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Revisiting 1-absorbing prime elements in multiplicative lattices

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Abstract. In this paper, we revisit the concept of 1-absorbing prime elements in C-lattices and obtain its characterizations. The main aim of this paper is to investigate some more properties of 1-absorbing prime elements in multiplicative lattices.
Key Words: 1-absorbing prime element, p-1-absorbing prime element, semiprimary element.
2020 MSC: 06F99, 06B23

Dedicated to the memory of my guide, Professor C. S. Manjarekar.

1 Introduction and preliminaries

The prime ideal, which is an important subject of ideal theory in commutative rings has been widely studied by many authors to obtain various generalizations of prime ideals. Since the first paper on absorbing ideals as in [3] by A. Badawi, much work in the literature has been devoted to studying different generalizations of absorbing ideals in commutative rings. Following the concept of 1-absorbing primary ideals in commutative rings as in [4] by A. Badawi and E. Y. Celikel, the authors in [14], introduced the idea of 1-absorbing prime ideals in commutative rings. Afterwards, this notion of 1-absorbing prime ideal was coined as OA-prime ideal and OA factorization was studied (see [10] and [8]). In an attempt of generalizing the notion of 1-absorbing prime ideal in commutative rings, E. A. Ugurlu in [13], introduced the notion of 1-absorbing prime submodule in ring modules.

R. P. Dilworth in [7] took a major step in giving a concrete abstract formulation of the ideal theory of commutative rings $\mathbb{R}$ with identity. He intended that the treatment should be purely ideal theoretic and therefore the system which was chosen for the study was multiplicative lattices. Following this, we find that researchers have extended different types of ideals in $\mathbb{R}$ to respective types of elements in multiplicative lattices. Moving alike, A. Reinhart and G. Ulucak in [11] extended the notion of 1-absorbing prime ideals in $\mathbb{R}$ to non-modular multiplicative lattices. In this paper, we revisit the concept of 1-absorbing prime elements in multiplicative lattices. The main aim of this paper is to investigate further properties of 1-absorbing prime elements in multiplicative lattices.

Before starting this work, let us review few definitions in multiplicative lattices. A multiplicative lattice L is a complete lattice in which there is a commutative, associative multiplication which distributes over arbitrary joins and has greatest element 1 as a multiplicative identity. 0 is the least element of L. An element $a \in L$ is called compact if for $X \subseteq L$, $a \leqslant \vee X$ implies the existence of a finite number of elements $a_1, a_2, \cdots, a_n$ in X such that $a \leqslant a_1 \vee a_2 \vee \cdots \vee a_n$. A multiplicative lattice is called a compactly generated lattice or simply a CG-lattice, if every element of it is a join of compact elements. By a C-lattice, we mean a (not necessarily modular) multiplicative lattice which is generated under joins by a multiplicatively closed subset C of compact elements. We note that, in a C-lattice, a finite product of compact elements is again compact.

R. P. Dilworth gave the following classical example of a multiplicative lattice. Let $\mathbb{R}$ be a commutative ring with unity and I, J be the ideals of $\mathbb{R}$. If $L(\mathbb{R}) = L$ denotes the set of all ideals of $\mathbb{R}$ then $< L; \wedge, \vee, \cdot >$ is a multiplicative lattice where the lattice operations defined are as follows:

① $I \wedge J = I \cap J$

② $I \vee J = I + J = (I \cup J)$

③ $I \cdot J = \{\sum_{finite} a_i \cdot b_i \in \mathbb{R} \mid a_i \in I \text{ and } b_i \in J\}$

$L(\mathbb{R})$ is known as the ideal lattice or the lattice of ideals of $\mathbb{R}$ and has following features:

1. Every principal ideal of $\mathbb{R}$ is a compact element of $L(\mathbb{R})$.
2. $L(\mathbb{R})$ is a compactly generated multiplicative lattice and a C-lattice.

Throughout this paper, L denotes a C-lattice, L_* denotes the set of all compact elements of L and Z_+ denotes the set of all natural numbers. The product of any two elements a and b of L will be denoted by ab instead of $a \cdot b$ for convenience.

