \beta. \tag{9}$$

Since the pairs $(F^A, \gamma_{(A,F)})$ and (G, β) are two solutions of the universal problem for homomorphisms of an algebra F into an algebra $A \otimes L$, then the map $\theta' \circ \theta$ is the unique endomorphism of the algebra F^A satisfying the relation (8) and the map $\theta \circ \theta'$ is the unique endomorphism of the algebra G satisfying the relation (9).

Since $id_A \otimes id_{F^A} = id_{A \otimes F^A}$ and $id_A \otimes id_G = id_{A \otimes G}$, then the endomorphisms id_{F^A} and id_G respectively verify relations (8) and (9). It follows from the uniqueness that $\theta' \circ \theta = id_{F^A}$ and $\theta \circ \theta' = id_G$. Therefore, the homomorphism θ is an isomorphism. □

4 Functorials properties

In this section, we describe the covariant functor and the contravariant functor in the categories corresponding to the prolongations of algebras by a local algebra.

Proposition 4.1. *Let F and G be two algebras over K, let A be a local algebra over K and let $\Psi : F \longrightarrow G$ be a homomorphism of algebras. Then, there exists a unique homomorphism of algebras Ψ^A from the algebra F^A into the algebra G^A such that the following diagram*

$$\begin{array}{ccc} G & \xrightarrow{\gamma_{(A,G)}} & A \otimes G^A \\ {\scriptstyle \Psi}\uparrow & & \uparrow{\scriptstyle id_A \otimes \Psi^A} \\ F & \xrightarrow{\gamma_{(A,F)}} & A \otimes F^A \end{array}$$

commutes, that is, $(id_A \otimes \Psi^A) \circ \gamma_{(A,F)} = \gamma_{(A,G)} \circ \Psi$.

Moreover, for all $\varphi \in A^*$ and $x \in F$, we have

$$\Psi^A(\overline{\varphi \otimes x}) = \overline{\varphi \otimes \Psi(x)}.$$

Proof. The existence and uniqueness of the homomorphism Ψ^A results from the Theorem 3.4, applied to the algebra G^A and to homomorphism $\gamma_{(A,G)} \circ \Psi : F \to A \otimes G^A$. For all $x \in F$, the homomorphism Ψ^A is such that

$$[(id_A \otimes \Psi^A) \circ \gamma_{(A,F)}](x) = (\gamma_{(A,F)} \circ \Psi)(x).$$

Thus, by (1), we have

$$(id_A \otimes \Psi^A)\left[\sum_i a_i \otimes (\overline{a_i^* \otimes x})\right] = \gamma_{(A,F)}[\Psi(x)].$$

This implies that,

$$\sum_i a_i \otimes \Psi^A(\overline{a_i^* \otimes x}) = \sum_i a_i \otimes (\overline{a_i^* \otimes \Psi(x)}),$$

that is

$$\sum_i \left[a_i \otimes \Psi^A(\overline{a_i^* \otimes x}) - a_i \otimes (\overline{a_i^* \otimes \Psi(x)}\right] = 0,$$

hence

$$\sum_i a_i \otimes \left[\Psi^A(\overline{a_i^* \otimes x}) - (\overline{a_i^* \otimes \Psi(x)}\right] = 0.$$

According to Proposition 3.3, we conclude that $\Psi^A(\overline{a_i^* \otimes x}) = (\overline{a_i^* \otimes \Psi(x)}$, for any $i \in I$. It follows that, $\Psi^A\overline{(\varphi \otimes x)} = \overline{\varphi \otimes \Psi(x)}$, for all $\varphi \in A^*$ and $x \in F$, since the elements $\overline{a_i^* \otimes x}$ generate F^A. □

Proposition 4.2. *For any local algebra A, the map $F \longrightarrow F^A$ is a covariant functor of the category of algebras over K in itself.*

Proof. Consider the composition $\Phi \circ \Psi : F \longrightarrow G \longrightarrow H$ of two homomorphisms of algebras. According to Proposition 4.1, we have

$$\begin{aligned}
\left[id_A \otimes (\Phi \circ \Psi)^A\right] \circ \gamma_{(A,F)} &= \gamma_{(A,H)} \circ (\Phi \circ \Psi) \\
&= (\gamma_{(A,H)} \circ \Phi) \circ \Psi \\
&= (id_A \otimes \Phi^A) \circ \gamma_{(A,G)} \circ \Psi \\
&= (id_A \otimes \Phi^A) \circ (id_A \otimes \Psi^A) \circ \gamma_{(A,F)} \\
&= [id_A \otimes (\Phi^A \circ \Psi^A)] \circ \gamma_{(A,F)}.
\end{aligned}$$

Since $(\Phi \circ \Psi)^A$ is the unique homomorphism verifying the relation

$$\left[id_A \otimes (\Phi \circ \Psi)^A\right] \circ \gamma_{(A,F)} = \gamma_{(A,H)} \circ (\Phi \circ \Psi).$$

We conclude that $(\Phi \circ \Psi)^A = \Phi^A \circ \Psi^A$.

On the other hand, since the elements $\overline{\varphi \otimes x}$, where $\varphi \in A^*$ and $x \in F$, generate F^A, then $(id_F)^A(\overline{\varphi \otimes x}) = \overline{\varphi \otimes id_F(x)} = \overline{\varphi \otimes x}$, for all $\varphi \in A^*$ and $x \in F$. It follows, according to the uniqueness of the homomorphism $(id_F)^A$, we deduce that $(id_F)^A = id_{F^A}$. □

Proposition 4.3. *Let A and B be two finite-dimensional algebras and $u : A \longrightarrow B$ be a homomorphism of algebras. For any algebra F, there exists a unique homomorphism $F^u : F^B \longrightarrow F^A$ such that*

$$(id_B \otimes F^u) \circ \gamma_{(B,F)} = (u \otimes id_{F^A}) \circ \gamma_{(A,F)}.$$

Furthermore, the homomorphism F^u is such that $F^u(\overline{\varphi \otimes x}) = \overline{{}^t u(\varphi) \otimes x}$, for all $\varphi \in B^*$ and $x \in F$.

Proof. The existence and uniqueness of the homomorphism F^u result from the Theorem 3.4 applied to the algebra F^A and to the homomorphism $(u \otimes id_{F^A}) \circ \gamma_{(A,F)} : F \xrightarrow{\gamma_{(A,F)}} A \otimes F^A \xrightarrow{u \otimes id_{F^A}} B \otimes F^A$. Then, we have

$$(id_B \otimes F^u) \circ \gamma_{(B,F)} = (u \otimes id_{F^A}) \circ \gamma_{(A,F)},$$

that is, for all $x \in F$,

$$[(id_B \otimes F^u) \circ \gamma_{(B,F)}](x) = [(u \otimes id_{F^A}) \circ \gamma_{(A,F)}](x).$$

Hence,

$$(id_B \otimes F^u)\left[\sum_j b_j \otimes (\overline{b_j^* \otimes x})\right] = (u \otimes id_{F^A})\left[\sum_i a_i \otimes (\overline{a_i^* \otimes x})\right]$$

implies that

$$\begin{aligned}\sum_j b_j \otimes F^u[(\overline{b_j^* \otimes x})] &= \sum_i u(a_i) \otimes (\overline{a_i^* \otimes x}) \\ &= \sum_i \left(\sum_j b_j^*[u(a_i)] \right) b_j \otimes (\overline{a_i^* \otimes x}) \\ &= \sum_i \left(\sum_j [{}^t u(b_j^*)](a_i) b_j \otimes (\overline{a_i^* \otimes x}) \right),\end{aligned}$$

that is,

$$\begin{aligned}\sum_j b_j \otimes F^u[(\overline{b_j^* \otimes x})] &= \sum_i \sum_j b_{j\otimes} \left(\overline{[{}^t u(b_j^*)](a_i) a_i^* \otimes x} \right) \\ &= \sum_j b_j \otimes \left(\sum_i \overline{[{}^t u(b_j^*)](a_i) a_i^* \otimes x} \right) \\ &= \sum_j b_j \otimes \overline{{}^t u(b_j^*) \otimes x}.\end{aligned}$$

Thus,

$$\sum_j b_j \otimes F^u[(\overline{b_j^* \otimes x})] = \sum_j b_j \otimes \overline{{}^t u(b_j^*) \otimes x},$$

that is

$$\sum_j b_j \otimes [F^u(\overline{b_j^* \otimes x}) - \overline{{}^t u(b_j^*) \otimes x}] = 0.$$

It follows that,

$$\sum_j [F^u(\overline{b_j^* \otimes x}) - \overline{{}^t u(b_j^*) \otimes x}](b_j \otimes 1_F) = 0.$$

According to Proposition 3.3, we have $F^u(\overline{b_j^* \otimes x}) - \overline{{}^t u(b_j^*) \otimes x} = 0$, for all $j \in J$, that is, $F^u(\overline{b_j^* \otimes x}) = \overline{u^t(b_j^*) \otimes x}$, for all $j \in J$. By linearity of F^u and ${}^t u$, we conclude that $F^u(\overline{\varphi \otimes x}) = \overline{{}^t u(\varphi) \otimes x}$, for all $\varphi \in B^*$ and $x \in F$. □

Lemma 4.4. *For any algebra F, the map $A \longrightarrow F^A$ is a contravariant functor from the category of finite-dimensional algebras over K in the category of algebras over K.*

Proof. Let $u : A \longrightarrow B$ and $v : B \longrightarrow C$ be two algebras homomorphisms. According to Proposition 4.3, we have

$$(id_B \otimes F^u) \circ \gamma_{(B,F)} = (u \otimes id_{F^A}) \circ \gamma_{(A,F)};$$

$$(id_C \otimes F^v) \circ \gamma_{(C,F)} = (v \otimes id_{F^B}) \circ \gamma_{(B,F)}$$

and

$$(id_C \otimes F^{v \circ u}) \circ \gamma_{(C,F)} = ((v \circ u) \otimes id_{F^A}) \circ \gamma_{(A,F)}.$$

Thus,

$$\begin{aligned}(id_C \otimes (F^u \circ F^v) \circ \gamma_{(C,F)} &= (id_C \otimes F^u) \circ (id_C \otimes F^v) \circ \gamma_{(C,F)} \\ &= (id_C \otimes F^u) \circ (v \otimes id_{F^B}) \circ \gamma_{(B,F)} \\ &= [(id_C \circ v) \otimes (F^u \otimes id_{F^B}) \circ \gamma_{(B,F)} \\ &= (v \otimes F^u) \circ \gamma_{(B,F)} \\ &= (v \otimes id_{F^A}) \circ (id_B \otimes F^u) \circ \gamma_{(B,F)} \\ &= (v \otimes id_{F^A}) \circ (u \otimes id_{F^A}) \circ \gamma_{(A,F)} \\ &= ((v \circ u) \otimes id_{F^A}) \circ \gamma_{(A,F)}.\end{aligned}$$

By uniqueness, we conclude that $F^{v\circ u} = F^u \circ F^v$. In particular, for $B = A$ and $u = id_A$, we have $(id_A \otimes F^{id_A}) \circ \gamma_{(A,F)} = (id_A \otimes id_{F^A}) \circ \gamma_{(A,F)}$. We conclude by uniqueness that $F^{id_A} = id_{F^A}$. □

Given an algebra L and a homomorphism $\tilde{u}$ from F^A into L, we know how to determine the homomorphism from F into $A \otimes F$ corresponding to $\tilde{u}$: it is the homomorphism $(id_A \otimes \tilde{u}\ ) \circ \gamma_{(A,F)}$. It is useful to determine the homomorphism from F^A onto L corresponding to F in $A \otimes L$.

Lemma 4.5. *Let L be an algebra and A be a finite-dimensional algebra. There exists a unique homomorphism β from $(A \otimes L)^A$ into L such that*

$$(id_A \otimes \beta) \circ \gamma_{(A,A\otimes L)} = id_{A\otimes L}. \tag{10}$$

Moreover, the homomorphism β is such that

$$\beta(\overline{\varphi \otimes l \otimes a}) = \varphi(a) \cdot l,$$

for all $\varphi \in A^*$, $a \in A$ and $l \in L$,

Proof. The existence and uniqueness of the homomorphism $\beta : (A\otimes L)^A \longrightarrow L$ result from the Theorem 3.4 applied to the algebra L and to the homomorphism $id_{A\otimes L}$. Thus, $(id_A \otimes \beta) \circ \gamma_{(A,A\otimes L)} = id_{A\otimes L}$ and for all $a \in A$ and $l \in L$, we have

$$[(id_A \otimes \beta) \circ \gamma_{(A,A\otimes L)}](a \otimes l) = a \otimes l.$$

It follows that,

$$(id_A \otimes \beta)[\sum_i a_i \otimes (\overline{a_i^* \otimes a \otimes l})] = \sum_i a_i^*(a) a_i \otimes l.$$

Hence,

$$\sum_i a_i \otimes \beta(\overline{a_i^* \otimes a \otimes l}) = \sum_i a_i \otimes a_i^*(a) \cdot l$$

implies that

$$\begin{aligned}0 &= \sum_i a_i \otimes [\beta(\overline{a_i^* \otimes a \otimes l}) - a_i^*(a) \cdot l] \\ &= \sum_i \Big[\beta(\overline{a_i^* \otimes a \otimes l}) - a_i^*(a) \cdot l\Big](a_i \otimes 1_{A\otimes L}).\end{aligned}$$

By the Proposition 3.3, we have $\beta(\overline{a_i^* \otimes a \otimes l}) - a_i^*(a) \cdot l = 0$, for all $i \in I$ that is, $\beta(\overline{a_i^* \otimes a \otimes l}) = a_i^*(a) \cdot l$, for all $i \in I$. By linearity, we conclude that $\beta(\overline{\varphi \otimes l \otimes a}) = \varphi(a) \cdot l$, for all $\varphi \in A^*, a \in A$ and $l \in L$. □

Proposition 4.6. *Let $\Phi : Hom(F^A, L) \longrightarrow Hom(F, A \otimes L), \tilde{u} \longmapsto (id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}$ be the bijection from $Hom(F^A, L)$ into $Hom(F, A \otimes L)$. Then the map $\Psi : Hom(F, A \otimes L) \longrightarrow Hom(F^A, L), u \longmapsto \beta \circ u^A$ is the inverse bijection of Φ.*

Proof. For all $u \in Hom(F, A \otimes L)$, we have

$$\begin{aligned}(\Phi \circ \Psi)(u) &= \Phi(\beta \circ u^A) \\ &= \Big(id_A \otimes (\beta \circ u^A)\Big) \circ \gamma_{(A,F)} \\ &= (id_A \otimes \beta) \circ (id_A \circ u^A) \circ \gamma_{(A,F)}\end{aligned}$$

According to Proposition 4.1 and by (10), we have

$$\begin{aligned}(\Phi \circ \Psi)(u) &= (id_A \otimes \beta) \circ \gamma_{(A,A\otimes L)} \circ u \\ &= id_{A\otimes L} \circ u = u.\end{aligned}$$

Thus, $\Phi \circ \Psi = id_{Hom(F,A\otimes F)}$.

Conversely, for all $\tilde{u} \in Hom(F^A, L)$, we have

$$\begin{aligned}(\Psi \circ \Phi)(\tilde{u}) &= \Psi[(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}] \\ &= \beta \circ [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A.\end{aligned}$$

Putting $\beta \circ [(id_A \otimes \tilde{u}) \circ \gamma]^A = v$, we have

$$\begin{aligned}\Phi(v) &= \Phi\Big[\beta \circ [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A\Big] \\ &= (id_A \otimes \Big[\beta \circ [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A\Big]) \circ \gamma_{(A,F)} \\ &= (id_A \otimes \beta) \circ \Big[id_A \otimes [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A\Big] \circ \gamma_{(A,F)}.\end{aligned}$$

According to Proposition 4.1, we have

$$\Phi(v) = (id_A \otimes \beta) \circ \gamma_{(A,A\otimes L)} \circ (id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}.$$

By (10), we get $\Phi(v) = (id_A \otimes \tilde{u}) \circ \gamma_{(A,F)} = \Phi(\tilde{u})$. Since Φ is a bijection, we have

$$\beta \circ [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A = \tilde{u}.$$

Therefore,

$$\Psi \circ \Phi = id_{Hom(F^A,L)}.$$

□

For $L = F^A$ and for $u = \gamma_{(A,F)} : F \longrightarrow A \otimes F^A$, the corresponding homomorphism is $id_{F^A} \in Hom(F^A, F^A)$. Indeed, for all $x \in F$,

$$\begin{aligned}[(id_A \otimes id_{F^A}) \circ \gamma_{(A,F)}](x) &= (id_A \otimes id_{F^A})\left[\sum_i a_i \otimes (\overline{a_i^* \otimes x})\right] \\ &= \sum_i a_i \otimes (\overline{a_i^* \otimes x}).\end{aligned}$$

That is, $(id_A \otimes id_{F^A}) \circ \gamma_{(A,F)} = \gamma_{(A,F)}$. By the uniqueness, we conclude that $\tilde{u} = \widetilde{\gamma_{(A,F)}} = id_{F^A}$.

Proposition 4.7. *Let A be a finite-dimensional algebra over K. Then the homomorphism*

$$\varphi : K \longrightarrow K^A, \lambda \longmapsto \overline{1_A^* \otimes \lambda}$$

is an isomorphism of algebras. Its inverse homomorphism $\tilde{u} : K^A \longrightarrow K$ is such that

$$(id_A \otimes \tilde{u}) \circ \gamma_{(A,K)} = u,$$

where $u : K \longrightarrow A = A \otimes K, \lambda \longmapsto \lambda \cdot 1_A = 1_A \otimes \lambda$ is the canonical homomorphism.

Proof. The map φ is well-defined by construction. Since, for all $\lambda_1, \lambda_2 \in K$, $\overline{[1_A^* \otimes (\lambda_1 + \lambda_2)]} = \overline{\left(1_A^* \otimes \lambda_1\right)} + \overline{\left(1_A^* \otimes \lambda_2\right)}$ and $\overline{1_A^* \otimes \lambda_1\lambda_2} = \lambda_1\overline{\left(1_A^* \otimes \lambda_2\right)}$, we have $\varphi(\lambda_1 + \lambda_2) = \varphi(\lambda_1) + \varphi(\lambda_2)$ and $\varphi(\lambda_1\lambda_2) = \lambda_1\varphi(\lambda_2)$. On the other hand, for all $\lambda \in K$, we have $\varphi(\lambda) = \lambda(\overline{1_A^* \otimes 1_K}) = \lambda\overline{1_A^*(1_A)} = \lambda 1_{K^A}$. So, since for all $\lambda_1, \lambda_2 \in K$, $\overline{1_A^* \otimes \lambda_1\lambda_2} = \lambda_1\lambda_2\overline{\left(1_A^* \otimes 1_K\right)} = (\lambda_1\lambda_2)\overline{1_A^*(1_A)}$, we $\varphi(\lambda_1\lambda_2) = (\lambda_1\lambda_2)1_{K^A} = (\lambda_1 \cdot 1_{K^A})(\lambda_2 \cdot 1_{K^A}) = \varphi(\lambda_1)\varphi(\lambda_2)$. The map φ is therefore a homomorphism of algebras.

Let $u : K \longrightarrow A = A \otimes K, \lambda \longmapsto \lambda \cdot 1_A = 1_A \otimes \lambda$ be the canonical homomorphism and let $\tilde{u} : K^A \longrightarrow K$ be the unique homomorphism of algebras such that $id_A \otimes \tilde{u}) \circ \gamma_{(A,K)} = u$. Since $\gamma_{(A,K)}$ does not depend of a choisen basis in A, consider the basis $(e_i)_{i\in I}$ of A. By the Proposition 3.2, we have for all $\lambda \in K$,

$$\gamma_{(A,K)}(\lambda) = \sum_i e_i \otimes (\overline{e_i^* \otimes \lambda}) = \sum_i e_i \otimes \lambda(\overline{e_i^* \otimes 1_K}) = \sum_i e_i \otimes \lambda e_i^*(1_A),$$

implies that

$$\gamma_{(A,K)}(\lambda) = 1_A \otimes \cdot\overline{1_A^*(\lambda \cdot 1_A)} = 1_A \otimes \lambda \cdot \overline{1_A^*(1_A)} = 1_A \otimes \overline{(1_A^* \otimes \lambda)}.$$

Thus, for all $\lambda \in K$,

$$[(id_A \otimes \varphi) \circ u](\lambda) = (id_A \otimes \varphi)[u(\lambda)] = (id_A \otimes \varphi)(1_A \otimes \lambda)$$

That is

$$[(id_A \otimes \varphi) \circ u](\lambda) = 1_A \otimes \overline{(1_A^* \otimes \lambda)} = \gamma_{(A,K)}(\lambda),$$

for all $\lambda \in K$. Hence, $(id_A \otimes \varphi) \circ u = \gamma_{(A,K)}$. Since $\gamma_{(A,K)} : K \longrightarrow A \otimes K^A$ is a homomorphism of algebras, the map $\varphi \otimes \tilde{u} : K^A \longrightarrow K \longrightarrow K^A$ is the unique homomorphism of algebras such that

$$[id_A \otimes (\varphi \circ \tilde{u})] \circ \gamma_{(A,K)} = \gamma_{(A,K)}. \tag{11}$$

Consider the map $(id_A \otimes id_{K^A}) \circ \gamma_{(A,K)} : K \longrightarrow A \otimes K^A \longrightarrow A \otimes K^A$. For all $\lambda \in K$, we have

$$\begin{aligned} [(id_A \otimes id_{K^A}) \circ \gamma_{(A,K)}](\lambda) &= (id_A \otimes id_{K^A})[\gamma_{(A,K)}(\lambda)] \\ &= (id_A \otimes id_{K^A})[1_A \otimes \overline{(1_A^* \otimes \lambda)}] \\ &= \gamma_{(A,K)}(\lambda). \end{aligned}$$

We deduce that

$$(id_A \otimes id_{K^A}) \circ \gamma_{(A,K)} = \gamma_{(A,K)}. \tag{12}$$

By (11) and (12), It follows by uniqueness that $\varphi \circ \tilde{u} = id_{K^A}$.

On the other hand, $\varphi \circ \tilde{u}$ is the unique homomorphism of algebras such that

$$[id_A \otimes (\tilde{u} \circ \varphi)] \circ u = u. \tag{13}$$

Since for all $\lambda \in K$,

$$[(id_A \otimes id_K) \circ u](\lambda) = (id_A \otimes id_K)[u(\lambda)] = (id_A \otimes id_K)(1_A \otimes \lambda),$$

that is $[(id_A \otimes id_K) \circ u](\lambda) = 1_A \otimes \lambda = u(\lambda)$, for any $\lambda \in K$. Hence

$$(id_A \otimes id_K) \circ u = u \tag{14}$$

By (13) and (14), we deduce by uniqueness that $\tilde{u} \circ \varphi = id_K$. Therefore φ is an isomorphism of algebras. This isomorphism allows us to identify K^A with K. □

Let $u : F \longrightarrow F = K \otimes F, x \longmapsto 1_K \otimes x$ be the canonical isomorphism.

Proposition 4.8. *Let F be an algebra over K and $u : F \longrightarrow F = K \otimes F$ be the canonical isomorphism. Then, the canonical homomorphism $\gamma_{(K,F)} : F \longrightarrow K \otimes F^K = F^K$ is an inverse isomorphism of the homomorphism $\tilde{u} : F^K \longrightarrow F$ such that $(id_K \otimes \tilde{u}) \circ \gamma_{(K,F)} = u$.*

Proof. Let $\gamma_{(K,F)} : F \longrightarrow K \otimes F^K$ be the canonical homomorphism and let $\gamma_{(K,F)} \otimes \tilde{u}$ be the unique homomorphism of algebras such that

$$[id_K \otimes (\gamma_{(K,F)} \circ \tilde{u})] \circ \gamma_{(K,F)} = \gamma_{(K,F)}. \tag{15}$$

Since for all $x \in F$, $\gamma_{(K,F)}(x) = 1_K \otimes \overline{(1_K^* \otimes x)}$ and

$$\begin{aligned} 1_K \otimes \overline{(1_K^* \otimes x)} &= id_K(1_K) \otimes id_{F^K}\left(\overline{1_K^* \otimes x}\right) \\ &= (id_K \otimes id_{F^K})[1_K \otimes \overline{(1_K^* \otimes x)}], \end{aligned}$$

then $\gamma_{(K,F)}(x) = (id_K \otimes id_{F^K})[\gamma_{(K,F)}(x)] = [(id_K \otimes id_{F^K}) \circ \gamma_{(K,F)}](x)$.Which implies

$$(id_K \otimes id_{F^K}) \circ \gamma_{(K,F)} = \gamma_{(K,F)}. \tag{16}$$

By (15) and (16), we conclude by uniqueness that $\gamma_{(K,F)} \circ \tilde{u} = id_{F^K}$.

On the other hand, since for all x in F,

$$\begin{aligned} [(id_K \otimes \gamma_{(K,F)}) \circ u](x) &= (id_K \otimes \gamma)(1_K \otimes x) \\ &= 1_K \otimes [1_K \otimes \overline{(1_K^* \otimes x)}], \end{aligned}$$

that is,

$$[(id_K \otimes \gamma_{(K,F)}) \circ u](x) = 1_K \cdot 1_K \otimes \overline{(1_K^* \otimes x)} = \gamma_{(K,F)}(x),$$

for all x in F. Hence, $\left(id_K \otimes \gamma_{(K,F)}\right) \circ u = \gamma_{(K,F)}$.

The homomorphism $\tilde{u} \circ \gamma_{(K,F)}$ is the unique homomorphism of algebras such that

$$[(id_K \otimes (\tilde{u} \circ \gamma_{(K,F)})] \circ u = u. \tag{17}$$

As $(id_K \otimes id_F) \circ u = u$ and since for all $x \in F$,

$$\begin{aligned} [(id_K \otimes id_F) \circ u](x) &= (id_K \otimes id_F)[u(x)] \\ &= (id_K \otimes id_F)(1_K \otimes x), \end{aligned}$$

implies that

$$[(id_K \otimes id_F) \circ u](x) = (1_K \otimes x) = u(x),$$

for all $x \in F$. Hence,

$$(id_K \otimes id_F) \circ u = u. \tag{18}$$

By (17) and (18), we deduce by uniqueness that $\tilde{u} \circ \gamma_{(K,F)} = id_F$.

It follows that $\gamma_{(K,F)}$ is an isomorphism of algebras with inverse $\tilde{u}$. This canonical isomorphism allows us to identify F with F^K. □

Proposition 4.9. *Let A and B be two algebras of finite dimensions. Then, there exists a unique isomorphism $\theta : F^{A\otimes B} \longrightarrow (F^A)^B$ such that*

$$(id_{A\otimes B} \otimes \theta) \circ \gamma_{(A\otimes B,F)} = \left(id_A \otimes \gamma_{(B,F^A)}\right) \circ \gamma_{(A,F)}.$$

Proof. By applying the Proposition 3.3 to the algebra $(F^A)^B$ and to the homomorphism $u = \left(id_A \otimes \gamma_{(B,F^A)}\right) \circ \gamma_{(A,F)} : F \longrightarrow A\otimes F^A \longrightarrow (A\otimes B)\otimes (F^A)^B$, we prove the existence and uniqueness of the homomorphism $\theta : F^{A\otimes B} \longrightarrow (F^A)^B$ such that

$$(id_{A\otimes B} \otimes \theta) \circ \gamma_{(A\otimes B,F)} = \left(id_A \otimes \gamma_{(B,F^A)}\right) \circ \gamma_{(A,F)}.$$

Furthermore, let L be an algebra and $\alpha : F \longrightarrow A\otimes B\otimes L$ be a homomorphism of algebras. Let $\bar{\alpha} : F^A \longrightarrow B\otimes L$ be the unique homomorphism of algebras such that

$$(id_A \otimes \bar{\alpha}) \circ \gamma_{(A,F)} = \alpha, \tag{19}$$

and let $\tilde{\alpha} : (F^A)^B \longrightarrow F^A$ be the unique homomorphism such that

$$(id_B \otimes \tilde{\alpha}) \circ \gamma_{(B,F^A)} = \bar{\alpha}. \tag{20}$$

By composing by tensor product on the left of the relation (20) by id_A, we have

$$id_A \otimes [(id_B \otimes \tilde{\alpha}) \circ \gamma_{(B,F^A)}] = id_A \otimes \bar{\alpha}.$$

It follows that,

$$(id_A \otimes [(id_B \otimes \tilde{\alpha}) \circ \gamma_{(B,F^A)}]) \circ \gamma_{(A,F)} = (id_A \otimes \bar{\alpha}) \circ \gamma_{(A,F)}$$

that is

$$(id_A \otimes id_B \otimes \tilde{\alpha}) \circ (id_A \otimes \gamma_{(B,F^A)}) \circ \gamma_{(A,F)} = [(id_A \otimes \bar{\alpha})] \circ \gamma_{(A,F)}. \tag{21}$$

Since $id_A \otimes id_B = id_{A\otimes B}$, $\left(id_A \otimes \gamma_{(B,F^A)}\right) \circ \gamma_{(A,F)} = u$ and by (19), the relation (21) becomes $(id_A \otimes id_B \otimes \tilde{\alpha}) \circ u = \alpha$. By uniqueness of the homomorphism $\tilde{\alpha}$, the pair $\left((F^A)^B, u\right)$ is therefore a solution of the universal problem for the algebras F and $A\otimes B\otimes L$, for any algebra L.
It follows that the homomorphism $\theta : F^{A\otimes B} \longrightarrow (F^A)^B$ is an isomorphism of algebras. □

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On S-(m,n)-absorbing ideals and S-(m,n)-absorbing prime ideals of commutative rings

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Abstract. Let R be a commutative ring with identity, S a multiplicative subset of R, and I a proper ideal of R disjoint from S. Let m and n be positive integers with $m > n$. In this paper, we introduce the concepts of S-(m,n)-absorbing ideals, S-(m,n)-absorbing prime ideals, and S-AB-(m,n)-absorbing ideals. These notions generalize several existing concepts, including (m,n)-absorbing ideals, (m,n)-absorbing prime ideals, AB-(m,n)-absorbing ideals, and S-n-absorbing ideals. We investigate the fundamental properties of these new classes of ideals and examine their behavior under various ring constructions, such as quotient rings, homomorphic images, and localizations. Furthermore, we explore the transfer of these properties to trivial ring extensions and amalgamated rings.

Key Words:S-(m,n)-absorbing ideal, (m,n)-absorbing ideal, S-n-absorbing ideal, S-prime ideal, trivial extension, amalgamated algebra.
2010 MSC: Primary 13A15; Secondary 13B99, 13B30.

1 Introduction

Throughout this paper, all rings considered are assumed to be commutative with nonzero identity. If R is a ring, we denote by $J(R)$ and $U(R)$ the Jacobson radical of R and the set of units of R, respectively. A nonempty subset S of R is called a *multiplicative subset* if $1 \in S$, $0 \notin S$, and for all $s_1, s_2 \in S$, we have $s_1 s_2 \in S$. For a multiplicative subset S of R, the localization of R at S is denoted by $S^{-1}R = \left\{\frac{a}{s} : a \in R, s \in S\right\}$. The *saturation* of S is defined as $S^* = \left\{r \in R : \frac{r}{1} \text{ is a unit in } S^{-1}R\right\}$; note that S^* is a multiplicative subset containing S.

In recent years, various generalizations of prime ideals have been introduced and extensively studied. Among these, Abadi and Moghimi [1] introduced the concept of (m,n)-*absorbing ideals*, where m and n are positive integers with $m > n$. A proper ideal I of R is called (m,n)-absorbing if, whenever $a_1 \cdots a_m \in I$ for some nonunit elements $a_1, \ldots, a_m \in R$, there exist n of the a_i's whose product lies in I. This line of generalization was extended by Badawi, El Khalfi, and Mahdou [8], who defined (m,n)-*absorbing prime ideals*: a proper ideal I of R is called (m,n)-absorbing prime if, whenever nonunit elements $a_1, \ldots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, then either $a_1 \cdots a_n \in I$ or $a_{n+1} \cdots a_m \in I$. They also introduced the notion of AB-(m,n)-*absorbing ideals*, where a proper ideal I of R is called AB-(m,n)-absorbing if, whenever $a_1 \cdots a_m \in I$ for some $a_1, \ldots, a_m \in R$, there exist n of the a_i's whose product belongs to I.

In 2020, Hamed and Malek [18] introduced the concept of *S-prime ideals* as a generalization of prime ideals. Given a multiplicative subset S and an ideal I of R disjoint from S, the ideal I is called S-prime if there exists an $s \in S$ such that, for all $a, b \in R$, $ab \in I$ implies $sa \in I$ or $sb \in I$.

Among the many generalizations of prime ideals, the notion of *n-absorbing ideals* introduced by Anderson and Badawi [4] plays a prominent role. For a natural number n and an ideal I of R, the ideal I is said to be n-absorbing if, whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, there exist n of the x_i's whose product belongs to I. The study of n-absorbing ideals has become an active area of research, with many results published on their properties (see, for example, the survey article [6] and its references).

More recently, Baek, Choi, and Lim [5] introduced the concept of *S-n-absorbing ideals*, which generalize both S-prime ideals and n-absorbing ideals. An ideal I disjoint from S is called S-n-absorbing if there exists $s \in S$ such that for all $r_1, \ldots, r_{n+1} \in R$, the condition $r_1 \cdots r_{n+1} \in I$ implies that

$$s \prod_{\substack{1 \le i \le n+1 \\ i \ne j}} r_i \in I$$

for some $1 \le j \le n+1$.

Considering these developments, it is natural to ask whether the (m,n)-absorbing type ideals and their S-generalizations can be studied within a unified framework. In this paper, we address this question by introducing new classes of ideals. Let R be a commutative ring with identity, S a multiplicative subset of R, and I a proper ideal of R disjoint from S. Let m and n be positive integers with $m > n$. We say that I is an S-(m,n)-absorbing ideal of R if there exists $s \in S$ such that, whenever nonunit elements $a_1, \ldots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, there exist n distinct indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$.

Similarly, I is an S-(m,n)-absorbing prime ideal if there exists $s \in S$ such that, whenever $a_1, \ldots, a_m \in R$ and $a_1 \cdots a_m \in I$, then there exist n distinct indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$. An ideal I is called an S-AB-(m,n)-absorbing ideal if, for any $a_1, \ldots, a_m \in R$ with $a_1 \cdots a_m \in I$, there exist $s \in S$ and n distinct indices $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$. In each of these cases, we say that I is associated to s.

It is easy to see that any (m,n)-absorbing prime ideal (resp., (m,n)-absorbing ideal, AB-(m,n)-absorbing ideal) disjoint from S, as well as any S-prime ideal, is an S-(m,n)-absorbing prime ideal (resp., S-(m,n)-absorbing ideal, S-AB-(m,n)-absorbing ideal) for any positive integers m, n. Furthermore, an S-n-absorbing ideal as defined in [5] is precisely an S-AB-$(n+1,n)$-absorbing ideal. We demonstrate that many interesting properties of (m,n)-absorbing type ideals and S-n-absorbing ideals continue to hold for these new classes. However, we also show that certain properties valid for the former do not extend seamlessly to the latter.

This paper is organized into three sections, including the introduction. In Section 2, we investigate the fundamental properties of S-(m,n)-absorbing prime ideals, S-(m,n)-absorbing ideals, and S-AB-(m,n)-absorbing ideals. In addition, we examine their behavior under various ring-theoretic constructions, such as homomorphic images and localizations (see Theorems 2.28 and 2.30). Moreover, Proposition 2.33 characterizes S-Noetherian rings in terms of the finiteness properties of S-(m,n)-absorbing ideals and S-AB-(m,n)-absorbing ideals.

Finally, Section 3, the concluding part of the paper, is devoted to an in-depth study of these ideals within the framework of trivial ring extensions and amalgamated algebras along an ideal (see Theorems 2.9 and 3.3). Several illustrative examples are provided throughout to clarify and support the theoretical results established in the paper.

Any undefined notation and terminology can be found in [22].

2 Main Results

The following definitions provide the foundation upon which this section is built.

Definition 2.1. Let R be a ring, S a multiplicative subset of R, and I a proper ideal of R such that $I \cap S = \emptyset$. Let m and n be positive integers with $m > n$. Then:

1. I is called an S-(m,n)-*absorbing ideal* of R if there exists $s \in S$ such that, whenever nonunit elements $a_1, \ldots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, there exist n distinct indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$.

2. I is said to be an S-(m,n)-*absorbing prime ideal* of R if there exists $s \in S$ such that, whenever nonunit elements $a_1, \ldots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, then either $sa_1 \cdots a_n \in I$ or $sa_{n+1} \cdots a_m \in I$.

3. I is defined as an S-AB-(m,n)-*absorbing ideal* of R if there exists $s \in S$ such that, whenever elements $a_1, \ldots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, there exist n distinct indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$.

In all of these cases, the ideal I is said to be *associated with* s.

Remark 2.2.
1. If S consists entirely of units of R, then the notions of (m,n)-absorbing ideals and S-(m,n)-absorbing ideals coincide. The same holds for (m,n)-absorbing prime ideals and S-(m,n)-absorbing prime ideals, as well as for AB-(m,n)-absorbing ideals and S-AB-(m,n)-absorbing ideals.

2. Every S-prime ideal is both an S-(m,n)-absorbing prime ideal and an S-(m,n)-absorbing ideal for any multiplicative subset S of R and for all positive integers m and n with $m > n$.

3. For any multiplicative subset S of R, every (m,n)-absorbing ideal is an S-(m,n)-absorbing ideal. Similarly, every (m,n)-absorbing prime ideal is an S-(m,n)-absorbing prime ideal, and every AB-(m,n)-absorbing ideal is an S-AB-(m,n)-absorbing ideal.

In general, the converses of statements (2) and (3) in the previous remark do not hold, as illustrated by the following example.

Example 2.3. Let $R = \mathbb{Z}[X]$ be the ring of polynomials with integer coefficients, and let $S = \{4^n \mid n \in \mathbb{N}\}$ be a multiplicative subset of R. Consider the ideal $I := 2X^2\mathbb{Z}[X]$. Let $f, g, h \in \mathbb{Z}[X]$ such that $fgh \in I$. Since $X^2\mathbb{Z}[X]$ is $X\mathbb{Z}[X]$-primary, it follows that X^2 divides at least one of the products fg, gh, or fh. Therefore, at least one of $4fg$, $4gh$, or $4fh$ belongs to I, showing that I is an S-$(3,2)$-absorbing ideal of R.

However, I is not an S-prime ideal of R. Indeed, we have $2X \cdot X \in I$, but for every $n \in \mathbb{N}$, neither 4^nX nor $4^n(2X)$ belongs to I. Moreover, I is not a $(3,2)$-absorbing ideal of R, since $2 \cdot X \cdot X \in I$, yet neither $2X$ nor X^2 belongs to I.

The following remark follows directly from the definition of S-(m,n)-absorbing prime ideals.

Remark 2.4. Let R be a ring with a multiplicative subset $S \subseteq R$, and let I be a proper ideal of R. Let m and n be positive integers with $m > n$. Then the following statements hold:

1. I is an S-$(2,1)$-absorbing prime ideal of R if and only if I is an S-prime ideal.

2. I is an S-$(3,2)$-absorbing prime ideal of R if and only if I is an S-1-absorbing prime ideal.

3. If I is an S-(m,n)-absorbing prime ideal, then I is also an S-$(m+1,n+1)$-absorbing prime ideal.

4. I is an S-(m,n)-absorbing prime ideal if and only if it is an S-$(m,m-n)$-absorbing prime ideal.

The following example shows that the converse of Remark 2.4(3) does not necessarily hold.

Example 2.5. Let $R = \mathbb{Z}[X]$ and let $S = \{2^n \mid n \in \mathbb{N}\}$. Then S is a multiplicative subset of R, and $P = 4X\mathbb{Z}[X]$ is an S-$(2,1)$-absorbing prime ideal of R. Indeed, let $f, g \in R$ such that $fg \in P$. Since $fg \in P \subseteq X\mathbb{Z}[X]$, it follows that X divides either f or g, so $4f \in P$ or $4g \in P$. Note that $P \cap S = \emptyset$.
However, P is not an S-$(2,1)$-absorbing prime ideal of R, because $2 \cdot 2X \in P$, yet $2 \notin P$ and $2X \notin P$.

The proof of the following proposition is straightforward and is therefore omitted.

Proposition 2.6. *Let R be a ring, S a multiplicative subset of R, I a proper ideal of R, and let m and n be positive integers with $m > n$. Then the following statements hold:*

1. *If I is an S-AB-(m,n)-absorbing ideal of R, then I is also an S-(m,n)-absorbing ideal.*
2. *If I is an S-(m,n)-absorbing prime ideal of R, and $k = \max\{n, m-n\}$, then I is an S-(m,k)-absorbing ideal of R.*
3. *Every S-n-absorbing ideal is an S-AB-(m,n)-absorbing ideal for any multiplicative subset S of R.*

The following example illustrates that the converses of statements (1), (2), and (3) in Proposition 2.6 do not hold.

Example 2.7. Let $A = \mathbb{Z}_2[[X,Y,Z,W]]$ and set $H = (X^2+Y^2, Y^2+Z^2)A$. Define $R = A/H$, and let x, y, z, w denote the images of X, Y, Z, W in R, respectively. Let $S = \{w^n \mid n \in \mathbb{N}\}$ be a multiplicative subset of R. Consider the ideal

$$I = \Big(H + (XYZ, X(Y+Z), Y(X+Z), X^2)A\Big)/H.$$

Note that $I \cap S \neq \emptyset$.

1. We show that I is an S-$(4,2)$-absorbing ideal of R, but not an S-AB-$(4,2)$-absorbing ideal. Indeed, $1 \cdot x \cdot y \cdot z \in I$, but for any $s \in S$, none of sxy, syz, or sxz belongs to I. Thus, I is not S-AB-$(4,2)$-absorbing. To show I is S-$(4,2)$-absorbing, let $T = \{x, y, z\}$, and let D be the set of nonunit elements of R. An argument similar to [8, Example 2.3(1)] shows that I is a $(4,2)$-absorbing ideal of R, and hence an S-$(4,2)$-absorbing ideal.
2. Let $a_1 = x$, $a_2 = y$, $a_3 = x$, and $a_4 = z$. Then $a_1a_2a_3a_4 \in I$, but for all $s \in S$, neither $sa_1a_2 \in I$ nor $sa_3a_4 \in I$. Hence, I is not an S-$(4,2)$-absorbing prime ideal.
3. As I is an S-$(4,2)$-absorbing ideal but not an S-AB-$(4,2)$-absorbing ideal, it follows that I is not an S-n-absorbing ideal of R.

An integral domain R is said to be a *valuation domain* if, for all $0 \neq x, y \in R$, either $x \mid y$ or $y \mid x$. Every valuation domain is a Bézout domain, i.e., a domain in which every finitely generated ideal is principal. Moreover, every local Bézout domain is a valuation domain by [10, Proposition 1.5]. The following theorem provides a necessary condition under which an S-(m,n)-absorbing ideal becomes an S-n-absorbing ideal.

Theorem 2.8. Let R be a Bézout ring, and let S be a multiplicative subset of R. Then an ideal I is an S-(m,n)-absorbing ideal if and only if I is an S-n-absorbing ideal. In particular, this holds when R is a valuation domain or a principal ideal domain (PID).

Proof. Let I be an S-(m,n)-absorbing ideal of R. We show that I is also an S-$(m-1,n)$-absorbing ideal. Suppose $a_1 \cdots a_{m-1} \in I$ for some nonunit elements $a_1, \ldots, a_{m-1} \in R$. Since R is a Bézout ring, the ideal $(a_1, \ldots, a_{m-1}) = xR$ for some $x \in R$. Then for each $1 \leq i \leq m-1$, there exists $y_i \in R$ such that $a_i = xy_i$. Hence,

$$a_1 \cdots a_{m-1} = x^{m-1} y_1 \cdots y_{m-1} \in I.$$

By assumption, there exists $s \in S$ such that one of the following holds:

- $sx^n \in I$, or
- $sy_{i_1} \cdots y_{i_n} \in I$ for some $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, m-1\}$, or
- $sx^t y_{i_1} \cdots y_{i_{n-t}} \in I$ for some $0 < t < n$ and corresponding indices.

In each case, it follows that $sa_{i_1} \cdots a_{i_n} \in I$, and thus I is an S-$(m-1,n)$-absorbing ideal. The "in particular" part follows directly. □

Remark 2.9. Let $S_1 \subseteq S_2$ be multiplicative subsets of R, and let I be an ideal of R disjoint from S_2. Clearly, if I is an S_1-(m,n)-absorbing ideal of R, then I is also an S_2-(m,n)-absorbing ideal. However, the converse does not hold in general.

To see this, consider the ideal $I = (X^2, 2X)$ in the ring $R = \mathbb{Z}[X]$, and let $S_1 = \{1\}$ and $S_2 = \{2^n \mid n \in \mathbb{N}\}$. Note that $X^2 \in I$, but $1 \cdot X \notin I$, so I is not an S_1-$(2,1)$-absorbing ideal. On the other hand, $(I : 2) = X\mathbb{Z}[X]$ is a prime ideal, and thus I is an S_2-prime ideal, making it an S_2-$(2,1)$-absorbing ideal.

Proposition 2.10. *Let R be a ring, and let $S_1 \subseteq S_2$ be multiplicative subsets of R such that for every $s \in S_2$, there exists $t \in S_2$ with $st \in S_1$. If I is an S_2-(m,n)-absorbing ideal of R, then I is also an S_1-(m,n)-absorbing ideal of R, for some integers $1 \leq n < m$.*

Proof. Assume I is an S_2-(m,n)-absorbing ideal. Then there exists $s_2 \in S_2$ such that, for nonunit elements $a_1, \ldots, a_m \in R$ with $a_1 \cdots a_m \in I$, there are distinct indices $i_1, \ldots, i_n$ such that

$$s_2 a_{i_1} \cdots a_{i_n} \in I.$$

By assumption, there exists $t \in S_2$ such that $s_1 := ts_2 \in S_1$. Multiplying both sides by t, we obtain

$$s_1 a_{i_1} \cdots a_{i_n} \in I,$$

showing that I is an S_1-(m,n)-absorbing ideal of R. □

Recall that if S is a multiplicative subset of a ring R with $1 \in S$, then the set

$$S^* = \left\{ r \in R : \frac{r}{1} \in U(S^{-1}R) \right\}$$

is called the *saturation* of S. It is easy to see that S^* is a multiplicative subset of R containing S. If $S = S^*$, then S is said to be *saturated*. Moreover, it is clear that $S^{**} = S^*$ (see [16]).

Proposition 2.11. *Let R be a ring, S a multiplicative subset of R, and I an ideal of R disjoint from S. Then I is an S-(m,n)-absorbing ideal of R if and only if I is an S^*-(m,n)-absorbing ideal of R.*

Proof. We first show that $S^* \cap I = \emptyset$. Suppose, for contradiction, that there exists $r \in S^* \cap I$. Since $\frac{r}{1} \in U(S^{-1}R)$, there exist $a \in R$ and $s \in S$ such that $\frac{r}{1} \cdot \frac{a}{s} = 1$. This implies there exists $t \in S$ such that $tra = ts$, hence $tra \in I \cap S$, since $r \in I$, $a \in R$, and $t \in S$. This contradicts the assumption that $I \cap S = \emptyset$. Thus, $S^* \cap I = \emptyset$.

Now, by taking $S_1 = S$ and $S_2 = S^*$, the result follows immediately from Proposition 2.10. □

Theorem 2.12. Let R be a ring and S a multiplicative subset of R. Then the following assertions hold:

1. Let I be an S-(m,n)-absorbing ideal of R, and let J be an ideal of R such that $J \cap S \neq \emptyset$. Then IJ is an S-(m,n)-absorbing ideal of R.
2. Let $R \subseteq T$ be a ring extension. If J is an S-(m,n)-absorbing ideal of T, then $J \cap R$ is an S-(m,n)-absorbing ideal of R.

Proof. (1) Suppose I is an S-(m,n)-absorbing ideal of R. Let $a_1,\dots,a_m \in R$ be such that $a_1 \cdots a_m \in IJ \subseteq I$. Then, by assumption, there exists $s \in S$ and n distinct indices $i_1,\dots,i_n \in \{1,\dots,m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$. Since $J \cap S \neq \emptyset$, there exists $t \in J \cap S$, and hence $sta_{i_1} \cdots a_{i_n} \in IJ$. Therefore, IJ is an S-(m,n)-absorbing ideal of R.

(2) Suppose J is an S-(m,n)-absorbing ideal of T, and let $a_1,\dots,a_m \in R$ such that $a_1 \cdots a_m \in J \cap R$. Since $a_1 \cdots a_m \in J$ and J is S-(m,n)-absorbing in T, there exists $s \in S$ and n distinct indices $i_1,\dots,i_n \in \{1,\dots,m\}$ such that $sa_{i_1} \cdots a_{i_n} \in J$. Since $a_{i_j} \in R$ for each j, it follows that $sa_{i_1} \cdots a_{i_n} \in J \cap R$. Thus, $J \cap R$ is an S-(m,n)-absorbing ideal of R. □

Theorem 2.13. Let R be a ring, S a multiplicative subset of R, and let m,n,k,ℓ be positive integers. Let I be a proper ideal of R such that $I \cap S = \emptyset$. The following statements hold:

1. I is an S-(m,n)-absorbing ideal of R if and only if there exists $s_0 \in S$ such that whenever $a_1 \cdots a_k \in I$ for nonunit elements $a_1,\dots,a_k \in R$ with $k \geq m$, there exist n distinct indices $i_1,\dots,i_n \in \{1,\dots,k\}$ such that $s_0 a_{i_1} \cdots a_{i_n} \in I$.
2. If I is an S-(m,n)-absorbing ideal of R, then I is also an S-(k,ℓ)-absorbing ideal of R for all $k \geq m$ and $\ell \geq n$. In particular, I is an S-$(m-1)$-absorbing ideal of R.
3. If each I_j is an S-(m_j,n_j)-absorbing ideal of R associated with $s_j \in S$ for $j \in \{1,\dots,p\}$, and $I = \bigcap_{j=1}^{p} I_j$ is disjoint from S, then I is an S-(m,n)-absorbing ideal of R, where $n = \sum_{j=1}^{p} n_j$ and $m = \max\{m_1,\dots,m_p,n+1\}$. Moreover, I is associated with $t = \prod_{j=1}^{p} s_j \in S$.

Proof. Parts (1) and (2) are routine and thus omitted.

(3) Assume that each I_j is an S-(m_j,n_j)-absorbing ideal associated with $s_j \in S$, for $j \in \{1,\dots,p\}$, and that $I = \bigcap_{j=1}^{p} I_j$ is disjoint from S. Set $n = \sum_{j=1}^{p} n_j$ and $m = \max\{m_1,\dots,m_p,n+1\}$. By part (2), each I_j is also an S-(m,n_j)-absorbing ideal associated with s_j.

Let $t = \prod_{j=1}^{p} s_j \in S$, and suppose $a_1,\dots,a_m \in R$ are nonunit elements such that $a_1 \cdots a_m \in I = \bigcap_{j=1}^{p} I_j$. Then $a_1 \cdots a_m \in I_j$ for each j. Since I_j is S-(m,n_j)-absorbing, there exists a subset $\Gamma_j \subseteq \{1,\dots,m\}$ with $|\Gamma_j| = n_j$ such that

$$s_j \left(\prod_{\gamma \in \Gamma_j} a_\gamma \right) \in I_j.$$

Define $\Gamma := \bigcup_{j=1}^{p} \Gamma_j$ and $P := \prod_{\gamma \in \Gamma} a_\gamma$. Then for each j, we have

$$s_j P = s_j \cdot \left(\prod_{\gamma \in \Gamma_j} a_\gamma \right) \cdot \left(\prod_{\gamma \in \Gamma \setminus \Gamma_j} a_\gamma \right) \in I_j.$$

Multiplying both sides by $\prod_{k \neq j} s_k$, we obtain $tP \in I_j$. Since this holds for all j, it follows that

$$tP \in \bigcap_{j=1}^{p} I_j = I.$$

Thus, I is an S-(m,n)-absorbing ideal associated with $t \in S$. □

Theorem 2.14. Let R be a ring, S a multiplicative subset of R, and I a proper ideal of R. Let m and n be positive integers, and suppose there exists some $i \in I$ such that $1+i \notin U(R)$. Then I is an S-AB-(m,n)-absorbing ideal of R if and only if I is an S-(m,n)-absorbing ideal of R.

Proof. Suppose I is an S-AB-(m,n)-absorbing ideal of R. Then, by definition, I is also an S-(m,n)-absorbing ideal.

Conversely, assume I is an S-(m,n)-absorbing ideal of R. Suppose $a_1a_2\cdots a_m \in I$ for some elements $a_1,a_2,\ldots,a_m \in R$. Without loss of generality, assume that for some integer $k \geq n+2$, the elements $a_k,a_{k+1},\ldots,a_m$ are units in R, and $a_1,a_2,\ldots,a_{k-1}$ are nonunits.

Then $a_1\cdots a_{k-1} \in I$. Since $1+i \notin U(R)$ for some $i \in I$, set $b_k = b_{k+1} = \cdots = b_m = i+1$. Then:

$$a_1\cdots a_{k-1}b_k\cdots b_m = a_1\cdots a_{k-1}(i+1)^{m-k+1} \in I.$$

Since I is an S-(m,n)-absorbing ideal, there exists $s \in S$ and n elements among the a_i's $(1 \leq i \leq k-1)$ and b_j's $(k \leq j \leq m)$ whose product lies in I.

Note that $i \in I$, $1 \notin I$, and $(i+1)^e \notin I$ for every integer $e \geq 1$. Thus, among the n elements, at least one must come from the a_i's. Suppose there exist h such elements $a_1,\ldots,a_h$ with $1 \leq h \leq n$ such that

$$sa_1\cdots a_h(i+1)^{n-h} \in I.$$

Note that $(i+1)^{n-h} = fi+1$ for some $f \in R$ (if $h=n$, take $f=0$). Then

$$sa_1\cdots a_h(fi+1) \in I,$$

which implies $sa_1\cdots a_h \in I$ (since $fi \in I$ and I is an ideal). Hence, I is an S-AB-(m,n)-absorbing ideal. □

Corollary 2.15. *Let $m,n \geq 1$ be positive integers with $m > n$, let R be a commutative ring with a multiplicative subset S, and let I be a proper ideal of R such that $I \not\subseteq \mathrm{Jac}(R)$. Then I is an S-AB-(m,n)-absorbing ideal of R if and only if I is an S-(m,n)-absorbing ideal of R.*

Proof. Since $I \not\subseteq \mathrm{Jac}(R)$, there exists $i \in I$ such that $1+i \notin U(R)$. The result follows directly from Theorem 2.14. □

Example 2.16. Let R be a semiprimitive ring (for example, $R=\mathbb{Z}$), and let $m,n \geq 1$ be positive integers with $m > n$. Let S be a multiplicative subset of R, and let I be a proper ideal of R. Then I is an S-AB-(m,n)-absorbing ideal of R if and only if I is an S-(m,n)-absorbing ideal of R.

Proposition 2.17. *Let R_1 and R_2 be commutative rings, and let $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$ be multiplicative subsets. Let I_1 and I_2 be ideals of R_1 and R_2, respectively. If $I_1 \times I_2$ is an $(S_1 \times S_2)$-(m,n)-absorbing ideal of $R_1 \times R_2$, then I_1 is an S_1-(m,n)-absorbing ideal of R_1, and I_2 is an S_2-(m,n)-absorbing ideal of R_2.*

Proof. Assume that $I_1 \times I_2$ is an $(S_1 \times S_2)$-(m,n)-absorbing ideal of $R_1 \times R_2$. Let $a_1,\ldots,a_m \in R_1$ be such that $a_1\cdots a_m \in I_1$. Then $(a_1\cdots a_m, 0) \in I_1 \times I_2$. By assumption, there exists $(s,t) \in S_1 \times S_2$ such that

$$(s,t)(a_{i_1}\cdots a_{i_n}, 0) = (sa_{i_1}\cdots a_{i_n}, 0) \in I_1 \times I_2,$$

for some n-tuple of indices. This implies $sa_{i_1}\cdots a_{i_n} \in I_1$. Hence, I_1 is an S_1-(m,n)-absorbing ideal of R_1. A similar argument applies to I_2. □

The converse of Proposition 2.17 does not hold in general, as shown by the following example.

Example 2.18. Let $R = \mathbb{Z} \times \mathbb{Z}$, with multiplicative subsets $S_1 = \{7^n \mid n \in \mathbb{N}\}$ and $S_2 = \{1\}$. Let $I_1 = 4\mathbb{Z}$ and $I_2 = 9\mathbb{Z}$. Then I_1 and I_2 are $(4,2)$-absorbing ideals of $\mathbb{Z}$. Hence, I_1 is S_1-$(4,2)$-absorbing and I_2 is S_2-$(4,2)$-absorbing.

Now consider $I = I_1 \times I_2 \subset R$. We claim that I is not an $(S_1 \times S_2)$-$(4,2)$-absorbing ideal of R. Let

$$x_1 = (2,1), \quad x_2 = (2,1), \quad x_3 = (1,3), \quad x_4 = (1,3).$$

Then $x_1, \ldots, x_4$ are nonunit elements of R and $x_1x_2x_3x_4 \in I$. However, for every $(s,t) \in S_1 \times S_2$, the product of (s,t) and any pairwise product $x_{i_1}x_{i_2}$ is not in I. Hence, I is not an $(S_1 \times S_2)$-$(4,2)$-absorbing ideal of R.

In view of Example 2.18, we now present the following result:

Theorem 2.19. Let $R = R_1 \times R_2$, where R_1 and R_2 are rings, and let $S = S_1 \times S_2$ be a multiplicative subset of R, where $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$ are multiplicative subsets. Let I_1 be an ideal of R_1 and I_2 an ideal of R_2, and set $I = I_1 \times I_2$. Then the following statements are equivalent:

1. I is an S-(m,n)-absorbing ideal of R;
2. I_1 is an S_1-AB-(m,n)-absorbing ideal of R_1;
3. I_2 is an S_2-AB-(m,n)-absorbing ideal of R_2.

Proof. (1) $\Rightarrow$ (2): Suppose I_1 is not an S_1-AB-(m,n)-absorbing ideal of R_1. Then there exist elements $x_1, \ldots, x_m \in R_1$ such that $x_1 \cdots x_m \in I_1$, but for all $s \in S_1$, no product of n of the x_i's lies in I_1 when multiplied by s.

Define $d_i = (x_i, 0) \in R$ for each $i = 1, \ldots, m$. Then $d_1 \cdots d_m = (x_1 \cdots x_m, 0) \in I$. Since I is assumed to be S-(m,n)-absorbing, there exists $(s_1, s_2) \in S = S_1 \times S_2$ such that the product of n of the d_i's multiplied by (s_1, s_2) lies in I. This implies that some product $s_1 x_{i_1} \cdots x_{i_n} \in I_1$, a contradiction. Hence, I_1 must be S_1-AB-(m,n)-absorbing.

(2) $\Rightarrow$ (3): By symmetry, the argument applies to I_2.

(3) $\Rightarrow$ (1): Since every S-AB-(m,n)-absorbing ideal is, in particular, an S-(m,n)-absorbing ideal, the conclusion follows. □

Let R be a ring with a multiplicative subset $S \subseteq R$, and let m, n be positive integers with $m > n$. In the following result, we show that if R admits an S-(m,n)-absorbing prime ideal that is not an S-$(m-1, n-1)$-absorbing prime ideal, then R must be a quasi-local ring. Moreover, if R is not quasi-local, then a proper ideal I of R is an S-(m,n)-absorbing prime ideal if and only if it is an S-prime ideal.

Theorem 2.20. Let S be a multiplicative subset of R, I a proper ideal of R, and let m, n be positive integers with $m > n$. Then the following statements hold:

1. Assume that R is not a quasi-local ring. Then the following conditions are equivalent:
   (a) I is an S-(m,n)-absorbing prime ideal of R;
   (b) I is an S-$(m-1, n-1)$-absorbing prime ideal of R;
   (c) I is an S-$(m-n+1, 1)$-absorbing prime ideal of R;
   (d) I is an S-prime ideal of R.
2. I is an S-$(n+1, n)$-absorbing prime ideal if and only if either I is an S-prime ideal, or R is quasi-local with maximal ideal M such that $SM^n \subseteq I$.

Proof. (1) $(a) \Leftrightarrow (b)$: By Remark 2.4(3), it suffices to prove $(a) \Rightarrow (b)$. Assume I is an S-(m,n)-absorbing prime ideal but not an S-$(m-1,n-1)$-absorbing prime ideal. Then there exist nonunit elements $a_1,\dots,a_{m-1} \in R$ and $s \in S$ such that:

$$a_1 \cdots a_{m-1} \in I, \quad sa_1 \cdots a_{n-1} \notin I, \quad \text{and} \quad sa_n \cdots a_{m-1} \notin I.$$

Let d be a nonunit of R. Then $da_1 \cdots a_{m-1} \in I$. Since I is S-(m,n)-absorbing prime and $sa_n \cdots a_{m-1} \notin I$, we must have $sda_1 \cdots a_{n-1} \in I$.

Now let c be a unit of R, and suppose $d+c$ is a nonunit. Since $(d+c)a_1 \cdots a_{m-1} \in I$ and $sa_n \cdots a_{m-1} \notin I$, we get:

$$s(d+c)a_1 \cdots a_{n-1} = sda_1 \cdots a_{n-1} + sca_1 \cdots a_{n-1} \in I.$$

Because $sda_1 \cdots a_{n-1} \in I$, it follows that $sa_1 \cdots a_{n-1} \in I$, contradicting the assumption. Hence, $d+c$ must be a unit. The conclusion follows from [7, Lemma 1].

$(b) \Leftrightarrow (c)$: This follows by induction on n.

$(c) \Leftrightarrow (d)$: Assume I is S-$(m-n+1,1)$-absorbing prime. By Remark 2.4(4), this implies I is also S-$(m-n+1,m-n)$-absorbing prime. Then the implication $(a) \Rightarrow (c)$ implies that I is S-prime. The converse is obvious since every S-prime ideal is S-(m,n)-absorbing prime for any $m > n$.

(2) We prove the "only if" direction. Suppose I is an S-$(n+1,n)$-absorbing prime ideal. If R is not quasi-local, then by (1) I is an S-prime ideal. Now assume R is quasi-local with maximal ideal M and $SM^n \nsubseteq I$. We show that I is also S-$(n,n-1)$-absorbing prime.

Assume the contrary: that I is not S-$(n,n-1)$-absorbing prime. Then there exist nonunit elements $a_1,\dots,a_n \in R$ such that:

$$a_1 \cdots a_n \in I, \quad sa_1 \cdots a_{n-1} \notin I, \quad sa_n \notin I$$

for some $s \in S$. Let $x_1,\dots,x_n \in SM$. Then

$$x_1 \cdots x_n a_1 \cdots a_n \in I.$$

Since I is S-$(n+1,n)$-absorbing prime and $sa_n \notin I$, we must have

$$sx_1 \cdots x_n a_1 \cdots a_{n-1} \in I.$$

Similarly, since $sa_1 \cdots a_{n-1} \notin I$, it must be that $x_1 \cdots x_n \in I$, implying $SM^n \subseteq I$, a contradiction. Thus I is S-$(n,n-1)$-absorbing prime. Repeating this process inductively, we conclude that I is S-prime. □

As a direct consequence of Theorem 2.20, we obtain the following result.

Corollary 2.21. *Let R_1 and R_2 be commutative rings, and let $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$ be multiplicative subsets. Let m and n be positive integers with $m > n$. Suppose that $I_1 \subseteq R_1$ and $I_2 \subseteq R_2$ are ideals. Then the following statements are equivalent:*

1. *$I_1 \times I_2$ is an $S_1 \times S_2$-(m,n)-absorbing prime ideal of $R_1 \times R_2$;*
2. *$I_1 \times I_2$ is an $S_1 \times S_2$-prime ideal of $R_1 \times R_2$;*
3. *Either I_1 is an S_1-prime ideal of R_1 and $I_2 = R_2$, or $I_1 = R_1$ and I_2 is an S_2-prime ideal of R_2.*

Proof. Since $R_1 \times R_2$ is not quasi-local, the conclusion follows directly from Theorem 2.20(1). □

Corollary 2.22. *Let R be a ring with a multiplicative subset S, and let m,n be positive integers such that $m > n$. Assume that I is an S-$(n+1,n)$-absorbing prime ideal of R that is not S-prime, and that $I \subseteq J$ for some proper ideal J of R. Then J is an S-(m,n)-absorbing prime ideal of R.*

Proof. Since I is an S-$(n+1,n)$-absorbing prime ideal that is not S-prime, it follows from Theorem 2.20(2) that R is a quasi-local ring with maximal ideal M such that $SM^n \subseteq I$. Since $SM^n \subseteq I \subseteq J$, the conclusion is immediate. □

Theorem 2.23. Let R be a ring, S a multiplicative subset of R, I a proper ideal of R, and let m,n be positive integers with $m > n$. If I is an S-(m,n)-absorbing prime ideal of R associated with some $s \in S$, then $\sqrt{I}$ is an S-prime ideal of R. In particular, if I is not an S-prime ideal, then R is quasi-local with maximal ideal M, and if $m = n+1$, then $\sqrt{I} = sM$.

Proof. Assume I is an S-(m,n)-absorbing prime ideal of R. Let $x,y \in R$ such that $xy \in \sqrt{I}$. Without loss of generality, assume x and y are nonunits. Then there exists a positive integer p such that $x^p y^p \in I$. Consider the product:

$$x^{n-1}x^p y^{m-n-1}y^p = a_1 \cdots a_m \in I,$$

where we define $a_i = x$ for $1 \le i \le n-1$, $a_n = x^p$, and $a_i = y$ for $n+1 \le i \le m-1$, with $a_m = y^p$. Since I is S-(m,n)-absorbing prime, we conclude that

$$sa_1 \cdots a_n = sx^{n+p-1} \in I \quad \text{or} \quad sa_{n+1} \cdots a_m = sy^{m-n+p-1} \in I.$$

Hence, $sx \in \sqrt{I}$ or $sy \in \sqrt{I}$, which shows that $\sqrt{I}$ is an S-prime ideal.

Now assume I is not S-prime. Then by Theorem 2.20(1), R is quasi-local with maximal ideal M. If $m = n+1$, then Theorem 2.20(2) implies $sM^n \subseteq I$, hence $\sqrt{I} = sM$. □

Theorem 2.24. Let S be a multiplicative set of a ring R, and let I be an ideal of R disjoint from S. Let m,n be positive integers such that $m > n \ge 2$, and suppose that I is an S-(m,n)-absorbing prime ideal of R. Let $d \in R \setminus I$ be a nonunit element of R. Then $(I : d) = \{x \in R \mid dx \in I\}$ is an S-$(m-1,n-1)$-absorbing prime ideal of R. In particular, for every proper ideal I of R with $J \not\subseteq I$, the colon ideal $(I : J)$ is also an S-$(m-1,n-1)$-absorbing prime ideal of R.

Proof. Suppose I is an S-(m,n)-absorbing prime ideal of R. Then there exists $s \in S$ such that for any nonunit elements $a_1,\ldots,a_m \in R$ with $a_1 \cdots a_m \in I$, we have either $sa_1 \cdots a_n \in I$ or $sa_{n+1} \cdots a_m \in I$.

Now assume $a_1 \cdots a_{m-1} \in (I : d)$ for some nonunit elements $a_1,\ldots,a_{m-1} \in R$. Then $da_1 \cdots a_{m-1} \in I$. Suppose $sa_1 \cdots a_{n-1} \notin (I : d)$, i.e., $dsa_1 \cdots a_{n-1} \notin I$. Since I is S-(m,n)-absorbing prime, we must have $sa_n \cdots a_{m-1} \in (I : d)$. Hence, $(I : d)$ is an S-$(m-1,n-1)$-absorbing prime ideal of R. □

Recall that a ring R is called a *chained ring* if the set of all ideals of R is linearly ordered by inclusion. Moreover, R is called an *arithmetical ring* if R_M is a chained ring for every maximal ideal M of R. We next determine the S-$(n+1,n)$-absorbing prime ideals in a chained ring.

Theorem 2.25. Let R be a chained ring with maximal ideal M, S a multiplicative set of R, and I an S-$(n+1,n)$-absorbing prime ideal of R disjoint from S, for some integer $n \ge 2$. If I is not an S-prime ideal of R, then $I = SM^k$ for some integer k with $2 \le k \le n$. Furthermore, if $k < n$, then I is also an S-$(k+1,k)$-absorbing prime ideal of R.

Proof. Assume I is an S-$(n+1,n)$-absorbing prime ideal of R that is not S-prime. Since R is a quasi-local ring with maximal ideal M, Theorem 2.20(2) implies that $SM^n \subseteq I$.

Let $k = \min\{i \mid M^i \subseteq I\}$. We claim that $I = SM^k$. Suppose for contradiction that $SM^k \subsetneq I$. Then there exist elements $a \in SM^{k-1} \setminus I$ and $b \in I \setminus SM^k$. Since R is chained, $b \in aR$, so $b = ar$ for some $r \in R$, and hence $r \in M$. Then $b = ar \in SM^k$, contradicting the choice of b. Thus, $I = SM^k$.

It is evident that SM^k is an S-$(k+1,k)$-absorbing prime ideal. □

Corollary 2.26. *Let R be an arithmetical ring with Jacobson radical M, and let I be an S-$(n+1,n)$-absorbing prime ideal of R for some integer $n \ge 2$. If I is not an S-prime ideal, then $I = SM^k$ for some k with $2 \le k \le n$. Furthermore, if $k < n$, then I is also an S-$(k+1,k)$-absorbing prime ideal of R.*

Proof. If R is not quasi-local, then I is S-prime by Theorem 2.20(1). Assume instead that R is quasi-local with maximal ideal M. Since R is an arithmetical quasi-local ring, it follows that R is a chained ring. Thus, by Theorem 2.25, $I = SM^k$ for some k with $2 \leq k \leq n$. □

Theorem 2.27. Let S be a multiplicative subset of R, and let I be a proper ideal of R. Then I is an S-(m,n)-absorbing prime ideal of R associated to some $s \in S$ if and only if, for any proper ideals $I_1,\ldots,I_m$ of R with $I_1 \cdots I_m \subseteq I$, we have $sI_1 \cdots I_n \subseteq I$ or $sI_{n+1} \cdots I_m \subseteq I$.

Proof. It suffices to prove the "if" direction. Suppose that I is an S-(m,n)-absorbing prime ideal associated to some $s \in S$, and let $I_1,\ldots,I_m$ be proper ideals of R such that $I_1 \cdots I_m \subseteq I$ and $sI_{n+1} \cdots I_m \nsubseteq I$. Then there exist elements $a_{n+1} \in I_{n+1},\ldots,a_m \in I_m$ such that $sa_{n+1} \cdots a_m \notin I$. For arbitrary $b_1 \in I_1,\ldots,b_n \in I_n$, we have $b_1 \cdots b_n a_{n+1} \cdots a_m \in I$. Since I is S-(m,n)-absorbing and $sa_{n+1} \cdots a_m \notin I$, it follows that $sb_1 \cdots b_n \in I$. Thus, $sI_1 \cdots I_n \subseteq I$. □

Let R and T be commutative rings with identity, and let S be a multiplicative subset of R. Let $f : R \to T$ be a ring homomorphism such that $f(S)$ does not contain zero. Then $f(S)$ is a multiplicative subset of $f(R)$.

Theorem 2.28. Let R and T be commutative rings with identity, and let S be a multiplicative subset of R. Suppose $f : R \to T$ is a ring homomorphism such that $f(S)$ contains no zero divisors. Then the following statements hold:

1. If J is an $f(S)$-(m,n)-absorbing ideal of T, then $f^{-1}(J)$ is an S-(m,n)-absorbing ideal of R.
2. If f is surjective and I is a proper ideal of R containing $\mathrm{Ker}(f)$, then I is an S-(m,n)-absorbing ideal of R if and only if $f(I)$ is an $f(S)$-(m,n)-absorbing ideal of T.

Proof. (1) Assume that $a_1 \cdots a_m \in f^{-1}(J)$ for some nonunit elements $a_1,\ldots,a_m \in R$. Then $f(a_1) \cdots f(a_m) \in J$. Since J is an $f(S)$-(m,n)-absorbing ideal, there exists $s \in S$ and n distinct indices $i_1,\ldots,i_n \in \{1,\ldots,m\}$ such that

$$f(s)f(a_{i_1}) \cdots f(a_{i_n}) \in J.$$

Thus, $sa_{i_1} \cdots a_{i_n} \in f^{-1}(J)$, and $f^{-1}(J)$ is an S-(m,n)-absorbing ideal of R.

(2) Suppose $f(I)$ is an $f(S)$-(m,n)-absorbing ideal of T. Since $I = f^{-1}(f(I))$ and $\mathrm{Ker}(f) \subseteq I$, it follows from part (1) that I is an S-(m,n)-absorbing ideal of R.

Conversely, assume that I is an S-(m,n)-absorbing ideal of R. Let $x_1 \cdots x_m \in f(I)$ for nonunit elements $x_1,\ldots,x_m \in T$. Then there exist $a_1,\ldots,a_m \in R$ such that $f(a_i) = x_i$ for $i = 1,\ldots,m$, and $a_1 \cdots a_m \in I$ since $\mathrm{Ker}(f) \subseteq I$. Since I is S-(m,n)-absorbing, there exists $s \in S$ and n distinct indices $i_1,\ldots,i_n$ such that $sa_{i_1} \cdots a_{i_n} \in I$, hence

$$f(s)x_{i_1} \cdots x_{i_n} = f(sa_{i_1} \cdots a_{i_n}) \in f(I).$$

Therefore, $f(I)$ is an $f(S)$-(m,n)-absorbing ideal of T. □

Corollary 2.29. *Let R be a ring, S a multiplicative subset of R, and $I \subseteq J$ proper ideals of R. Let $f : R \to R/I$ be the canonical surjection. Assume that $a + I$ is a nonunit in R/I for every nonunit element $a \in R$. Then J is an S-(m,n)-absorbing ideal of R if and only if J/I is an $f(S)$-(m,n)-absorbing ideal of R/I.*

Let R be a ring and I an ideal of R. The set

$$Z_I(R) := \{x \in R : xy \in I \text{ for some } y \notin I\}$$

is known as the set of zero-divisors modulo I (see [21]).

Theorem 2.30. Let R be a ring, and let $S, S' \subseteq R$ be multiplicative subsets. Let I be an ideal of R disjoint from S. If I is an S'-(m,n)-absorbing ideal of R with $I \cap S = \emptyset$, then $S^{-1}I$ is an $S^{-1}S'$-(m,n)-absorbing ideal of $S^{-1}R$. The converse holds if $Z_I(R) \cap S = \emptyset$.

Proof. If S' is a multiplicative subset of R, then $S^{-1}S'$ is a multiplicative subset of $S^{-1}R$. Since $I \cap S = \emptyset$, $S^{-1}I$ is a proper ideal of $S^{-1}R$ and $S^{-1}I \cap S^{-1}S' = \emptyset$.

Assume $\frac{a_1}{s_1} \cdots \frac{a_m}{s_m} \in S^{-1}I$ for nonunit elements $a_1, \ldots, a_m \in R$ and $s_1, \ldots, s_m \in S$. Then $ta_1 \cdots a_m \in I$ for some $t \in S$. Since I is S'-(m,n)-absorbing, there exists $s' \in S'$ and distinct indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $s'ta_{i_1} \cdots a_{i_n} \in I$. Hence,

$$\frac{s'}{1} \cdot \frac{a_{i_1}}{s_{i_1}} \cdots \frac{a_{i_n}}{s_{i_n}} \in S^{-1}I,$$

proving $S^{-1}I$ is $S^{-1}S'$-(m,n)-absorbing.

Conversely, suppose $S^{-1}I$ is $S^{-1}S'$-(m,n)-absorbing and $Z_I(R) \cap S = \emptyset$. Let $a_1 \cdots a_m \in I$ for nonunit elements $a_1, \ldots, a_m \in R$. Then $\frac{a_1}{1} \cdots \frac{a_m}{1} \in S^{-1}I$. By assumption, there exists $\frac{s'}{s} \in S^{-1}S'$ and indices $i_1, \ldots, i_n$ such that

$$\frac{s'}{s} \cdot \frac{a_{i_1}}{1} \cdots \frac{a_{i_n}}{1} \in S^{-1}I.$$

Hence, for some $v \in S$, we get $vs'a_{i_1} \cdots a_{i_n} \in I$. Since $v \notin Z_I(R)$, it follows that $s'a_{i_1} \cdots a_{i_n} \in I$. Therefore, I is S'-(m,n)-absorbing. □

Theorem 2.31. Let S be a multiplicative subset of R, I an ideal of R, and m, n positive integers with $m > n$. Then there exists an S-(m,n)-absorbing ideal of R that is minimal among all S-(m,n)-absorbing ideals of R containing I. In particular, R has a minimal S-(m,n)-absorbing ideal.

Proof. Let V denote the set of all S-(m,n)-absorbing ideals of R that contain I. Since any maximal ideal containing I is S-(m,n)-absorbing, V is non-empty. Order V by reverse inclusion: for $I_1, I_2 \in V$, define $I_1 \leq I_2$ if and only if $I_1 \supseteq I_2$.

Let $C = \{I_\lambda\}_{\lambda \in \Lambda}$ be a non-empty chain in V and let $J = \bigcap_{\lambda \in \Lambda} I_\lambda$. Clearly $J \neq R$ and $J \in V$. Let $a_1, \ldots, a_m \in R$ be nonunit elements with $a_1 \cdots a_m \in J$, and suppose that for every $s \in S$, no n-subproduct of the a_i's multiplied by s lies in J. Since C is a chain, there exists some I_λ such that this also fails in I_λ, contradicting that I_λ is S-(m,n)-absorbing. Hence J is S-(m,n)-absorbing.

By Zorn's Lemma, V has a maximal element with respect to inclusion, which is a minimal S-(m,n)-absorbing ideal of R containing I. □

Corollary 2.32. *Let S be a multiplicative subset of R, I an ideal of R disjoint from S, and n a positive integer. Then there exists a minimal S-n-absorbing ideal of R containing I.*

As a generalization of Noetherian rings, Anderson and Dumitrescu introduced the concept of S-Noetherian rings. Recall from [3] that a ring R is called an S-Noetherian ring if every ideal I of R is S-finite; that is, there exists $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. The following result characterizes the S-Noetherian property in terms of S-AB-(m,n)-absorbing ideals and S-(m,n)-absorbing ideals.

Proposition 2.33. *Let S be a multiplicative subset of a ring R. Then the following statements are equivalent:*

1. *R is an S-Noetherian ring;*
2. *Every S-AB-(m,n)-absorbing ideal is S-finite;*
3. *Every S-(m,n)-absorbing ideal is S-finite.*

Proof. (1) $\Rightarrow$ (2): Suppose that R is S-Noetherian. Then every ideal of R is S-finite. In particular, every S-AB-(m,n)-absorbing ideal is S-finite.

(2) $\Rightarrow$ (3): Assume that every S-AB-(m,n)-absorbing ideal is S-finite. Let I be an S-(m,n)-absorbing ideal of R. Then I is also an S-AB-(m,n)-absorbing ideal, so by assumption, I is S-finite.

(3) $\Rightarrow$ (1): Assume that every S-(m,n)-absorbing ideal is S-finite. In particular, every S-prime ideal is S-finite. Thus, by [18, Theorem 3], R is an S-Noetherian ring. □

3 S-(m,n)-absorbing Ideals in Trivial Extension Rings and Amalgamated Algebras

Let E be a unitary A-module. The idealization of E in A, denoted by $A \propto E = A \oplus E$, is a commutative ring with componentwise addition and multiplication defined by

$$(a_1, e_1)(a_2, e_2) = (a_1 a_2, a_1 e_2 + a_2 e_1)$$

for all $a_1, a_2 \in A$ and $e_1, e_2 \in E$ [2]. For an ideal I of A and a submodule F of E, the set $I \propto F = I \oplus F$ is not necessarily an ideal of $A \propto E$; it is an ideal if and only if $IE \subseteq F$ [2, Theorem 3.1]. If S is a multiplicative subset of A, then the set $S \propto E = \{(s,e) \mid s \in S,\ e \in E\}$ is clearly a multiplicative subset of $A \propto E$. For background on trivial ring extensions, see [14, 20]. We now present some properties of S-(m,n)-absorbing ideals in the context of the trivial extension.

Theorem 3.1. Let A be a ring, E an A-module, S a multiplicative subset of A, I an ideal of A, and F a submodule of E such that $IE \subseteq F$. Then the following statements hold:

1. If $I \propto F$ is an $S \propto E$-(m,n)-absorbing ideal of $A \propto E$, then I is an S-(m,n)-absorbing ideal of A.
2. $I \propto E$ is an $S \propto E$-(m,n)-absorbing ideal of $A \propto E$ if and only if I is an S-(m,n)-absorbing ideal of A.

Proof. (1) Suppose $I \propto F$ is an $S \propto E$-(m,n)-absorbing ideal of $A \propto E$, and let $a_1, \ldots, a_m$ be nonunit elements of A such that $a_1 \cdots a_m \in I$. Then,

$$(a_1, 0) \cdots (a_m, 0) = (a_1 \cdots a_m, 0) \in I \propto F.$$

Hence, there exists $(s,t) \in S \propto E$ and n distinct indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that

$$(s,t)(a_{i_1}, 0) \cdots (a_{i_n}, 0) \in I \propto F.$$

Therefore, $s a_{i_1} \cdots a_{i_n} \in I$, proving the assertion.

(2) By part (1), it suffices to prove the "if" direction. Let $(a_1, e_1), \ldots, (a_m, e_m)$ be nonunit elements of $A \propto E$ such that

$$(a_1, e_1) \cdots (a_m, e_m) \in I \propto E.$$

Then clearly $a_1 \cdots a_m \in I$. Since I is S-(m,n)-absorbing, there exists $s \in S$ and n distinct indices $i_1, \ldots, i_n$ such that

$$s a_{i_1} \cdots a_{i_n} \in I.$$

Note that

$$(s, 0)(a_{i_1}, e_{i_1}) \cdots (a_{i_n}, e_{i_n}) = (s a_{i_1} \cdots a_{i_n}, c)$$

for some $c \in E$. Hence, $(s, 0)(a_{i_1}, e_{i_1}) \cdots (a_{i_n}, e_{i_n}) \in I \propto E$, proving the claim. □

The following example illustrates Theorem 3.1.

Example 3.2. Let A be a ring, E an A-module, S a multiplicative subset of A, and F a submodule of E such that $IE \subseteq F$. Let m, n be positive integers. If I is an S-prime ideal of A (e.g., $A = \mathbb{Z}[X]$, $E = F = 4X\mathbb{Z}[X]$, $I = 8X\mathbb{Z}[X]$, and $S = \{2^k \mid k \in \mathbb{Z}\}$), then by Remark 2.2, I is an S-(m,n)-absorbing ideal of A. Hence, by Theorem 3.1, $I \propto F$ is an $S \propto E$-(m,n)-absorbing ideal of $A \propto E$.

Let A and B be commutative rings, J an ideal of B, and $f : A \to B$ a ring homomorphism. Define the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A,\ j \in J\}.$$

The ring $A \bowtie^f J$ is called the *amalgamation of A with B along J with respect to f*. This construction generalizes the amalgamated duplication of a ring along an ideal (cf. [11, 13, 12, 15]).

Let I be an ideal of A, S a multiplicative subset of A, and K an ideal of $f(A) + J$. Define:

$$\begin{aligned} S' &:= \{(s, f(s)) \mid s \in S\}, \\ I \bowtie^f J &:= \{(i, f(i) + j) \mid i \in I,\ j \in J\}, \\ S^{\bowtie^f} &:= \{(s, f(s) + j) \mid s \in S,\ j \in J\}, \\ \overline{K}^f &:= \{(a, f(a) + j) \mid a \in A,\ j \in J,\ f(a) + j \in K\}. \end{aligned}$$

Clearly, S' and $S^{\bowtie^f}$ are multiplicatively closed subsets of $A \bowtie^f J$, and both $I \bowtie^f J$ and $\overline{K}^f$ are ideals of $A \bowtie^f J$. Our next result characterizes when these ideals are S-(m,n)-absorbing in $A \bowtie^f J$.

Theorem 3.3. Let A and B be commutative rings, $f : A \to B$ a ring homomorphism, J an ideal of B, I an ideal of A such that $S \cap I = \emptyset$, and K an ideal of $f(A)+J$ disjoint from $f(S)$. Let S be a multiplicative subset of A, and let m, n be positive integers such that $m > n$. Then the following statements hold:

1. I is an S-(m,n)-absorbing ideal of A if and only if $I \bowtie^f J$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$.

2. K is an $f(S)$-(m,n)-absorbing ideal of $f(A) + J$ if and only if $\overline{K}^f$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$.

Proof. (1) Define $\varphi : A \to (A \bowtie^f J)/(\{0\} \bowtie^f J)$ by $\varphi(a) = (a, f(a)) + (\{0\} \bowtie^f J)$. Then φ is a ring isomorphism and $\varphi(S) = S^{\bowtie^f}/(\{0\} \bowtie^f J)$. By Theorem 2.28(1), I is an S-(m,n)-absorbing ideal of A if and only if $\varphi(I) = (I \bowtie^f J)/(\{0\} \bowtie^f J)$ is an $\varphi(S)$-(m,n)-absorbing ideal of $(A \bowtie^f J)/(\{0\} \bowtie^f J)$. Applying Corollary 2.29, this is equivalent to $I \bowtie^f J$ being an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$.

(2) Define $\psi : A \bowtie^f J \to f(A) + J$ by $\psi(a, f(a) + j) = f(a) + j$. Then ψ is an epimorphism with $\mathrm{Ker}(\psi) = f^{-1}(J) \times \{0\}$, $\psi(\overline{K}^f) = K$, and $\psi(S^{\bowtie^f}) = f(S)$.