For $a, b \in L$, $(a : b) = \vee\{x \in L \mid xb \leqslant a\}$. Note that $c \leqslant (a : b)$ if and only if $cb \leqslant a$ and $a \leqslant b$ implies $(a : c) \leqslant (b : c)$ $\forall a, b, c \in L$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L_* \mid x^n \leqslant a$, for some $n \in Z_+\}$. An element $q \in L$ is said to be proper if $q < 1$. An element $m < 1$ in L is said to be maximal if $m < x \leqslant 1$ imply $x = 1$. A proper element $q \in L$ is said to be prime if $ab \leqslant q$ imply either $a \leqslant q$ or $b \leqslant q$ $\forall$ $a, b \in L$. In a C-lattice, an element q is prime if and only if $ab \leqslant q$ imply either $a \leqslant q$ or $b \leqslant q$ $\forall$ $a, b \in L_*$. An element $q \in L$ is called semiprimary if $\sqrt{q}$ is a prime element. A proper element $q \in L$ is said to be primary if $ab \leqslant q$ imply either $a \leqslant q$ or $b^n \leqslant q$ for some $n \in Z_+$ where $a, b \in L_*$. A proper element $q \in L$ is called p-prime if q is prime and $p = \sqrt{q}$ is prime. A proper element $q \in L$ is called p-primary if q is primary and $p = \sqrt{q}$ is prime. In a C-lattice L, $\sqrt{a} = \wedge\{p \in L \mid p \geqslant a$ is a minimal prime over $a\} = \wedge\{p \in L \mid p$ is prime and $a \leqslant p\}$. (See also Theorem 3.6 of [12]). Note that $q \leqslant \sqrt{q}$ for every $q \in L$. According to [9], a proper element $q \in L$ is said to be 2-absorbing if for every $a, b, c \in L$, $abc \leqslant q$ imply either $ab \leqslant q$ or $bc \leqslant q$ or $ca \leqslant q$. According to [5], a proper element $q \in L$ is said to be 2-absorbing primary if for every $a, b, c \in L$, $abc \leqslant q$ imply either $ab \leqslant q$ or $bc \leqslant \sqrt{q}$ or $ca \leqslant \sqrt{q}$. According to [5], a proper element $q \in L$ is called p-2-absorbing primary if q is 2-absorbing primary and $p = \sqrt{q}$ is 2-absorbing. For general background and terminology in multiplicative lattices, the reader may consult [1] and [2].

The content of the paper is as follows: Initially, we revisit the definition of a 1-absorbing prime element in L. We obtain characterizations of a 1-absorbing prime element of L (See Theorems 2.5, 2.8, 2.9). We show that if $q \in L$ is 1-absorbing prime, then q is semiprimary in Theorem 2.6-④ and hence we define a p-1-absorbing prime element (See definition 2.10). In Theorems 2.11, 2.13, 2.14 and 2.15, we obtain results regarding the meet of 1-absorbing prime elements in L. In Theorem 2.16, we show that if a proper element $q \in L$ is 1-absorbing prime, then for every $x \nleqslant q$, $(q : x)$ is prime. Then we investigate under which condition(s), converse of Theorem 2.16 holds true (See Theorems 2.18 and 2.26). In an attempt of highlighting the future scope of this paper, we conclude this paper by defining the notion of "weakly 1-absorbing prime element" and "n-1-absorbing prime element" ($n \geqslant 1$ in Z_+) of L in the last section.

2 Results on 1-absorbing prime element in L

We begin with introducing (revisiting) the notion of 1-absorbing prime elements in C-lattices.

Definition 2.1. (See also Definition 2.1 of [11]). A proper element $q \in L$ is called a 1-absorbing prime element if for all proper elements $a, b, c \in L$ such that the product of a, b, c need not collapse to one of the element among them, $abc \leqslant q$ imply either $ab \leqslant q$ or $c \leqslant q$.