Suppose K is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$. Let $x_1, \ldots, x_m$ be nonunit elements of $A \bowtie^f J$ such that $x_1 \cdots x_m \in \overline{K}^f$. Then $\psi(x_1 \cdots x_m) = \psi(x_1) \cdots \psi(x_m) \in K$, and each $\psi(x_i)$ is a nonunit in $f(A) + J$. By assumption, there exists $f(s) \in f(S)$ and distinct indices $i_1, \ldots, i_n$ such that $f(s)\psi(x_{i_1}) \cdots \psi(x_{i_n}) \in K$. Thus,

$$\psi\left((s, f(s))x_{i_1} \cdots x_{i_n}\right) \in K,$$

which implies $(s, f(s))x_{i_1} \cdots x_{i_n} \in \overline{K}^f$. Hence, $\overline{K}^f$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal.

Conversely, assume $\overline{K}^f$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$, and let $y_1, \ldots, y_m$ be nonunit elements of $f(A) + J$ such that $y_1 \cdots y_m \in K$. Since ψ is surjective, there exist $x_1, \ldots, x_m \in A \bowtie^f J$ with

$\psi(x_j) = y_j$ for all j. As x_j are preimages of nonunits, they are also nonunits. Then $x_1 \cdots x_m \in \overline{K}^f$. By assumption, there exists $(s, f(s)) \in S^{\bowtie^f}$ and distinct indices $i_1, \ldots, i_n$ such that

$$(s, f(s))x_{i_1} \cdots x_{i_n} \in \overline{K}^f.$$

Applying ψ yields $f(s)y_{i_1} \cdots y_{i_n} \in K$. Hence, K is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$. □

Example 3.4. Let $A = \mathbb{Z}[X]$, $B = k[X]$ with k a field, J an ideal of B, and let $S = \{5^k : k \in \mathbb{N}\}$ be a multiplicative subset of A. Let m, n be positive integers. Consider the ideal $I = 5X\mathbb{Z}[X]$. It is easy to verify that I is an S-prime ideal of A, and hence I is an S-(m,n)-absorbing ideal of A. Therefore, by Theorem 3.3(1), $I \bowtie^f J$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$.

Let I be a proper ideal of A. The (amalgamated) duplication of A along I (see [9]) is a special case of amalgamation and is defined as

$$A \bowtie I := \{(a, a+i) \mid a \in A,\ i \in I\}.$$

The next corollary is an immediate consequence of Theorem 3.3 regarding the transfer of the S-(m,n)-absorbing ideal property to duplications.

Corollary 3.5. *Let S be a multiplicative subset of a ring A, J an ideal of A, and let m, n be positive integers with $m > n$. Define $S^{\bowtie} = \{(s,s) \mid s \in S\}$, which is a multiplicative subset of $A \bowtie J$. Then:*

1. *I is an S-(m,n)-absorbing ideal of A if and only if $I \bowtie J$ is an $S^{\bowtie}$-(m,n)-absorbing ideal of $A \bowtie J$.*
2. *K is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$ if and only if $\overline{K}$ is an $S^{\bowtie}$-(m,n)-absorbing ideal of $A \bowtie J$.*

Corollary 3.6. *Let A and B be rings, J an ideal of B, and $f : A \to B$ a ring homomorphism. Consider the amalgamation of A and B along J with respect to f. Let S be a multiplicative subset of A, and let I be an ideal of A such that $S \cap I = \emptyset$. Let K be an ideal of $f(A)+J$ disjoint from $f(S)$, and let m, n be positive integers with $m > n$. Then:*

1. *Every $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$ containing $\{0\} \times J$ is of the form $I \bowtie^f J$, where I is an S-(m,n)-absorbing ideal of A.*
2. *Every $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$ is of the form $\overline{K}$, where K is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$.*

Proof. Let L be an ideal of $A \bowtie^f J$ disjoint from $S^{\bowtie^f}$. Define two ring epimorphisms:

$$\pi_1 : A \bowtie^f J \to A, \quad \pi_1(a, f(a)+j) = a,$$

$$\pi_2 : A \bowtie^f J \to f(A)+J, \quad \pi_2(a, f(a)+j) = f(a)+j.$$

(1) By Theorem 3.3(1), $I \bowtie^f J$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal for any S-(m,n)-absorbing ideal I of A. Conversely, assume L is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$ containing $\{0\} \times J$. Then it is straightforward to verify that $L = \pi_1(L) \bowtie^f J$. Since $\mathrm{Ker}(\pi_1) = \{0\} \times J$, it follows from Theorem 2.28 that $\pi_1(L)$ is an S-(m,n)-absorbing ideal of A.
(2) Similarly, by Theorem 3.3(2), $\overline{K}$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$ for any $f(S)$-(m,n)-absorbing ideal K of $f(A)+J$. Now suppose L is an $S^{\bowtie^f}$-(m,n)-absorbing ideal containing $f^{-1}(J) \times \{0\}$. Then one can check that $L = \overline{\pi_2(L)}^f$. Since $\mathrm{Ker}(\pi_2) = f^{-1}(J) \times \{0\}$, it follows again from Theorem 2.28 that $\pi_2(L)$ is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$, as desired. □

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Graded S-r-ideals

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Abstract. Let G be a group, R a G-graded commutative ring with nonzero unity 1, and $S \subseteq h(R)$ a multiplicative subset of R. This article introduces the concept of graded S-r-ideal and graded S-pr-ideal. A proper graded ideal P of R is termed a *graded S-r-ideal* (resp., *graded S-pr-ideal*) if there exists an $s \in S$ such that for any nonunit elements $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$, then $sy \in P$ (resp., $sy \in Grad(P)$) with $Grad(P)$ being the graded radical of P. In this paper, we begin by providing counterexamples illustrating graded S-r-ideals and graded S-pr-ideals that do not qualify as graded S-prime ideals. Additionally, we present an example of a graded S-prime ideal that does not meet the criteria to be considered a graded S-r-ideal or a graded S-pr-ideal. Moreover, we investigate some fundamental properties of graded S-r-ideals (resp., graded S-pr-ideals), discuss their forms in finite direct products, and study their presence in Nagata's idealization ring.
Key Words: Graded S-r-ideals, Graded S-pr-ideals, Nagata's idealization.
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1 Introduction

Let G be a multiplication group with identity e. A ring R is called a G-graded ring if $R = \bigoplus_{g\in G} R_g$ with the property $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. The set of all homogeneous elements of R is $h(R) = \bigcup_{g\in G} R_g$.
Let P be an ideal of a G-graded ring R. Then P is called a graded ideal if $P = \bigoplus_{g\in G} P_g$, i.e, for $x \in P$, $x = \sum_{g\in G} x_g$ where $x_g \in P$ for all $g \in G$. An ideal of a G-graded ring is not necessary graded ideal (see [2]). Let P be a proper graded ideal of R. Then the graded radical of P is denoted by $Grad(P)$ and it is defined as written below:
$Grad(P) = \{x = \sum_{g\in G} x_g \in R \mid \text{for all } g \in G, \text{there exists } n_g \in \mathbb{N} \text{ such that } x_g^{n_g} \in P\}$.
Note that $Grad(P)$ is always a graded ideal of R (see [15]). The graded radical of P is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in P$. Note that, if r is a homogeneous element of (R, G), then $r \in Grad(P)$ if and only if $r^n \in P$ for some $n \in \mathbb{N}$.

Lemma 1.1. *([14]) Let I, J be graded ideals of R. Then*

1. $I \subseteq Grad(I)$.

2. $Grad(Grad(I)) = Grad(I)$.

3. $Grad(I) = R$ *iff* $I = R$.

4. $Grad(IJ) = Grad(I \cap J) = Grad(I) \cap Grad(J)$.

Throughout this paper, all rings are to be commutative with $1 \neq 0$. A graded ideal P of a graded ring R is called *graded prime*, $ab \in P$ where $a, b \in h(R)$ implies $a \in P$ or $b \in P$ (see [3]). In recent years, prime ideals and graded prime ideals play a very important role in the Commutative graded rings theory. There are several ways to generalize this concept of a prime ideals graded prime ideals, for example, in [19], Yavuz and all, introduced the notion of weakly S-2-prime ideals of commutative rings. Let S a multiplicative subset of R. A proper ideal Q of R with $Q \cap S = \emptyset$ is called a weakly S-2-prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ with $0 \neq ab \in Q$, we have $sa^2 \in Q$ or $sb^2 \in Q$. In [16], H. Saber and all, further generalized this concept by introducing graded S-prime ideals. In their influential paper, a proper graded ideal I of R is termed *graded S-prime* if $ab \in I$ for some $a, b \in h(R)$ implies either $sa \in I$ or $sb \in I$ for some $s \in S$. In [12] Issoual and Ugurlu introduced the notion of graded 1-absorbing primary ideals and graded weakly 1-absorbing primary ideals. Subsequently, in 2021 the authors in ([4]) introduced the notion of graded S-primary ideals. A graded ideal I of R is called *graded S-primary* if $ab \in I$ for some $a, b \in h(R)$ implies either $sa \in I$ or $sb^n \in I$ for some $s \in S$ and some positive integer n. As usual, if R is a graded ring and $a \in R$, then $ann(a)$ denotes the set of annihilator of a in R. $ann(a) = \{r \in R \mid ra = 0\}$. In commutative algebra, graded r-ideal have an important role. There have been lots of studies on this issue (See [2]). Recall that a proper graded ideal I is called *graded r-ideal* (resp., *graded pr-ideal*), if for all $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$, then $x \in I$ (resp., $y^n \in I$ for some positive integer n). It's clear that any graded r-ideal (or graded pr-ideal) disjoint from S automatically qualifies as a graded S-r-ideal (or graded S-pr-ideal). However, Example 2.24, illustrates a counterexample showing that a graded S-r-ideal (or graded S-pr-ideal) does not necessarily retain its status as a graded r-ideal (or graded pr-ideal) in general.

The main aim of this paper is to expand upon the concept of graded ideals by introducing and investigating graded S-r-ideals (and graded S-pr-ideals) within graded rings. This approach serves to generalize essentially all the results pertaining to graded r-ideals and graded pr-ideals. Let R be a graded ring and $S \subseteq h(R)$ a multiplicative subset of R. A proper graded ideal P of a graded ring R disjoint from S is said to be a *graded S-r-ideal* (resp., *graded S-pr-ideal*) of R if whenever nonunit elements $x, y \in h(R)$ such that $xy \in P$ and $ann(a) = 0$, then $sy \in P$ (resp., $sy \in Grad(P)$). In the opening section, we commence with counterexamples illustrating graded S-r-ideals and graded S-pr-ideals that do not belong to be a graded S-prime ideals. Moreover, we offer an illustration of a graded S-prime ideal that does not satisfy the requirements to be considered a graded S-r-ideal or a graded S-pr-ideal (see Example 2.2 and 2.3).

Among many results in paper, we investigate graded S-r-ideal (resp., graded S-pr-ideal) within the context of graded homomorphisms Proposition 2.12. Also, Theorem 2.11, proves that if P is a graded S-r-ideal of R, then $(P : a)$ is a graded S-r-ideal of R for all $a \in h(R) - P$. Additionally, Theorem 2.20 and Theorem 2.22, allows us to identify all graded S-r-ideal (resp., graded S-pr-ideal) within the Cartesian product of two cross products. Let R be a graded ring, then $zd(R)$ denotes the set of zero divisors of R and $Reg(R) = R - zd(R)$. Consider $S \subset Reg(R)$ in Proposition 2.14, we show that if I be an graded S-prime ideal of R, then I is a graded S-r-ideal of R if and only if $(I : t) \cap h(R) \subseteq zd(R)$ for some $t \in S$.

Finally, in the current section, our focus lies in investigating the graded S-r-ideal (resp., graded S-pr-ideal) and graded $S(+)M$-r-ideals (resp., graded $S(+)M$-pr-ideals) of $R(+)M$.

2 Basic results

Definition 2.1. Let R be a graded ring and $S \subseteq h(R)$ a multiplicative subset of R. A proper graded ideal P of a graded ring R disjoint from S is said to be a *graded S-r-ideal* (resp., *graded S-pr-ideal*) of R if whenever nonunit elements $x, y \in h(R)$ such that $xy \in P$ and $ann(a) = 0$, then $sy \in P$ (resp., $sy \in Grad(P)$).

Let R be a graded ring, $S \subseteq h(R)$ a multiplicative subset of R, and P a graded ideal of R disjoint from S. According to the definition of graded S-prime ideals given in [16], P is considered a *graded S-prime* ideal of R if there exists an element $s \in S$ such that for all $x, y \in h(R)$ such that $xy \in P$, then $sx \in P$ or $sy \in P$. In the following, we will provide an example of a graded S-prime ideal which is neither a graded S-r-ideal nor a graded S-pr-ideal of R.

Example 2.2. Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider the multiplicative subset $S = \{2^n \mid n \in \mathbb{N}\}$ and the graded ideal $I = 5R$ of R. To see more details to show I is graded prime ideal of R (see [4], Example 1); so S-prime ideal of R since $I \cap S = \emptyset$. But I is neither an S-r-ideal nor an S-pr-ideal of R. Indeed, we have $5 \cdot 2 \in I$ with $ann(5) = 0$, but $2^n \cdot 2 \notin Grad(I)$ for all positive integer n. This shows that I is neither an S-r-ideal nor an S-pr-ideal of R.

Now, we give an example of graded S-r-ideal which is not graded S-prime ideal.

Example 2.3. Consider $R = \mathbb{Z}_6[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}_6$ and $R_1 = i\mathbb{Z}_6$. Consider the multiplicative subset $S = \{\overline{1}, \overline{5}\}$ and the graded ideal $P = \{\overline{0}\}$ of R. Let $x, y \in h(R)$ such that $xy \in P$ and $ann(x) = \overline{0}$, then $y = \overline{0} \in P$ which implies that P is a graded r-ideal of R; so is a graded S-r-ideal of R since $P \cap S = \emptyset$. But P is not a graded S-prime ideal of R. Indeed, $\overline{2} \cdot \overline{3} = \overline{0} \in P$, but $s2 \notin P$ and $s3 \notin P$ for all $s \in S$.

Let R and S are two G-graded rings, then $R \times S$ is a G-graded ring by $(R \times S)_g = R_g \times S_g$ for all $g \in G$, (see [16]). Also, if P be an ideal of a G-graded ring R and K be an ideal of a G-graded ring S. Then $P \times K$ is a graded ideal of $R \times S$ if and only if P is a graded ideal of R and K is a graded ideal of S ([7]).

Example 2.4. Consider $R = \mathbb{Z}_2[i] \times \mathbb{Z}_2[i] \times \mathbb{Z}_2[i]$ and $G = \mathbb{Z}_2$. As $A = \mathbb{Z}_2[i]$ is a G-graded ring by $A_0 = \mathbb{Z}_2$ and $A_1 = i\mathbb{Z}_2$, R is G-graded by $R_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $R_1 = i\mathbb{Z}_2 \times i\mathbb{Z}_2 \times i\mathbb{Z}_2$. Consider the multiplicative subset $S = \{(\overline{1}, \overline{1}, \overline{0}), (\overline{1}, \overline{1}, \overline{1})\}$ and the graded ideal $P = \overline{0} \times \overline{0} \times i\mathbb{Z}_2$ of R. Let $x, y \in h(R)$ such that $xy \in P$ and $ann(x) = \overline{0}$. If $x \in R_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is boolean, $x^2 = x$; so $x - 1 \in ann(x) = 0$, thus $x = 1$. This implies that $y \in P$. If $x \in R_1 = i\mathbb{Z}_2 \times i\mathbb{Z}_2 \times i\mathbb{Z}_2$, then there exits $a, b, c \in \mathbb{Z}_2$ such that $x = (ia, ib, ic)$. Thus $x^2 = (-a^2, -b^2, -c^2) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is boolean, $x^4 = (-a^2, -b^2, -c^2)^2 = (-a^2, -b^2, -c^2) = x^2$. So $x^3 - x \in ann(x) = 0$, thus $x^3 = x$ which implies that $x^2 - 1 \in ann(x) = 0$; so $x^2 = 1$. As $xy \in P$, $x^2 y \in P$. This implies that $y \in P$. Hence P is a graded S-r-ideal of R. Now, we have $(\overline{0}, \overline{1}, \overline{1}), (\overline{1}, \overline{0}, \overline{1}) \in h(R)$, $(\overline{0}, \overline{1}, \overline{1})(\overline{1}, \overline{0}, \overline{1}) = (\overline{0}, \overline{0}, \overline{1}) \in P$ and $s(\overline{0}, \overline{1}, \overline{1}) \notin P$ and $s(\overline{1}, \overline{0}, \overline{1}) \notin P$ for all $s \in S$. This implies that P Does not have a graded S-prime ideal of R.

Lemma 2.5. *([11]) Let I be a proper graded ideal of a G-graded ring R. Let $c \in h(R)$, then $(I : c) = \{r \in R \mid cr \in I\}$ is a graded ideal of R.*

Theorem 2.6. Let R be a G-graded ring, $S \subseteq h(R)$ a multiplicative subset of R and P be a graded ideal of R disjoint from S. Then P is a graded S-r-ideal if and only if for all $a \in h(R)$ if $ann(a) = (0)$, then $(P : a) \subseteq (P : s)$ for some $s \in S$.

Proof. Let $a \in h(R)$ such that $ann(a) = (0)$ and let $x \in (P : a)$. As $(P : a)$ is a graded ideal of R, $x_g \in (P : a)$ for all $g \in G$, then $ax_g \in P$ for all $g \in G$. Since P is an S-r-ideal, $sx_g \in P$ for some $s \in S$; so $x \in (P : s)$. Conversely, let $a, b \in h(R)$ with $ab \in P$ and $ann(a) = (0)$. Then $b \in (P : a) \subseteq (P : s)$ for some $s \in S$; so $sb \in P$. Hence P is a S-r-ideal of R. □

Let R be a G-graded ring, and $S \subseteq h(R)$ a multiplicatively subset of R. Then $S^* = \{a_g \in h(R) \mid \frac{a_g}{1}$ is a unit of $S^{-1}R\}$ denotes the saturation of S, $S^* \subseteq h(R)$ is a multiplicative subset of R containing S (see [17]).

Proposition 2.7. *Let R be a G-graded ring, and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then*

1. *If $S \subseteq T$ are multiplicatively subsets of R such that $P \cap T = \emptyset$ and P is a graded S-r-ideal (resp., S-pr-ideal) of R, then P is a graded T-r-ideal (resp., T-pr-ideal) of R. If for any $s \in T$, there exists $t \in T$ satisfying $st \in S$, then P is a graded T-r-ideal (resp., T-pr-ideal) of R implies P is a graded S-r-ideal (resp., S-pr-ideal) of R.*
2. *P is a graded S-r-ideal (resp., S-pr-ideal) of R if and only if P is a graded S^*-r-ideal (resp., S^*-pr-ideal) of R.*
3. *If $S \subseteq Reg(R)$, then P is a graded S-r-ideal of R if and only if $(P : s)$ is a graded r-ideal for some $s \in S$.*
4. *If $S \subseteq Reg(R)$, then P is a graded S-pr-ideal of R if and only if $(P : s)$ is a graded pr-ideal for some $s \in S$.*

Proof. (1). It is clear that if P is a graded S-r-ideal of R, then P is a graded T-r-ideal of R. Now, assume that P is graded T-r-ideal of R, there exists $s \in T$ such that if $xy \in P$ for all $x, y \in h(R)$ with $ann(x) = 0$, then $sy \in P$. By assumption $w = ts \in S$ for some $t \in T$. So $wy = tsy \in P$. Consequently, P is a graded S-r-ideal of R.

(2). Assume that P is a graded S-r-ideal of R. First, we want to show that $P \cap S^* = \emptyset$. Suppose there exists $a_g \in P \cap S^*$. Then $\frac{a_g}{1}$ is a unit of $S^{-1}R$, it follows that there exist $b_h \in h(R)$ and $s_i \in S$ such that $\frac{a_g}{1}\frac{b_h}{s_i} = 1$. Hence $t_j s_i = t_j a_g b_h$ for some $t_j \in S$. So $t_j s_i = t_j a_g b_h \in P \cap S$, a contradiction. Therefore $P \cap S^* = \emptyset$. Now, by (1) we get P is a graded S^*-r-ideal of R. Conversely, assume that P is a graded S^*-r-ideal of R. Let $xy \in P$ with $ann(x) = 0$ for some $x, y \in h(R)$. Then there exists $a_g \in S^*$ such that $a_g y \in P$. Since $\frac{a_g}{1}$ is a unit of $S^{-1}R$, there exist $b_j \in h(R)$ and $s, t \in S$ such that $ts = ta_g b_j$. It follows that $(ts)y = ta_g b_j y \in P$. Therefore, P is a graded S-r-ideal of R.

(3). Assume that P is a graded S-r-ideal of R. Then for all $x, y \in h(R)$ if $xy \in P$ with $ann(x) = 0$, then $sy \in P$ for some $s \in S$. Now, let $a, b \in h(R)$ such that $ab \in (P : s)$ with $ann(a) = 0$. Then $sab \in P$. It follows that $s^2 y \in P$; so $y \in (P : s)$ since $ann(s^2) = 0$. This implies that $(P : s)$ is a graded r-ideal of R. Conversely, assume that $(P : s)$ is a graded r-ideal of R for some $s \in S$ and let $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$. Then $xy \in (P : s)$; so $y \in (P : s)$. hence $sy \in P$.

(4). Assume that P is a graded S-pr-ideal of R. Then for all $x, y \in h(R)$ if $xy \in P$ with $ann(x) = 0$, then $sy \in Grad(P)$ for some $s \in S$. Now, let $a, b \in h(R)$ such that $ab \in (P : s)$ with $ann(a) = 0$. Then $sab \in P$. It follows that $sy \in Grad(P)$, then $s^m y^m \in P$ for some positive integer m. As $s \in Reg(R)$, $ann(s^m) = 0$; so $y^m \in (P : s)$. This implies that $(P : s)$ is a graded pr-ideal of R. Conversely, assume that $(P : s)$ is a graded pr-ideal of R for some $s \in S$ and let $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$. Then $xy \in Grad(P : s)$; so $y^n \in (P : s)$ for some positive integer n which implies that $sy^n \in P$. Hence $sy \in Grad(P)$. □

Let R be a G-graded ring, and $S \subseteq h(R)$ a multiplicatively subset of R. Note that $S^{-1}R$ is a G-graded ring with $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$. If P is a graded ideal of R and $P \cap S = \emptyset$, then $S^{-1}P \neq S^{-1}R$, $S^{-1}Grad(P) = Grad(S^{-1}P)$ and $S^{-1}P$ is a graded ideal of $S^{-1}R$ (see [6] and [13]).

Proposition 2.8. *Let R be a graded ring $S \subseteq h(R)$ a multiplicative subset of R and P a graded ideal of R. If P is a graded S-r-ideal (resp., S-pr-ideal) of R, then $S^{-1}P$ is a graded r-ideal (resp., pr-ideal) of $S^{-1}R$.*

Proof. Assume that P is a graded S-r-ideal of R. Let $\frac{x}{\alpha}, \frac{y}{\beta}, \in h(S^{-1}R)$ such that $\frac{x}{\alpha}\frac{y}{\beta} \in S^{-1}P$ with $ann_{S^{-1}R}(\frac{x}{\alpha}) = 0$. Then there exists $s \in S$ such that $sxy \in P$ with $ann_R(x) = 0$. Since P is a graded

S-r-ideal of R, there exists $s_1 \in S$ such that $s_1 sy \in P$. Therefore, $\frac{y}{\beta} = \frac{s_1 sy}{s_1 s\beta} \in S^{-1}P$. Hence, $S^{-1}P$ is a graded r-ideal of $S^{-1}R$. □

Proposition 2.9. *Let R be a G-graded ring and $S \subseteq Reg(R)$ a multiplicative subset of R. Then P is a graded S-r-ideal if and only if $S^{-1}P$ is a graded r-ideal of $S^{-1}R$ and $(S^{-1}P) \cap h(R) = (P :_{h(R)} s)$ for some $s \in S$.*

Proof. Assume that P is a graded S-r-ideal of R. By Proposition 2.8, $S^{-1}P$ is a graded r-ideal of $S^{-1}R$. Now, we show that there exists an $s \in S$ such that $(S^{-1}P) \cap h(R) = (P :_{h(R)} s)$. As P is a graded S-r-ideal of R, there exists an $s \in S$ such that for all $x, y \in h(R)$, if $xy \in P$ with $ann(x) = 0$, then $sy \in P$. Let $x \in (P :_{h(R)} s)$. Then $sx \in P$. As $S \subseteq Reg(R)$, the graded mapping $\varphi : R \to S^{-1}R$ defined by $\varphi(r) = \frac{r}{1}$ is an injective rings homomorphism. Thus $x = \frac{x}{1} = \frac{sx}{s} \in S^{-1}P$; so $x \in (S^{-1}P) \cap h(R)$. Conversely, let $x \in (S^{-1}P) \cap h(R)$. Then $x = \frac{\alpha}{t}$ for some $\alpha \in P$ and some $t \in S$. So $xt = \alpha \in P$ with $ann(t) = 0$, then $sx \in P$ since P is a graded S-r-ideal. Hence $x \in (P :_{h(R)} s)$.

Now, assume that $S^{-1}P$ is a graded r-ideal of $S^{-1}R$ and $(S^{-1}P) \cap h(R) = (P :_{h(R)} s)$ for some $s \in S$. Let $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$. Then $\frac{x}{1}\frac{y}{1} \in S^{-1}P$ and $ann(\frac{x}{1}) = 0$; so $\frac{y}{1} \in S^{-1}P$ which implies that $y = \frac{y}{1} \in S^{-1}P \cap h(R) = (P :_{h(R)} s)$ for some $s \in S$. Hence $sy \in P$.

□

Proposition 2.10. *Let R be a G-graded ring, $S \subseteq h(R)$ a multiplicative subset of R and P a graded ideal of R disjoint from S. Then P is a graded S-r-ideal (resp., S-pr-ideal) if and only if whenever I, J are graded ideals of R such that $IJ \subseteq P$ and $I \cap Reg(h(R)) \neq \emptyset$, then $sJ \subseteq P$ (resp., $sJ \subseteq Grad(P)$) for some $s \in S$.*

Proof. Suppose that P is a graded S-r-ideal of R. Let I, J be two graded ideals of R such that $IJ \subseteq P$ and $I \cap Reg(h(R)) \neq \emptyset$. There exists $a \in I \cap Reg(h(R))$. Let $g \in G$ and $b \in Jg$. Then $ann(a) = 0$ and $ab \in IJg \subseteq IJ \subseteq P$. Since P is a graded S-r-ideal, $sb \in P$ for some $s \in S$. So $sJg \subseteq P$ for all $g \in G$ which implies that $sJ \subseteq P$. Conversely, let $a, b \in h(R)$ such that $ab \in P$ with $ann(a) = \{0\}$. Then $I = \langle a \rangle$ and $J = \langle b \rangle$ are graded ideals of R such that $IJ \subseteq P$ and $a \in I \cap Reg(h(R))$. By assumption, $sJ \subseteq P$ for some $s \in S$. So $sb \in P$. Hence, P is a graded S-r-ideal of R. □

Theorem 2.11. Let R be a graded ring, $S \subseteq h(R)$ a multiplicative subset of R and P a graded ideal of R disjoint from S. If P is a graded S-r-ideal of R, then $(P : a)$ is a graded S-r-ideal of R for all $a \in h(R) - P$.

Proof. Put $a \in h(R) - P$, then $(P : a)$ is a graded ideal of R. Let $x, y \in h(R)$ such that $xy \in (P : a)$ with $ann(x) = 0$. Then $axy \in P$; so $say \in P$ for some $s \in S$ since P is graded S-r-ideal and $ann(a) = 0$. Thus $sy \in (P : a)$. □

For G-graded rings R_1 and R_2, a G-graded ring homomorphism $f : R_1 \to R_2$ is a ring homomorphism such that $f((R_1)_g) \subseteq (R_2)_g$ for every $g \in G$ (see[11]). By [18, Remark 2.6], we have the following results.

1. If I_2 is a graded ideal of R_2 and $f : R_1 \to R_2$ is a graded ring homomorphism, then $Grad(f^{-1}(I_2)) = f^{-1}(Grad(I_2))$.
2. If I_1 is a graded ideal of R_1 such that $Ker(f) \subseteq I_1$ and f is a graded epimorphism, then $f(Grad(I_1)) = Grad(f(I_1))$.

Proposition 2.12. *Let R, R' be two G-graded rings, $f : R \to R'$ be a graded isomorphism with $f(1) = 1$ and $S \subseteq h(R)$ a multiplicatively subset of R such that $f(S)$ does not contain zero. Then*

1. *Assume that for every nonunits homogeneous elements r of $h(R)$, we have $f(r)$ is nonunit element of $h(R')$. If P' is a graded $(f(S))$-r-ideal of R', then $f^{-1}(P')$ is a graded S-r-ideal of R.*

2. *If P is a graded S-r-ideal of R, then $f(P)$ is a graded $(f(S))$-r-ideal of R'.*

Proof. (1). Assume that P' is a graded $(f(S))$-r-ideal of R'. First, we want to show that $f^{-1}(P')\cap S=\emptyset$. Suppose on the contrary that there exists $s\in f^{-1}(P')\cap S$. Then $f(s)\in P'\cap f(S)$, which is a contradiction since $P'\cap f(S)=\emptyset$. Now, let $x,y\in h(R)$ such that $xy\in f^{-1}(P')$ with $ann(x)=0$. Since f is bijective, $ann(f(x))=0$. We have $f(xy)=f(x)f(y)\in P'$. Since P' is a graded $(f(S))$-r-ideal of R', there exists $s\in S$ such that $f(s)f(y)=f(sy)\in P'$. Hence $sy\in f^{-1}(P')$.

(2). First, we show that $f(P)\cap f(S)=\emptyset$. Assume not. Then there exists $u\in f(P)\cap f(S)$, which implies $u=f(x)=f(s)$ for some $x\in P$ and $s\in S$. Hence, $x-s\in Ker(f)=(0)\subseteq P$ and $s\in P\cap S$, which is a contradiction. Now, let $x',y'\in h(R')$ such that $x'y'\in f(P)$ with $ann(x')=0$. Say $x':=f(x)$ and $y':=f(y)$ for some $x,y\in h(R)$ with $ann(x)=0$. This yields that there exists an $s\in S$ such that $sy\in P$. Thus, clearly we have $f(s)y'\in f(P)$. □

Let R be a G-graded ring and $S\subseteq h(R)$. Then R/I is a G-graded ring by $(R/I)_g=(R_g+I)/I$ for all $g\in G$. Moreover, if P and I are graded ideals of R, then P/I is graded ideal of R/I and $\overline{S}=\{s+I\mid s\in S\}$ is a multiplicative subset of R/I (see [16]).

Proposition 2.13. *Let R be a G-graded ring and $S\subseteq h(R)$. Let I be a graded r-ideal of R disjoint from S and P a graded ideal of R such that $I\subseteq P$. Then P is a graded S-r-ideal of R if and only if P/I is a graded $\overline{S}$-r-ideal of R/I.*

Proof. Note that $P\cap S=\emptyset$ if and only if $(P/I)\cap\overline{S}=\emptyset$. Now, assume that P is a graded S-r-ideal of R. Let $a+I,b+I\in h(R/I)$ such that $(a+I)(b+I)\in P/I$ with $ann(a+I)=0$. Then $(ab+I)\in P/I$; so $ab\in P$. As P is a graded S-r-ideal of R and $ann(a)=0$, there exists $s\in S$ such that $sb\in P$. This implies that $(sb+I)\in P/I$. Conversely, since P/I is a graded $\overline{S}$-r-ideal of R/I, there exists $t\in S$ such that for all $a+I,b+I\in h(R)$ if $(a+I)(b+I)\in P/I$ with $ann(a+I)=\overline{0}$, then $\overline{s}(b+I)\in P/I$. Let $xy\in J$ for all $x,y\in h(R)$ with $ann(x)=0$. Then $(x+I)(y+I)\in P/I$, since I is graded r-ideal, $ann(x+I)=\overline{0}$. So $(ty+I)=(t+I)(y+I)\in P/I$. This implies that $ty\in P$. □

Proposition 2.14. *Let R be a graded ring and $S\subseteq h(R)$ a multiplicative subset of R. Then the following are hold.*

1. *If I is a graded S-r-ideal of R, then $I\cap h(R)\subseteq zd(R)$.*
2. *Let I be an graded S-prime ideal of R. If $S\subset Reg(R)$, then I is a graded S-r-ideal of R if and only if $(I:t)\cap h(R)\subseteq zd(R)$ for some $t\in S$.*

Proof. (1). Let I be a graded S-r-ideal of R. Then for all $x,y\in h(R)$ such that $xy\in I$ with $ann(x)=0$, then $sy\in I$ for some $s\in S$. Assume that $I\cap h(R)\not\subseteq zd(R)$. There exists an $a\in I\cap h(R)$ such that so $ann(a)=0$. Since $a\in I$, $s\cdot 1\in I$; so $s\in I\cap S$ contradiction. Hence $I\cap h(R)\subseteq zd(R)$.

(2). Assume that I is a graded S-prime ideal. There exists an $t\in S$ such that for all $x,y\in h(R)$ such that $xy\in I$, then $tx\in I$ or $ty\in I$. If I is a graded S-r-ideal, then by Proposition 2.7, $(I:t)$ is a graded r-ideal. As $I\cap S=\emptyset$, $(I:t)\cap S=\emptyset$; so $(I:t)$ is a graded S-r-ideal of R. Then by (1), $(I:t)\cap h(R)\subseteq zd(R)$. Conversely, A assume that $(I:t)\cap h(R)\subseteq zd(R)$. Let $x,y\in h(R)$ such that $xy\in I$ with $ann(x)=0$. We have $tx\in I$ or $ty\in I$. If $tx\in I$, then $x\in(I:t)\cap h(R)\subseteq zd(R)$, contradiction. So $ty\in I$. This implies that I is a graded S-r-ideal of R. □

Lemma 2.15. *Let R be a graded ring, $S\subseteq h(R)$ a multiplicative subset of R and P a graded ideal of R. Then P is a graded S-pr-ideal of a graded ring R, if and only if $Grad(P)$ is a graded S-r-ideal of R.*

Proof. Assume that P is a graded S-pr-ideal. There exists $s\in S$ such that for all $a,b\in h(R)$ if $ab\in P$ with $ann(a)=0$, then $sb\in Grad(P)$. Let $x,y\in h(R)$ such that $xy\in Grad(P)$ with $ann(x)=0$. Then $(xy)^n\in P$. So $sx^{n-1}y^n\in P$ which implies that $s^2x^{n-2}y^n\in P$ as $ann(x)=0$. We continue this process,

we obtain $s^n y^n \in P$. This implies that $sy \in Grad(P)$. Hence $Grad(P)$ is a graded S-r-ideal of R. Conversely, Assume that $Grad(P)$ is a graded S-r-ideal. There exists $s \in S$ such that for all $a, b \in h(R)$ if $ab \in Grad(P)$ with $ann(a) = 0$, then $sb \in Grad(P)$. Let $x, y \in h(R)$ such that $xy \in P \subseteq Grad(P)$ with $ann(x) = 0$, then $sy \in Grad(P)$ for some $s \in S$. This implies that P is graded S-pr-ideal of R. □

Proposition 2.16. *Let R be a graded ring and $S \subseteq h(R)$ a multiplicative subset of R. Let P be an graded S-primary ideal of R. If $S \subset Reg(R)$, then P is a graded S-pr-ideal of R if and only if $(P : s) \cap h(R) \subseteq zd(R)$ for some $s \in S$.*

Proof. Let P be a graded S-primary ideal of R. Then for all $x, y \in h(R)$ if $xy \in P$, then $sx \in P$ or $sy \in Grad(P)$ for some $s \in S$. Assume that P is a graded S-pr-ideal of R, then $Grad(P)$ is a graded S-r-ideal of R. So $(Grad(P) : s) \cap h(R) \subseteq zd(R)$. As $P \subseteq Grad(P)$, $(P : s) \cap h(R) \subseteq (Grad(P) : s) \cap h(R) \subseteq zd(R)$. Conversely, assume that $(P : s) \cap h(R) \subseteq zd(R)$. Let $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$, then $sx \in P$ or $sy \in Grad(P)$. If $sa \in P$, then $a \in (P : s) \cap h(R) \subseteq zd(R)$ contradiction. Then $sb \in Grad(P)$. This implies that P is a graded S-pr-ideal of R. □

Note that in R a G-graded ring and I, J are graded ideals of R, then $I \cap J$ is a graded ideal of R and $Grad(I \cap J) = Grad(I) \cap Grad(J)$ (see [9] and [14]).

Theorem 2.17. Let R be a graded ring, $S \subseteq h(R)$ a multiplicative subset of R and I, J are a graded S-r-ideals (resp., S-pr-ideals) of R disjoint from S. Then $I \cap J$ is graded S-r-ideal (resp., S-pr-ideal) of R.

Proof. First, since $I \cap S = \emptyset$ and $J \cap S = \emptyset$, $I \cap J \cap S = \emptyset$. Let $x, y \in h(R)$ such that $xy \in I \cap J$ with $ann(x) = 0$. Then $xy \in I$. Since I is a graded S-r-ideal, $sy \in I$ for some $s \in S$. Similarly, $ty \in J$ for some $t \in S$ and hence $sty \in I \cap J$. Therefore, $I \cap J$ is a graded S-r-ideal of R. □

Theorem 2.18. Let I be a graded S-r-ideal (resp., S-pr-ideal) of R. For a graded ideal J of R with $J \cap S \neq \emptyset$, $I \cap J$ is graded S-r-ideal (resp., S-pr-ideal) of R.

Proof. Since $I \cap S = \emptyset$, clearly we have $(I \cap J) \cap S = \emptyset$. Suppose that I is a graded S-r-ideal of R. There exists an $s \in S$ such that for all $x, y \in h(R)$, if $xy \in I$ with $ann(x) = 0$, then $sy \in I$. Let $xy \in I \cap J \subseteq I$ with $ann(x) = 0$. Then $sy \in I$. Choose $t \in J \cap S$. This means $w = st \in S$. Thus $wy \in I$ which implies that $wy \in I \cap J$. So $I \cap J$ is a graded S-r-ideal of R. □

Theorem 2.19. Let R be a G-graded ring, $S \subseteq h(R)$ a multiplicative subset of R and I, J be graded prime ideals of R which are not comparable disjoint from S. If $I \cap J$ is a graded S-r-ideal of R, then I and J are graded S-r-ideals of R.

Proof. Let $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$. Suppose that $a \in J - I$. Then $xya \in I \cap J$. Since $I \cap J$ is graded S-r-ideal, $sya \in I \cap J$ for some $s \in S$. Then $sya \in I$. Since I is a graded prime ideal of R and $a \notin I$, $sy \in I$. Hence, I is a graded S-r-ideal of R. Similarly, J is a graded S-r-ideal of R. □

Theorem 2.20. Let $R = R_1 \times R_2$ where each R_i is a G-graded ring for all $i \in \{1, 2\}$, $P = P_1 \times P_2$ a graded ideal of R where each P_i is a graded ideal of R_i and $S = S_1 \times S_2 \subseteq h(R)$ a multiplicative subset of R such that $S_1 \cap P_1 = \emptyset$ and $S_2 \cap P_2 = \emptyset$. Then the following are equivalents.

1. $P = P_1 \times P_2$ is a graded S-r-ideal of R.
2. $P_1 = R_1$ and P_2 is a graded S_2-r-ideal of R_2 or $P_2 = R_2$ and P_1 is a graded S_1-r-ideal of R_1 or either P_1 is a graded S_1-r-ideal of R_1 and P_2 is a graded S_2-r-ideal of R_2.

Proof. (1)⇒(2). Suppose that $P = P_1 \times P_2$ is a graded S-r-ideal of R and $P_2 = R_2$. We will show that P_1 is a graded S_1-r-ideal of R_1. Let $x, y \in h(R_1)$ such that $xy \in P_1$ with $ann(x) = 0$. Note that $ann(a,1) = (0,0)$ and $(a,1)(b,0) = (ab,0) \in P$. Since P is a graded S-r-ideal of R, there exists an $s = (s_1, s_2) \in S$ such that $s(b,0) \in P$; so $s_1 b \in P_1$. Hence P_1 is a graded S_1-r-ideal of R_1. Similarly, one can easily show that P_2 is a graded S_2-r-ideal of R_2 and $P_1 = R_1$. Assume that P_1, I_2 are proper ideals, similarly, P_1, P_2 be graded S_i-r-ideals of R_i respectively.

(2)⇒(1). Assume that P_1, P_2 be S_i-r-ideals of R_i for every $i = 1, 2$. Now, let $(a_1, a_2), (b_1, b_2) \in h(R)$ such that $(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2) \in P_1 \times P_2$ with $ann(a_1, a_2) = (0,0)$. Then $ann(a_1) = ann(a_2) = 0$. Note that $a_i b_i \in P_i$ with $ann(a_i) = 0$. So $s_i b_i \in P_i$ for some $s_i \in S$ since P_1, P_2 be graded S_i-r-ideals of R_i for every $i = 1, 2$. Put $s = (s_1, s_2) \in S$. Then $(s_1, s_2)(b_1, b_2) = (s_1 b_1, s_2 b_2) \in P$. This implies that P is a graded S-r-ideal of R. In other cases, one can similarly prove that P is a graded S-r-ideal of R. □

Let R be a G-graded ring. Then the graded radical of R is denoted by $Grad(R)$ and equal R (see [14]).

Lemma 2.21. *([5]) Let $R = R_1 \times R_2$ where each R_i is a G-graded ring for all $i \in \{1, 2\}$. Then the following hold.*

1. *P_1 is a graded ideal of R_1, if and only if $Grad(P_1 \times R_2) = Grad(P_1) \times R_2$.*
2. *P_2 is a graded ideal of R_2, if and only if $Grad(R_1 \times P_2) = R_1 \times Grad(P_2)$.*
3. *P_1 is a graded ideal of R_1 and P_2 is a graded ideal of R_2, if and only if $Grad(P_1 \times P_2) = Grad(P_1) \times Grad(P_2)$.*

By analogy, in Theorem 2.20, it is straightforward to demonstrate the preceding theorem.

Theorem 2.22. Let $R = R_1 \times R_2$ where each R_i is a G-graded ring for all $i \in \{1, 2\}$, $P = P_1 \times P_2$ a graded ideal of R where each P_i is a graded ideal of R_i and $S = S_1 \times S_2 \subseteq h(R)$ a multiplicative subset of R such that $S_1 \cap P_1 = \emptyset$ and $S_2 \cap P_2 = \emptyset$. Then the following are equivalents.

1. $P = P_1 \times P_2$ is a graded S-pr-ideal of R.
2. $P_1 = R_1$ and P_2 is a graded S_2-pr-ideal of R_2 or $P_2 = R_2$ and P_1 is a graded S_1-pr-ideal of R_1 or either P_1 is a graded S_1-pr-ideal of R_1 and P_2 is a graded S_2-pr-ideal of R_2.

Corollary 2.23. *Let $R = R_1 \times R_2$ where each R_i is a G-graded ring for all $i \in \{1, 2\}$ and $P = P_1 \times P_2$ a graded ideal of R where each P_i is a graded ideal of R_i. Then the following are equivalents.*

1. *$P = P_1 \times P_2$ is a graded r-ideal (resp., pr-ideal) of R.*
2. *$P_1 = R_1$ and P_2 is a graded r-ideal (resp., pr-ideal) of R_2 or $P_2 = R_2$ and P_1 is a graded r-ideal (resp., pr-ideal) of R_1 or either P_1 is a graded r-ideal of R_1 and P_2 is a graded r-ideal (resp., pr-ideal) of R_2.*

In the case $S = \{1\}$, a proper graded ideal P of a graded ring R is said to be a *graded r-ideal* (resp., *pr-ideal*) of R if whenever nonunit elements $x, y \in h(R)$ such that $xy \in P$ and $ann(x) = 0$, then $y \in P$ (resp., $y \in Grad(P)$) [2]. It is recognizable that a graded r-ideal (resp., pr-ideal) of R disjoint with S is a graded S-r-ideal (resp., S-pr-ideal) of R. However, a graded S-r-ideal (resp., S-pr-ideal) of R needs not be a graded r-ideal (resp., pr-ideal) of R, see the following example.

Example 2.24. Let R_1, R_2 be two graded rings. Let P_1 be a proper non graded pr-ideal ring and P_2 be a proper graded S_2-r-ideal of R_2, where S_2 a multiplicative subset of R_2. We consider $R = R_1 \times R_2$ and a multiplicative subset $S := (S_1 \cup \{0\} \times S_2)$, where S_1 be a multiplicative subset of R_1. Then S is a multiplicative subset of R. By Corollary 2.23, $P_1 \times P_2$ is neither an graded r-ideal nor graded

pr-ideal of R. Now, we will show that $P_1 \times P_2$ is an graded S-r-ideal of R, and so graded S-pr-ideal of R. Since P_2 is a graded S_2-r-ideal, there exists an $s_2 \in S_2$ such that for any $x, y \in h(R_2)$ such that $xy \in P_2$ with $ann(x) = 0$, then $s_2 y \in P_2$. Let $(a,x),(b,y) \in h(R_1 \times R_2)$ such that $(a,x),(b,y) \in P_1 \times P_2$ with $ann((a,x)) = (0,0)$, then $(xy \in P_2$ with $ann(x) = 0$. So $s_2 y \in P_2$. Take $s := (0,s_2) \in S$. Then for any $(0,s_2)(b,y) = (0,s_2 y) \in \{0\} \times P_2 \subseteq P_1 \times P_2$. This implies that $P_1 \times P_2$ is a graded S-r-ideal.

Assume that M is an R-module. Then M is said to be G-graded if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$ where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. It is clear that M_g is an R_e-submodule of M for all $g \in G$. We assume that $h(M) \bigcup_{g \in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be graded R-submodule if $N = \bigoplus_{g \in G} (N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g \in G} x_g$ where $x_g \in N$ for all $g \in G$. It is known that an R-submodule of a graded R-module need not be graded. For more terminology (see [10] and [13]). Let G be an abelian group and M be a G-graded R-module. Then $X = R(+)M$ is G-graded by $X_g = R_g(+)M_g$ for all $g \in G$. Note that, X_g is an additive subgroup of X for all $g \in G$. Also, for $g, h \in G$, $X_g X_h = (R_g(+)M_g)(R_h(+)M_h) = (R_g R_h, R_g M_h + R_h M_g) \subseteq (R_{gh}, M_{gh} + M_{hg}) \subseteq (R_{gh}, M_{gh}) = X_{gh}$ as G is abelian [18]. Moreover, if P is an ideal of R and N is an R-submodule of M such that $PM \subseteq N$, then $P(+)N$ is a graded ideal of $R(+)M$ if and only if P is a graded ideal of R and N is a graded R-submodule of M [18, Proposition 3.3]. Consequently, $h(R(+)M) = \{(a,x); a \in h(R), x \in h(M)\}$. Clearly, if S is a multiplicative subset of $h(R)$, then $S(+)\{0\}$ and $S(+)h(M)$ are multiplicative subset of $h(R(+)M)$.

Lemma 2.25. *([8], [11] and [18]) Let $R = \bigoplus_{g \in G} R_g$ be G-graded ring and $M = \bigoplus_{g \in G} M_g$ be a G-graded R-module. Let I be a graded ideal of R and N a graded submodule of M such that $IM \subseteq N$. Then the following hold:*

1. *$I(+)N$ is a graded ideal of $R(+)M$.*
2. *$Grad(I(+)N) = Grad(I)(+)M$.*

Theorem 2.26. Let G be an abelian group, R be a G-graded ring, M be a G-graded R-module, S be a multiplicative subset of $h(R)$ and I be a graded ideal of R disjoint with S. Then the following are hold.

1. I is a graded S-r-ideal of R, then $I(+)M$ is an $(S(+)M)$-r-ideal of $R(+)M$.
2. If $Z(M) = zd(R)$ and $I(+)M$ an $S(+)M$-r-ideal, then I is an S-r-ideal of R.

Proof. (1). First, $I(+)M \cap (S(+)M) = \emptyset$, as $I \cap S = \emptyset$. Now, let $(x,\alpha),(y,\beta) \in h(R(+)M)$ such that $(x,\alpha)(y,\beta) = (xy, x\beta + y\alpha) \in I(+)M$ with $ann(x,\alpha) = (0,0)$. Then $xy \in I$ with $ann(x) = 0$; so $s\alpha \in I$ for some $s \in S$ since I is a graded S-r-ideal of R. This implies that $(s,0)(y,\beta) = (sy, s\alpha) \in I(+)M =$. Hence $I(+)M$ is a graded $(S(+)\{0\})$-r-ideal of $R(+)M$. As $(S(+)\{0\}) \subseteq S(+)h(M)$, then by Proposition 2.7, $I(+)M$ is a graded $(S(+)M)$-r-ideal of $R(+)M$.

(2). Let $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$. Then $(x,0),(y,0) \in h(R(+)M)$ with $(x,0)(y,0) = (xy,0) \in I(+)M$. As $Z(M) = zd(R)$, $ann(x,0) = (0,0)$. This implies that $(s,m)(y,0) = (sy, ym) \in I(+)M$ for some $(s,m) \in S(+)h(M)$. Thus $sy \in I$; so I is a graded S-r-ideal of R.

□

Example 2.27. Consider the $G = \mathbb{Z}_2$, $R = \mathbb{Z}[i]$ a G-graded ring and $M = R$ a G-graded R-module, by $R_0 = M_0 = \mathbb{Z}$ and $R_1 = M_1 = i\mathbb{Z}$. Let $S = \{1, -1\}$ a multiplicative subset of R, $I = \{0\}$ a graded ideal of R and $N = 6R$ a graded submodule of M. Note that I is a graded S-r-ideal of R. But $I(+)N$ note have a graded $(S(+)M)$-r-ideal of $R(+)M$. Indeed, we have $(1,2)(0,3) \in h(R(+)M)$, $(1,2)(0,3) = (0,6) \in I(+)N$ and $ann(1,2) = (0,0)$ but $s(0,3) \notin I(+)N$ for all $s \in S(+)M$. This shows that $I(+)N$ does not have a graded $(S(+)M)$-r-ideal of R.

Theorem 2.28. Let G be an abelian group, R be a G-graded ring, M be a G-graded R-module, S be a multiplicative subset of $h(R)$, I be a graded ideal of R disjoint with S and N be a graded submodule of M with $IM \subseteq N$. Then the following are hold.

1. I is a graded S-pr-ideal of R, then $I(+)N$ is an $(S(+)M)$-pr-ideal of $R(+)M$.
2. If $Z(M) = zd(R)$ and $I(+)N$ an $S(+)M$-pr-ideal, then I is an S-pr-ideal of R.

Proof. (1). First, $I(+)M \cap (S(+)M) = \emptyset$, as $I \cap S = \emptyset$. Now, let $(x,\alpha),(y,\beta) \in h(R(+)M)$ such that $(x,\alpha)(y,\beta) = (xy, x\beta + y\alpha) \in I(+)N$ with $ann(x,\alpha) = (0,0)$. Then $xy \in I$ with $ann(x) = 0$; so $s\alpha \in Grad(I)$ for some $s \in S$ since I is a graded S-pr-ideal of R. This implies that $(s,0)(y,\beta) = (sy, s\alpha) \in Grad(I)(+)M = Grad(I(+)N)$. Hence $I(+)N$ is a graded $(S(+)\{0\})$-r-ideal of $R(+)M$. As $(S(+)\{0\}) \subseteq S(+)h(M)$, then by Proposition 2.7, $I(+)M$ is a graded $(S(+)M)$-pr-ideal of $R(+)M$.

(2). Let $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$. Then $(x,0),(y,0) \in h(R(+)M)$ with $(x,0)(y,0) = (xy,0) \in I(+)N$. As $Z(M) = zd(R)$, $ann(x,0) = (0,0)$. This implies that $(s,m)(y,0) = (sy, ym) \in Grad(I(+)N) = Grad(I)(+)M$ for some $(s,m) \in S(+)h(M)$. Thus $sy \in Grad(I)$; so I is a graded S-pr-ideal of R. □

Example 2.29. Consider the $G = \mathbb{Z}_2$, $R = \mathbb{Z}[i]$ a G-graded ring and $M = R$ a G-graded R-module, by $R_0 = M_0 = \mathbb{Z}$ and $R_1 = M_1 = i\mathbb{Z}$. Let $S = \{1, -1\}$ a multiplicative subset of R, $I = \{0\}$ a graded ideal of R and $N = 6R$ a graded submodule of M. Note that P is a graded S-pr-ideal of R. Then by Theorem 2.28, $I(+)N$ is a graded $(S(+)M)$-pr-ideal of $R(+)M$. But by Example 2.27, $I(+)N$ does not have a graded $(S(+)M)$-r-ideal of R.

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Revisiting 1-absorbing prime elements in multiplicative lattices

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Abstract. In this paper, we revisit the concept of 1-absorbing prime elements in C-lattices and obtain its characterizations. The main aim of this paper is to investigate some more properties of 1-absorbing prime elements in multiplicative lattices.
Key Words: 1-absorbing prime element, p-1-absorbing prime element, semiprimary element.
2020 MSC: 06F99, 06B23

Dedicated to the memory of my guide, Professor C. S. Manjarekar.

1 Introduction and preliminaries

The prime ideal, which is an important subject of ideal theory in commutative rings has been widely studied by many authors to obtain various generalizations of prime ideals. Since the first paper on absorbing ideals as in [3] by A. Badawi, much work in the literature has been devoted to studying different generalizations of absorbing ideals in commutative rings. Following the concept of 1-absorbing primary ideals in commutative rings as in [4] by A. Badawi and E. Y. Celikel, the authors in [14], introduced the idea of 1-absorbing prime ideals in commutative rings. Afterwards, this notion of 1-absorbing prime ideal was coined as OA-prime ideal and OA factorization was studied (see [10] and [8]). In an attempt of generalizing the notion of 1-absorbing prime ideal in commutative rings, E. A. Ugurlu in [13], introduced the notion of 1-absorbing prime submodule in ring modules.

R. P. Dilworth in [7] took a major step in giving a concrete abstract formulation of the ideal theory of commutative rings $\mathbb{R}$ with identity. He intended that the treatment should be purely ideal theoretic and therefore the system which was chosen for the study was multiplicative lattices. Following this, we find that researchers have extended different types of ideals in $\mathbb{R}$ to respective types of elements in multiplicative lattices. Moving alike, A. Reinhart and G. Ulucak in [11] extended the notion of 1-absorbing prime ideals in $\mathbb{R}$ to non-modular multiplicative lattices. In this paper, we revisit the concept of 1-absorbing prime elements in multiplicative lattices. The main aim of this paper is to investigate further properties of 1-absorbing prime elements in multiplicative lattices.

Before starting this work, let us review few definitions in multiplicative lattices. A multiplicative lattice L is a complete lattice in which there is a commutative, associative multiplication which distributes over arbitrary joins and has greatest element 1 as a multiplicative identity. 0 is the least element of L. An element $a \in L$ is called compact if for $X \subseteq L$, $a \leqslant \vee X$ implies the existence of a finite number of elements $a_1, a_2, \cdots, a_n$ in X such that $a \leqslant a_1 \vee a_2 \vee \cdots \vee a_n$. A multiplicative lattice is called a compactly generated lattice or simply a CG-lattice, if every element of it is a join of compact elements. By a C-lattice, we mean a (not necessarily modular) multiplicative lattice which is generated

under joins by a multiplicatively closed subset C of compact elements. We note that, in a C-lattice, a finite product of compact elements is again compact.

R. P. Dilworth gave the following classical example of a multiplicative lattice. Let $\mathbb{R}$ be a commutative ring with unity and I, J be the ideals of $\mathbb{R}$. If $L(\mathbb{R}) = L$ denotes the set of all ideals of $\mathbb{R}$ then $< L; \wedge, \vee, \cdot >$ is a multiplicative lattice where the lattice operations defined are as follows:

① $I \wedge J = I \cap J$

② $I \vee J = I + J = (I \cup J)$

③ $I \cdot J = \{ \sum\limits_{finite} a_i \cdot b_i \in \mathbb{R} \mid a_i \in I \text{ and } b_i \in J \}$

$L(\mathbb{R})$ is known as the ideal lattice or the lattice of ideals of $\mathbb{R}$ and has following features:

1. Every principal ideal of $\mathbb{R}$ is a compact element of $L(\mathbb{R})$.

2. $L(\mathbb{R})$ is a compactly generated multiplicative lattice and a C-lattice.

Throughout this paper, L denotes a C-lattice, L_* denotes the set of all compact elements of L and Z_+ denotes the set of all natural numbers. The product of any two elements a and b of L will be denoted by ab instead of $a \cdot b$ for convenience.

For $a, b \in L$, $(a : b) = \vee\{x \in L \mid xb \leqslant a\}$. Note that $c \leqslant (a : b)$ if and only if $cb \leqslant a$ and $a \leqslant b$ implies $(a : c) \leqslant (b : c)$ $\forall a, b, c \in L$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L_* \mid x^n \leqslant a,$ for some $n \in Z_+\}$. An element $q \in L$ is said to be proper if $q < 1$. An element $m < 1$ in L is said to be maximal if $m < x \leqslant 1$ imply $x = 1$. A proper element $q \in L$ is said to be prime if $ab \leqslant q$ imply either $a \leqslant q$ or $b \leqslant q$ $\forall$ $a, b \in L$. In a C-lattice, an element q is prime if and only if $ab \leqslant q$ imply either $a \leqslant q$ or $b \leqslant q$ $\forall$ $a, b \in L_*$. An element $q \in L$ is called semiprimary if $\sqrt{q}$ is a prime element. A proper element $q \in L$ is said to be primary if $ab \leqslant q$ imply either $a \leqslant q$ or $b^n \leqslant q$ for some $n \in Z_+$ where $a, b \in L_*$. A proper element $q \in L$ is called p-prime if q is prime and $p = \sqrt{q}$ is prime. A proper element $q \in L$ is called p-primary if q is primary and $p = \sqrt{q}$ is prime. In a C-lattice L, $\sqrt{a} = \wedge\{p \in L \mid p \geqslant a$ is a minimal prime over $a\} = \wedge\{p \in L \mid p$ is prime and $a \leqslant p\}$. (See also Theorem 3.6 of [12]). Note that $q \leqslant \sqrt{q}$ for every $q \in L$. According to [9], a proper element $q \in L$ is said to be 2-absorbing if for every $a, b, c \in L$, $abc \leqslant q$ imply either $ab \leqslant q$ or $bc \leqslant q$ or $ca \leqslant q$. According to [5], a proper element $q \in L$ is said to be 2-absorbing primary if for every $a, b, c \in L$, $abc \leqslant q$ imply either $ab \leqslant q$ or $bc \leqslant \sqrt{q}$ or $ca \leqslant \sqrt{q}$. According to [5], a proper element $q \in L$ is called p-2-absorbing primary if q is 2-absorbing primary and $p = \sqrt{q}$ is 2-absorbing. For general background and terminology in multiplicative lattices, the reader may consult [1] and [2].

The content of the paper is as follows: Initially, we revisit the definition of a 1-absorbing prime element in L. We obtain characterizations of a 1-absorbing prime element of L (See Theorems 2.5, 2.8, 2.9). We show that if $q \in L$ is 1-absorbing prime, then q is semiprimary in Theorem 2.6-④ and hence we define a p-1-absorbing prime element (See definition 2.10). In Theorems 2.11, 2.13, 2.14 and 2.15, we obtain results regarding the meet of 1-absorbing prime elements in L. In Theorem 2.16, we show that if a proper element $q \in L$ is 1-absorbing prime, then for every $x \nleqslant q$, $(q : x)$ is prime. Then we investigate under which condition(s), converse of Theorem 2.16 holds true (See Theorems 2.18 and 2.26). In an attempt of highlighting the future scope of this paper, we conclude this paper by defining the notion of "weakly 1-absorbing prime element" and "n-1-absorbing prime element" ($n \geqslant 1$ in Z_+) of L in the last section.

2 Results on 1-absorbing prime element in L

We begin with introducing (revisiting) the notion of 1-absorbing prime elements in C-lattices.

Definition 2.1. (See also Definition 2.1 of [11]). A proper element $q \in L$ is called a 1-absorbing prime element if for all proper elements $a, b, c \in L$ such that the product of a, b, c need not collapse to one of the element among them, $abc \leqslant q$ imply either $ab \leqslant q$ or $c \leqslant q$.

Without loss of generality, we define above notion of 2.1 in the following way.

Definition 2.2. A proper element $q \in L$ is called a 1-absorbing prime element if for every $a, b, c \in L$, $abc \leqslant q$ imply either $ab \leqslant q$ or $c \leqslant q$.

Note that in view of the concept of 2-absorbing element of L, a prime element of L is nothing but 1-absorbing element of L.

Clearly, every 1-absorbing element of L is a 1-absorbing prime element of L. Its converse need not be true as seen in the Example 2.2 of [11].

Theorem 2.3. If a proper element $q \in L$ is 1-absorbing prime, then q is a 2-absorbing element of L.

Proof. Let $abc \leqslant q$, $ab \nleqslant q$ and $bc \nleqslant q$ where $a, b, c \in L$. Since q is 1-absorbing prime, $abc \leqslant q$, $ab \nleqslant q$ gives $c \leqslant q$ and hence $ac \leqslant q$. Therefore, q is a 2-absorbing element of L. □

The converse of Theorem 2.3 need not be true as seen in the Example 2.3 of [11].

Now before obtaining the characterization of a 1-absorbing prime element of L, we state the following essential lemma which is outcome of Lemma 2.3.13 from [6].

Lemma 2.4. *Let a_1, $a_2 \in L$. Suppose $b \in L$ satisfies the following property:*
(*). If $h \in L_*$ with $h \leqslant b$, then either $h \leqslant a_1$ or $h \leqslant a_2$.
Then either $b \leqslant a_1$ or $b \leqslant a_2$.

We present Lemma 2.4 of [11] in the following result in an alternative way.

Theorem 2.5. Let $q \in L$ be a proper element of L. Then the following statements are equivalent:
①. q is a 1-absorbing prime element of L.
②. for every $a, b, c \in L$ such that $q \leqslant c$, $abc \leqslant q$ implies either $ab \leqslant q$ or $c = q$.
③. for every $a, b \in L$ such that $ab \nleqslant q$, $(q : ab) = q$.
④. for every $x, y, z \in L_*$, $xyz \leq q$ implies either $xy \leqslant q$ or $z \leqslant q$.

Proof. ①⟹②. It is obvious.

②⟹③. Suppose ② holds. Let $h \in L_*$ be such that $h \leqslant (q : ab)$ and $ab \nleqslant q$ where $a, b \in L$. Then $hab \leqslant q$. Clearly, $ab(h \vee q) \leqslant q$. Let $u = h \vee q$. Now as $q \leqslant u$, $abu \leqslant q$ and $ab \nleqslant q$, by hypothesis ②, it follows that $u = q$ and so $h \leqslant q$. Hence by Lemma 2.4, we have $(q : ab) \leqslant q$. Consequently, $(q : ab) = q$.

③⟹④. Suppose ③ holds. Let $xyz \leqslant q$ and $xy \nleqslant q$ where $x, y, z \in L_*$. Then as $L_* \subseteq L$, by hypothesis ③, it follows that $(q : xy) = q$ and we are done, since $z \leqslant (q : xy)$.

④⟹①. Suppose ④ holds. Let $abc \leqslant q$ and $ab \nleqslant q$ where $a, b, c \in L$. Since $abc \leqslant q$, there exist $a', b', c' \in L_*$ with $a' \leqslant a$, $b' \leqslant b$ and $c' \leqslant c$ such that $a'b'c' \leqslant q$. Since $ab \nleqslant q$, there exists $a_1, b_1 \in L_*$ with $a_1 \leqslant a$ and $b_1 \leqslant b$ such that $a_1 b_1 \nleqslant q$. Let $a_2 = a_1 \vee a'$ and $b_2 = b_1 \vee b'$. We show $c \leqslant q$. Choose $c_\alpha \in L_*$ such that $c_\alpha \leqslant c$. Now since $a_2 b_2 (c' \vee c_\alpha) \leqslant q$ and $a_2 b_2 \nleqslant q$ where $a_2, b_2, (c' \vee c_\alpha) \in L_*$, by hypothesis ④, we have $(c' \vee c_\alpha) \leqslant q$ which implies $c_\alpha \leqslant q$. Hence by Lemma 2.4, we have $c \leqslant q$. Therefore, q is a 1-absorbing prime element of L. □

In the following result, we present Proposition 2.5 (1) of [11] in a different way.

Theorem 2.6. Let a proper element $q \in L$ be 1-absorbing prime.
①. For each $x \in L$, $x \leqslant \sqrt{q}$ if and only if $x^2 \leqslant q$.
②. If $x, y \leqslant \sqrt{q}$ where $x, y \in L$, then $xy \leqslant q$.
③. $(\sqrt{q})^2 \leqslant q$.
④. q is a semiprimary element of L.

Proof. ①. Let $h \in L_*$ be such that $h \leqslant \sqrt{q}$. Then there exists a least positive integer k such that $h^k \leqslant q$. So $hhh^{k-2} \leqslant q$ with $h^{k-2} \not\leqslant q$. Since q is 1-absorbing prime, we get $hh = h^2 \leqslant q$.

Suppose $x \leqslant \sqrt{q}$ where $x \in L$. Let $a, b \in L_*$ be such that $a \leqslant x$ and $b \leqslant x$. Since $a \leqslant \sqrt{q}$ and $b \leqslant \sqrt{q}$, it follows that $a^2 \leqslant q$ and $b^2 \leqslant q$. Hence $a(a \vee b)b \leqslant q$. As q is 1-absorbing prime, we get either $a(a \vee b) \leqslant q$ or $b \leqslant q$. Consequently, $ab \leqslant q$. As $x^2 = \vee\{ab \mid a, b \in L_*, a \leqslant x, b \leqslant x\}$, it follows that $x^2 \leqslant q$. The converse is obvious.

②. Let $x, y \in L$ be such that $x \leqslant \sqrt{q}$ and $y \leqslant \sqrt{q}$. By ①, $x^2 \leqslant q$ and $y^2 \leqslant q$. Hence $x(x \vee y)y \leqslant q$. As q is 1-absorbing prime, we get either $x(x \vee y) \leqslant q$ or $y \leqslant q$. Consequently, $xy \leqslant q$.

③. It follows from ②, since $(\sqrt{q})^2 = \vee\{ab \mid a, b \in L_*, a \leqslant \sqrt{q}, b \leqslant \sqrt{q}\}$.

④. Let $a, b \leqslant L_*$ be such that $ab \leqslant \sqrt{q}$. Then by ①, $(ab)^2 \leqslant q$. Since q is 1-absorbing prime and $aab^2 \leqslant q$, we get either $aa = a^2 \leqslant q$ or $b^2 \leqslant q$. Hence either $a \leqslant \sqrt{q}$ or $b \leqslant \sqrt{q}$ and we are done. □

Corollary 2.7. *If q is a p-prime element of L, then $p^2 \leqslant q$.*

Proof. The proof is obvious. □

The following characterization of a 1-absorbing prime element in L is an outcome of the Theorem 2.6.

Theorem 2.8. A proper element $q \in L$ is a 1-absorbing prime element of L if and only if q is a semiprimary element of L and $(\sqrt{q})^2 \leqslant q$.

Proof. If q is a 1-absorbing prime element of L, then by Theorem 2.6 ③ and ④, we are done. Conversely, assume that $(\sqrt{q})^2 \leqslant q$ and q is a semiprimary element of L. Let $xyz \leqslant q$ where $x, y, z \in L_*$. If either $x \leqslant q$ or $yz \leqslant q$, then we are done. So assume that $x \not\leqslant q$ and $yz \not\leqslant q$. Now, q being semiprimary, $\sqrt{q}$ is prime. Also, $q \leqslant \sqrt{q}$. So $xyz \leqslant \sqrt{q}$ gives $x \leqslant \sqrt{q}$ and $yz \leqslant \sqrt{q}$. Thus $x, y \leqslant \sqrt{q}$ or $x, z \leqslant \sqrt{q}$. Since $\vee\{ab \mid a, b \in L_*, a \leqslant \sqrt{q}, b \leqslant \sqrt{q}\} = (\sqrt{q})^2 \leqslant q$, it follows that either ($xy \leqslant q$ with $z \not\leqslant q$) or ($xz \leqslant q$ with $y \not\leqslant q$) and hence q is a 1-absorbing prime element of L. □

Theorem 2.9. Let q be a p-primary element of L. Then q is a 1-absorbing prime element of L if and only if $p^2 \leqslant q$. In particular, m^2 is a 1-absorbing prime element of L for each maximal element $m \in L$.

Proof. The proof is obvious. □

Similar to the concept of p-prime element of L, we define p-1-absorbing prime element of L.

Definition 2.10. If a proper element $q \in L$ is 1-absorbing prime, then by Theorem 2.6-④, q is semiprimary (i.e. $p = \sqrt{q}$ is a prime element of L). In this case, we say that $q \in L$ is a p-1-absorbing prime element of L.

In the following 4 successive results, we obtain results regarding the meet of 1-absorbing prime elements in L.

Theorem 2.11. If $q_1, q_2, ..., q_n$ are p-1-absorbing prime elements of L for some prime element $p \in L$, then $q = \bigwedge_{i=1}^{n} q_i$ is a p-2-absorbing primary element of L.

Proof. Clearly, $\bigwedge_{i=1}^{n} q_i = q \neq 1$ and by property (p4) of [12], we have $\sqrt{q} = \bigwedge_{i=1}^{n} \sqrt{q_i} = p$. Let $abc \leqslant q$, $ac \nleqslant q$ and $bc \nleqslant \sqrt{q}$ where $a, b, c \in L$. Then $ac \nleqslant q_i$ for some i $(1 \leqslant i \leqslant n)$ and $bc \nleqslant \sqrt{q_j}$ for some j $(1 \leqslant j \leqslant n)$. Without loss of generality, assume $i \neq j$. As q_i is a p-1-absorbing prime element, $(ac)b \leqslant q \leqslant q_i$ and $ac \nleqslant q_i$, we have $b \leqslant q_i \leqslant \sqrt{q_i}$ and hence $ab \leqslant \sqrt{q_i} = p = \sqrt{q}$. (Also considering q_j as a p-1-absorbing prime element, $(bc)a \leqslant q \leqslant q_j$ and $bc \nleqslant q_j$, we have $a \leqslant q_j \leqslant \sqrt{q_j}$ and hence $ab \leqslant \sqrt{q_j} = p = \sqrt{q}$). The proof is complete as p is a 2-absorbing element because every prime element of L is a 2-absorbing element of L. □

The following lemma is a statement made by C. Jayaram et. al. in [9].

Lemma 2.12. *If p_1 and p_2 are the prime elements of L, then $p_1 \wedge p_2$ is a 2-absorbing element of L.*

Theorem 2.13. If q_1 is a p_1-1-absorbing prime element of L and q_2 is a p_2-1-absorbing prime element of L for some prime elements $p_1,\ p_2 \in L$, then $q_1 \wedge q_2$ is a 2-absorbing primary element of L.

Proof. Suppose $h = q_1 \wedge q_2$. Let $abc \leqslant h$, $ac \nleqslant \sqrt{h}$ and $bc \nleqslant \sqrt{h}$ where $a, b, c \in L$. Then $a, b, c \nleqslant \sqrt{h}$. As $p_1 = \sqrt{q_1}$ and $p_2 = \sqrt{q_2}$, by property (p4) of [12], we have $\sqrt{h} = p_1 \wedge p_2$ and so by Lemma 2.12 , $\sqrt{h}$ is a 2-absorbing element of L. Hence $ab \leqslant \sqrt{h}$. Since p_1 is prime and $ab \leqslant \sqrt{h} \leqslant p_1$, we assume that $a \leqslant p_1$. Also, since p_2 is prime, $ab \leqslant \sqrt{h} \leqslant p_2$ and $a \nleqslant \sqrt{h}$, we must have $a \nleqslant p_2$ and $b \leqslant p_2$. As $b \leqslant p_2$ and $b \nleqslant \sqrt{h}$, we must have $b \nleqslant p_1$. If $a \leqslant q_1$ and $b \leqslant q_2$, then $ab \leqslant h$ and we are done. So assume that $a \nleqslant q_1$. As q_1 is 1-absorbing prime, $abc \leqslant h \leqslant q_1$ and $a \nleqslant q_1$, we have $bc \leqslant q_1 \leqslant \sqrt{q_1} = p_1$. Since $b \leqslant p_2$ and $bc \leqslant p_1$, we have $bc \leqslant \sqrt{h}$, a contradiction. Thus we must have $a \leqslant q_1$. Similarly, if we assume that $b \nleqslant q_2$, then we get a contradiction to $ac \nleqslant \sqrt{h}$ and hence we must have $b \leqslant q_2$. Therefore, $ab \leqslant h$ and the proof is complete. □

Theorem 2.14. If q_1 and q_2 are two 1-absorbing prime elements of L, then $q_1 \wedge q_2$ is a 2-absorbing element of L.

Proof. Let $abc \leqslant (q_1 \wedge q_2)$ but $ac \nleqslant (q_1 \wedge q_2)$ and $bc \nleqslant (q_1 \wedge q_2)$ where $a, b, c \in L$. Then $ac \nleqslant q_i$ for some $i \in \{1, 2\}$ and $bc \nleqslant q_j$ for some $j \in \{1, 2\}$. If $i = j$, then as $a(bc) \leqslant q_j$, $bc \nleqslant q_j$ and q_j is 1-absorbing prime, we have $a \leqslant q_j$ and hence $ac \leqslant q_i$, which contradicts $ac \nleqslant q_i$. So let $i \neq j$; say $ac \nleqslant q_1$ and $bc \nleqslant q_2$. As $(bc)a \leqslant q_2$, $bc \nleqslant q_2$ and q_2 is 1-absorbing prime, we have $a \leqslant q_2$ and hence $ab \leqslant q_2$. Also, as $(ac)b \leqslant q_1$, $ac \nleqslant q_1$ and q_1 is 1-absorbing prime, we have $b \leqslant q_1$ and hence $ab \leqslant q_1$. Thus $ab \leqslant (q_1 \wedge q_2)$ and we are done. □

The next result shows that the meet of a family of ascending chain of 1-absorbing prime elements of L is again 1-absorbing prime.

Theorem 2.15. Let $\{q_i \mid i \in Z_+\}$ be an ascending chain of prime elements (or 1-absorbing prime elements) in L. Then $\underset{i \in Z_+}{\wedge} q_i$ is a 1-absorbing prime element of L.

Proof. Clearly, $(\underset{i \in Z_+}{\wedge} q_i) \neq 1$. Let $abc \leqslant (\underset{i \in Z_+}{\wedge} q_i)$ and $ab \nleqslant (\underset{i \in Z_+}{\wedge} q_i)$ where $a, b, c \in L$. Then $ab \nleqslant q_j$ for some $j \in Z_+$ but $abc \leqslant q_j$ implies $c \leqslant q_j$ as $q_j \in L$ is a prime element (or a 1-absorbing prime element). Now let $q_i \neq q_j$. Then as $\{q_i\}$ is a chain, we have either $q_i < q_j$ or $q_j < q_i$. If $q_i < q_j$, then as q_i is a prime element (or a 1-absorbing prime element), $abc \leqslant q_i$ and $ab \nleqslant q_i$, we have $c \leqslant q_i$. If $q_j < q_i$, then $c \leqslant q_j \leqslant q_i$. Thus $c \leqslant (\underset{i \in Z_+}{\wedge} q_i)$ and we are done. □

In the following result, we revisit Proposition 2.5 (2) of [11].

Theorem 2.16. If a proper element $q \in L$ is 1-absorbing prime, then for every $x \in L$ such that $x \nleqslant q$, $(q : x)$ is a prime element of L.

Proof. Let $x \in L$ be such that $x \nleqslant q$. Then $(q : x) \neq 1$. Let $ab \leqslant (q : x)$ and $a \nleqslant (q : x)$ where $a, b \in L$. Since q is 1-absorbing prime, $axb \leqslant q$, $ax \nleqslant q$ gives $b \leqslant q \leqslant (q : x)$. Therefore, $(q : x)$ is a prime element of L. □

Corollary 2.17. *Let q be a 1-absorbing prime element of L. Then the following statements hold true:*
①. If $x, y \in L$ such that $x \nleqslant q$ and $xy \nleqslant q$, then $(q : xy) = (q : x) = q$.
②. If $x, y \in L$ such that $x \nleqslant q$ and $(q : x) < (q : y)$, then $(q : (ax \vee by)) = (q : x)$, $\forall$ $a, b \in L$ such that $ab \nleqslant (q : x)$. In particular, $(q : (x \vee y)) = (q : x)$.

Proof. ①. Let $x, y \in L$ be such that $x \nleqslant q$ and $xy \nleqslant q$. To show $(q : xy) \leqslant (q : x)$, consider $z \in L$ such that $z \leqslant (q : xy)$. Then $yz \leqslant (q : x)$ and $y \nleqslant (q : x)$ which implies $z \leqslant (q : x)$, because as $x \nleqslant q$, by Theorem 2.16, $(q : x)$ is prime. Hence $(q : xy) \leqslant (q : x)$. Further, as $xy \leqslant x$, we have $(q : x) \leqslant (q : xy)$ and hence $(q : xy) = (q : x)$. By Theorem 2.5 ③, as $xy \nleqslant q$, we have $(q : xy) = q$ and we are done.
②. Let $x, y \in L$ be such that $x \nleqslant q$ and $(q : x) < (q : y)$. Let $a, b \in L$ be such that $ab \nleqslant (q : x)$. Clearly, $(q : x) \leqslant (q : (ax \vee by))$. Assume that $(q : (ax \vee by)) \neq (q : x)$. Then there exists $z \leqslant (q : (ax \vee by))$ such that $z \nleqslant (q : x)$ for some $z \in L$. So $zax \leqslant q$ and hence $za \leqslant (q : x)$. As $x \nleqslant q$, by Theorem 2.16, $(q : x)$ is prime. It follows that either $z \leqslant (q : x)$ or $a \leqslant (q : x)$, a contradiction. Therefore, $(q : (ax \vee by)) = (q : x)$. For the particular case, put $a = b = 1$ and we are done. □

We will investigate under which condition(s), the converse of Theorem 2.16 holds true.

Theorem 2.18. Let q be an element of L such that $q \neq \sqrt{q}$ and $\sqrt{q}$ is a prime element of L. If $(q : x)$ is a prime element of L for each $x \leqslant \sqrt{q}$ with $x \nleqslant q$ such that $(q : x) < (q : a)$ $\forall$ $a \in L$ where $x \neq a$ and $a \neq 1$, then q is a 1-absorbing prime element of L.

Proof. Consider $xyz \leqslant q$ where $x, y, z \in L$. Since, $\sqrt{q}$ is prime, at least one of the elements x, y, z is less than or equal to $\sqrt{q}$. Without loss of generality, assume that $x \leqslant \sqrt{q}$. If $x \leqslant q$, then we are done. So suppose $x \nleqslant q$. As $(q : x)$ is a prime element of L and $yz \leqslant (q : x)$, we have either $y \leqslant (q : x)$ or $z \leqslant (q : x)$. It follows that either $y \leqslant (q : z)$ or $z \leqslant (q : y)$, since $(q : x) < (q : a)$ $\forall$ $a \in L$ where $x \neq a$ and $a \neq 1$. Therefore in any case, $yz \leqslant q$ and we are done. □

Theorem 2.19. Let q be a 1-absorbing prime element of L and let p and p' be prime elements of L. If $\sqrt{q} = (p \wedge p')$, then $(q : x)$ is a prime element of L and $\sqrt{(q : x)} = \sqrt{q}$, for all $x \nleqslant (p \vee p')$.

Proof. Let $x \nleqslant (p \vee p')$. Then $x \nleqslant \sqrt{q} = (p \wedge p')$ and hence $x \nleqslant q$. So by Theorem 2.16, $(q : x)$ is a prime element of L. Now to show $\sqrt{(q : x)} \leqslant \sqrt{q}$, let $y \leqslant \sqrt{(q : x)}$ where $y \in L_*$. Then there exists $k \in Z_+$ such that $y^k \leqslant (q : x)$. As $(q : x)$ is prime, it follows that $y \leqslant (q : x)$ and hence $xy \leqslant \sqrt{q}$. Since, by Theorem 2.6 ④, $\sqrt{q}$ is prime and $x \nleqslant \sqrt{q}$, we have $y \leqslant \sqrt{q}$ and thus $\sqrt{(q : x)} \leqslant \sqrt{q}$. Clearly, $\sqrt{q} \leqslant \sqrt{(q : x)} \leqslant \sqrt{q}$ and we are done. □

Lemma 2.20. *Let q be a 1-absorbing prime element of L and p_1, p_2 be two distinct minimal primes over q. If $x_1, x_2 \in L_*$ are such that $x_1 \leqslant p_1$, $x_1 \nleqslant p_2$, $x_2 \leqslant p_2$ and $x_2 \nleqslant p_1$, then $x_1 x_2 \leqslant q$.*

Proof. Let $x_1, x_2 \in L_*$ be such that $x_1 \leqslant p_1$, $x_1 \nleqslant p_2$, $x_2 \leqslant p_2$ and $x_2 \nleqslant p_1$. By Lemma 3.5 in [1], there exist $c_1, c_2 \in L_*$, $c_2 \nleqslant p_1$, and $c_1 \nleqslant p_2$ such that $c_2 x_1^n \leqslant q$ and $c_1 x_2^m \leqslant q$, for some $m, n \in Z_+$. As q is 1-absorbing prime and $(c_2 x_1^{n-1}) x_1 \leqslant q$, it follows that either $c_2 x_1 \leqslant q$ or $x_1 \leqslant q$. If $x_1 \leqslant q$, then $x_1 \leqslant p_2$, a contradiction. So the only possibility is $c_2 x_1 \leqslant q$. Similarly, $c_1 x_2^m \leqslant q$ will lead to $c_1 x_2 \leqslant q$. Clearly, $(c_1 \vee c_2) x_1 x_2 \leqslant q$. Since $(c_1 \vee c_2) \nleqslant p_1$, $(c_1 \vee c_2) \nleqslant p_2$ and p_1, p_2 are primes, we conclude that $(c_1 \vee c_2) x_1 \nleqslant p_2$, $(c_1 \vee c_2) x_2 \nleqslant p_1$. Hence $(c_1 \vee c_2) x_1 \nleqslant q$ and $(c_1 \vee c_2) x_2 \nleqslant q$ because $q \leqslant p_1$ and $q \leqslant p_2$ as p_1, p_2 are minimal primes over q. Moreover, $(c_1 \vee c_2) \nleqslant q$ because if $(c_1 \vee c_2) \leqslant q$, then $c_2 \leqslant (c_1 \vee c_2) \leqslant p_1$, a contradiction to $c_2 \nleqslant p_1$. Considering $((c_1 \vee c_2) x_1) x_2 \leqslant q$, $(c_1 \vee c_2) x_1 \nleqslant q$ and q is 1-absorbing prime,

we get $x_2 \leqslant q$ which contradicts $x_2 \nleqslant p_1$. Considering $((c_1 \vee c_2)x_2)x_1 \leqslant q$, $(c_1 \vee c_2)x_2 \nleqslant q$ and q is 1-absorbing prime, we get $x_1 \leqslant q$ which contradicts $x_1 \nleqslant p_2$. Considering $(c_1 \vee c_2)(x_1x_2) \leqslant q$, $(c_1 \vee c_2) \nleqslant q$ and q is 1-absorbing prime, we get $x_1x_2 \leqslant q$. □

Lemma 2.21. *If q is a 1-absorbing prime element of L, then there are at most two prime elements of L that are minimal over q.*

Proof. Let $A = \{p_i \mid p_i \text{ is a prime element of } L \text{ that is minimal over } q\}$. Assume that A has at least three elements. Let $p_1,\ p_2 \in A$ be two distinct prime elements. Then there exists $x_1,\ x_2 \in L_*$ such that $x_1 \leqslant p_1$, $x_1 \nleqslant p_2$ and $x_2 \leqslant p_2$, $x_2 \nleqslant p_1$. By Lemma 2.20, we have $x_1x_2 \leqslant q$. Now assume that there exists $p_3 \in A$ distinct from p_1 and p_2. Then we can choose $y_i \in L_*$ such that $y_i \leqslant p_i,\ y_i \nleqslant p_j$ for $i \neq j$ where $i, j = 1, 2, 3$. By Lemma 2.20, we have $y_1y_2 \leqslant q \leqslant p_3$, a contradiction. Therefore, A has at most two elements. □

Theorem 2.22. If q is a 1-absorbing prime element of L, then one of the following statements hold true:

①. $\sqrt{q} = p$ is a prime element of L such that $p^2 \leqslant q$.

②. $\sqrt{q} = p_1 \wedge p_2$ and $p_1p_2 \leqslant q$, where $p_1,\ p_2$ are the only distinct prime elements of L that are minimal over q.

Proof. By Lemma 2.21, we have either $\sqrt{q} = p$ is a prime element of L or $\sqrt{q} = p_1 \wedge p_2$ where $p_1,\ p_2$ are the only distinct prime elements of L that are minimal over q. If $\sqrt{q} = p$ is a prime element of L, then by Theorem 2.6, $p^2 \leqslant q$, so the condition ① holds.

Now assume that $\sqrt{q} = p_1 \wedge p_2$ where $p_1,\ p_2$ are the only distinct prime elements of L that are minimal over q. We show that $p_1p_2 \leqslant q$. Observe that $p_1p_2 = \vee\{xy \mid x, y \in L_*, x \leqslant p_1, y \leqslant p_2\}$. Following 3 cases arise:

Case 1. If $x,\ y \leqslant p_1 \wedge p_2 = \sqrt{q}$, then by Theorem 2.6-②, $xy \leqslant q$.

Case 2. If $x,\ y \in L_*$ are such that $x \leqslant p_1$, $x \nleqslant p_2$, and $y \leqslant p_2$, $y \nleqslant p_1$, then by Lemma 2.20, $xy \leqslant q$.

Case 3. Let $x,\ y \in L_*$ be such that $x \leqslant \sqrt{q}$, and $y \leqslant p_2$, $y \nleqslant p_1$. Choose $y_1 \in L_*$ such that $y_1 \leqslant p_1$ and $y_1 \nleqslant p_2$. Then by Lemma 2.20, $y_1y \leqslant q$. Clearly, $(x \vee y_1) \leqslant p_1$ but $(x \vee y_1) \nleqslant p_2$. So by Lemma 2.20, $(x \vee y_1)y \leqslant q$ and hence $xy \leqslant q$. Similarly, one can show that if $y \leqslant \sqrt{q}$, $x \leqslant p_1$ and $x \nleqslant p_2$, then $xy \leqslant q$.

Thus, in any case, $xy \leqslant q$. Consequently, we get $p_1p_2 \leqslant q$, so the condition ② holds. □

Lemma 2.23. *Let q be a 1-absorbing prime element of L such that $\sqrt{q} = p$ is a prime element of L and assume that $q \neq p$. Then the following statements hold true:*

①. *For each $x \leqslant p$ and $x \nleqslant q$, $(q : x)$ is a prime element of L such that $p \leqslant (q : x)$.*

②. *Either $(q : x) \leqslant (q : y)$ or $(q : y) \leqslant (q : x)$, for all $x,\ y \leqslant p$ and $x,\ y \nleqslant q$.*

Proof. ①. Let $x \leqslant p$ and $x \nleqslant q$. By Theorem 2.16, $(q : x)$ is prime. Now, by Theorem 2.6 ③, $p^2 \leqslant q$. Clearly, $xp \leqslant p^2$ and hence $p \leqslant (q : x)$.

②. Let $x,\ y \leqslant p$ and $x,\ y \nleqslant q$. Choose any $z \in L_*$ such that $z \leqslant (q : x)$ and $z \nleqslant (q : y)$. As $x \leqslant p$ and $x \nleqslant q$, by ①, we have $p \leqslant (q : x)$. As $y \leqslant p$ and $y \nleqslant q$, by ①, we have $p \leqslant (q : y)$. So $z \nleqslant p$ and hence $z \nleqslant q$. To show $(q : y) \leqslant (q : x)$, consider $w \in L_*$ such that $w \leqslant (q : y)$. If $w \leqslant p$, then as $p \leqslant (q : x)$, we have $w \leqslant (q : x)$ and we are done. So assume that $w \nleqslant p$. Clearly, $(x \vee y)zw \leqslant q$ and $zw \nleqslant q$. Moreover, if $(x \vee y)z \leqslant q$, then $yz \leqslant q$ which contradicts $z \nleqslant (q : y)$. Considering $(x \vee y)(zw) \leqslant q$, $zw \nleqslant q$ and q is 1-absorbing prime, we get $(x \vee y) \leqslant q$ which is a contradiction, since $x, y \nleqslant q$. Considering $((x \vee y)z)w \leqslant q$, $(x \vee y)z \nleqslant q$ and q is 1-absorbing prime, we get $w \leqslant q$ which implies $w \leqslant (q : x)$, since $q \leqslant (q : x)$ and we are done. Considering $((x \vee y)w)z \leqslant q$, $z \nleqslant q$ and q is 1-absorbing prime, we get $((x \vee y)w) \leqslant q$ which implies $xw \leqslant q$ and hence $w \leqslant (q : x)$. □

Lemma 2.24. *Let q be a 1-absorbing prime element of L such that $q \neq \sqrt{q} = p_1 \wedge p_2$ where p_1 and p_2 are the only non-zero distinct prime elements of L that are minimal over q. Then the following statements hold true:*

①. For each $x \leqslant \sqrt{q}$ and $x \nleqslant q$, $(q : x)$ is a prime element of L such that $p_1 \leqslant (q : x)$ and $p_2 \leqslant (q : x)$.

②. Either $(q : x) \leqslant (q : y)$ or $(q : y) \leqslant (q : x)$, for all $x,\ y \leqslant \sqrt{q}$ and $x,\ y \nleqslant q$.

Proof. ①. Let $x \leqslant \sqrt{q}$ and $x \nleqslant q$. By Theorem 2.16, $(q : x)$ is prime. Further, by Theorem 2.22, $p_1p_2 \leqslant q$. So we have $xp_1 \leqslant q$ and $xp_2 \leqslant q$. Thus $p_1 \leqslant (q : x)$ and $p_2 \leqslant (q : x)$.

②. The proof is similar to the proof of Lemma 2.23 ② and hence omitted. □

Lemma 2.25. *Let q be a 1-absorbing prime element of L such that $q \neq \sqrt{q} = p_1 \wedge p_2$ where p_1 and p_2 are the only non-zero distinct prime elements of L that are minimal over q. Then the following statements hold true:*

①. For each $x \leqslant \sqrt{q}$ and $x \nleqslant q$, $(q : x)$ is a prime element of L such that $p_1p_2 \leqslant q$.

②. If $(q : x)$ is proper and either $x \leqslant p_1$ or $x \leqslant p_2$, then $(q : x)$ is a prime element of L.

Proof. ①. Let $x \leqslant \sqrt{q}$ and $x \nleqslant q$. By Theorem 2.16, $(q : x)$ is prime. Further, by Theorem 2.22, $p_1p_2 \leqslant q$.

②. Let $x \leqslant p_1$ and $x \nleqslant p_2$. Since by Theorem 2.22, $p_1p_2 \leqslant q$, we have $xp_2 \leqslant q$. Hence $(q : x) = p_2$ is a prime element of L. Similarly, if $x \leqslant p_2$ and $x \nleqslant p_1$, then $(q : x) = p_1$ is a prime element of L. If $x \leqslant p_1$ and $x \leqslant p_2$ and $x \nleqslant q$, then by Theorem 2.16, $(q : x)$ is a prime element of L. □

Theorem 2.26. Let $q \in L$ such that $q \neq \sqrt{q} = p_1 \wedge p_2$ where p_1 and p_2 are the only non-zero distinct prime elements of L that are minimal over q. Let $(q : x)$ be proper element of L such that either $x \leqslant p_1$ or $x \leqslant p_2$. If $(q : x) < (q : a)\ \forall a \in L$ where $x \neq a$ and $a \neq 1$, then q is a 1-absorbing prime element of L.

Proof. Consider $xyz \leqslant q$ where $x, y, z \in L$. Without loss of generality, assume that $x \leqslant p_1$. If $x \leqslant q$, then we are done. So suppose $x \nleqslant q$. By Lemma 2.25, $(q : x)$ is a prime element of L. So as $yz \leqslant (q : x)$, we have either $y \leqslant (q : x)$ or $z \leqslant (q : x)$. It follows that either $y \leqslant (q : z)$ or $z \leqslant (q : y)$, since $(q : x) < (q : a)\ \forall\ a \in L$ where $x \neq a$ and $a \neq 1$. Therefore in any case, $yz \leqslant q$ and we are done. □

3 Future scope

In an attempt to push ahead and give future scope to the concept of "1-absorbing prime element" in L, we define the following notions and conclude the paper.

Definition 3.1. A proper element $q \in L$ is called a weakly 1-absorbing prime element if for all proper elements $a, b, c \in L$ such that the product of a, b, c need not collapse to one of the element among them, $0 \neq abc \leqslant q$ imply either $ab \leqslant q$ or $c \leqslant q$.

Clearly, every 1-absorbing prime element of L is weakly 1-absorbing prime.

Definition 3.2. Let $n \geqslant 1$ in Z_+. A proper element $q \in L$ is said to be n-1-absorbing prime if for all proper elements $x_1, \cdots, x_{n+1} \in L$ such that the product of $x_1, \cdots, x_{n+1}$ need not collapse to one of the element among them, $x_1 \cdots x_{n+1} \leqslant q$ imply either $x_1 \cdots x_n \leqslant q$ or $x_{n+1} \leqslant q$.

Clearly, "2-1-absorbing prime element" of L is "1-absorbing prime element" of L.

Obviously, "1-1-absorbing prime element" of L is "prime element" of L.

Conflict of Interests. The author reports that there are no competing interests to declare.

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Positive-additive Nearly functional equation in C^*-algebras by two approaches

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Abstract. In this paper, we introduce the following positive-additive functional equation in C^*-algebras

$$f\left(x+3x^{\frac{2}{3}}\sqrt[3]{y}+3y^{\frac{2}{3}}\sqrt[3]{x}+y\right)=f(x)+3f(x)^{\frac{2}{3}}\sqrt[3]{f(y)}+3f(y)^{\frac{2}{3}}\sqrt[3]{f(x)}+f(y).$$

Using fixed point methods, we prove the stability of the positive-additive functional equation in C^*-algebras. Moreover, we prove the Hyers-Ulam stability of the positive-additive functional equation in C^*-algebras by the direct method of Hyers and Ulam.

Key Words: Hyers-Ulam stability, C^*-algebra, fixed point, positive-additive functional equation.
2020 MSC: 46L05, 47H10, 39B52.

1 Introduction and preliminaries

The stability problem of functional equations was originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [16] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [29] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. (Th.M. Rassias) let f be an approximately additive mapping from a normed vector space E into a Banach space E', i.e., f satisfies the inequality

$$\frac{|f(x+y)-f(x)-f(y)|}{\|x\|^r+\|y\|^r}\leq\epsilon$$

for all $x,y\in E$, where ϵ and r are constants with $\epsilon>0$ and $0\leq r<1$. Then the mapping $L:E\to E'$ defined by $L(x):=\lim_{n\to\infty}2^{-n}f(2^nx)$ is the unique additive mapping which satisfies

$$\frac{|f(x)-L(x)|}{|x|^r}\leq\frac{2\epsilon}{2-2^r}$$

for all $x\in E$.

The paper of Th.M. Rassias [29] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th.M. Rassias' approach. J.M.

Rassias [26]–[28] followed the innovative approach of the Th.M. Rassias' theorem [29] in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\cdot\|y\|^q$ for $p,q\in\mathbb{R}$ with $p+q\neq 1$. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6, 9, 10, 17, 19]).

Definition 1.2. [8] Let A be a C^*-algebra and $x\in A$ a self-adjoint element, i.e., $x^*=x$. Then x is said to be *positive* if it is of the form yy^* for some $y\in A$.

The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone (see [8]).

It is well-known that for a positive element x and a positive integer n there exists a unique positive element $y\in A^+$ such that $x=y^n$. We denote y by $x^{\frac{1}{n}}$ (see [14]).

In this paper, we introduce the following functional equation

$$f\left(x+3x^{\frac{2}{3}}\sqrt{y}+3y^{\frac{2}{3}}\sqrt{x}+y\right)=f(x)+3f(x)^{\frac{2}{3}}\sqrt{f(y)}+3f(y)^{\frac{2}{3}}\sqrt{f(x)}+f(y) \tag{1}$$

in the set of for all $x,y\in A^+$, which is called a *positive-additive functional equation*. Each solution of the positive-additive functional equation is called a *positive-additive mapping*.

Note that the function $f(x)=cx,\quad c\geq 0$, in the set of non-negative real numbers is a solution of the functional equation (1.1).

Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;

(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(3) $d(x,z)\leq d(x,y)+d(y,z)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.3. [7, 30, 32] Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either

$$d(J^nx,J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^nx,J^{n+1}x)<\infty,\qquad \forall n\geq n_0$;

(2) the sequence $\{J^nx\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y=\{y\in X\mid d(J^{n_0}x,y)<\infty\}$;

(4) $d(y,y^*)\leq\frac{1}{1-L}d(y,Jy)$ for all $y\in Y$.

In 1991, Baker [2] used the Banach fixed point theorem to give generalized Hyers-Ulam stability results for a nonlinear functional equation. In 2003, Radu [25] applied the fixed point alternative theorem to prove the generalized Hyers-Ulam stability. Miheţ [20] applied the Luxemburg-Jung fixed point theorem in generalized metric spaces to study the generalized Hyers-Ulam stability for two functional equations in a single variable and L. Găvruta [11] used the Matkowski's fixed point theorem to obtain a new general result concerning the generalized Hyers-Ulam stability of a functional equation in a single variable. For a new method, see [13]. In 1996, G. Isac and Th.M. Rassias [18] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 15], [22]–[24], [31]).

This paper is organized as follows: In Section 2, using the fixed point method, we prove the Hyers-Ulam stability of the functional equation (1) in C^*-algebras. In Section 3, using the direct method, we prove the Hyers-Ulam stability of the functional equation (1) in C^*-algebras.

Throughout this paper, let A^+ and B^+ be the sets of positive elements in C^*-algebras A and B, respectively. Assume that m is a fixed integer greater than 1.

2 Stability of the positive-additive functional equation (1): fixed point approach

In this section, we investigate the positive-additive functional equation (1) in C^*-algebras.

Lemma 2.1. *Let $T : A^+ \to B^+$ be a positive-additive mapping satisfying* (1). *Then T satisfies $T(8^n x) = 8^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$.*

Proof. Putting $x = y$ in (1), we obtain $T(8x) = 8T(x)$ for all $x \in A^+$. So one can show that $T(8^n x) = 8^n T(x)$ for all $x \in A^+$ and all $n \in \mathbb{Z}$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the positive-additive functional equation (1) in C^*-algebras. Note that the fundamental ideas in the proofs of the main results in this section are contained in [3, 4, 5].

Theorem 2.2. Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi\left(\frac{x}{8}, \frac{y}{8}\right) \leq \frac{L}{8}\varphi(x, y) \tag{2}$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying

$$\left\| f\left(x + 3x^{\frac{2}{3}}\sqrt{y} + 3y^{\frac{2}{3}}\sqrt{x} + y\right) - f(x) - 3f(x)^{\frac{2}{3}}\sqrt{f(y)} - 3f(y)^{\frac{2}{3}}\sqrt{f(x)} - f(y)\right\| \leq \varphi(x, y) \tag{3}$$

for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $A : A^+ \to A^+$ satisfying (1) and

$$\|f(x) - A(x)\| \leq \frac{L\varphi(x, x)}{8(1 - L)} \tag{4}$$

for all $x \in A^+$.

Proof. Letting $y = x$ in (3), we get

$$\|f(8x) - 8f(x)\| \leq \varphi(x, x) \tag{5}$$

for all $x \in A^+$. Consider the set

$$X := \{g : A^+ \to B^+\}$$

and introduce the generalized metric on X:

$$d(g, h) = \inf\{\mu \in (0, +\infty) : \|g(x) - h(x)\| \leq \mu\varphi(x, x), \ \ \forall x \in A^+\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (X, d) is complete (see [21]). Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 8g\left(\frac{x}{8}\right)$$

for all $x \in A^+$. Let $g, h \in X$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon\varphi(x, x)$$

for all $x \in A^+$. Hence

$$\|Jg(x) - Jh(x)\| = \|8g\left(\frac{x}{8}\right) - 8h\left(\frac{x}{8}\right)\| \leq L\varepsilon\varphi(x, x)$$

for all $x \in A^+$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leq L\varepsilon$. This means that

$$d(Jg,Jh) \leq L\varepsilon d(g,h)$$

for all $g,h \in X$.

It follows from (5) that

$$\left\| f(x) - 8f\left(\frac{x}{8}\right)\right\| \leq \frac{L}{8}\varphi(x,x)$$

for all $x \in A^+$. So $d(f,Jf) \leq \frac{L}{8}$.

By Theorem 1.2, there exists a mapping $A : A^+ \to B^+$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{8}\right) = \frac{1}{8}A(x) \tag{6}$$

for all $x \in A^+$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in X : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (6) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - A(x)\| \leq \mu\varphi(x,x)$$

for all $x \in A^+$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n f\left(\frac{x}{8^n}\right) = A(x)$$

for all $x \in A^+$;

(3) $d(f,A) \leq \frac{1}{1-L}d(f,Jf)$, which implies the inequality

$$d(f,A) \leq \frac{L}{8(1-L)}.$$

This implies that the inequality (4) holds. By (2) and (3),

$$\begin{aligned}
&\left\|A\left(x + 3x^{\frac{2}{3}}\sqrt{y} + 3y^{\frac{2}{3}}\sqrt{x} + y\right) - A(x) - 3A(x)^{\frac{2}{3}}\sqrt{A(y)} - 3A(y)^{\frac{2}{3}}\sqrt{A(x)} + A(y)\right\| \\
&= \lim_{n\to+\infty}\left\|8^n\left[f\left(\frac{x}{8^n} + 3\left(\frac{x}{8^n}\right)^{\frac{2}{3}}\sqrt{\frac{y}{8^n}} + 3\left(\frac{y}{8^n}\right)^{\frac{2}{3}}\sqrt{\frac{x}{8^n}} + \frac{y}{8^n}\right) - f\left(\frac{x}{8^n}\right)\right.\right. \\
&\left.\left. -3f\left(\frac{x}{8^n}\right)^{\frac{2}{3}}\sqrt{f\left(\frac{y}{8^n}\right)} - 3f\left(\frac{y}{8^n}\right)^{\frac{2}{3}}\sqrt{f\left(\frac{x}{8^n}\right)} - f\left(\frac{y}{8^n}\right)\right]\right\| \\
&\leq \lim_{n\to+\infty} 8^n\varphi\left(\frac{x}{8^n},\frac{y}{8^n}\right) \\
&\leq \lim_{n\to+\infty} 8^n \times \frac{L^n}{8^n}\varphi(x,y)
\end{aligned}$$

for all $x,y \in A^+$. So

$$\left\|A\left(x + 3x^{\frac{2}{3}}\sqrt{y} + 3y^{\frac{2}{3}}\sqrt{x} + y\right) - A(x) - 3A(x)^{\frac{2}{3}}\sqrt{A(y)} - 3A(y)^{\frac{2}{3}}\sqrt{A(x)} - A(y)\right\| = 0$$

for all $x,y \in A^+$. Thus the mapping $A : A^+ \to B^+$ is positive-additive, as desired. □

Corollary 2.3. *Let $p > 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping such that*

$$\begin{aligned} &\left\| f\left(x + 3x^{\frac{2}{3}}\sqrt{y} + 3y^{\frac{2}{3}}\sqrt{x} + y\right) - f(x) - 3f(x)^{\frac{2}{3}}\sqrt{f(y)} - 3f(y)^{\frac{2}{3}}\sqrt{f(x)} - f(y)\right\| \\ &\leq \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}} \end{aligned} \tag{7}$$

for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying (1) *and*

$$\|f(x) - A(x)\| \leq \frac{(2\theta_1 + \theta_2)\|x\|^p}{8^p - 8}$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 8^{1-p}$ and we get the desired result. □

Theorem 2.4. Let $\varphi : A^+ \times A^+ \to [0, \infty)$ be a function such that there exists an $\alpha < 1$ with

$$\varphi(x, y) \leq 8L\varphi\left(\frac{x}{8}, \frac{y}{8}\right)$$

for all $x, y \in A^+$. Let $f : A^+ \to B^+$ be a mapping satisfying (3). Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying (1) and

$$\|f(x) - A(x)\| \leq \frac{\varphi(x, x)}{8 - 8L}$$

for all $x \in A^+$.

Proof. Let (X, d) be the generalized metric space defined in the proof of Theorem 2.2. Consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{8}g(8x)$$

for all $x \in A^+$. It follows from (5) that

$$\left\| f(x) - \frac{f(8x)}{8} \right\| \leq \frac{1}{8}\varphi(x, x)$$

for all $x \in A^+$. So $d(f, Jf) \leq \frac{1}{8}$. The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $0 < p < 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (7). *Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying* (1) *and*

$$\|f(x) - A(x)\| \leq \frac{2\theta_1 + \theta_2}{8 - 8^p}\|x\|^p$$

for all $x \in A^+$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$ for all $x, y \in A^+$. Then we can choose $L = 8^{p-1}$ and we get the desired result. □

3 Stability of the positive-additive functional equation (1): direct method

In this section, using the direct method of Hyers and Ulam, we prove the Hyers-Ulam stability of the positive-additive functional equation (1) in C^*-algebras.

Theorem 3.1. Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0,\infty)$ satisfying (3) and

$$\widetilde{\varphi}(x,y) := \sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{8^j}, \frac{y}{8^j}\right) \quad < \quad \infty \tag{8}$$

for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $A : A^+ \to A^+$ satisfying (1) and

$$\|f(x) - A(x)\| \leq \frac{1}{8}\widetilde{\varphi}(x,y) \tag{9}$$

for all $x \in A^+$.

Proof. It follows from (5) that

$$\left\| f(x) - 8f\left(\frac{x}{8}\right)\right\| \leq \varphi\left(\frac{x}{8}, \frac{x}{8}\right)$$

for all $x \in A^+$. Hence

$$\left\| 8^l f\left(\frac{x}{8^l}\right) - 8^k f\left(\frac{x}{8^k}\right)\right\| \quad \leq \quad \frac{1}{8}\sum_{j=l+1}^{k} 8^j \varphi\left(\frac{x}{8^j}, \frac{x}{8^j}\right) \tag{10}$$

for all nonnegative integers k and l with $k > l$ and all $x \in A^+$. It follows from (8) and (10) that the sequence $\left\{8^j f\left(\frac{x}{8^j}\right)\right\}$ is Cauchy for all $x \in A^+$. Since B^+ is complete, the sequence $\left\{8^j f\left(\frac{x}{8^j}\right)\right\}$ converges. So one can define the mapping $A : A^+ \to B^+$ by

$$A(x) := \lim_{j\to\infty} 8^j f\left(\frac{x}{8^j}\right)$$

for all $x \in A^+$. By (3) and (8),

$$\begin{aligned}
&\left\| A\left(\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^3\right) - \left(A(x)^{\frac{1}{3}} + A(y)^{\frac{1}{3}}\right)^3\right\| \\
&= \lim_{j\to\infty} 8^j \left\| f\left(\frac{\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^3}{8^j}\right) - \left(\left(8^j f\left(\frac{x}{8^j}\right)\right)^{\frac{1}{3}} + \left(8^j f\left(\frac{y}{8^j}\right)\right)^{\frac{1}{3}}\right)^3\right\| \\
&\leq \lim_{j\to\infty} 8^j \varphi\left(\frac{x}{8^j}, \frac{y}{8^j}\right) = 0
\end{aligned}$$

for all $x, y \in A^+$. So

$$A\left(\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^3\right) - \left(A(x)^{\frac{1}{3}} + A(y)^{\frac{1}{3}}\right)^3 = 0$$

for all $x, y \in A^+$. Hence the mapping $A : A^+ \to B^+$ is positive-additive. Moreover, letting $l = 0$ and passing the limit $k \to \infty$ in (10), we get (9). So there exists a positive-additive mapping $A : A^+ \to B^+$ satisfying (1) and (9).

Now, let $A' : A^+ \to B^+$ be another positive-additive mapping satisfying (1) and (9). Then we have

$$\begin{aligned} \|A(x) - A'(x)\| &= 8^q \left\| A\left(\frac{x}{8^q}\right) - A'\left(\frac{x}{8^q}\right) \right\| \\ &\leq 8^q \left\| A\left(\frac{x}{8^q}\right) - f\left(\frac{x}{8^q}\right) \right\| + 8^q \left\| A'\left(\frac{x}{8^q}\right) - f\left(\frac{x}{8^q}\right) \right\| \\ &\leq 2 \cdot 8^{q-1} \widetilde{\varphi}\left(\frac{x}{8^q}, \frac{x}{8^q}\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in A^+$. So we can conclude that $A(x) = A'(x)$ for all $x \in A^+$. This proves the uniqueness of T. □

Corollary 3.2. *Let $p > 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (7)*. Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying* (1) *and*

$$\|f(x) - A(x)\| \leq \frac{2\theta_1 + \theta_2}{8^p - 8} \|x\|^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.1. Then we get the desired result. □

Theorem 3.3. Let $f : A^+ \to B^+$ be a mapping for which there exists a function $\varphi : A^+ \times A^+ \to [0, \infty)$ satisfying (3) such that

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{\varphi(8^j x, 8^j y)}{8^j} < \infty$$

for all $x, y \in A^+$. Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying (1) and

$$\|f(x) - A(x)\| \leq \frac{1}{8} \widetilde{\varphi}(x, x)$$

for all $x \in A^+$.

Proof. It follows from (5) that

$$\left\| f(x) - \frac{f(8x)}{8} \right\| \leq \frac{1}{8} \varphi(x, x)$$

for all $x \in A^+$. The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $0 < p < 1$ and θ_1, θ_2 be non-negative real numbers, and let $f : A^+ \to B^+$ be a mapping satisfying* (7)*. Then there exists a unique positive-additive mapping $A : A^+ \to B^+$ satisfying* (1) *and*

$$\|f(x) - A(x)\| \leq \frac{2\theta_1 + \theta_2}{8 - 8^p} \|x\|^p$$

for all $x \in A^+$.

Proof. Define $\varphi(x, y) = \theta_1(\|x\|^p + \|y\|^p) + \theta_2 \cdot \|x\|^{\frac{p}{2}} \cdot \|y\|^{\frac{p}{2}}$, and apply Theorem 3.3. Then we get the desired result. □

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A new class of graded prime ideals

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Abstract. In this paper, we introduce and investigate a new class of graded ideals in graded commutative rings, called graded weakly 2-absorbing quasi-primary ideals. We explore basic properties of these ideals and examine their relationships with existing classes of graded ideals. We also study their behavior under graded ring homomorphisms, quotient rings, localization, and idealization of graded modules. Several structural results and characterizations are presented, highlighting the flexibility and generality of this new class in the broader context of graded ideal theory.

Key Words: graded prime ideal, graded weakly prime ideal, graded 2-absorbing ideal, graded weakly 2-absorbing quasi primary ideal.

2010 MSC: 13A15, 16W50, 13A02.

1 Introduction

Graded ring structures have attracted considerable attention in recent decades due to their deep connections with algebraic geometry and module theory. Graded versions of classical ideal-theoretic concepts often provide deeper insights into ring-theoretic properties and allow for more refined classifications. In particular, generalizations of prime and primary ideals to graded settings have led to a rich framework in which ideal-theoretic behavior can be better understood in terms of homogeneous components. A substantial portion of graded ideal theory focuses on extending the concepts of primality and primary decomposition to the graded setting. In this regard, graded prime ideals and graded primary ideals have been studied extensively [4], [8], [9], [10], [12], [13]. More recently, a number of generalizations of prime-like ideals have continued to expand this landscape [1], [2], and [11]. These developments reflect a growing interest in the graded analogues of multiplicative structures and the pursuit of finer classifications that preserve essential algebraic properties.

Motivated by these trends, this paper introduces and investigates a new class of graded ideals, termed graded weakly 2-absorbing quasi-primary ideals. This class unifies and extends both graded weakly 2-absorbing ideals and graded 2-absorbing quasi-primary ideals. We show that this generalization is nontrivial and exhibits behavior consistent with other known classes, while also offering novel structural characteristics.

Throughout this paper, all rings given are assumed to be commutative with identity, and all modules are unital. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring, where G is a group and each R_g is an additive subgroup of R, satisfying $R_g R_h \subseteq R_{gh}$ for all $g,h \in G$. An R-module M is said to be a G-graded R-module if $M = \bigoplus_{g\in G} M_g$, where each M_g is an additive subgroup of M and $R_g M_h \subseteq M_{gh}$ for $g,h \in G$. In this context, the elements of R_g are called homogeneous elements of degree g, and the set of all homogeneous elements is denoted by $h(R) = \bigcup_{g\in G} R_g$. For simplicity, we will refer to these structures as graded rings and graded modules.

Graded prime ideals have been studied thoroughly in the literature due to their significant role in the extension of classical ideal theory (see [1-4, 6–11, 13-14]). The notion of 2-absorbing ideals, introduced by Badawi in [5], was later extended to the graded context by Naghani and Moghimi [7]. A graded ideal I of a graded ring R is said to be a graded 2-absorbing ideal if for any homogeneous elements $r,s,t \in h(R)$, the condition $rst \in I$ implies that at least one of $rs \in I$, $rt \in I$, or $st \in I$ holds. Following this, graded 2-absorbing primary ideals were defined in [3] as ideals $I \subseteq R$ such that $rst \in I$ implies that $rs \in I$, or $rt \in Gr(I)$, or $st \in Gr(I)$, where $Gr(I)$ denotes the graded radical of I, defined by $Gr(I) = \left\{r \in R : r_g^{n_g} \in I \text{ for some } n_g \geq 0, \text{ where } g \in G\right\}$. In [13], this class was further generalized through the definition of graded 2-absorbing quasi-primary ideals. A graded ideal I of R is called a graded 2-absorbing quasi-primary ideal if for all homogeneous elements $r,s,t \in h(R)$, the condition $rst \in I$ implies that $rs \in Gr(I)$, or $rt \in Gr(I)$, or $st \in Gr(I)$.

The main objective of this paper is to introduce a new class of graded ideals, which we call graded weakly 2-absorbing quasi-primary ideals, and to study their fundamental properties. We examine the relationship between these ideals and other well-known classes of graded ideals, such as graded weakly 2-absorbing ideals and graded 2-absorbing quasi-primary ideals. One of our key results shows that if R is a reduced graded ring and I is a graded weakly 2-absorbing quasi-primary ideal of R, then the graded radical $Gr(I)$ is a graded weakly 2-absorbing ideal. In addition, we provide characterizations of graded weakly 2-absorbing quasi-primary ideals under graded ring homomorphisms and quotient constructions. Theorem 2.7 addresses their behavior under localization. In Theorem 2.9, we study these ideals in the setting of cross products, and in Theorem 2.12, we examine them within the idealization of graded modules.

2 Characterizations of Graded Weakly 2-absorbing Quasi Primary Ideals

Definition 2.1. *A graded ideal I of R is called a graded weakly 2-absorbing quasi primary ideal if $0 \neq abc \in I$ whenever $ab \in Gr(I)$ or $ac \in Gr(I)$ or $bc \in Gr(I)$ for each $a,b,c \in h(R)$.*

Theorem 2.2. For any ideal of reduced ring R, if I is a graded weakly 2-absorbing quasi primary ideal of R, then $Gr(I)$ is a graded weakly 2-absorbing ideal of R.

Proof. Suppose that I is a graded weakly 2-absorbing quasi primary ideal of R. Let $a,b,c \in h(R)$ such that $0 \neq abc \in Gr(I)$. Then there exist $n \in \mathbb{N}$ such that $0 \neq (abc)^n = a^n b^n c^n \in I$. As I is a graded weakly 2-absorbing quasi primary ideal of R then $a^n b^n \in Gr(I)$ or $a^n c^n \in Gr(I)$ or $b^n c^n \in Gr(I)$ and so $ab \in Gr(I)$ or $ac \in Gr(I)$ or $bc \in Gr(I)$. Hence the proof is completed. □

Remark 2.3. *For given rings R_1 and R_2, a graded ring homomorphism $\varphi : R_1 \to R_2$ is a ring homomorphism with property that $\varphi(R_{1,g}) \subseteq R_{2,g}$ for every $g \in G$.*

Remark 2.4. *(i) If Q_2 is a graded ideal of R_2 and $\varphi : R_1 \to R_2$ is a G-graded ring homomorphism, then $Gr(\varphi^{-1}(Q_2)) = \varphi^{-1}(Gr(Q_2))$.*

(ii) If Q_1 is a graded ideal of R_1 such that $Ker\varphi \subseteq Q_1$, then $\varphi(Gr(Q_1)) = Gr(\varphi(Q_1))$.

Theorem 2.5. Suppose that R_1 and R_2 are two graded rings and $\varphi : R_1 \to R_2$ is a graded ring homomorphism. Then the followings hold:

(i) If φ is a graded monomorphism and Q_2 is a graded weakly 2-absorbing quasi primary ideal of R_2, then $\varphi^{-1}(Q_2)$ is a graded weakly 2-absorbing quasi primary ideal of R_1.

(ii) If φ is a graded epimorphism and Q_1 is a graded weakly 2-absorbing quasi primary ideal of R_1 containing $Ker\varphi$, then $\varphi(Q_1)$ is a graded weakly 2-absorbing quasi primary ideal of R_2.

Proof. (i) Let $r,s,t \in h(R_1)$ such that $0 \neq rst \in \varphi^{-1}(Q_2)$. Then we have $0 \neq \varphi(rst) = \varphi(r)\varphi(s)\varphi(t) \in Q_2$. Since Q_2 is a graded weakly 2-absorbing quasi primary ideal of R_2, we get $\varphi(r)\varphi(s) \in Gr(Q_2)$ or $\varphi(r)\varphi(t) \in Gr(Q_2)$ or $\varphi(s)\varphi(t) \in Gr(Q_2)$ and so $rs \in \varphi^{-1}(Gr(Q_2)) = Gr(\varphi^{-1}(Q_2))$ or $rt \in \varphi^{-1}(Gr(Q_2)) = Gr(\varphi^{-1}(Q_2))$ or $st \in \varphi^{-1}(Gr(Q_2)) = Gr(\varphi^{-1}(Q_2))$ and this completes the proof.

(ii) Let $r',s',t' \in h(R_2)$ such that $0 \neq r's't' \in \varphi(Q_1)$. Since φ is epimorphism then there exist $r,s,t \in h(R_1)$ such that $\varphi(r) = r'$, $\varphi(s) = s'$, $\varphi(t) = t'$. Then $0 \neq \varphi(rst) = r's't' \in \varphi(Q_1)$. Since $Q_1 \supseteq Ker\varphi$, we conclude that $0 \neq rst \in Q_1$. This implies $rs \in Gr(Q_1)$ or $rt \in Gr(Q_1)$ or $st \in Gr(Q_1)$ and hence $\varphi(rs) = r's' \in \varphi(Gr(Q_1)) = Gr(\varphi(Q_1))$ or $\varphi(rt) = r't' \in \varphi(Gr(Q_1)) = Gr(\varphi(Q_1))$ or $\varphi(st) = s't' \in \varphi(Gr(Q_1)) = Gr(\varphi(Q_1))$. Therefore $\varphi(Q_1)$ is a graded weakly 2-absorbing quasi primary ideal of R_2. □

Corollary 2.6. *Let $J \subseteq I$ be ideals of R with $I \neq R$. Then the followings hold:*

(i) If J is a graded ideal of R and I is a graded weakly 2-absorbing quasi primary ideal of R, then I/J is a graded weakly 2-absorbing quasi primary ideal of R/J.

(ii) If J and I/J are graded weakly 2-absorbing quasi primary ideals, then I is a graded weakly 2-absorbing quasi primary ideal of R.

Proof. (i) It follows from Theorem 2.5 (i).

(ii) Let $0 \neq abc \in I$ for some $a,b,c \in h(R)$. Thus $(a+J)(b+J)(c+J) \in I/J$. If $abc \in J$, then either $ab \in Gr(J) \subseteq Gr(I)$ or $ac \in Gr(J) \subseteq Gr(I)$ or $bc \in Gr(J) \subseteq Gr(I)$. So let $abc \notin J$. Then $0 \neq (a+J)(b+J)(c+J) \in I/J$. Since I/J is a graded weakly 2-absorbing quasi primary ideal, it follows either $(a+J)(b+J) \in Gr(I/J)$ or $(a+J)(c+J) \in Gr(I/J)$ or $(b+J)(c+J) \in Gr(I/J)$. Therefore $ab \in Gr(I)$ or $ac \in Gr(I)$ or $bc \in Gr(I)$. This completes the proof. □

Theorem 2.7. Let R be a graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. For a graded ideal I of R, the followings hold:

(i) If I is a graded weakly 2-absorbing quasi primary ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a graded weakly 2-absorbing quasi primary ideal of $S^{-1}R$.

(ii) If S is the subset of regular elements and $S^{-1}I$ is a graded weakly 2-absorbing quasi primary ideal of $S^{-1}R$ and $S \cap G\text{-}Z_I(R) = \emptyset$, then I is a graded weakly 2-absorbing quasi primary ideal of R.

Proof. (i) Let $0 \neq \frac{a_1}{s_1}\frac{a_2}{s_2}\frac{a_3}{s_3} \in S^{-1}I$ for some $\frac{a_i}{s_i} \in h(S^{-1}R)$, where $1 \leq i \leq 3$. Then we have $0 \neq (ta_1)a_2a_3 \in I$ for some $t \in S$. As Q is a graded weakly 2-absorbing quasi primary ideal of R, $ta_1a_2 \in Gr(I)$ or $ta_1a_3 \in Gr(I)$ or $a_2a_3 \in Gr(I)$. This means $\frac{a_1}{s_1}\frac{a_2}{s_2} = \frac{ta_1a_2}{ts_1s_2} \in S^{-1}(Gr(I)) = Gr(S^{-1}I)$ or $\frac{a_1}{s_1}\frac{a_3}{s_3} \in S^{-1}(Gr(I)) = Gr(S^{-1}I)$ or $\frac{a_2}{s_2}\frac{a_3}{s_3} \in S^{-1}(Gr(I)) = Gr(S^{-1}I)$.

(ii) Let $a,b,c \in h(R)$ such that $0 \neq abc \in I$. Then we have $0 \neq \frac{abc}{1} = \frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{-1}I$ and this implies $\frac{a}{1}\frac{b}{1} \in Gr(S^{-1}I) = S^{-1}(Gr(I))$ or $\frac{a}{1}\frac{c}{1} \in S^{-1}(Gr(I))$ or $\frac{b}{1}\frac{c}{1} \in S^{-1}(Gr(I))$ hence $S^{-1}I$ is a graded weakly 2-absorbing quasi primary ideal of $S^{-1}R$. Suppose that $\frac{a}{1}\frac{b}{1} \in Gr(S^{-1}I) = S^{-1}(Gr(I))$. Then we have $v(ab) \in Gr(I)$ for some $v \in S$ and this yields $v^n(a^nb^n) \in I$ for some $n \in \mathbb{N}$. Since $v^n \in S$ and $S \cap G\text{-}Z_I(R) = \emptyset$, we come to the conclusion that $a^nb^n \in Q$ and so $ab \in Gr(I)$. In other cases, we also get $ac \in Gr(I)$ or $bc \in Gr(I)$. Thus I is a graded weakly 2-absorbing quasi primary ideal of R. □

A G-graded ring $R = \bigoplus_{g\in G} R_g$ is said to be a *cross product* if R_g contains a unit for every $g \in G$ [3].

Lemma 2.8. *Suppose that $R = R_1 \times R_2$ where R_1 and R_2 are cross products. Let I_1 be an ideal of R_1. The followings are equivalent:*

(i) $I_1 \times R_2$ is a graded weakly 2-absorbing quasi primary ideal of R.

(ii) I_1 is a graded 2-absorbing quasi primary ideal of R_1.

(iii) $I_1 \times R_2$ is a graded 2-absorbing quasi primary ideal of R.

Proof. $(i) \Rightarrow (ii)$: Let $abc \in I_1$ for some $a,b,c \in h(R_1)$. Since R_2 is a cross product, there exists a unit homogeneous element $i \in R_2$ such that $\deg(i) = \deg(a)$. As R_2 is a cross product, there exist unit homogeneous elements $j,k \in R_2$ such that $\deg(b) = \deg(j)$ and $\deg(c) = \deg(k)$. Then we have $(0,0) \neq (a,i)(b,j)(c,k) \in I_1 \times R_2$. As $I_1 \times R_2$ is a graded weakly 2-absorbing quasi primary ideal of R, we have $(a,i)(b,j) \in Gr(I_1 \times R_2)$ or $(a,i)(c,k) \in Gr(I_1 \times R_2)$ or $(b,j)(c,k) \in Gr(I_1 \times R_2)$. This completes the proof.

$(ii) \Rightarrow (iii)$: Follows from [13] Lemma 2.13.

$(iii) \Rightarrow (i)$: It is clear. □

Theorem 2.9. Suppose that $R = R_1 \times R_2$ where R_1 and R_2 are cross products. Let I_i be a nonzero graded ideal of R_i for each $i = 1,2$. The following statements are equivalent:

(i) $I_1 \times I_2$ is a graded weakly 2-absorbing quasi primary ideal of R.

(ii) $I_1 = R_1$ and I_2 is a graded 2-absorbing quasi primary ideal of R_2 or I_1 is a graded 2-absorbing quasi primary ideal of R_1 and $I_2 = R_2$ or I_1, I_2 are graded quasi primary ideals of R_1 and R_2, respectively.

(iii) $I_1 \times I_2$ is a graded 2-absorbing quasi primary ideal of R.

Proof. $(i) \Rightarrow (ii)$: Suppose that $I_1 \times I_2$ is a graded weakly 2-absorbing quasi primary ideal of R.

Case 1: Let $I_1 = R_1$. In this case, let us show that I_2 is a graded 2-absorbing quasi primary ideal of R_2. Let $x_2y_2z_2 \in I_2$ for some $x_2,y_2,z_2 \in h(R_2)$. Since R_1 is a cross product, there exist unit homogeneous elements $i,j,k \in R_1$ such that $\deg(i) = \deg(x_2)$, $\deg(j) = \deg(y_2)$ and $\deg(k) = \deg(z_2)$. Then $(0,0) \neq (i,x_2)(j,y_2)(k,z_2) \in I_1 \times I_2$. As $I_1 \times I_2$ is a graded weakly 2-absorbing quasi primary ideal of R, we have $(i,x_2)(j,y_2) \in Gr(I_1 \times I_2)$ or $(i,x_2)(k,z_2) \in Gr(I_1 \times I_2)$ or $(j,y_2)(k,z_2) \in Gr(I_1 \times I_2)$. Then $x_2y_2 \in Gr(I_2)$ or $x_2z_2 \in Gr(I_2)$ or $y_2z_2 \in Gr(I_2)$. Therefore, I_2 is a graded 2-absorbing quasi primary ideal of R_2.

Case 2: Let take $I_2 = R_2$. The rest can be shown as Case 1.

Case 3: Let $I_1 \neq R_1$ and $I_2 \neq R_2$. Now, we will show that I_1, I_2 are graded quasi primary ideals. Let $ab \in I_2$ for some $a,b \in h(R_2)$. Since I_1 is a nonzero graded ideal, there exists $0 \neq i \in h(I_1)$. As R_1, R_2 are cross products, there exist unit homogeneous elements $r \in R_2$ and $j,k \in R_1$ such that $\deg(i) = \deg(r)$, $\deg(j) = \deg(a)$ and $\deg(k) = \deg(b)$. Then note that $(0,0) \neq (i,r)(j,a)(k,b) = (ijk, rab) \in I_1 \times I_2$. Since $I_1 \times I_2$ is a graded weakly 2-absorbing quasi primary ideal of R, then $(i,r)(j,a) = (ij,ra) \in Gr(I_1 \times I_2)$ or $(i,r)(k,b) = (ik,rb) \in Gr(I_1 \times I_2)$ or $(j,a)(k,b) = (jk,ab) \in Gr(I_1 \times I_2)$ which implies that $ra \in Gr(I_2)$ or $rb \in Gr(I_2)$ or $jk \in Gr(I_1)$. As r,j,k are units, we have $a \in Gr(I_2)$ or $b \in Gr(I_2)$ or $jk \in Gr(I_1)$. Since $I_1 \neq R_1$, it follows that $jk \notin Gr(I_1)$. So that $a \in Gr(I_2)$ or $b \in Gr(I_2)$. Therefore I_2 is a graded quasi primary ideal of R_2. Similarly one can show that I_1 is a graded quasi primary ideal of R_1.

$(ii) \Rightarrow (iii)$: **Case 1**: Assume that $I_1 = R_1$ and I_2 is a graded 2-absorbing quasi primary ideal of R_2. Then by Lemma 2.8 $I_1 \times I_2$ is a graded 2-absorbing quasi primary ideal of R.

Case 2: Suppose that $I_2 = R_2$ and I_1 is a graded 2-absorbing quasi primary ideal of R_1. Again by Lemma 2.8 $I_1 \times I_2$ is a graded 2-absorbing quasi primary ideal of R.

Case 3: Assume that I_1 and I_2 are graded quasi primary ideals of R_1 and R_2, respectively.
Let $(a_1,a_2)(b_1,b_2)(c_1,c_2) \in I_1 \times I_2$ for some homogeneous elements $a_i,b_i,c_i \in h(R_i)$ and for $i = 1,2$. Then $a_ib_ic_i \in I_i$. Since I_i is a graded quasi primary ideal, $Gr(I_i)$ is a graded prime ideal. As $a_ib_ic_i \in Gr(I_i)$, we have $a_i \in Gr(I_i)$ or $b_i \in Gr(I_i)$ or $c_i \in Gr(I_i)$. Without loss of generality, it may be assumed that $a_1 \in Gr(I_1)$ and $b_2 \in Gr(I_2)$. In this case $(a_1,a_2)(b_1,b_2) = (a_1b_1, a_2b_2) \in Gr(I_1) \times Gr(I_2) = Gr(I_1 \times I_2)$. Hence, $I_1 \times I_2$ graded 2-absorbing quasi primary ideal of R.

$(iii) \Rightarrow (i)$: It is clear. □

Proposition 2.10. *Let $I_1, I_2, ..., I_n$ be a graded weakly 2-absorbing quasi primary ideal of R where $Gr(I_i) = P$ for each $i \in 1,2,...,n$. Then $\bigcap_{i=1}^{n} I_i$ is a graded weakly 2-absorbing quasi primary ideal of R.*

Proof. First note that $Gr(\bigcap_{i=1}^{n} I_i) = \bigcap_{i=1}^{n} Gr(I_i) = P$. (See [9], Proposition 2.4). Let $0 \neq abc \in \bigcap_{i=1}^{n} I_i$ for some $a,b,c \in h(R)$. Then for each $i \in 1,2,...,n$ we have $0 \neq abc \in I_i$. Since I_i is a graded weakly 2-absorbing quasi primary ideal, then $ab \in Gr(I_i) = P$ or $ac \in P$ or $bc \in P$. So $\bigcap_{i=1}^{n} I_i$ is a graded weakly 2-absorbing quasi primary ideal of R. □

Remark 2.11. *Recall that the idealization of M, $R(+)M = \{(a,m) : r \in R, m \in M\}$ is a commutative ring with component wise addition and multiplication $(a,m_1)(b,m_2) = (ab, bm_1 + am_2)$ for each $a,b \in R$ and $m_1, m_2 \in M$ [6].*

Theorem 2.12. Let G be an Abelian group, I be a graded ideal of a graded ring R, and M be a graded R-module. Then the followings are equivalent:

(i) $I(+)M$ is a graded weakly 2-absorbing quasi primary ideal of $R(+)M$.

(ii) I is a graded weakly 2-absorbing quasi primary ideal of R whenever $abc = 0$ and $ab, ac, bc \in Gr(I)$ for some $a,b,c \in h(R)$ where $deg(a) = s, deg(b) = u, deg(c) = t$. Then $abM_t = acM_u = bcM_s = 0$.

Proof. $(i) \Rightarrow (ii)$: $0 \neq abc \in I$ for some $a,b,c \in h(R)$. Then $(0,0) \neq (a,0)(b,0)(c,0) \in I(+)M$. By [13] Corollary 3.5 we know that $Gr(I(+)M) = Gr(I)(+)M$. As $I(+)M$ is a graded weakly 2-absorbing quasi primary ideal of $R(+)M$, we have $(a,0)(b,0) \in Gr(I)(+)M$ or $(a,0)(c,0) \in Gr(I)(+)M$ or $(b,0)(c,0) \in Gr(I)(+)M$. Then $ab \in Gr(I)$ or $ac \in Gr(I)$ or $bc \in Gr(I)$. Now we will show that $abM_t = 0$. Suppose to the contrary that $abM_t \neq 0$. Then there is a homogeneous element $m \in h(M)$ such that $abm \neq 0$ and $deg(m) = t$. Then we have $(0,0) \neq (a,0)(b,0)(c,m) = (0,abm) \in I(+)M$. As $I(+)M$ is a graded weakly 2-absorbing quasi primary ideal of $R(+)M$, $(a,0)(b,0) \in Gr(I(+)M)$ or $(a,0)(c,m) \in Gr(I(+)M)$ or $(b,0)(c,m) \in Gr(I(+)M)$. Then we get $ab \in Gr(I)$ or $ac \in Gr(I)$ or $bc \in Gr(I)$ which is a contradiction. Hence $abM_t = 0$. Similarly one can show that $acM_u = 0$ and $bcM_s = 0$.

$(ii) \Rightarrow (i)$: Let $(0,0) \neq (a,m_1)(b,m_2)(c,m_3) \in I(+)M$ where $deg(a) = s, deg(b) = u, deg(c) = t$.

Case 1: Asume that $abc \neq 0$. In this case since I is a graded weakly 2-absorbing quasi primary ideal of R and $abc \in I$, we get $(a,m_1)(b,m_2) \in Gr(I(+)M)$ or $(a,m_1)(c,m_3) \in Gr(I(+)M)$ or $(b,m_2)(c,m_3) \in Gr(I(+)M)$. Hence $I(+)M$ is a graded weakly 2-absorbing quasi primary ideal of $R(+)M$.

Case 2: Assume that $abc = 0$. If $ab \in Gr(I)$ or $ac \in Gr(I)$ or $bc \in Gr(I)$, then we are done. Now suppose that $ab, ac, bc \notin Gr(I)$. In this case by assumption $abm_3 = acm_2 = bcm_1 = 0$. This gives $(a,m_1)(b,m_2)(c,m_3) = (abc, abm_3 + acm_2 + bcm_1) = (0,0)$ which is a contradiction. □

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Isosimple N-injective modules

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Abstract. In this paper, we introduce and study the concept of isosimple N-injective modules as a generalization of N-injective modules. We show that the theory of isosimple N-injective modules resemble that of N-injective. Furthermore, we provide a number of characterizations of this new class of modules.

Key Words: Isosimple modules; N-injective modules; isosimple N-injective modules.
2020 MSC: Primary 16D50, 16D60; Secondary 16D70.

1 Introduction

Let R be an associative ring with identity and M be an unital right R-module. Recall [12], M is isosimple if every nonzero submodule of M is isomorphic to M. Simultaneously, in [7, 8, 10], authors defined such type of modules to be *iso-retractable* modules and studied several characterizations and properties of this class of modules. An isosimple submodule N of M means N is isosimple as an R-module. Further, recall [2] that a module M is called N-injective if, for every submodule K of N, every homomorphism from K to M extends to N and M is called simple N-injective if, for every simple submodule K of N, every homomorphism from K to M extends to N. Several authors have studied these notions (see [2, 9, 3, 4, 6, 14, 19]). The notion of simple N-injective modules generalizes the notion of N-injective modules. Inspired by the notion of simple N-injective modules, in this paper, we introduce and study the "isosimple" version of N-injective modules as a proper generalization of N-injective modules. A natural way to extend the notion of N-injective module to isosimple N-injective module is to replace "arbitrary submodule" by "isosimple submodule". We say that an R-module M is isosimple N-injective if for every isosimple submodule K of N, every homomorphism from K to M extends to N. The class of isosimple N-injective modules lies between the class of N-injective modules and the class of simple N-injective modules.

Section 2 discusses the main objective of the paper which is devoted to the study of isosimple N-injective modules. We extend several results on N-injective modules to isosimple N-injective modules. For instants, if M is isosimple N-injective and B is a submodule of N, then M is isosimple B-injective (see Proposition 2.5). Also, every isosimple summand of an isosimple direct injective right R-module M is isosimple M-injective (see Proposition 2.11). Moreover, for R-modules M and N with $X = M \oplus N$, N is isosimple M-injective if and only if for each isosimple submodule K of X with $K \cap M = K \cap N = 0$, there exists a submodule C of X such that $K \leq C$ and $N \oplus C = X$. Finally, we provide a connection between isosimple direct injective modules and isosimple N-injective modules (see Proposition 2.11).

Unless otherwise stated, all rings are associative with identity and all modules are unital right

modules. We denote $N \subseteq^{\oplus} M$ if N is a direct summand of M, $N \cong M$ if N is isomorphic to M, $Soc(M)$ denotes socle of M and $E(M)$ denotes the injective hull of M. For undefined terms and notions, we refer [3, 5, 15] and [18].

2 Isosimple N-injective modules

We begin this section by introducing the concept of isosimple N-injective modules as a generalization of N-injective modules.

Definition 2.1. Let R be a ring and N be an R-module. An R-module M is said to be isosimple N-injective if for every isosimple submodule S of N, any homomorphism $f : S \to M$ can be extended to a homomorphism $g : N \to M$.

It is clear from the definition that every N-injective module is isosimple N-injective, and every isosimple N-injective module is simple N-injective. The following example shows that there is a simple N-injective module which is not isosimple N-injective.

Example 2.2. Consider two $\mathbb{Z}$-modules $M = N = \mathbb{Z}$ and an isosimple submodule $K = 2\mathbb{Z}$ of N. Define a homomorphism $f : K \to M$ by $f(x) = \frac{x}{2}$, for all $x \in K$. Then there is no homomorphism $g : N \to M$ that extends f. This implies that M is not an isosimple N-injective module. However, M is simple N-injective because N has no nonzero simple submodules.

Now, we characterize isosimple N-injective modules.

Theorem 2.3. Let N and M be R-modules and $X = N \oplus M$. Then the following statements are equivalent:

1. M is isosimple N-injective.
2. For each isosimple submodule K of X with $K \cap M = 0$, there exists a submodule C of X such that $K \leq C$ and $M \oplus C = X$.

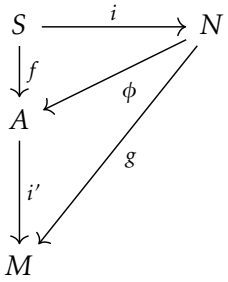
Proof. (1) $\Rightarrow$ (2). Let K be an isosimple submodule of X with $K \cap M = 0$. Let $\pi_N : N \oplus M \to N$ and $\pi_M : N \oplus M \to M$ be canonical projections. Notice that $M \oplus K = M \oplus \pi_N(K)$ and $\pi_N(K)$ is an isosimple submodule of N. Let $k \in K$. Write $k = n + m$, where $n \in N, m \in M$. Define a homomorphism $\phi : \pi_N(K) \to \pi_M(K)$ by $\phi(n) = m$. Then by [12, Lemma 3.1], ϕ is a monomorphism. Since M is isosimple N-injective, ϕ can be extended to a homomorphism $\overline{\phi} : N \to M$ such that $\overline{\phi} \circ i = \phi$, where $i : \pi_N(K) \to N$ be the inclusion map. Consider a submodule $C = \{n - \phi(n) : n \in N\}$ of X. Then $X = M \oplus C$ and $K \leq C$.

(2) $\Rightarrow$ (1). Let $\phi : A \to M$ be a homomorphism, where A is an isosimple submodule of N. Take the submodule $K = \{a - \phi(a) : a \in A\}$ of X. Then K is a submodule of $A \oplus \phi(A)$. This implies that $\pi_N(K) = A$, because $\phi(A) \leq M$ and $K \cap M = 0$. Consequently, $M \oplus K = M \oplus \pi_N(K) = M \oplus A$, and K is an isosimple submodule of X. By assumption, there exists a submodule C of X containing K such that $M \oplus C = X$. Let $\pi : M \oplus C \to M$ be a canonical projection, then $\pi|_N$ extends ϕ. This implies that M is an isosimple N-injective R-module. □

Proposition 2.4. *Direct summand of an isosimple N-injective module is isosimple N-injective.*

Proof. Let M be an isosimple N-injective R-module and A be a direct summand of M. Let S be an isosimple submodule of N and $f : S \to A$ be a homomorphism. Consider the following diagram of

R-modules and R-homomorphisms

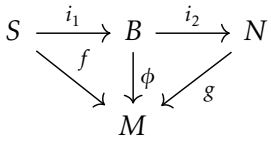


where $i : S \to N$ is inclusion map and $i' : A \to M$ is an injection map. Since M is isosimple N-injective, the map $i' \circ f : S \to M$ can be extended to a homomorphism $g : N \to M$ such that $g \circ i = i' \circ f$. Let $\pi : M \to A$ be the canonical projection and let $\phi = \pi \circ g$. Then ϕ is a homomorphism from N to A and $\phi \circ i = \pi \circ g \circ i = \pi \circ i' \circ f = I_A \circ f = f$, where I_A is the identity map on A. Thus, ϕ extends f and so A is isosimple N-injective. □

In the following, we prove that if a module M is isosimple N-injective, then it is isosimple B-injective, for every submodule B of N.

Proposition 2.5. *Let M be an isosimple N-injective R-module and B be a submodule of N. Then M is an isosimple B-injective R-module.*

Proof. Let S be an isosimple submodule of B (and so an isosimple submodule of N) and $f : S \to M$ be a homomorphism. Consider the following diagram of R-modules and R-homomorphisms

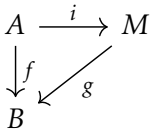


where i_1, i_2 are inclusion maps. Since M is isosimple N-injective, f can be extended to a homomorphism $g : N \to M$ such that $g \circ i_2 \circ i_1 = f$. Let $\phi = g \circ i_2$. Then $\phi : B \to M$ is a homomorphism and $\phi \circ i_1 = g \circ i_2 \circ i_1 = f$. Therefore, ϕ extends f and so M is isosimple B-injective. □

Recall [16], an R-module M is said to be an *iso $C2$-module* if whenever A and B are two isosimple submodules with $A \cong B$ and $B \subseteq^{\oplus} M$, then $A \subseteq^{\oplus} M$. Also, an R-module M is said to be *quasi isosimple injective* if M is isosimple M-injective. The following proposition gives a class of examples of isosimple direct injective modules.

Proposition 2.6. *Every quasi isosimple injective R-module is an iso $C2$-module.*

Proof. Let M be a quasi isosimple injective R-module. Let $A \cong B$ be two isosimple submodule of M and $B \subseteq^{\oplus} M$. By Proposition 2.4, B is an isosimple M-injective R-module. Consider the following diagram of R-modules and R-homomorphisms



where $i : A \to M$ is the inclusion map. Let $f : A \to B$ be an isomorphism. Since B is isosimple M-injective, f can be extended to a homomorphism $g : M \to B$ such that $g \circ i = f$. Since f is an isomorphism and $(f^{-1} \circ g) \circ i = f^{-1} \circ f = I_A$, where I_A is the identity map on A. Thus, the map $i : A \to M$ splits and so $A \subseteq^{\oplus} M$. Therefore, M is an iso $C2$-module. □

Proposition 2.7. *Let M be a quasi isosimple injective R-module and $M = A \oplus B$, where A is isosimple. If $f : A \to B$ is a homomorphism, then $Imf \subseteq^{\oplus} B$.*

Proof. Suppose M is a quasi isosimple injective R-module. Then by [16, Proposition 2.15], M is an iso $C2$-module. Therefore, by [16, Proposition 3.1], M is an iso $C3$-module and so $Imf \subseteq^{\oplus} B$ by [16, Theorem 3.2]. □

Proposition 2.8. *Let M be an isosimple R-module. If M is a pseudo isosimple N-injective module, for any R-module N. Then any monomorphism $f : M \to N$ splits.*

Proof. Let $f : M \to N$ be a monomorphism and $f^{-1} : f(M) \to M$ be the inverse of f. Since M is pseudo isosimple N-injective, the monomorphism f^{-1} can be extended to a homomorphism $g : N \to M$ such that $g \circ i = f^{-1}$, where $i : f(M) \to N$ is the inclusion map. Clearly, $g \circ f = id_M$. Thus the map f split. □

Proposition 2.9. *Let N be an isosimple R-module. If a module M is a isosimple N-injective R-module, then any nonzero homomorphism $f : M \to N$ splits.*

Proof. Let $f : M \to N$ be a nonzero homomorphism and $f^{-1} : f(M) \to M$ be the inverse of f. Since M is isosimple N-injective, the homomorphism $f^{-1} : f(M) \to M$ can be extended to a homomorphism $g : N \to M$ such that $g \circ i = f^{-1}$, where $i : f(M) \to N$ is the inclusion map. Clearly, $g \circ f = id_M$. Thus the map f split. □

Proposition 2.10. *Let M be an isosimple R-module. Then M is injective if and only if M is pseudo isosimple N-injective, for every R-module N.*

Proof. Suppose M is pseudo isosimple N-injective, for all N. Then by Proposition 2.8, every monomorphism $f : M \to N$ splits. This implies that M is injective. The converse is obvious. □

The following result shows that isosimple summand of an iso $C2$-module N is isosimple N-injective.

Proposition 2.11. *Let N be an iso $C2$-module. Then every isosimple summand of N is pseudo isosimple N-injective.*

Proof. Let A be an isosimple summand of N and $f : K \to A$ be a homomorphism, where K is an isosimple submodule of N. Consider the following diagram of R-modules and R-homomorphisms

$$\begin{array}{ccc} K & \xrightarrow{\ i\ } & N \\ \downarrow{\scriptstyle f} & & \\ A & & \end{array}$$

If $f = 0$, then the projection map $\pi : N \to A$ be an extension of f. If $f \neq 0$ and f is a monomorphism. Then $K \cong f(K) \cong A$ (since A is isosimple). Since N is an iso $C2$-module and $K \cong A$, K is a direct summand of N. Then the homomorphism $f \circ p : N \to A$ extends f, where $p : N \to K$ be the canonical projection. Thus A is pseudo isosimple N-injective. □

Question. Is the assumption "isosimple summand" necessary in the Proposition 2.11 ?

Yes, this is necessary condition to prove Proposition 2.11. In support, we provide the following example.

Example 2.12. [17, Example 2.13] Consider $R = \mathbb{Z}_8$ and $N = \mathbb{Z}_8 \oplus 2\mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus 2\mathbb{Z}_8$. By [16, Example 2.3], N is an iso $C2$-module. Consider an isosimple submodule $K = 4\mathbb{Z}_8 \oplus \bar{0} \oplus \bar{0} \oplus \bar{0}$ of N and a direct summand $A = \bar{0} \oplus 2\mathbb{Z}_8 \oplus \bar{0} \oplus \bar{0}$ of N. Let $f : K \to A$ be a homomorphism defined by $f((\bar{4}, \bar{0}, \bar{0}, \bar{0})) = (\bar{0}, \bar{4}, \bar{0}, \bar{0})$. We show that there is no homomorphism $g : N \to A$ that extends f. For this, let $g : N \to A$ be an arbitrary homomorphism. Then all possible values of $g((\bar{1}, \bar{0}, \bar{0}, \bar{0}))$ are $(\bar{0}, \bar{0}, \bar{0}, \bar{0}), (\bar{0}, \bar{2}, \bar{0}, \bar{0}), (\bar{0}, \bar{4}, \bar{0}, \bar{0}), (\bar{0}, \bar{6}, \bar{0}, \bar{0})$. Then $g((\bar{4}, \bar{0}, \bar{0}, \bar{0})) = g((\bar{1}, \bar{0}, \bar{0}, \bar{0})) + g((\bar{1}, \bar{0}, \bar{0}, \bar{0})) + g((\bar{1}, \bar{0}, \bar{0}, \bar{0})) + g((\bar{1}, \bar{0}, \bar{0}, \bar{0})) = (\bar{0}, \bar{0}, \bar{0}, \bar{0})$, for all possible values of $g((\bar{1}, \bar{0}, \bar{0}, \bar{0}))$. This implies that $f((\bar{4}, \bar{0}, \bar{0}, \bar{0})) \neq g((\bar{4}, \bar{0}, \bar{0}, \bar{0}))$. Consequently, g is not an extension of f. Thus, the direct summand A of N is not an isosimple N-injective module, as desired. Here we observe that A is not isosimple.

Proposition 2.13. *If $N \oplus M$ is a quasi isosimple injective R-module, then M is isosimple N-injective.*

Proof. Let $f : A \to M$ be a homomorphism , where A is an isosimple submodule of N. Let $\phi : A \to N \oplus M$ be a homomorphism defined by $\phi(n) = (n, f(n))$. Consider the following diagram of R-modules and R-homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & N \\ {\scriptstyle f}\downarrow & \overset{\phi}{\searrow} & \downarrow{\scriptstyle \overline{\phi}} \\ M & \xleftarrow[\pi_M]{} & N \oplus M \end{array}$$

Since $M \oplus N$ is quasi isosimple injective R-module, $M \oplus N$ is an isosimple N-injective R-module by Proposition 2.5. Therefore, ϕ can be extended to a homomorphism $\overline{\phi} : N \to N \oplus M$ such that $\overline{\phi}|_A = \phi$, i.e., $\overline{\phi} \circ i = \phi$, where $i : A \to N$ be the inclusion map. Clearly, $f = \pi_M \circ \overline{\phi} \circ i$. This implies that $\pi_M \circ \overline{\phi}$ extends f. Thus M is an isosimple N-injective R-module. □

Corollary 2.14. *If $M \oplus M$ is a quasi isosimple injective R-module. Then M is quasi injective.*

Proof. Follows from Proposition 2.13 for $M = N$. □

In general, quotient of an isosimple N-injective module need not be isosimple N-injective. In the following, we provide a sufficient condition on N so that quotient of isosimple N-injective modules becomes isosimple N-injective.

Theorem 2.15. Let N be a projective isosimple R-module. Then

1. Every quotient module of an isosimple N-injective R-module is an isosimple N-injective R-module.
2. Every quotient of an injective R-module is an isosimple N-injective R-module.

Proof. 1. Let E be an isosimple N-injective R-module and $\pi : E \to B$ be a surjective homomorphism, where B is an R-module. We have to prove that B is an isosimple N-injective R-module. Consider the following diagram of R-modules and R-homomorphisms

$$\begin{array}{ccccc} S & \xrightarrow{\ i\ } & N & & \\ {\scriptstyle g}\downarrow & \overset{f}{\swarrow}\ \overset{h}{\searrow} & \downarrow{\scriptstyle g'} & & \\ E & \xrightarrow{\ \pi\ } & B & \longrightarrow & 0 \end{array}$$

Let $f : S \to B$ be a homomorphism, where S is an isosimple submodule of N. Since $S \cong N$, S is projective. So there exists a homomorphism $g : S \to E$ such that $\pi \circ g = f$. Since f is a monomorphism, g is a monomorphism. Also, since E is an isosimple N-injective R-module, g can be extended to a

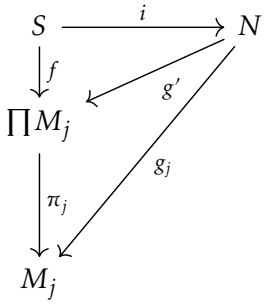
homomorphism $h : N \to E$ such that $g = h \circ i$. Let $g' = \pi \circ h$, then $g' = \pi \circ h : N \to B$ is a homomorphism that extends f, i.e., $\pi \circ h \circ i = f$. Thus B is an isosimple N-injective R-module.

2. Directly follows by (1). □

It is well known that the direct product of N-injective modules is N-injective if and only if each component is N-injective. It is natural to ask whether it is true for the case of isosimple N-injective R-modules. This issue is addressed in the following result.

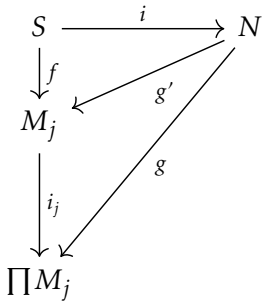
Proposition 2.16. *Let N be an R-module and $\{M_j\}$ be a family of R-modules. Then $M = \prod M_j$ is an isosimple N-injective R-module if and only if each M_j is an isosimple N-injective R-module.*

Proof. Suppose M_j is isosimple N-injective. Let S be an isosimple submodule of N. Let $f : S \to \prod M_j$ be a homomorphism. Consider the following diagram of R-modules and R-homomorphisms



Since M_j is isosimple N-injective, the map $\pi_j \circ f : S \to M_j$ can be extended to a homomorphism $g_j : N \to M_j$ such that $\pi_j \circ f = g_j \circ i$, where $i : S \to N$ is the inclusion map. Let $g' = \sum i_j \circ g_j$, where $i_j : M_j \to \prod M_j$ is the injection map, then $g' \circ i = \sum i_j \circ g_j \circ i = \sum i_j \circ \pi_j \circ f = Id_M \circ f = f$. Thus, g' extends f, and so $\prod M_j$ is an isosimple N-injective R-module.

Conversely, suppose $\prod M_j$ is an isosimple N-injective R-module. Let S be an isosimple submodule of N. Let $f : S \to M_j$ be a homomorphism. Consider the following diagram of R-modules and R-homomorphisms



Since $\prod M_j$ is isosimple N-injective, the map $i_j \circ f$ can be extended to a homomorphism $g : N \to \prod M_j$ such that $i_j \circ f = g \circ i$. Let $g' = \pi_j \circ g$, where $\pi_j : \prod M_j \to M_j$ is the projection map, then $g' \circ i = \pi_j \circ g \circ i = \pi_j \circ i_j \circ f = Id_{M_j} \circ f = f$. Thus g' extends f, and so M_j is isosimple N-injective. □

We end this section by proving finite generation of $soc(M)$ under a mild condition.

Proposition 2.17. *Let M be a finitely generated R-module. If the direct sum of isosimple M-injective R-modules is isosimple M-injective, then $Soc(M)$ is finitely generated.*

Proof. Let $Soc(M) = \oplus S_i$, where S_i's are simple submodules of M. Let $E(S_i)$ be the injective hull of S_i, for all $i \in I$. Let $i : \oplus S_i \to \oplus E(S_i)$ be the inclusion map. Since $\oplus E(S_i)$ is isosimple M-injective, $\oplus E(S_i)$ is $Soc(M)$-injective. Therefore, there exists a homomorphism $g : M \to \oplus E(S_i)$ that extends

i. Since M is finitely generated, $g(M) \subseteq \oplus_{i\in F} E(S_i)$, for some finite subset F of I. This implies that $Soc(M) = i(Soc(M)) = g(Soc(M)) \subseteq \oplus E(S_i)$. Since $\oplus E(S_i)$ has finitely generated socle, $Soc(M)$ is finitely generated. □

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Nil-M-Noetherian and nil-M-Artinian modules

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Nil-M-Noetherian and nil-M-Artinian modules

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Abstract. Let R be a commutative ring with identity and M be an R-module. The aim of this paper is to introduce and investigate the notions of nil-M-Noetherian and nil-M-Artinian modules as generalizations of Noetherian and Artinian modules. Also, in this regard we introduce nil versions of some algebraic concepts.
Key Words: Nil-M-cyclic, nil-M-Noetherian, nil-M-Artinian, nil-M-finitely generated, nil-M-finitely cogenerated.
2020 MSC: 13C13, 13C99.

1 Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. Also, $Nil(R)$ will denote the set of all nilpotent elements of R.

In [13], the authors introduced and investigated the notion of Ω-prime ideals as a generalization of prime ideals. A proper ideal P of R is said to be a Ω-prime ideal if $ab \in P$, for each $a, b \in R$, then either $a \in P + Nil(R)$ or $b \in P + Nil(R)$ [13]. In [7] (resp. [8]), the present author introduced the notion of nil-prime (resp. nil-primary) ideals as a generalization of prime (resp. primary) ideals and studied some basic properties of this class of ideals. A proper ideal P of R is said to be a *nil-prime ideal* if there exists $x \in Nil(R)$ such that whenever $ab \in P$, then $a \in P$ or $b \in P$ or $a + x \in P$ or $b + x \in P$ for each $a, b \in R$ (for generalizations of this notion for modules, see [9]). Also, a proper ideal P of R is said to be a *nil-primary ideal* if there exists $x \in Nil(R)$ such that whenever $ab \in P$ for some $a, b \in R$, then $a \in P$ or $b^n \in P$ or $a + x \in P$ or $b^n + x \in P$ for some $n \in \mathbb{N}$.

For a submodule N of an R-module M, a non-empty subset K of M, and a non-empty subset J of R, the residuals of N by K and J are defined as $(N :_R K) = \{a \in R : aK \subseteq N\}$ and $(N :_M J) = \{m \in M : Jm \subseteq N\}$, respectively. In particular, we use $Ann_R(M)$ to denote $(0 :_R M)$.

In algebra, some generalizations and weakenings for the concepts of Noetherian and Artinian modules have arisen. In this regard, you can refer to [11, 6]. Motivated by nil-prime ideals, the purpose of this paper is to introduce and investigate the notions of nil-M-Noetherian and nil-M-Artinian modules as generalizations of Noetherian and Artinian modules. Also, in this regard we introduce the notions of nil-M-cyclic, nil-M-finitely generated and nil-M-finitely cogenerated modules.

2 Main Results

Definition 2.1. We say that an R-module M is a *nil-M-Noetherian module* if there is $r \in \sqrt{Ann_R(M)}$ such that for each ascending chain

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$$

of submodules of M there is a positive integer t such that $N_{t+i} \subseteq N_t + rM$ for all $i \geq 0$. Moreover, we say that the ring R is a *nil-R-Noetherian ring* if R as an R-module is a nil-R-Noetherian module.

Definition 2.2. We say that an R-module M is a *nil-M-Artinian module* if there is $r \in \sqrt{Ann_R(M)}$ such that for each descending chain

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq ... \supseteq N_k \supseteq \dots$$

of submodules of M there is a positive integer t such that $N_t \cap (0 :_M r) \subseteq N_{t+i}$ for all $i \geq 0$. Also, we say that the ring R is a *nil-R-Artinian ring* if R as an R-module is a nil-R-Artinian module.

Remark 2.3. As $0 \in \sqrt{Ann_R(M)}$, we have that every Noetherian (resp. Artinian) R-module is a nil-M-Noetherian (resp. nil-M-Artinian) R-module. This motivates the Question 2.4. If $\sqrt{Ann_R(M)} = 0$, then the notions of Noetherian R-modules and nil-M-Noetherian R-modules (resp. Artinian R-modules and nil-M-Artinian R-modules) are equal. For example, for a positive integer n which is square-free (i.e., n has not a square factor), consider the ring $\mathbb{Z}_n$ of integers modulo n. Then $\sqrt{Ann_{\mathbb{Z}_n}(\mathbb{Z}_n)} = 0 = Nil(\mathbb{Z}_n)$. Also, $\sqrt{Ann_{\mathbb{Z}}(\mathbb{Z})} = 0 = Nil(\mathbb{Z})$ and $\sqrt{Ann_{F[x]}(F[x]))} = 0 = Nil(F[x])$, where F is a field.

Question 2.4. Let M be an R-module. Is every nil-M-Noetherian (resp. nil-M-Artinian) module a Noetherian (resp. Artinian) module?

Definition 2.5. We say that a submodule N of an R-module M is a *nil-M-cyclic* if there exist $r \in \sqrt{Ann_R(M)}$ and $x \in M$ such that $N = Rx + rM$.

Let M be an R-module. Since $0 \in \sqrt{Ann_R(M)}$, clearly each cyclic submodule of M is a nil-M-cyclic submodule of M. But Example 2.6 shows that the converse is not true in general.

Example 2.6. The submodule $N = \mathbb{Z}x + 2\mathbb{Z}_4[x,y]$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_4[x,y]$ is a nil-M-cyclic since $2 \in \sqrt{Ann_{\mathbb{Z}}(M)}$ and $x \in \mathbb{Z}_4[x,y]$. But N is not a cyclic submodule of M.

Definition 2.7. We say that a submodule N of an R-module M is a *nil-M-finitely generated* (resp. *weakly nil-M-finitely generated*) if there exist $r \in \sqrt{Ann_R(M)}$ and $x_1, x_2, \dots, x_t \in M$ such that $N = Rx_1 + Rx_2 + \dots + Rx_t + rM$ (resp. $N + rM = Rx_1 + Rx_2 + \dots + Rx_t + rM$).

Example 2.8. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \oplus \cdots$. Then $6 \in \sqrt{Ann_{\mathbb{Z}}(M)}$. For each finitely generated submodule N of M we have $N + 6M$ is a nil-M-finitely generated submodule of M which is not a finitely generated submodule of M.

Remark 2.9. Let M be an R-module. Clearly, every nil-M-finitely generated submodule of M is a weakly nil-M-finitely generated submodule of M. If N is a weakly nil-M-finitely generated submodule of M such that $(N :_R M)$ is prime ideal of R, then $\sqrt{Ann_R(M)} \subseteq \sqrt{(N :_R M)} = (N :_R M)$. This in turn implies that N is a nil-M-finitely generated submodule of M.

Proposition 2.10. *Let S be a multiplicatively subset of R and let M be an R-module. If N is a nil-M-finitely generated submodule of M, then $S^{-1}N$ is a nil-$S^{-1}M$-finitely generated submodule of $S^{-1}M$.*

Proof. Assume that N is a nil-M-finitely generated submodule of M. Then there exist $r \in \sqrt{Ann_R(M)}$ and $x_1, x_2, \dots, x_t \in M$ such that $N = Rx_1 + Rx_2 + \dots + Rx_t + rM$. By using the proof of [14, Lemma 9.12(ii)] and [14, Lemma 5.31(iv)], we have $S^{-1}(\sqrt{Ann_R(M)}) \subseteq \sqrt{Ann_{S^{-1}R}(S^{-1}M)}$. Therefore, $r/1 \in \sqrt{Ann_{S^{-1}R}(S^{-1}M)}$ and $x_1/1, x_2/1, \dots, x_t/1 \in S^{-1}M$ such that

$$S^{-1}N = (S^{-1}R)(x_1/1) + (S^{-1}R)(x_2/1) + \dots + (S^{-1}R)(x_t/1) + r/1(S^{-1}M).$$

□

A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. By [10], every submodule of M is an intersection of completely irreducible submodules of M. Thus the intersection of all completely irreducible submodules of M is zero [10].

Definition 2.11. We say that a submodule N of an R-module M is a *nil-M-completely irreducible submodule* if there exist $r \in \sqrt{Ann_R(M)}$ and completely irreducible submodule L of M such that $N = L \cap (0 :_M r)$.

Definition 2.12. For a submodule N of an R-module M, we say that M/N is a *nil-M-finitely cogenerated* (resp. *weakly nil-M-finitely cogenerated*) if there exists $r \in \sqrt{Ann_R(M)}$ such that $N = \cap_{i\in I} N_i$ implies that $N = \cap_{i\in J} N_i \cap (0 :_M r)$ (resp. $\cap_{i\in J} N_i \cap (0 :_M r) = N \cap (0 :_M r)$) for some finite subset J of I.

Proposition 2.13. *If for a submodule N of an R-module M we have M/N is a nil-M-finitely cogenerated (resp. weakly nil-M-finitely cogenerated) module, then there exist $r \in \sqrt{Ann_R(M)}$ and completely irreducible submodules $L_1, L_2, \ldots, L_t$ of M such that $N = L_1 \cap L_2 \cap \cdots \cap L_t \cap (0 :_M r)$ (resp. $L_1 \cap L_2 \cap \cdots \cap L_t \cap (0 :_M r) = N \cap (0 :_M r)$).*

Proof. This follows from the fact that every submodule of M is an intersection of completely irreducible submodules of M. □

Remark 2.14. Let M be an R-module. Clearly, every nil-M-finitely cogenerated homomorphic image of M is a weakly nil-M-finitely cogenerated homomorphic image of M. It is easy to see that, if M/N is a weakly nil-M-finitely cogenerated homomorphic image of M such that $Ann_R(N)$ is prime ideal of R, then M/N is a nil-M-finitely cogenerated homomorphic image of M.

Proposition 2.15. *Let M be an R-module. Then we have the following.*

(a) If for a submodule N of M and $r \in \sqrt{Ann_R(M)}$ we have $N + rM = M$, then $N = M$.

(b) If for a submodule N of M and $r \in \sqrt{Ann_R(M)}$ we have $N \cap (0 :_M r) = 0$, then $N = 0$.

Proof. (a) Suppose that for a submodule N of M and $r \in \sqrt{Ann_R(M)}$ we have $N + rM = M$. Then there exists $t \in \mathbb{N}$ such that $r^t M = 0$. We have $rN + r^2M = rM$. This implies that $rN + r^2M + N = rM + N$ and so $N + r^2M = M$. Continuing this way, we get that $M = N + r^tM = N + 0 = N$.

(b) Suppose that for a submodule N of M and $r \in \sqrt{Ann_R(M)}$ we have $N \cap (0 :_M r) = 0$. Then there exists $t \in \mathbb{N}$ such that $r^t M = 0$. Clearly, $N \cap (0 :_M r) = (0 :_N r)$. Hence, $(0 :_N r) = 0$. Therefore,

$$(0 :_N r^2) = ((0 :_N r) :_N r) = (0 :_N r) = N \cap (0 :_M r) = 0.$$

Continuing this way, we have $0 = (0 :_N r^t) = N \cap (0 :_M r^t) = N \cap M = N$. □

Corollary 2.16. *Let M be an R-module. Then we have the following.*

(a) M is a nil-M-cyclic module if and only if it is a cyclic module.

(b) M is a weakly nil-M-finitely generated module if and only if it is a finitely generated module.

(c) The submodule 0 of M is a nil-M-completely irreducible submodule if and only if it is a completely irreducible submodule.

(d) M is a weakly nil-M-finitely cogenerated module if and only if it is a finitely cogenerated module.

Proof. (a) Let M be a nil-M-cyclic R-module. Then there exist $r \in \sqrt{Ann_R(M)}$ and $x \in M$ such that $M = Rx + rM$. This implies that $M = Rx$ by Proposition 2.15 (a). Thus M is a cyclic module. Since $0 \in \sqrt{Ann_R(M)}$, the converse is clear.

(b) Let M be a weakly nil-M-finitely generated R-module. Then there exist $r \in \sqrt{Ann_R(M)}$ and $x_1, x_2, \ldots, x_t \in M$ such that $M = M + rM = Rx_1 + Rx_2 + \cdots + Rx_t + rM$. Thus $M = Rx_1 + Rx_2 + \cdots + Rx_t$ by Proposition 2.15 (a). So, M is a finitely generated module. Since $0 \in \sqrt{Ann_R(M)}$, the converse is clear.

(c) Assume that the submodule 0 of M is a nil-M-completely irreducible submodule. Then there exist $r \in \sqrt{Ann_R(M)}$ and completely irreducible submodule L of M such that $0 = L \cap (0 :_M r)$. This implies that $L = 0$ by Proposition 2.15 (b). So, 0 is a completely irreducible submodule of M. Since $0 \in \sqrt{Ann_R(M)}$, the converse is clear.

(d) Suppose that M is a weakly nil-M-finitely cogenerated R-module and $0 = \cap_{i \in I} N_i$. Then there exist $r \in \sqrt{Ann_R(M)}$ and finite subset J of I such that $\cap_{i \in J} N_i \cap (0 :_M r) = 0 \cap (0 :_M r) = 0$. It follows that $\cap_{i \in J} N_i = 0$ by Proposition 2.15 (b). Therefore, M is a finitely cogenerated module. Since $0 \in \sqrt{Ann_R(M)}$, the converse is clear. □

Theorem 2.17. Let M be an R-module. Consider the following statements:

(a) Every submodule of M is nil-M-finitely generated;

(b) M is a nil-M-Noetherian module;

(c) There exists $r \in \sqrt{Ann_R(M)}$ such that any non-empty collection Σ of submodules of M with the form $K + rM$ has a maximal element $N + rM$;

(d) Every submodule of M is weakly nil-M-finitely generated.

Then $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, and $(c) \Rightarrow (d)$.

Proof. $(a) \Rightarrow (b)$ Assume that the following is an increasing sequence of submodules of M

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots.$$

Consider $N := \bigcup_{i \geq 1} N_i$. Then N is a submodule of M, hence it is nil-M-finitely generated by part (a). Thus there exist $r \in \sqrt{Ann_R(M)}$ and $u_1, \ldots, u_t \in M$ such that $N = Ru_1 + \cdots + Ru_t + rM$. Then we can find s such that $u_j \in N_s$ for all $j = 1, 2, \ldots, t$. In this case we have $N + rM \subseteq N_s + rM$. Therefore, for all $i \geq 0$ we have $N_{s+i} \subseteq N \subseteq N + rM \subseteq N_s + rM$, as needed.

$(b) \Rightarrow (c)$ First note that by part (b), there is $r \in \sqrt{Ann_R(M)}$ such that for each ascending chain

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$$

of submodules of M there is a positive integer t such that $N_{t+i} \subseteq N_t + rM$ for all $i \geq 0$. Suppose that there exists a non-empty collection Σ of submodules of M with the form $N + rM$ such that it has no maximal element. Let us choose $N_1 + rM \in \Sigma$. Since this is not maximal, there is $N_2 + rM \in \Sigma$ such that $N_1 + rM \subset N_2 + rM$, and we continue in this way to construct the following infinite strictly increasing sequence of submodules of M:

$$N_1 + rM \subset N_2 + rM \subset N_3 + rM \subset \ldots.$$

By part (b), there is a positive integer t such that

$$N_{t+1} + rM \subseteq N_t + rM + rM = N_t + rM,$$

which is a contradiction.

$(c) \Rightarrow (d)$ Suppose that there exists $r \in \sqrt{Ann_R(M)}$ such that any non-empty collection Σ of submodules of M with the form $K+rM$ has a maximal element. Let N be a submodule of M and consider the family Σ of all submodules of M with the form $K + rM$, where $K = Rx_1 + \cdots + Rx_t$ for some $x_i \in N$. Clearly, Σ is non-empty. By part (c), Σ has a maximal element $\acute{N} + rM$, where $\acute{N} = Rx_1 + \cdots + Rx_n$ for some $x_i \in N$. If $\acute{N} + rM \neq N + rM$, then there is $u \in N + rM \setminus \acute{N} + rM$ and the submodule $Ru + \acute{N} + rM$ strictly containing $\acute{N} + rM$, a contradiction. Thus $\acute{N} + rM = N + rM$, as needed. □

Definition 2.18. (a) We say that a submodule N of an R-module M is a *nil-M-Noetherian submodule* if there is $r \in \sqrt{Ann_R(M)}$ such that for each ascending chain

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$$

of submodules of N there is a positive integer t such that $N_{t+i} \subseteq N_t + rM$ for all $i \geq 0$.

(b) We say that a quotient module M/N of an R-module M is a *nil-M-Noetherian quotient module* if there is $r \in \sqrt{Ann_R(M)}$ such that for each ascending chain

$$N_1/N \subseteq N_2/N \subseteq N_3/N \subseteq \ldots \subseteq N_k/N \subseteq \ldots$$

of submodules of M/N there is a positive integer t such that $N_{t+i}/N \subseteq (N_t + rM)/N$ for all $i \geq 0$.

Recall that a submodule N of an R-module M is said to be *pure* in M if $rM \cap N = rN$ for each $r \in R$ [12]. M is said to be *fully pure* if every submodule of M is pure [2].

Proposition 2.19. *Let M be a nil-M-Noetherian R-module. Then we have the following.*

(a) Any pure submodule N of M is a nil-N-Noetherian R-module.

(b) Any quotient module M/N of M is a nil-M/N-Noetherian R-module.

(c) Any submodule N of M is a nil-M-Noetherian R-module.

(d) Any quotient module M/N of M is a nil-M-Noetherian R-module.

Proof. (a) Let N be a pure submodule of M and

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_t \subseteq \ldots$$

be an ascending chain of submodules of N. Then since this chain is also an ascending chain of submodules of M, there is a positive integer s such that $K_{s+i} \subseteq K_s + rM$ for all $i \geq 0$, where $r \in \sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(N)}$. As N is pure, $N \cap rM = rN$. Therefore, for $r \in \sqrt{Ann_R(N)}$ we get that

$$K_{s+i} = K_{s+i} \cap N \subseteq (K_s + rM) \cap N = (K_s \cap N) + (rM \cap N) = K_s + rN,$$

for all $i \geq 0$. Thus N is a nil-N-Noetherian R-module.

(b) Let N be a submodule of M and

$$H_1/N \subseteq H_2/N \subseteq H_3/N \subseteq \ldots \subseteq H_t/N \subseteq \ldots$$

be an ascending chain of submodules of M/N. Then

$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \ldots \subseteq H_t \subseteq \ldots$$

is an ascending chain of submodules of M. Thus there is a positive integer t such that $H_{t+i} \subseteq H_t + rM$ for all $i \geq 0$, where $r \in \sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(M/N)}$. This implies that for all $i \geq 0$, $H_{t+i}/N \subseteq H_t/N + r(M/N)$, where $r \in \sqrt{Ann_R(M/N)}$. Hence, M/N is a nil-M/N-Noetherian R-module.

(c) This follows from the fact that any increasing sequence of submodules of N is also an increasing sequence of submodules of M.

(d) This follows from the fact that any increasing sequence of submodules of M/N corresponds to a sequence of submodules of M containing N, So M/N is a nil-M-Noetherian module. □

Theorem 2.20. Let M be an R-module. Consider the following statements:

(a) Every homomorphic image of M is nil-M-finitely cogenerated R-module;

(b) M is a nil-M-Artinian module;

(c) There exists $r \in \sqrt{Ann_R(M)}$ such that any non-empty collection Σ of submodules of M with the form $K \cap (0 :_M r)$ has a minimal element $N \cap (0 :_M r)$.

(d) Every homomorphic image of M is weakly nil-M-finitely cogenerated.

Then $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, and $(c) \Rightarrow (d)$.

Proof. $(a) \Rightarrow (b)$ Suppose that the following is a descending chain of submodules of M

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \ldots \supseteq N_k \supseteq \ldots.$$

Consider $N := \bigcap_{i\geq 1} N_i$. Then N is a submodule of M, hence M/N is nil-M-finitely generated by part (a). Thus there exist $r \in \sqrt{Ann_R(M)}$ and $N_{j_1}, \ldots, N_{j_t}$ of the above chain such that $N = N_{j_1} \cap \ldots \cap N_{j_t} \cap (0 :_M r)$. Then we can find s such that $N_{j_s} \subseteq N_{j_i}$ for all $i = 1, \ldots, t$. Therefore, $N = N_{j_s} \cap (0 :_M r)$. Now, for all $i \geq 0$ we have $N_{j_s} \cap (0 :_M r) = N \subseteq N_{j_{s+i}}$, as needed.

$(b) \Rightarrow (c)$ First note that by part (b), there is $r \in \sqrt{Ann_R(M)}$ such that for each descending chain

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \ldots \supseteq N_k \supseteq \ldots$$

of submodules of M there is a positive integer t such that $N_t \cap (0 :_M r) \subseteq N_{t+i}$ for all $i \geq 0$. Suppose that there exists a non-empty collection Σ of submodules of M with the form $N \cap (0 :_M r)$ such that it has no minimal element. Let us choose $N_1 \cap (0 :_M r) \in \Sigma$. Since this is not minimal, there is $N_2 \cap (0 :_M r) \in \Sigma$ such that $N_2 \cap (0 :_M r) \subset N_1 \cap (0 :_M r)$, and we continue in this way to construct the following infinite strictly descending chains of submodules of M:

$$N_1 \cap (0 :_M r) \supset N_2 \cap (0 :_M r) \supset N_3 \cap (0 :_M r) \supset \ldots.$$

By part (b), there is a positive integer t such that

$$N_t \cap (0 :_M r) = N_t \cap (0 :_M r) \cap (0 :_M r) \subseteq N_{t+1} \cap (0 :_M r),$$

which is a contradiction.

$(c) \Rightarrow (d)$ Suppose that there exists $r \in \sqrt{Ann_R(M)}$ such that any non-empty collection Σ of submodules of M with the form $K \cap (0 :_M r)$ has a minimal element. Let N be a submodule of M and $N = \cap_{i\in I} N_i$. Consider the family Σ of all submodules of M with the form $\cap_{i\in J} N_i \cap (0 :_M r)$, where J is a finite subset of I. Clearly, Σ is non-empty. By part (c), Σ has a minimal element $\cap_{i\in H} N_i \cap (0 :_M r)$, where H is a finite subset of I. By the minimality, we get that $N \cap (0 :_M r) = \cap_{i\in I} N_i \cap (0 :_M r) = \cap_{i\in H} N_i \cap (0 :_M r)$, as needed. □

Corollary 2.21. *Let M be an R-module. Then we have the following.*

(a) If M is a nil-M-Noetherian R-module, then M is a finitely generated R-module.

(b) If M is a nil-M-Artinian R-module, then M is a finitely cogenerated R-module.

Proof. (a) Assume that M is a nil-M-Noetherian R-module. Then M is a weakly nil-M-finitely generated module by Theorem 2.17. Now the result follows from Corollary 2.16 (b).

(b) Suppose that M is a nil-M-Artinian R-module. Then M is a weakly nil-M-finitely cogenerated module by Theorem 2.20. Now the result follows from Corollary 2.16 (d) □

Definition 2.22. (a) We say that a submodule N of an R-module M is a *nil-M-Artinian submodule* if there is $r \in \sqrt{Ann_R(M)}$ such that for each descending chain

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq ... \supseteq N_k \supseteq \dots$$

of submodules of N there is a positive integer t such that $N_t \cap (0 :_M r) \subseteq N_{t+i}$ for all $i \geq 0$.

(b) We say that a quotient module M/N of an R-module M is a *nil-M-Artinian quotient module* if there is $r \in \sqrt{Ann_R(M)}$ such that for each descending chain

$$N_1/N \supseteq N_2/N \supseteq N_3/N \supseteq ... \supseteq N_k/N \supseteq \dots$$

of submodules of M/N there is a positive integer t such that $(N_t \cap (0 :_M r))/N \subseteq N_{t+i}/N$ for all $i \geq 0$.

A submodule N of an R-module M is said to be *copure* in M if $(N :_M I) = N + (0 :_M I)$ for each ideal I of R [3]. M is said to be *fully copure* if every submodule of M is copure [2].

Proposition 2.23. *Let M be a nil-M-Artinian R-module. Then we have the following.*

(a) Any submodule N of M is a nil-N-Artinian R-module.

(b) Any quotient module M/N of M is a nil-M/N-Artinian R-module, where N is a copure submodule of M.

(c) Any submodule N of M is a nil-M-Artinian R-module.

(d) Any quotient module M/N of M is a nil-M-Artinian R-module.

Proof. (a) Let N be a submodule of M and

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots \supseteq K_t \supseteq \dots$$

be a descending chain of submodules of N. Then since this chain is also a descending chain of submodules of M, there is a positive integer s such that $K_s \cap (0 :_M r) \subseteq K_{s+i}$ for all $i \geq 0$, where $r \in \sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(N)}$. This implies that $K_s \cap (0 :_N r) = N \cap K_s \cap (0 :_M r) \subseteq N \cap K_{s+i} = K_{s+i}$ for $r \in \sqrt{Ann_R(N)}$.

(b) Let N be a copure submodule of M and

$$H_1/N \supseteq H_2/N \supseteq H_3/N \supseteq \dots \supseteq H_t/N \supseteq \dots$$

be a descending chain of submodules of M/N. Then we have the following descending chain of submodules of M.

$$H_1 \supseteq H_2 \supseteq H_3 \supseteq \dots \supseteq H_t \supseteq \dots.$$

Thus there is a positive integer t such that $H_t \cap (0 :_M r) \subseteq H_{t+i}$ for all $i \geq 0$, where $r \in \sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(M/N)}$. Since N is copure, $(N :_M r) = N + (0 :_M r)$. Thus we have

$$H_t \cap (N :_M r) = H_t \cap (N + (0 :_M r)) = (H_t \cap (0 :_M r)) + (N \cap H_t)$$

$$= (H_t \cap (0 :_M r)) + N \subseteq H_{t+i} + N = H_{t+i}.$$

This implies that for all $i \geq 0$, $H_t/N \cap (0 :_{M/N} r) \subseteq H_{t+i}/N$, where $r \in \sqrt{Ann_R(M/N)}$. Hence, M/N is a nil-M/N-Artinian R-module.

(c) This follows from the fact that any descending chain of submodules of N is also a descending chain of submodules of M.

(d) This follows from the fact that any descending chain of submodules of M/N corresponds to a descending chain of submodules of M containing N, So M/N is a nil-M-Artinian module. □

Theorem 2.24. Let M be an R-module. Then we have the following.

(a) If M is a fully pure nil-M-Noetherian R-module, then M is a Noetherian R-module.

(b) If M is a fully copure nil-M-Artinian R-module, then M is an Artinian R-module.

Proof. (a) Assume that M is a fully pure nil-M-Noetherian R-module. Then by Proposition 2.19 (a), every submodule N of M is a nil-N-Noetherian R-module. Now, by Corollary 2.21 (a), every submodule N of M is a finitely generated. Thus M is a Noetherian R-module.

(b) Assume that M is a fully copure nil-M-Artinian R-module. Then by Proposition 2.23 (b), for every submodule N of M we have M/N is a nil-M/N-Artinian R-module. Now, by Corollary 2.21 (b), M/N is a finitely generated. Hence, M is an Artinian R-module. □

Recall that an R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [5]. An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [1]. It is easy to see that M is a multiplication (resp. comultiplication) module if and only if $N = (N :_R M)M$ (resp. $N = (0 :_M Ann_R(N))$) for each submodule N of M.

Proposition 2.25. *Let M be an R-module. Then we have the following.*

(a) If R is a nil-R-Noetherian ring and M is a multiplication R-module, then M is a nil-M-Noetherian module.

(b) If R is a nil-R-Artinian ring and M is a faithful comultiplication R-module, then M is a nil-M-Noetherian module.

Proof. (a) Let

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_t \subseteq \ldots$$

be an ascending chain of submodules of M. Then we have

$$(K_1 :_R M) \subseteq (K_2 :_R M) \subseteq (K_3 :_R M) \subseteq \ldots \subseteq (K_t :_R M) \subseteq \ldots$$

is an ascending chain of ideals of R. Thus by assumption, there exist $r \in \sqrt{0} \subseteq \sqrt{Ann_R(M)}$ and a positive integer t such that $(N_{t+i} :_R M) \subseteq (N_t :_R M) + rR$ for all $i \geq 0$. This implies that

$$N_{t+i} = (N_{t+i} :_R M)M \subseteq (N_t :_R M)M + rM = N_t + rM$$

for all $i \geq 0$, as needed.

(b) Let

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_t \subseteq \ldots$$

be an ascending chain of submodules of M. Then we have

$$Ann_R(K_1) \supseteq Ann_R(K_2) \supseteq Ann_R(K_3) \supseteq \ldots \supseteq Ann_R(K_t) \supseteq \ldots$$

is a descending chain of ideals of R. Thus by assumption, there exist $r \in \sqrt{0} \subseteq \sqrt{Ann_R(M)}$ and a positive integer t such that $Ann_R(N_t) \cap (0 :_R r) \subseteq Ann_R(N_{t+i})$ for all $i \geq 0$. As M is faithful, $Ann_R(r) = Ann_R(rM)$. Thus by using [4, Proposition 3.3], we have

$$N_{t+i} = (0 :_M Ann_R(N_{t+i})) \subseteq (0 :_M Ann_R(N_t) \cap (0 :_R r)) =$$

$$(0 :_M Ann_R(N_t) \cap Ann_R(rM)) = N_t + rM.$$

for all $i \geq 0$, as needed. □

Definition 2.26. Let M be an R-module and f be an endomorphism of M. We say that f is *nil-epic* if there exists $r \in \sqrt{Ann_R(M)}$ such that $(0 :_M r) \subseteq Imf$. Also, we say that f is *nil-monic* if there exists $r \in \sqrt{Ann_R(M)}$ such that $Kerf \subseteq rM$.

Theorem 2.27. Let M be an R-module and f be an endomorphism of M.

(a) If M is a nil-Artinian module, then there exists $r \in \sqrt{Ann_R(M)}$ such that $(0 :_M r) \subseteq Imf^n + Kerf^n$ for some n, whence if f is monic, then f is nil-epic.

(b) If M is a nil-Noetherian module, then there exists $r \in \sqrt{Ann_R(M)}$ such that $Imf^n \cap Kerf^n \subseteq rM$ for some n, whence if f is epic, then f is nil-monic.

Proof. (a) Clearly, we have

$$Imf \supseteq Imf^2 \supseteq \cdots.$$

Thus by assumption, there exists $r \in \sqrt{Ann_R(M)}$ such that $Imf^n \cap (0 :_M r) \subseteq Imf^{2n}$ for some positive integer n. Now, let $x \in (0 :_M r)$. Then $rx = 0$ and so $rf(x) = 0$. Hence, $f(x) \in (0 :_M r)$. This implies that $f^n(x) \in (0 :_M r)$. Therefore, $f^n(x) \in Imf^n \cap (0 :_M r) \subseteq Imf^{2n}$. Thus $f^n(x) = f^{2n}(y)$. Clearly,

$$x = f^n(y) + (x - f^n(y)) \in Imf^n + Kerf^n.$$

Finally, if f is monic, then $Kerf^n = 0$. This implies that $(0 :_M r) \subseteq Imf^n \subseteq Imf$.

(b) Since

$$Kerf \subseteq Kerf^2 \subseteq \cdots,$$

by assumption, there exists $r \in \sqrt{Ann_R(M)}$ such that $Kerf^{2n} \subseteq Kerf^n + rM$ for some positive integer n. Now, let $x \in Imf^n \cap Kerf^n$. Then $x = f^n(y)$ and $f^n(x) = 0$. Thus $f^{2n}(y) = 0$. Therefore, $y \in Kerf^{2n} \subseteq Kerf^n + rM$. Thus

$$x = f^n(y) \in f^n(Kerf^n) + f^n(rM) = rf^n(M) \subseteq rM.$$

Finally, if f is epic, then $Imf^n = M$. It follows that $Kerf \subseteq Kerf^n \subseteq rM$, as needed. □

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On S-J-filters of a bounded lattice

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Abstract. Let $\mathcal{L}$ be a bounded lattice and S denote the join subset of $\mathcal{L}$. In this paper, we introduce the concept of S-J-filters as a new generalization of J-filters. Many of the basic properties and characterizations of this concept are investigated. We study S-J-filters under various contexts of constructions such as the stability of S-J-join subsets in some lattice-theoretic constructions, homomorphic image lattices, quotient lattices, cartesian product lattices, S-finite filters, SJ-finite filters, SJ-Noetherian lattices and filterlization lattices.

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1 Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. Recently, the study of algebraic structures, using the properties of lattice theory, tends to a useful research topic. Associating a lattice with an algebraic structure allows us to obtain characterizations and representations of special classes of algebraic structures in terms of lattices and vice versa (see for instance [5, 6, 8-14]).

Over the years, several types of ideals have been developed in order to let us fully understand the structures of rings in general. The notion of prime ideal plays a key role in the theory of commutative algebra, and it has been widely studied (see, for example, [1]). Recall from [2], a prime ideal P of R is a proper ideal having the property that $ab \in P$ implies either $a \in P$ or $b \in P$ for each $a, b \in R$. In [19], Mohamadian defined a proper ideal I of R as an r-ideal if whenever $a, b \in R$ with $ab \in I$ and $\text{ann}(a) = 0$ imply that $b \in I$. He investigated the behavior of r-ideals and compare them with other classical ideals such as prime and maximal ideals. In [21], Tekir et al., defined and studied some subclass of r-ideals, namely, the class of n-ideals. A proper ideal I of a ring R is called an n-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$, then $b \in I$. Khashan and Bani-Ata generalized the concept of n-ideals in [17] (also see [16] and [18]). A proper ideal I of a ring R is called a J-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then $b \in I$, where $J(R)$ is the Jacobson radical of R. In [4], Abouhalaka et al. defined an ideal I of a ring R disjoint from S is called an S-J-ideal, if there exists an (a fixed) $s \in S$ such that for all $a, b \in R$, if $ab \in I$, then either $sa \in J(R)$ or $sb \in I$, where S is a multiplicative subset of R. Let $\mathcal{L}$ be a bounded distributive lattice and $J(\mathcal{L})$ denote the Jacobson radical of $\mathcal{L}$. J-filters was introduced in some detail in [12]. A proper filter P of $\mathcal{L}$ is called a J-filter if whenever $x \vee y \in P$ with $x \notin J(\mathcal{L})$, then $y \in P$ for every $x, y \in \mathcal{L}$. We say that a subset $S \subseteq \mathcal{L}$ is join if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$. In this paper, we continue this approach and present S-J-filters as a generalization of J-filters. A proper filter F of $\mathcal{L}$ disjoint from S is called an S-J-filter (resp. a weakly S-J-filter) of $\mathcal{L}$, if there exists an (a fixed) $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in F$ (resp. $1 \neq x \vee y \in P$), then either $s \vee x \in J(R)$ or $s \vee y \in F$. The S-J-filter was introduced and studied in [12] in a very special case when

$S = \{0\}$.
Among many other results in this paper, the first, preliminaries section contains elementary observations needed later on. In the third section of this paper, we collect some basic properties of S-J-filters. At first, we give the definition of S-J-filters (Definition 3.1) and we provide an example of lattices for which their S-J-filters and J-ideals are the same (Proposition 3.3). Next, we provide an example (Example 3.4) of a S-J-filter of $\mathcal{L}$ that is not a J-filter (resp. is not prime filter). Many equivalent characterizations of S-J-filters for any lattice are presented in Theorem 3.7, Theorem 3.10 and Proposition 3.13. It is proved in Proposition 3.19 that an intersection of a nonempty family of S-J-filters of $\mathcal{L}$ is an S-J-filter and we provide some condition under which the converse of Proposition 3.19 is true (Theorem 3.20). Theorem 3.22 proves that S-J-filters and S-J-primary filters that contained in the Jacobson radical are the same.

Section 4 is dedicated to the investigation of the stability of S-J-join subsets in some lattice-theoretic constructions. Let X be a nonempty subset of $\mathcal{L}$ with $\mathcal{L} \setminus J(\mathcal{L}) \subseteq X$. Then X is called an S-J-join subset of $\mathcal{L}$ if there exists $s \in S$ such that for all $x, y \in \mathcal{L}$ with $s \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s \vee y \in X$ we have $s \vee x \vee y \in X$. The concept of S-J-join subset of $\mathcal{L}$ introduced in definition 4.2 is of some fundamental importance in the subject. one crucial example of the idea is $\mathcal{L} \setminus (s \vee P)$, where P is an S-J-filter of $\mathcal{L}$ for some $s \in S$ (Theorem 4.3). Similar to the prime ideals (see [20, Theorem 3.44]), it is shown in Theorem 4.5 that if P is a filter of $\mathcal{L}$ and X is an S-J-join subset of $\mathcal{L}$ such that $s \vee X \subseteq X$ and $P \cap X = \emptyset$ fo some $s \in S$, then there exists an S-J-filter K containing P such that $K \cap X = \emptyset$. Also, similar to minimal prime ideals (see [15, Theorem 2.1]), it is proved in Theorem 4.6 that if Q is an S-J-filter of $\mathcal{L}$, P is a filter of $\mathcal{L}$ with $P \subseteq s \vee Q$ for some $s \in S$ and $s \vee Q \in \min_{SJ}(P)$, then for each $x \in s \vee Q$, there is some $y \notin s \vee Q$ such that $s \vee x \vee y \in P$ (also, see Corollary 4.8 and Corollary 4.9).

In Section 5, we investigate the properties of S-J-filters similar to prime ideals. In particular, we investigate the behavior of S-J-filters under homomorphism, in factor lattices, $\vee$-lattices, cartesian product lattices, S-finite filters, SJ-finite filters, SJ-Noetherian lattices and filterlization lattices (see Theorem 5.1, Theorem 5.3, Theorem 5.5, Proposition 5.8, Theorem 5.13 and Theorem 5.14).

2 Preliminaries

A poset $(\mathcal{L}, \leq)$ is a lattice if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the meet and $\vee$ the join). A lattice $\mathcal{L}$ is complete when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in $\mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\mathcal{L}$). A non-empty subset F of a lattice $\mathcal{L}$ is called a filter, if for $a \in F$, $b \in \mathcal{L}$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\mathcal{L}$ is a lattice with 0 and 1, then $1 \in F$, $\{1\}$ is a filter of £ and $0 \in F$ if and only if $F = \mathcal{L}$). A proper filter F of $\mathcal{L}$ is called prime if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter F of $\mathcal{L}$ is said to be maximal if G is a filter in $\mathcal{L}$ such that $F \subseteq G$ with $F \neq G$, then $G = \mathcal{L}$. We define the Jacobson radical of $\mathcal{L}$, denoted by $J(\mathcal{L})$, to be the intersection of all the maximal filters of $\mathcal{L}$. A proper filter P of a lattice $\mathcal{L}$ is called a J-filter if whenever $x, y \in \mathcal{L}$ with $x \vee y \in P$ and $x \notin J(\mathcal{L})$, then $y \in P$. Assume that P is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ disjoint with P. We say that P is an S-prime filter of $\mathcal{L}$ if there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$ if $x \vee y \in P$, then $s \vee x \in P$ or $s \vee y \in P$. Let D be a subset of a lattice $\mathcal{L}$. Then the filter generated by D, denoted by $T(D)$, is the intersection of all filters that is containing D. A filter F is called finitely generated if there is a finite subset D of F such that $F = T(D)$. A lattice $\mathcal{L}$ with 1 is called $\mathcal{L}$-domain if $a \vee b = 1$ $(a, b \in \mathcal{L})$, then $a = 1$ or $b = 1$ (so $\mathcal{L}$ is $\mathcal{L}$-domain if and only if $\{1\}$ is a prime filter of $\mathcal{L}$). If $x \in \mathcal{L}$, then a complement of x in $\mathcal{L}$ is an element $y \in \mathcal{L}$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice $\mathcal{L}$ is complemented if every element of $\mathcal{L}$ has a complement in $\mathcal{L}$. If $\mathcal{L}$ and $\mathcal{L}'$ are lattices, then a lattice homomorphism $f : \mathcal{L} \to \mathcal{L}'$ is a map from $\mathcal{L}$ to $\mathcal{L}'$ satisfying $f(x \vee y) = f(x) \vee f(y)$ and

$f(x \wedge y) = f(x) \wedge f(y)$ for $x, y \in \mathcal{L}$. Let X be a nonempty subset of $\mathcal{L}$ and $a \in \mathcal{L}$. By $a \vee X$, we mean the set $\{a \vee x : x \in X\}$. Assume that $(\mathcal{L}_1, \leq_1), (\mathcal{L}_2, \leq_2)$ are lattices and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2)$ and $x \wedge_c y = (x_1 \wedge y_1, x_2 \wedge y_2)$. In this case, we say that $\mathcal{L}$ is a decomposable lattice. An element x of $\mathcal{L}$ is called identity join of a lattice $\mathcal{L}$, if there exists $1 \neq y \in \mathcal{L}$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\mathcal{L}$ is denoted by $\mathrm{I}(\mathcal{L})$. A lattice $\mathcal{L}$ is called presimplifiable if $\mathrm{I}(\mathcal{L}) \subseteq J(\mathcal{L})$. First we need the following easy observation proved in [5, 6, 8, 9].

Lemma 2.1. *Let $\mathcal{L}$ be a lattice.*

(1) A non-empty subset F of $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \mathcal{L}$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \mathcal{L}$.

(2) If F_1, F_2 are filters of $\mathcal{L}$ and $a \in \mathcal{L}$, then $F_1 \vee F_2 = \{f_1 \vee f_2 : f_1 \in F_1, f_2 \in F_2\}$ and $a \vee F_1 = \{a \vee f_1 : f_1 \in F_1\}$ are filters of $\mathcal{L}$ and $F_1 \vee F_2 = F_1 \cap F_2$.

(3) Assume that A is an arbitrary non-empty subset of $\mathcal{L}$ and let $a \in \mathcal{L}$. Then $T(A) = \{x \in \mathcal{L} : a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x$ for some $a_i \in A$ $(1 \leq i \leq n)\}$ and $T(\{a\}) = T(a) = \{x \vee a : x \in \mathcal{L}\}$.

(4) If $\mathcal{L}$ is distributive, F, G are filters of $\mathcal{L}$ and $y \in \mathcal{L}$, then $(G :_{\mathcal{L}} F) = \{x \in \mathcal{L} : x \vee F \subseteq G\}$, $(F :_{\mathcal{L}} T(y)) = (F :_{\mathcal{L}} y) = \{a \in \mathcal{L} : a \vee y \in F\}$ and $(1 :_{\mathcal{L}} y) = \{x \in \mathcal{L} : x \vee y = 1\}$ are filters of $\mathcal{L}$.

(5) If $\{F_i\}_{i \in \Delta}$ is a chain of filters of $\mathcal{L}$, then $\bigcup_{i \in \Delta} F_i$ is a filter of $\mathcal{L}$.

(6) If $\mathcal{L}$ is distributive and F_1, F_2 are filters of $\mathcal{L}$, then $F_1 \wedge F_2 = \{f_1 \wedge f_2 : f_1 \in F_1, f_2 \in F_2\}$ is a filter of $\mathcal{L}$ and $F_1, F_2 \subseteq F_1 \wedge F_2$.

(7) If $f : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism with $f(1) = 1$, then $\mathrm{Ker}(f) = \{x \in \mathcal{L} : f(x) = 1\}$ is a filter of $\mathcal{L}$.

For undefined notations or terminologies in lattice theory, we refer the reader to [5, 6].

3 Properties of S-J-filters

In this paper, by $\mathcal{L}$ we mean a bounded distributive lattice, and S will denote a join subset of $\mathcal{L}$. We remind the reader of the following definition.

Definition 3.1. A filter P of $\mathcal{L}$ disjoint from S is called an S-J-filter of $\mathcal{L}$, if there exists an (a fixed) $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in P$, then either $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P$.

Remark 3.2. (1) Due to the symmetry between x and y, we can show that if P is an S-J-filter with $s \vee x, s \vee y \notin P$, then $s \vee x, s \vee y \in J(\mathcal{L})$.

(2) If $S = \{0\}$, then the J-filters and S-J-filters of $\mathcal{L}$ are the same.

(3) By definitions, it is clear that $J(\mathcal{L})$ is an S-J-filter of $\mathcal{L}$ if and only if $J(\mathcal{L})$ is S-prime.

(4) By definitions, it is clear that every J-filter is S-J-filter.

Referring to [12, Theorem 3.4] states that $\mathcal{L}$ is quasi-local if and only if every proper filter is a J-filter. The next result determines the class of lattices for which their J-filters and S-J-filters are the same.

Proposition 3.3. *If $\mathcal{L}$ is a quasi-local lattice with unique maximal filter M, then every proper filter of $\mathcal{L}$ is an S-J-filter.*

Proof. Let P be a proper filter of $\mathcal{L}$ and let $x, y \in \mathcal{L}$ such that $x \vee y \in P$ and $s \vee x \notin J(\mathcal{L}) = M$ for $s \in S$. Then $M \wedge T(s \vee x) = \mathcal{L}$ by maximality of M. If $T(s \vee x) \neq \mathcal{L}$, then $T(s \vee x) \subseteq M$ by [11, Lemma 2.1] gives $M = \mathcal{L}$, a contradiction. Therefore, $T(s \vee x) = \mathcal{L}$. So $0 = s \vee x \vee z$ for some $z \in \mathcal{L}$. Then $x = 0$ gives $y \in P$ and so $s \vee y \in P$. Thus, P is an S-J-filter. □

Example 3.4. (1) Let $\mathcal{L}_1 = \{0, a, b, c, 1\}$ be a lattice with the relations $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$. Consider $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_1$, $P = \{1, c\} \times \{1\} = \{(1,1), (c,1)\}$ and $S = \{0, c\} \times \{0, c\} = \{(0,0), (0,c), (c,0), (c,c)\}$. Then P is a filter of $\mathcal{L}$ with $P \cap S = \emptyset$, $J(\mathcal{L}_1) = \{a, c, 1\} \cap \{b, c, 1\} = \{1, c\}$ and $J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_1) = \{1, c\} \times \{1, c\} = \{(1,1), (1,c), (c,1), (c,c)\}$. Now, we show that P is an S-J-filter. Let $(x, y) \vee_c (x', y') = (x \vee x', y \vee y) \in P$ for some $(x, y), (x', y') \in \mathcal{L}$. Then by the definition of $\mathcal{L}_1$, $y \vee y' = 1$ gives either $y = 1$ or $y' = 1$. Without loss of generality, we can assume that $y = 1$. Then $(c, c) \vee_c (x, 1) = (c \vee x, 1) \in J(\mathcal{L})$ since $c \vee x = c$ or $c \vee x = 1$. Therefore, P is an S-J-filter of $\mathcal{L}$.

On the other hand, P is not a J-filter since $(a, b) \vee (b, 1) = (c, 1) \in P$ but neither $(a, b) \in P$ nor $(b, 1) \in J(\mathcal{L})$. Thus an S-J-filter need not be a J-filter.

(2) The collection of ideals of $\mathbb{Z}$, the ring of integers, form a lattice under set inclusion which we shall denote by $\mathcal{L}$ with respect to the following definitions: $m\mathbb{Z} \vee n\mathbb{Z} = (m, n)\mathbb{Z}$ and $m\mathbb{Z} \wedge n\mathbb{Z} = [m, n]\mathbb{Z}$ for all ideals $m\mathbb{Z}$ and $n\mathbb{Z}$ of $\mathbb{Z}$, where (m, n) and $[m, n]$ are greatest common divisor and least common multiple of m, n, respectively. Note that $\mathcal{L}$ is a distributive complete lattice with least element the zero ideal and the greatest element $\mathbb{Z}$. By [8, Theorem 2.9 (ii)], $\mathcal{L} \setminus \{0\}$ is the only maximal filter of $\mathcal{L}$ and so $\mathcal{L}$ is a quasi-local lattice. It follows from Proposition 3.3 that every proper filter of $\mathcal{L}$ is an S-J-filter. Consider $P = \{\mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}, \cdots\}$. Since $5\mathbb{Z} \vee 7\mathbb{Z} = \mathbb{Z} \in P$ with $5\mathbb{Z}, 7\mathbb{Z} \notin F$, we infer that P is not a prime (maximal) filter.

Proposition 3.5. *If P is an S-J-filter of $\mathcal{L}$, then $s \vee P \subseteq J(\mathcal{L})$ for some $s \in S$. In particular, if $J(\mathcal{L}) = \{1\}$, then $s \vee P = \{1\}$ for some $s \in S$.*

Proof. We keep in mind that there exists a fixed $s \in S$ that satisfies the S-J-filter condition. On the contrary, assume that $s \vee P$ is not a subset of $J(\mathcal{L})$. Then there exists $p \in P$ such that $s \vee p \notin J(\mathcal{L})$. Now $p = p \vee 0 \in P$ and $s \vee p \notin J(\mathcal{L})$ implies that $s = s \vee 0 \in P \cap S$, a contradiction. Hence, $s \vee P \subseteq J(\mathcal{L})$. The "in particular" statement is clear. □

Corollary 3.6. *If P is a J-filter of $\mathcal{L}$, then $P \subseteq J(\mathcal{L})$. In particular, if $J(\mathcal{L}) = \{1\}$, then $P = \{1\}$.*

Proof. Take $S = \{0\}$ in Proposition 3.5. □

In the following theorem we give one other characterizations of S-J-filters.

Theorem 3.7. If P is a filter of $\mathcal{L}$ with $S \cap P = \emptyset$, then the following assertions are equivalent:

(1) P is an S-J-filter of $\mathcal{L}$;

(2) There exists $s \in S$ such that for all F, G two filters of $\mathcal{L}$, if $F \vee G \subseteq P$, then either $s \vee F \subseteq J(\mathcal{L})$ or $s \vee G \subseteq P$.

Proof. (1) $\Rightarrow$ (2) Since P is an S-J-filter, we infer that there is an element $s \in S$ such that for all $a, b \in \mathcal{L}$, if $a \vee b \in P$, then $s \vee a \in J(\mathcal{L})$ or $s \vee b \in P$. On the contrary, assume that for all $t \in S$, there are F_t, G_t two filters of $\mathcal{L}$ with $F_t \vee G_t \subseteq P$, but $t \vee F_t \nsubseteq J(\mathcal{L})$ and $t \vee G_t \nsubseteq P$. So there exist F_s, G_s two filters of $\mathcal{L}$ with $F_s \vee G_s \subseteq P$, but $s \vee F_s \nsubseteq J(\mathcal{L})$ and $s \vee G_s \nsubseteq P$, as $s \in S$. This shows that there exist $a_s \in F_s$ and $b_s \in G_s$ such that $a_s \vee b_s \in P$, $s \vee a_s \notin J(\mathcal{L})$ and $s \vee b_s \notin P$ which is impossible, as P is an S-J-filter, i.e. (2) holds.

(2) $\Rightarrow$ (1) Let $a, b \in \mathcal{L}$ such that $a \vee b \in P$. Set $F = T(a)$ and $G = T(b)$. Then $F \vee G \subseteq P$ gives there exits $s \in S$ such that $s \vee a \in s \vee F \subseteq J(\mathcal{L})$ or $s \vee b \in s \vee G \subseteq P$ by (2), i.e. (1) holds. □

Corollary 3.8. *If P is a filter of $\mathcal{L}$, then P is a J-filter of $\mathcal{L}$ if and only if for all filters F, G of $\mathcal{L}$, $F \vee G \subseteq P$ implies that $F \subseteq J(\mathcal{L})$ or $G \subseteq P$.*

Proof. Take $S = \{0\}$ in Theorem 3.7. □

Proposition 3.9. *Let P be a filter of $\mathcal{L}$ disjoint from S. If there exists an $s \in S$ such that $(P :_{\mathcal{L}} s)$ is a J-filter, then P is an S-J-filter of $\mathcal{L}$.*

Proof. Let $x \vee y \in P$ for some $x, y \in \mathcal{L}$. Then $x \vee y \in (P :_{\mathcal{L}} s)$ gives $x \in J(\mathcal{L})$ or $y \in (P :_{\mathcal{L}} s)$ which implies that $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P.$, as needed. □

The following theorem provides some condition under which the converse of Proposition 3.9 is true.

Theorem 3.10. Suppose that P is a filter of $\mathcal{L}$ disjoint from S and $J(\mathcal{L})$ is a J-filter of $\mathcal{L}$ such that $J(\mathcal{L}) \cap S = \emptyset$. Then P is an S-J-filter of $\mathcal{L}$ if and only if there exists $s \in S$ such that $(P :_{\mathcal{L}} s)$ is a J-filter of $\mathcal{L}$.

Proof. One side is clear by Proposition 3.9. To see the other side, assume that P is an S-J-filter of $\mathcal{L}$. Suppose that $s \in S$ satisfies S-J-filter condition for P. Let $x \vee y \in (P :_{\mathcal{L}} s)$ for some $x, y \in \mathcal{L}$ and $x \notin J(\mathcal{L})$. Then $x \vee (s \vee y) \in P$ gives either $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P$. If $s \vee x \in J(\mathcal{L})$, then $x \in J(\mathcal{L})$ since $J(\mathcal{L}) \cap S = \emptyset$, a contradiction. Therefore, $s \vee y \in P$ and so $y \in (P :_{\mathcal{L}} s)$. □

Proposition 3.11. *Let P be a filter of $\mathcal{L}$ disjoint from S. Then, P is an S-J-filter if and only if for some $s \in S$, $(P :_{\mathcal{L}} a) \subseteq (J(\mathcal{L}) :_{\mathcal{L}} s)$ for all $a \notin (P :_{\mathcal{L}} s)$.*

Proof. Suppose that P is an S-J-filter of $\mathcal{L}$. Let $s \in S$ satisfies S-J-filter condition for P. Now, let $x \in (P :_{\mathcal{L}} a)$, where $a \notin (P :_{\mathcal{L}} s)$ (so $a \vee s \notin P$). Then by assumption, $a \vee x \in P$ gives $s \vee x \in J(\mathcal{L})$, i.e. $x \in (J(\mathcal{L}) :_{\mathcal{L}} s)$. Conversely, assume that $x \vee y \in P$ for some $x, y \in \mathcal{L}$. If $s \vee y \notin P$ (i.e. $y \notin (P :_{\mathcal{L}} s)$), then by the hypothesis, $x \in (P :_{\mathcal{L}} y) \subseteq (J(\mathcal{L}) :_{\mathcal{L}} s)$. This shows that $s \vee x \in J(\mathcal{L})$. □

Corollary 3.12. *P is a J-filter of $\mathcal{L}$ if and only if $(P :_{\mathcal{L}} a) \subseteq J(\mathcal{L})$ for all $a \notin P$.*

Proof. Take $S = \{0\}$ in Proposition 3.11. □

Proposition 3.13. *Let P be a filter of $\mathcal{L}$ disjoint from S. Then, P is an S-J-filter if and only if for some $s \in S$, $(P :_{\mathcal{L}} a) \subseteq (P :_{\mathcal{L}} s)$ for all $a \notin (J(\mathcal{L}) :_{\mathcal{L}} s)$.*

Proof. Suppose that P is an S-J-filter of $\mathcal{L}$ and let $s \in S$ satisfies S-J-filter condition for P. Now, let $x \in (P :_{\mathcal{L}} a)$, where $a \notin (J(\mathcal{L}) :_{\mathcal{L}} s)$ (so $s \vee a \notin J(\mathcal{L})$). Then by assumption, $a \vee x \in P$ gives $s \vee x \in P$, i.e. $x \in (P :_{\mathcal{L}} s)$. Conversely, assume that $x \vee y \in P$ for some $x, y \in \mathcal{L}$. If $s \vee x \notin J(\mathcal{L})$ (i.e. $x \notin (J(\mathcal{L}) :_{\mathcal{L}} s)$), then by the hypothesis, $y \in (P :_{\mathcal{L}} x) \subseteq (P :_{\mathcal{L}} s)$. This shows that $s \vee y \in P$. □

Corollary 3.14. *P is a J-filter of $\mathcal{L}$ if and only if $(P :_{\mathcal{L}} a) = P$ for all $a \notin J(\mathcal{L})$.*

Proof. Take $S = \{0\}$ in Proposition 3.13. □

Corollary 3.15. *If P is a proper filter of a lattice $\mathcal{L}$, then the following statements are equivalent:*
(1) P is a J-filter of $\mathcal{L}$;
(2) $P = (P :_{\mathcal{L}} x)$ for every $x \notin J(\mathcal{L})$;
(3) For filters G and K of $\mathcal{L}$, $G \vee K \subseteq P$ and G is not subset of $J(\mathcal{L})$ gives $K \subseteq P$;
(4) $(P :_{\mathcal{L}} x) \subseteq J(\mathcal{L})$ for every $x \notin P$.

Proof. This is a direct consequence of Corollary 3.8, Corollary 3.12 and Corollary 3.14. □

Proposition 3.16. *If P is an S-J-filter of $\mathcal{L}$ and $a \notin P$, then $(P :_{\mathcal{L}} a)$ is an S-J-filter.*

Proof. Since P is an S-J-filter, we conclude that there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in P$, then $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P$. It suffices to show that there exists $u \in S$ such that for all filters F, G of $\mathcal{L}$, if $F \vee G \subseteq (P :_{\mathcal{L}} a)$, then either $u \vee F \subseteq J(\mathcal{L})$ or $u \vee G \subseteq (P :_{\mathcal{L}} a)$ by Theorem 3.7. On the contrary, assume that for all $t \in S$, there are F_t, G_t two filters of $\mathcal{L}$ with $F_t \vee G_t \subseteq P$, but $t \vee F_t$ is not a subset of $J(\mathcal{L})$ and $t \vee G_t$ is not a subset of $(P :_{\mathcal{L}} a)$. So there exist F_s, G_s two filters of $\mathcal{L}$ with $F_s \vee G_s \subseteq (P :_{\mathcal{L}} a)$, but $s \vee F_s$ is not a subset of $J(\mathcal{L})$ and $s \vee G_s$ is not a subset of $(P :_{\mathcal{L}} a)$, as $s \in S$. This shows that there exist $a_s \in F_s$ and $b_s \in G_s$ such that $a_s \vee b_s \vee a \in P$, $s \vee a_s \notin J(\mathcal{L})$ and $s \vee (b_s \vee a) \notin P$ which is impossible, as P is an S-J-filter, i.e. $(P :_{\mathcal{L}} a)$ is an S-J-filter. □

An S-J-filter P of $\mathcal{L}$ is called a maximal S-J-filter if there is no S-J-filter which contains P properly.

Theorem 3.17. If P is a maximal S-J-filter of $\mathcal{L}$, then P is a prime filter. If, in particular, $P = (J(\mathcal{L}) :_{\mathcal{L}} s)$ for some $s \in S$, then the converse is true.

Proof. Suppose that P is a maximal S-J-filter and let $x, y \in \mathcal{L}$ such that $x \vee y \in P$ and $x \notin P$. Then $(P :_{\mathcal{L}} x)$ is S-J-filter by Proposition 3.15 and $P \subseteq (P :_{\mathcal{L}} x)$. By Maximality of P, we have $y \in (P :_{\mathcal{L}} x) = P$ and hence P is a pime. Suppose that $P = (J(\mathcal{L}) :_{\mathcal{L}} s)$ for some $s \in S$ and let $a \vee b \in P$ for some $a, b \in \mathcal{L}$. Since P is prime, we conclude that either $s \vee a \in P$ or $s \vee b \in J(\mathcal{L})$. So P is an S-J-filter of $\mathcal{L}$. Let Q be any S-J-filer of $\mathcal{L}$. Then by Proposition 3.5, $Q \subseteq (J(\mathcal{L}) :_{\mathcal{L}} s)$, which shows the maximality of P. □

Theorem 3.18. If $J(\mathcal{L}) = \{1\}$, then the following statements are equivalent:
(1) $\mathcal{L}$ is an $\mathcal{L}$-domain;
(2) The filter $\{1\}$ is the only S-J-filter of $\mathcal{L}$;
(3) $(1 :_{\mathcal{L}} x \vee y) = (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$ for every $x, y \in \mathcal{L}$.

Proof. (1) $\Rightarrow$ (2) If P is a S-J-filter of $\mathcal{L}$, then $s \vee P = \{1\}$ for some $s \in S$ by Proposition 3.5. If $p \in P$, then $s \vee p = 1$ gives $p = 1$ and so $P = \{1\}$, as $\mathcal{L}$ is a $\mathcal{L}$-domain. Clearly, $P = \{1\}$ is an S-J-filter.

(2) $\Rightarrow$ (3) Assume that $s \in S$ satisfies S-J-filter condition for $\{1\}$. At first, we show that $(1 :_{\mathcal{L}} z)$ is a S-J-filter for each $1 \neq z \in \mathcal{L}$. Let $a, b \in \mathcal{L}$ such that $a \vee b \in (1 :_{\mathcal{L}} z)$ and $s \vee a \notin J(\mathcal{L})$. Then $\{1\}$ is an S-J-filter gives $s \vee b \vee z = 1$; hence $s \vee b \in (1 :_{\mathcal{L}} z)$. Therefore, by our hypothesis, we have $(1 :_{\mathcal{L}} z) = \{1\}$ for each $1 \neq z \in \mathcal{L}$. This immediately implies that $(1 :_{\mathcal{L}} x \vee y) = (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$ for every $x, y \in \mathcal{L}$.

(3) $\Rightarrow$ (1) Let $x \vee y = 1$, where $x, y \in \mathcal{L}$. Then $\mathcal{L} = (1 :_{\mathcal{L}} x \vee y) = (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$ implies that $0 \in (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$. This means that $x = 1$ or $y = 1$, i.e. (1) holds. □

Proposition 3.19. *If $\{P_i\}_{i \in \Delta}$ is a nonempty family of S-J-filters of $\mathcal{L}$, then $P = \cap_{i \in \Delta} P_i$ is an S-J-filter of $\mathcal{L}$.*

Proof. Clearly, $P \cap S = \emptyset$. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P$ and $s \vee x \notin J(\mathcal{L})$ for some $s \in S$. Then $a \vee b \in P_i$ for each $i \in \Delta$ gives $s \vee b \in P_i$ for all $i \in \Delta$, as they are S-J-filters; hence $s \vee b \in P$ and so the result holds. □

The following theorem provides some condition under which the converse Proposition 3.19 is true.

Theorem 3.20. Let $P_1, \cdots, P_n$ be prime filters of $\mathcal{L}$ which are not comparable. Then $\cap_{i=1}^{n} P_i$ is an S-J-filter of $\mathcal{L}$ if and only if P_i is an S-J-filter for $i \in \{1, \cdots, n\}$.

Proof. One side is clear by Proposition 3.19. To see the other side, assume that $s \in S$ satisfies S-J-filter condition for $\cap_{i=1}^{n} P_i$. Let $x \vee y \in P_i$ for some $x, y \in \mathcal{L}$ and take $z \in (\cap_{i \neq j} P_j) \setminus P_i$. Therefore, $x \vee y \vee z \in \cap_{i=1}^{n} P_i$. Since $\cap_{i=1}^{n} P_i$ is an S-J-filter, we conclude that either $s \vee x \in J(\mathcal{L})$ or $s \vee y \vee z \in \cap_{i=1}^{n} P_i$ and so $s \vee y \vee z \in P_i$. This implies that $s \vee y \in P_i$, i.e. P_i is a S-J-filter of $\mathcal{L}$. □

For a filter P of a lattice $\mathcal{L}$, the Jacobson radical of P, denoted by $J(P)$, is defined as the intersection of all maximal filters of $\mathcal{L}$ containing P.

Definition 3.21. A filter P of $\mathcal{L}$ with $P \cap S = \emptyset$ is called an S-J-primary, if there exists an (a fixed) $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in P$, then either $s \vee x \in J(P)$ or $s \vee y \in P$.

In the following result, we show that S-J-filters and S-J-primary filters that contained in $J(\mathcal{L})$ are the same.

Theorem 3.22. If P is a proper filter of a lattice $\mathcal{L}$ such that $P \subseteq J(\mathcal{L})$, then P is an S-J-filter if and only if P is S-J-primary.

Proof. Suppose P is an S-J-filter of $\mathcal{L}$. Assume that $s \in S$ satisfies S-J-filter condition and let $x \vee y \in P$ with $s \vee x \notin J(P)$. Since $J(\mathcal{L}) \subseteq J(P)$, we conclude thats $\vee x \notin J(\mathcal{L})$. Now, P is an S-J-filter gives $s \vee y \in P$ and so P is an S-J-primary filter of $\mathcal{L}$. Conversely, assume that P is a S-J-primary filter of $\mathcal{L}$ and let $a \vee b \in P$ with $s \vee a \notin J(\mathcal{L})$ for some $s \in S$. Now, $P \subseteq J(\mathcal{L})$ implies that $J(P) \subseteq J(J(\mathcal{L}) = J(\mathcal{L})$ by [12, Lemma 3.17]. Therefore, $s \vee a \notin J(P)$ and so $s \vee b \in P$, as P is S-J-primary. Hence, P is an S-J-filter. □

4 properties of S-J-join subsets

We continue this section with the investigation of the stability of S-J-join subsets in some ring-theoretic constructions. Let us begin this section with the following lemma.

Lemma 4.1. *If P is an S-J-filter of $\mathcal{L}$, then $s \vee P$ is an S-J-filter of $\mathcal{L}$ for some $s \in S$.*

Proof. If $s \vee x \in s \vee P$ for some $x \in P$, then $s \vee x \in P$, as P is a filter; hence $s \vee P \subseteq P$. Suppose that $s \in S$ satisfies S-J-filter condition for P and let $a \vee b \in s \vee P \subseteq P$ for some $a, b \in \mathcal{L}$. Since P is an S-J-filter, we conclude that either $s \vee a \in J(\mathcal{L})$ or $s \vee b \in P$ which implies that either $s \vee a \in J(\mathcal{L})$ or $s \vee b = s \vee (s \vee b) \in s \vee P$. Therefore, $s \vee P$ is an S-J-filter of $\mathcal{L}$. □

Definition 4.2. Let $\mathcal{T}$ be a nonempty subset of $\mathcal{L}$ with $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{T}$. Then $\mathcal{T}$ is called a S-J-join subset of $\mathcal{L}$ if there exists $s \in S$ such that for all $x, y \in \mathcal{L}$ with $s \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s \vee y \in \mathcal{T}$ we have $s \vee x \vee y \in \mathcal{T}$.

In the following proposition, we describe the relation between S-J-filters and S-J-join-subsets of $\mathcal{L}$.

Theorem 4.3. If P is a proper filter of $\mathcal{L}$, then the following statements are equivalent:
(1) P is an S-J-filter of $\mathcal{L}$;
(2) $\mathcal{L} \setminus (s \vee P)$ is a J-join-subset of $\mathcal{L}$ for some $s \in S$.

Proof. (1) $\Rightarrow$ (2) We keep in mind that there exists a fixed $s \in S$ that satisfies the S-J-filter condition for P. Since $s \vee P \subseteq J(\mathcal{L})$ by Proposition 3.4, we conclude that $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{L} \setminus (s \vee P)$. Let $s \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s \vee y \in \mathcal{L} \setminus (s \vee P)$. We show that $s \vee x \vee y \in \mathcal{L} \setminus (s \vee P)$. On the contrary, assume that $s \vee x \vee y \in s \vee P$. Since $s \vee P$ is an S-J-filter by Lemma 4.1 and $s \vee x \notin J(\mathcal{L})$, we infer that $s \vee y \in s \vee P$ which is a contradiction. Thus, $s \vee x \vee y \in \mathcal{L} \setminus (s \vee P)$ and so $\mathcal{L} \setminus (s \vee P)$ is an S-J-join subset of $\mathcal{L}$.
(2) $\Rightarrow$ (1) Let $x, y \in \mathcal{L}$ and $x \vee y \in P$ (so $s \vee x \vee y \in s \vee P$) with $s \vee x \notin J(\mathcal{L})$. Then we have $s \vee y \in s \vee P \subseteq P$ since otherwise we would have $s \vee x \vee y \in \mathcal{L} \setminus (s \vee P)$ which is impossible by (2), i.e. (1) holds. □

Proposition 4.4. *For S-J-filters $P_1, \cdots, P_n$ of $\mathcal{L}$, let $s_i \in S$ satisfies S-J-filter condition for P_i, where $i \in \{1, \cdots, n\}$. Then $\mathcal{T} = \cap_{i=1}^{n} (\mathcal{L} \setminus (s_i \vee P_i))$ is an S-J-join subset of $\mathcal{L}$.*

Proof. It is clear that $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{T}$. Set $s = \vee_{i=1}^{n} s_i$. Let $x, y \in \mathcal{L}$ such that $s \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s \vee y \in \mathcal{T}$. Then for each $i \in \{1, \cdots, n\}$, $s_i \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s_i \vee y \in \mathcal{L} \setminus (s_i \vee P_i)$, as $J(\mathcal{L})$, $s_i \vee P_i$ are filters of $\mathcal{L}$. This shows that $s_i \vee x \vee y \in \mathcal{L} \setminus (s_i \vee P_i)$ for each $i \in \{1, \cdots, n\}$ by Proposition 4.3. We claim that $s \vee x \vee y = ((\vee_{i \neq j}^{n} s_j) \vee x) \vee (s_i \vee y) \in \mathcal{L} \setminus (s_i \vee P_i)$ for all $i \in \{1, \cdots, n\}$. Otherwise, $s_i \vee P_i$ is an S-J-filter gives either $s \vee x \in J(\mathcal{L})$ or $s_i \vee y \in s_i \vee P_i$ which is impossible. This implies that $s \vee x \vee y \in \mathcal{T}$, as needed. □

The following Theorem is a lattice counterpart of Theorem 3.44 in [20] describing the structure of S-J-filters.

Theorem 4.5. Suppose P is a filter of $\mathcal{L}$ and let $\mathcal{T}$ be a S-J-join subset of $\mathcal{L}$ such that $s \in S$ satisfies S-J-join subset condition for $\mathcal{T}$, $s \vee \mathcal{T} \subseteq \mathcal{T}$ and $P \cap \mathcal{T} = \emptyset$. Then there exists an S-J-filter Q containing P such that $Q \cap \mathcal{T} = \emptyset$.

Proof. Let Δ be the set of all filters G of $\mathcal{L}$ such that $P \subseteq G$ and $G \cap \mathcal{T} = \emptyset$. Since $P \in \Delta$, we conclude that $\Delta \neq \emptyset$. Clearly, $(\Delta, \subseteq)$ is a partial order. Let $\{P_i\}_{i \in \Lambda}$ be a chain of elements of Δ and set $H = \bigcup_{i \in \Lambda} P_i$. It is clear that H is a filter of $\mathcal{L}$ and $H \in \Delta$. Thus H is an upper bound of the chain $\{P_i\}_{i \in \Lambda}$; hence by Zorn's Lemma, Δ has a maximal element Q containing P such that $Q \cap \mathcal{T} = \emptyset$. It remains to show that Q is an S-J-filter of $\mathcal{L}$. Assume that $s \in S$ satisfies S-J-join subset condition for $\mathcal{T}$. On the contrary, let $x \vee y \in Q$ such that $s \vee x \notin J(\mathcal{L})$ and $s \vee y \notin Q$ (so $y \notin Q$). Since $Q \subsetneq (Q :_{\mathcal{L}} x)$, we infer that $(Q :_{\mathcal{L}} x) \cap \mathcal{T} \neq \emptyset$ by maximality of Q and so there exists $t \in \mathcal{T}$ with $s \vee t \in s \vee \mathcal{T} \subseteq \mathcal{T}$ such that $t \vee x \in Q$. Since $s \vee x \notin J(\mathcal{L})$ and $s \vee t \in \mathcal{T}$, we conclude that $s \vee t \vee x \in Q \cap \mathcal{T}$ which is a contradiction. Therefore, Q is an S-J-filter of $\mathcal{L}$ with $Q \cap \mathcal{T} = \emptyset$ and $P \subseteq Q$. □

If P is a filter in $\mathcal{L}$, the set of all minimal S-J-filters (resp. the set of all minimal J-filters) over P will be denoted by $\min_{SJ}(P)$ (resp. $\min_J(P)$) (set $\min_{SJ}(\{1\}) = \min_{SJ}(\mathcal{L})$ and $\min_J(\{1\}) = \min_J(\mathcal{L})$)).

The following theorem is a weakly S-J-filters counterpart of Theorem 2.1 in [15] describing the structure of such filters.

Theorem 4.6. Suppose that Q is an S-J-filter of $\mathcal{L}$ such that $s \in S$ satisfies S-J-filter condition for Q and let P be a filter of $\mathcal{L}$ with $P \subseteq s \vee Q$. If $s \vee Q \in \min_{SJ}(P)$, then for each $x \in s \vee Q$, there is some $y \notin s \vee Q$ such that $s \vee x \vee y \in P$.

Proof. Set $X = \mathcal{L} \setminus (s \vee Q)$. At first, we show that $s \vee X \subseteq X$. Let $s \vee x \in s \vee X$ for some $x \in X$ but $s \vee x \notin X$ (so $s \vee x \in s \vee Q$). Since $\mathcal{L} \setminus J(\mathcal{L}) \subseteq X$ by Proposition 3.4, then $s \vee x \notin J(\mathcal{L})$. Now, $(s \vee x) \vee 0 \in s \vee Q$ and $s \vee (s \vee x) = s \vee x \notin J(\mathcal{L})$ gives $s = s \vee 0 \in (s \vee Q) \cap S$, a contradiction, as $s \vee Q$ is an S-J-filter by Lemma 4.1. So $s \vee X \subseteq X$.

Next, we prove that $\mathcal{L} \setminus (s \vee Q)$ is a S-J-join subset that is maximal with $(\mathcal{L} \setminus (s \vee Q)) \cap P = \emptyset$. Since $(\mathcal{L} \setminus (s \vee Q)) \cap P = \emptyset$, the set Δ of all S-J-join subsets, say $\mathcal{T}'$, with $\mathcal{T}' \cap P = \emptyset$ is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ. Now Δ is easily seen to be inductive under inclusion, so by Zorn's Lemma Δ has a maximal element $\mathcal{T}$ with $P \cap \mathcal{T} = \emptyset$ and $s \vee \mathcal{T} \subseteq \mathcal{T}$. By Theorem 4.5, there is a filter K of $\mathcal{L}$ containing P that is maximal with respect to being disjoint from $\mathcal{T}$ which is S-J-filter. Note that $K \cap (\mathcal{L} \setminus (s \vee Q)) = \emptyset$ which implies that $K = Q$. Hence, $\mathcal{T} = \mathcal{L} \setminus (s \vee Q)$. Therefore, $\mathcal{L} \setminus (s \vee Q)$ has the stated property. Let $1 \neq a \in Q$ and set

$$\mathcal{T} = \{s \vee b) \vee (\wedge_{j=1}^{n} a) : s \vee b \in \mathcal{L} \setminus (s \vee Q),\ n = 0, 1, \cdots\}$$

(note that $\wedge_{i=1}^{0} a$ is interpreted as 0, and clearly, $\wedge_{i=1}^{n} a = a$). Since Q is a S-J-filter of $\mathcal{L}$, we conclude that $\mathcal{L} \setminus (s \vee Q)$ is a S-J-join subset of $\mathcal{L}$ by Proposition 3.7; so $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{L} \setminus (s \vee Q) \subseteq \mathcal{T}$. Let $u, v \in \mathcal{L}$ such that $s \vee u \notin J(\mathcal{L})$ and $s \vee v \in \mathcal{T}$. We show that $s \vee u \vee v \in \mathcal{T}$. By the hypothesis, $s \vee v = (s \vee b) \vee a$ for some $s \vee b \notin s \vee Q$ which implies that $s \vee u \vee v = (s \vee b \vee u) \vee a$. Then $s \vee b \vee u \notin s \vee Q$ (otherwise, either $s \vee u \in J(\mathcal{L})$ or $s \vee b \in s \vee Q$ by Lemma 4.1, a contradiction) and so $s \vee u \vee v \in \mathcal{T}$. Then $\mathcal{T}$ is an S-J-join subset that properly contains $\mathcal{L} \setminus (s \vee Q)$; so $P \cap \mathcal{T} \neq \emptyset$ by maximality of $\mathcal{L} \setminus s \vee Q$. So there is some $s \vee b \in \mathcal{L} \setminus (s \vee Q)$ such that $s \vee a \vee b \in P$. □

Corollary 4.7. *Suppose that Q is an S-J-filter of $\mathcal{L}$ such that $s \in S$ satisfies S-J-filter condition for Q. If $s \vee Q \in \min_{SJ}(P)$, then for each $x \in s \vee Q$, there is some $y \notin s \vee Q$ such that $s \vee x \vee y = 1$.*

Proof. Take $P = \{1\}$ in Theorem 4.6. □

Corollary 4.8. *Suppose that Q is a J-filter of $\mathcal{L}$ and let P be a filter of $\mathcal{L}$ with $P \subseteq s \vee Q$. If $Q \in \min_J(P)$, then for each $x \in Q$, there is some $y \notin Q$ such that $x \vee y \in P$.*

Proof. Take $S = \{0\}$ in Theorem 4.6. □

Corollary 4.9. *Suppose that Q is a J-filter of $\mathcal{L}$. If $Q \in \min_J(\mathcal{L})$, then for each $x \in Q$, there is some $y \notin Q$ such that $x \vee y = 1$.*

Proof. Take $S = \{0\}$ in Corollary 4.7. □

Theorem 4.10. Suppose P is an S-J-filter of a presimplifiable lattice $\mathcal{L}$ and let $s \in S$ satisfies S-J-filter condition for P. If P is prime, $s \vee P \in \min_{SJ}(\mathcal{L})$ and $c \in \mathcal{L}$ has a complement in $\mathcal{L}$, then $Q = (s \vee P) \wedge (1 :_{\mathcal{L}} c)$ is an S-J-filter.

Proof. Let $x, y \in \mathcal{L}$ such that $x \vee y \in Q$ with $s \vee x \notin J(\mathcal{L})$. Then $x \vee y = p \wedge b$ for some $p \in s \vee P$ and $b \vee c = 1$. By Corollary 4.7, there exists $q \notin s \vee P$ such that $s \vee p \vee q = 1$. Therefore, $(s \vee x) \vee (c \vee q \vee y) = (c \vee s \vee q) \vee (x \vee y) = (c \vee s \vee q) \vee (p \wedge b) = (c \vee s \vee q \vee p) \wedge (c \vee s \vee q \vee b) = 1 \vee 1 = 1$. Since $s \vee x \notin J(\mathcal{L})$, we conclude that $s \vee x \notin \mathrm{I}(\mathcal{L})$; so $c \vee q \vee y = 1 \in P$ which implies that $c \vee y \in P$, as P is prime. By assumption, there is an element $c' \in \mathcal{L}$ such that $c \vee c' = 1$ and $c \wedge c' = 0$. Now, $s \vee y = (s \vee y) \vee (c \wedge c') = (s \vee y \vee c) \wedge (s \vee y \vee c') \in Q$, and therefore Q is an S-J-filter. □

5 S-J-filters in related lattices

We begin this section with the investigation of the stability of S-J-filters in various lattice-theoretic constructions.

Theorem 5.1. Let $\mathcal{L}$ be a complemented lattice. If $f : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism such that $f(0) = 0$ and $f(1) = 1$, then the following hold:

(1) If f is epimorphism and P is an S-J-filter of $\mathcal{L}$ with $\ker(f) \subseteq P$, then $f(P)$ is an $f(S)$-J-filter of $\mathcal{L}'$.

(2) If K is an $f(S)$-J-filter of $\mathcal{L}'$ with $\ker(f) \subseteq J(\mathcal{L})$, then $f^{-1}(K)$ is an S-J-filter.

Proof. At first, we need to prove that $f(S) \cap f(P) = \emptyset$. Otherwise, there are elements $p \in P$ and $t \in S$ such that $f(p) = f(t)$. By assumption, $p \vee p' = 1$ and $p \wedge p' = 0$ for some $p' \in \mathcal{L}$. Since $f(t \vee p') = f(t) \vee f(p') = f(p \vee p') = 1$, we conclude that $t \vee p' \in \ker(f) \subseteq P$ which implies that $t = t \vee (p \wedge p') = (t \vee p) \wedge (t \vee p') \in P \cap S$, as P is a filter, a contradiction. Also, it is clear that $f(S)$ is a join subset of $\mathcal{L}'$.

(1) Let $x, y \in \mathcal{L}'$ such that $x \vee y \in f(P)$. Since f is an epimorphism, there exist $a, b \in \mathcal{L}$ such that $x = f(a)$ and $y = f(b)$. Then $x \vee y = f(a \vee b) \in f(P)$ and so $f(a \vee b) = f(c)$ for some $c \in P$. By the hypothesis, $c \vee c' = 1$ and $c \wedge c' = 0$ for some $c' \in \mathcal{L}$. Since $f(a \vee b \vee c') = f(a \vee b) \vee f(c') = 1$, we conclude that $a \vee b \vee c' \in \ker(f) \subseteq P$; hence $a \vee b = (a \vee b) \vee (c \wedge c') = (a \vee b \vee c) \wedge (a \vee b \vee c') \in P$, as P is a filter. Then there exists $s \in S$ such that either $s \vee a \in J(\mathcal{L})$ or $s \vee b \in P$, as P is an S-J-filter. Therefore, either $f(s) \vee x = f(s \vee a) \in f(J(\mathcal{L})) \subseteq J(\mathcal{L}')$ or $f(s) \vee y = f(s \vee b) \in f(P)$, as required.

(2) Since K is an $f(S)$-J-filter of $\mathcal{L}'$, we infer that there exists $s \in S$ such that for all $x', y' \in \mathcal{L}'$ with $x' \vee y' \in K$ we have $f(s) \vee x' \in J(\mathcal{L}')$ or $f(s) \vee y' \in K$. Now, let $a, b \in \mathcal{L}$ with $a \vee b \in f^{-1}(K)$ and $s \vee a \notin J(\mathcal{L})$. Then $f(a) \vee f(b) = f(a \vee b) \in K$. First, we show that $f(s \vee a) \notin J(\mathcal{L}')$. On the contrary, assume that $f(s \vee a) \in J(\mathcal{L}')$. Let M be any maximal filter of $\mathcal{L}$. Then $f(M)$ is a maximal filter of $\mathcal{L}'$. Since $f(s \vee a) \in J(\mathcal{L}') \subseteq f(M)$, we conclude that $f(s \vee a) = f(m)$ for some $m \in M$. By assumption, $m \vee m' = 1$ and $m \wedge m' = 0$ for some $m' \in \mathcal{L}$. As $f(s \vee a \vee m') = f(m) \vee f(m') = 1$, we conclude that $s \vee a \vee m' \in \ker(f) \subseteq M$; hence $s \vee a = (s \vee a) \vee (m \wedge m') = (s \vee a \vee m) \wedge (s \vee a \vee m') \in M$, as M is a filter. Therefore, $s \vee a \in J(\mathcal{L})$ which is a contradiction. Since K is an $f(S)$-J-filter, $f(a) \vee f(b) \in K$ and $f(s) \vee f(a) \notin J(\mathcal{L}')$, we get $f(s) \vee f(b) = f(s \vee b) \in K$ and so $s \vee b \in f^{-1}(K)$. □

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If P is a filter of a lattice $(\mathcal{L}, \leq)$, we define a relation on $\mathcal{L}$, given by $x \sim y$ if and only if there exist $a, b \in P$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $\mathcal{L}$, and we denote the equivalence class of a by $a \wedge P$ and these collection of all equivalence classes by $\mathcal{L}/P$. We set up a partial order $\leq_Q$ on $\mathcal{L}/P$ as follows: for each $a \wedge P, b \wedge P \in \mathcal{L}/P$, we write $a \wedge P \leq_Q b \wedge P$ if and only if $a \leq b$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}/P, \leq_Q)$ is a lattice with $(a \wedge P) \vee_Q (b \wedge P) = (a \vee b) \wedge P$ and $(a \wedge P) \wedge_Q (b \wedge P) = (a \wedge b) \wedge P$ for all elements $a \wedge P, b \wedge P \in \mathcal{L}/P$. Note that $p \wedge P = P$ if and only if $p \in P$ (see [10, Remark 4.2 and Lemma 4.3]). An inspection will show that if P is a filter of $\mathcal{L}$, then $S_Q = \{s \wedge P : s \in S\}$ is a join subset of $\mathcal{L}/P$.

Corollary 5.2. *Suppose $\mathcal{L}$ is a complemented lattice and let P, Q be two filters of $\mathcal{L}$ with $P \subseteq Q$. Then the followings hold:*

(1) If Q is an S-J-filter of $\mathcal{L}$, then Q/P is an S_Q-J-filter of $\mathcal{L}/P$;
(2) If Q/P is an S_Q-J-filter of $\mathcal{L}/P$ and $P \subseteq J(\mathcal{L})$, then Q is an S-J-filter of $\mathcal{L}$;
(3) If Q/P is an S_Q-J-filter of $\mathcal{L}/P$ and P is a J-filter of $\mathcal{L}$, then Q is an S-J-filter.

Proof. (1) Let $v : \mathcal{L} \to \mathcal{L}/P$ be the natural epimorphism defined by $v(x) = x \wedge P$ for $x \in \mathcal{L}$. Then $\ker(v) = \{x \in \mathcal{L} : x \wedge P = 1 \wedge P\} = P \subseteq Q$ by [10, Lemma 4.3] and so by Theorem 4.1 (1) and [10, Lemma 4.3] that $v(Q) = \{x \wedge P : x \in Q\} = Q/P$ is an S_Q-J-filter of $\mathcal{L}/P$, where $v(S) = S_Q$.

(2) Consider the natural epimorphism $v : \mathcal{L} \to \mathcal{L}/P$ defined by $v(x) = x \wedge P$ for $x \in \mathcal{L}$. Since $P \subseteq J(\mathcal{L})$, we conclude that $v^{-1}(Q/P) = Q$ is an S-J-filter of $\mathcal{L}$ by Theorem 5.1 (2).

(3) Since P is a J-filter of $\mathcal{L}$, we conclude that $P \subseteq J(\mathcal{L})$ by Corollary 3.5. Now the assertion follows from (2). □

If F is a proper filter of $\mathcal{L}$, then by a $\vee$-factorization of F we mean an expression of F as an intersection $\bigcap_{i=1}^{n} F_i$ of S-J-filters. We call $\mathcal{L}$ a $\vee$-lattice if every proper filter has a $\vee$-factorization.

Theorem 5.3. Let P be a proper filter of a complemented lattice $\mathcal{L}$. If $\mathcal{L}$ is an $\vee$-lattice, then $\mathcal{L}/P$ is an $\vee$-lattice.

Proof. Let Q be a proper filter of $\mathcal{L}/P$. Then $Q = K/P$ for some proper filter K of $\mathcal{L}$ by [10, Lemma 4.3 (5)]. Let $K = \cap_{i=1}^{n} P_i$ be a $\vee$-factorization of K. Then $Q = (\cap_{i=1}^{n} P_i)/P = \cap_{i=1}^{n} P_i/P$ by [10, Lemma 4.3 (7)]. By Corollary 5.2 (1), we conclude that P_i/P is an Q_S-J-filter of $\mathcal{L}/P$ for each $i \in \{1, 2, \cdots, n\}$. It means that $\mathcal{L}/P$ is an $\vee$-lattice. □

Lemma 5.4. *Suppose $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice. Then $J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$.*

Proof. If P is a proper filter of $\mathcal{L}$, then P is a maximal filter of $\mathcal{L}$ if and only if $P = P_1 \times \mathcal{L}_2$ for some maximal filter P_1 of $\mathcal{L}_1$ or $P = \mathcal{L}_1 \times P_2$ for some maximal filter P_2 of $\mathcal{L}_2$ by [12, Lemma 4.16]. Since $(P_i \times \mathcal{L}_2) \cap (\mathcal{L}_1 \times P_j) = P_i \times P_j$ and $(P_i \times P_j) \cap (P_i' \times P_j') = (P_i \cap P_i') \times (P_j \cap P_j')$, we infer that $J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$. □

We next show that S-J-filters are really only of interest in indecomposable lattices.

Theorem 5.5. Assume that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice and let P_1, P_2 be filters of $\mathcal{L}_1$, $\mathcal{L}_2$, respectively. If S_i is a join subset of $\mathcal{L}_i$, $i = 1, 2$, then the following hold:
(1) $P_1 \times \mathcal{L}_2$ is an $(S_1 \times S_2)$-J-filter of $\mathcal{L}$ if and only if P_1 is an S_1-J-filter of $\mathcal{L}_1$ and $J(\mathcal{L}_2) \cap S_2 \neq \emptyset$.
(2) $\mathcal{L}_1 \times P_2$ is an $(S_1 \times S_2)$-J-filter of $\mathcal{L}$ if and only if P_2 is an S_2-J-filter of $\mathcal{L}_2$ and $J(\mathcal{L}_1) \cap S_1 \neq \emptyset$.

Proof. (1) It is easy to see that $(P_1 \times \mathcal{L}_2) \cap S = \emptyset$ if and only if $P_1 \cap S_1 = \emptyset$ and $S = S_1 \times S_2$ is a join subset of $\mathcal{L}$. Suppose that $P = P_1 \times \mathcal{L}_2$ is an S-J-filter of $\mathcal{L}$ and let $(s_1, s_2) \in S$ satisfies S-J-filter condition for P. Since $(1, 0) \vee_c (0, 1) = (1, 1) \in P$, we conclude that either $(s_1, s_2) \vee_c (0, 1) = (s_1, 1) \in P$ or $(s_1, s_2) \vee_c (1, 0) = (1, s_2) \in J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$ by Lemma 5.4. This shows that either $s_2 \in J(\mathcal{L}_2)$ (so $J(\mathcal{L}_2) \cap S_2 \neq \emptyset$) or $s_1 \in P_1 \cap S_1$ which is impossible. Let $p \vee p' \in P_1$ for some $p, p' \in \mathcal{L}_1$. Then $(p, 0) \vee_c (p', 0) = (p \vee p', 0) \in P$ gives $(s_1, s_2) \vee_c (p, 0) = (s_1 \vee p, s_2) \in J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$ or $(s_1, s_2) \vee_c (p', 0) = (s_1 \vee p', s_2) \in P_1 \times \mathcal{L}_2$; hence either $s_1 \vee p \in J(\mathcal{L})$ or $s_1 \vee p' \in P_1$. Therefore, P_1 is an S_1-J-filter.

Conversely, suppose P_1 is an S_1-J-filter of $\mathcal{L}_1$ and $J(\mathcal{L}_2) \cap S_2 \neq \emptyset$. Assume $s_1 \in S_1$ satisfies S_1-J-filter condition for P_1 and let $(p, q), (p', q') \in \mathcal{L}$ such that $(p, q) \vee_c (p', q') = (p \vee p', q \vee q') \in P$. Then $s_1 \vee p \in J(\mathcal{L}_1)$ or $s_1 \vee p' \in P_1$, as $p \vee p' \in P_1$. By the hypothesis, there exists $s_2 \in S_2$ such that $s_2 \in J(\mathcal{L}_2)$. This shows that $(p, q) \vee_c (s_1, s_2) = (s_1 \vee p, s_2 \vee q) \in J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$ or $(s_1, s_2) \vee_c (p', q') = (s_1 \vee p', s_2 \vee q') \in P$. Hence, P is an S-J-filter of $\mathcal{L}$.

(2) The proof is similar to that in case (1) and we omit it. □

Lemma 5.6. *If P is a finitely generated filter of $\mathcal{L}$ and $a \in \mathcal{L} \setminus P$, then $a \vee P$ is a finitely generated filter.*

Proof. Let $P = T(A)$, where $A = \{p_1, \cdots, p_n\} \subseteq P$. An inspection will show that $a \vee P = T(B)$, where $B = \{a \vee p_1, \cdots, a \vee p_n\} \subseteq a \vee P$. □

We say that a filter P of $\mathcal{L}$ is S-finite if $s \vee P \subseteq F \subseteq P$ for some finitely generated filter F of $\mathcal{L}$ and some $s \in S$.

Proposition 5.7. *For the lattice $\mathcal{L}$, the following hold:*
(1) Let P be an intersection of maximal filters of $\mathcal{L}$. If P is a finitely generated S-J-filter, then $J(\mathcal{L})$ is S-finite.
(2) If $J(\mathcal{L})$ is finitely generated, then it is S-finite. In particular, if P is a finitely generated maximal S-J-filter, then $J(\mathcal{L})$ is S-finite.

Proof. (1) Since P is S-J-filter, we conclude that $J(\mathcal{L}) \subseteq P \subseteq (J(\mathcal{L}) :_{\mathcal{L}} s)$ for some $s \in S$ by Proposition 3.4. Therefore, $s \vee J(\mathcal{L}) \subseteq s \vee P \subseteq J(\mathcal{L})$ and since $s \vee P$ is finitely generated by Lemma 5.6, we infer that $J(\mathcal{L})$ is S-finite.
(2) This is a direct consequence of (1). □

Proposition 5.8. *Let P, Q and K be filters of $\mathcal{L}$ such that $s \vee K \not\subseteq J(\mathcal{L})$ for all $s \in S$. Then the following hold:*
(1) Let P and Q be S-J-filters of $\mathcal{L}$. If P is a finitely generated with $K \vee P = K \vee Q$, then Q is a S-finite filter.
(2) Let P and Q be S-J-filters of $\mathcal{L}$. If Q a finitely generated with $K \vee P = K \vee Q$, then P is a S-finite filter.
(3) If $K \vee P$ is a finitely generated S-J-filter, then P is S-finite.

Proof. (1) Assume that $s \in S$ satisfies S-J-filter condition for Q. Since $K \vee P = K \vee Q \subseteq Q$ and $s \vee K \not\subseteq J(\mathcal{L})$, we conclude that $s \vee P \subseteq Q$ by Theorem 3.6, as Q is an S-J-filter. Similarly, There exists $t \in S$ such that $t \vee Q \subseteq P$. Therefore, $(s \vee t) \vee Q \subseteq s \vee P \subseteq Q$. For $s' = s \vee t \in S$, $s' \vee Q \subseteq s \vee P \subseteq Q$, where $s \vee P$ is finitely generated by Lemma 5.6. Hence, Q is S-finite.
(2) Similar to the proof of the part (1), one can show that P is S-finite.
(3) Since $s \vee K \not\subseteq J(\mathcal{L})$ and $K \vee P$ is S-J-filter, we infer that $s \vee P \subseteq K \vee P \subseteq P$ and so P is S-finite. □

Corollary 5.9. *Let P, Q and K be filters of $\mathcal{L}$ such that $K \not\subseteq J(\mathcal{L})$. Then the following hold:*
(1) Let P and Q be J-filters of $\mathcal{L}$. If P is a finitely generated with $K \vee P = K \vee Q$, then Q is a S-finite filter.
(2) Let P and Q be J-filters of $\mathcal{L}$. If Q a finitely generated with $K \vee P = K \vee Q$, then P is a S-finite filter.
(3) If $K \vee P$ is a finitely generated J-filter, then P is S-finite.

Proof. Take $S = \{0\}$ in Proposition 5.8. □

Definition 5.10. (1) A proper filter P of $\mathcal{L}$ is called SJ-finite if $s \vee P \subseteq G \subseteq P \subseteq J(\mathcal{L})$ for some finitely generated filter K of $\mathcal{L}$ and some $s \in S$.
(2) We say that $\mathcal{L}$ is SJ-Noetherian if each proper filter of $\mathcal{L}$ is SJ-finite.

Lemma 5.11. *If $P \subseteq J(\mathcal{L})$ is a filter of $\mathcal{L}$ which is maximal among all non-SJ-finite filters of $\mathcal{L}$, then P is a J-filter of $\mathcal{L}$.*

Proof. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P$ and $a \notin J(\mathcal{L})$ (so $a \notin P$). We show that $b \in P$. On the contrary, assume that $b \notin P$. By an argument like that in [13, Lemma 2.20], there exist $s, t \in S$, $p_1, \cdots, p_n \in P$ and $b_1, \cdots, b_k \in (P :_{\mathcal{L}} a)$ such that $(s \vee t) \vee P \subseteq T(A) \subseteq P$, where $A = \{p_1 \vee t, \cdots, p_n \vee t, a \vee b_1, \cdots, a \vee b_k\} \subseteq P$; hence P is S-finite, a contradiction. Thus P is a J-filter of $\mathcal{L}$. □

Proposition 5.12. *$\mathcal{L}$ is SJ-Noetherian if and only if every J-filter of $\mathcal{L}$ (disjoint from S) is S-finite.*

Proof. One side is clear. To see the other side, suppose that P is SJ-finite for each J-filter P of $\mathcal{L}$ disjoint from S. On the contrary, Suppose that $\mathcal{L}$ is not SJ-Noetherian. Then the set Δ of all non-SJ-finite filters of $\mathcal{L}$ is inductively ordered under inclusion. By Zorn's Lemma, choose Q maximal in Δ. Then Lemma 5.11 implies that Q is a J-filter of $\mathcal{L}$. If $Q \cap S \neq \emptyset$, then $s \vee Q \subseteq T(s) \subseteq Q$ for every $s \in Q \cap S$ gives Q is SJ-finite, a contradiction. Thus $Q \cap S = \emptyset$. Now, by the hypothesis, Q is SJ-finite which is impossible since $Q \in \Delta$. Thus $\mathcal{L}$ is SJ-Noetherian. □

In the following theorem we give an SJ-version of Cohen's Theorem (Theorem 2 in [7])

Theorem 5.13. For a lattice $\mathcal{L}$, the following assertions are equivalent:
(1) $\mathcal{L}$ is SJ-Noetherian;
(2) Every S-J-filter of $\mathcal{L}$ is SJ-finite;
(3) Every J-filter of $\mathcal{L}$ is SJ-finite.

Proof. (1) $\Rightarrow$ (2) By assumption, each proper filter of $\mathcal{L}$ is SJ-finite.
(2) $\Rightarrow$ (3) Let P be a J-filter of $\mathcal{L}$. If $P \cap S \neq \emptyset$, then $s \vee P \subseteq T(s) \subseteq P$ for every $s \in P \cap S$ gives P is SJ-finite. If $P \cap S = \emptyset$, then P is an S-J-filter of $\mathcal{L}$; hence by the hypothesis, P is S-finite.
(3) $\Rightarrow$ (1) Follows from Proposition 5.12. □

Similar to the idealization of a module over a ring (Huckaba in [14]), the remaining part of this section mainly devoted to investigation the stability of S-J-filters in the filterlization of a lattice and a filter. Let $\mathcal{M}$ be a filter of lattice $\mathcal{L}$. The filterlization $\mathcal{L}(+)\mathcal{M} = \{(a,m) : a \in \mathcal{L}, m \in \mathcal{M}\}$ of $\mathcal{M}$ in $\mathcal{L}$ is a lattice with respect to the following definitions: (1) $(a,m) = (a',m')$ if $a = a'$ and $m = m'$, (2) $(a,m_1) \wedge_F (b,m_2) = (a \wedge b, m_1 \wedge m_2)$ and (3) $(a,m_1) \vee_F (b,m_2) = (a \vee b, (a \vee m_2) \wedge (b \vee m_1))$. Under these definitions $\mathcal{L}(+)\mathcal{M}$ becomes a lattice. An easy inspection will show that $S_{\mathcal{M}} = S(+)\mathcal{M}$ is a join subset of $\mathcal{L}(+)\mathcal{M}$ and $J(\mathcal{L}(+)\mathcal{M}) = J(\mathcal{L})(+)\mathcal{M}$. In addition, if F is a filter of $\mathcal{L}$, then for a subfilter N of $\mathcal{M}$, $F(+)N$ is a filter of $\mathcal{L}(+)\mathcal{M}$ if and only if $F \vee \mathcal{M} \subseteq N$. Here we clarify the relationships between S-J-filters in a lattice $\mathcal{L}$ and in a filterlization lattice $\mathcal{L}(+)\mathcal{M}$.

Theorem 5.14. Let P be a filter of a lattice $\mathcal{L}$ and K be a subfilter of $\mathcal{M}$. Then the following hold:
(1) If $P(+)K$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$, then P is an S-J-filter of $\mathcal{L}$.
(2) $P(+)\mathcal{M}$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$ if and only if P is an S-J-filter of $\mathcal{L}$.

Proof. (1) Let $P(+)K$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$. So $S_{\mathcal{M}} \cap (P(+)K) = \emptyset$ gives $S \cap P = \emptyset$. Let $x, y \in \mathcal{L}$ such that $x \vee y \in P$. Then $(x,1) \vee_F (y,1) = (x \vee y, 1) \in P(+)K$ implies that there exists $(s,m) \in S_{\mathcal{M}}$ such that either $(s,m) \vee_F (x,1) = (s \vee x, x \vee m) \in J(\mathcal{L}(+)\mathcal{M}) = J(\mathcal{L})(+)\mathcal{M}$ or $(s,m) \vee_F (y,1) = (s \vee y, y \vee m) \in P(+)K$. Hence, either $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P$ and so P is an S-J-filter of $\mathcal{L}$.
(2) Suppose that $P(+)\mathcal{M}$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$. Then P is an S-J-filter of $\mathcal{L}$ by (1). To see the other side, suppose P is an S-J-filter of $\mathcal{L}$ and let $s \in S$ satisfies S-J-filter condition for P. Note that $S_{\mathcal{M}} \cap (P(+)\mathcal{M}) = \emptyset$, as $S \cap P = \emptyset$. Let $(p,m) \vee_F (p',m') = (p \vee p', (p \vee m') \wedge (p' \vee m) \in P(+)\mathcal{M}$ for some $(p,m),(p',m') \in \mathcal{L}(+)\mathcal{M}$. By the hypothesis, since $p \vee p' \in P$, we conclude that either $s \vee p \in J(\mathcal{L})$ or $s \vee p' \in P$ which implies either $(s,m) \vee_F (p,m) \in J(\mathcal{L})(+)\mathcal{M} = J(\mathcal{L}(+)\mathcal{M})$ or $(s,m) \vee_F (p',m') \in P(+)\mathcal{M}$. This shows that $P(+)\mathcal{M}$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$. □

Corollary 5.15. *Let P be a filter of a lattice $\mathcal{L}$ and K be a subfilter of $\mathcal{M}$. Then the following hold:*
(1) If $P(+)K$ is a J-filter of $\mathcal{L}(+)\mathcal{M}$, then P is an J-filter of $\mathcal{L}$.
(2) $P(+)\mathcal{M}$ is an J-filter of $\mathcal{L}(+)\mathcal{M}$ if and only if P is an J-filter of $\mathcal{L}$.

Proof. Take $S = \{0\}$ in Theorem 5.14. □

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