Without loss of generality, we define above notion of 2.1 in the following way.

Definition 2.2. A proper element $q \in L$ is called a 1-absorbing prime element if for every $a, b, c \in L$, $abc \leqslant q$ imply either $ab \leqslant q$ or $c \leqslant q$.

Note that in view of the concept of 2-absorbing element of L, a prime element of L is nothing but 1-absorbing element of L.

Clearly, every 1-absorbing element of L is a 1-absorbing prime element of L. Its converse need not be true as seen in the Example 2.2 of [11].

Theorem 2.3. If a proper element $q \in L$ is 1-absorbing prime, then q is a 2-absorbing element of L.

Proof. Let $abc \leqslant q$, $ab \nleqslant q$ and $bc \nleqslant q$ where $a, b, c \in L$. Since q is 1-absorbing prime, $abc \leqslant q$, $ab \nleqslant q$ gives $c \leqslant q$ and hence $ac \leqslant q$. Therefore, q is a 2-absorbing element of L. □

The converse of Theorem 2.3 need not be true as seen in the Example 2.3 of [11].

Now before obtaining the characterization of a 1-absorbing prime element of L, we state the following essential lemma which is outcome of Lemma 2.3.13 from [6].

Lemma 2.4. *Let $a_1,\ a_2 \in L$. Suppose $b \in L$ satisfies the following property:*
$(*)$. If $h \in L_*$ with $h \leqslant b$, then either $h \leqslant a_1$ or $h \leqslant a_2$.
Then either $b \leqslant a_1$ or $b \leqslant a_2$.

We present Lemma 2.4 of [11] in the following result in an alternative way.

Theorem 2.5. Let $q \in L$ be a proper element of L. Then the following statements are equivalent:
①. q is a 1-absorbing prime element of L.
②. for every $a, b, c \in L$ such that $q \leqslant c$, $abc \leqslant q$ implies either $ab \leqslant q$ or $c = q$.
③. for every $a, b \in L$ such that $ab \nleqslant q$, $(q : ab) = q$.
④. for every $x, y, z \in L_*$, $xyz \leq q$ implies either $xy \leqslant q$ or $z \leqslant q$.

Proof. ①$\Longrightarrow$②. It is obvious.
②$\Longrightarrow$③. Suppose ② holds. Let $h \in L_*$ be such that $h \leqslant (q : ab)$ and $ab \nleqslant q$ where $a, b \in L$. Then $hab \leqslant q$. Clearly, $ab(h \vee q) \leqslant q$. Let $u = h \vee q$. Now as $q \leqslant u$, $abu \leqslant q$ and $ab \nleqslant q$, by hypothesis ②, it follows that $u = q$ and so $h \leqslant q$. Hence by Lemma 2.4, we have $(q : ab) \leqslant q$. Consequently, $(q : ab) = q$.
③$\Longrightarrow$④. Suppose ③ holds. Let $xyz \leqslant q$ and $xy \nleqslant q$ where $x, y, z \in L_*$. Then as $L_* \subseteq L$, by hypothesis ③, it follows that $(q : xy) = q$ and we are done, since $z \leqslant (q : xy)$.
④$\Longrightarrow$①. Suppose ④ holds. Let $abc \leqslant q$ and $ab \nleqslant q$ where $a, b, c \in L$. Since $abc \leqslant q$, there exist $a', b', c' \in L_*$ with $a' \leqslant a$, $b' \leqslant b$ and $c' \leqslant c$ such that $a'b'c' \leqslant q$. Since $ab \nleqslant q$, there exists $a_1, b_1 \in L_*$ with $a_1 \leqslant a$ and $b_1 \leqslant b$ such that $a_1 b_1 \nleqslant q$. Let $a_2 = a_1 \vee a'$ and $b_2 = b_1 \vee b'$. We show $c \leqslant q$. Choose $c_\alpha \in L_*$ such that $c_\alpha \leqslant c$. Now since $a_2 b_2 (c' \vee c_\alpha) \leqslant q$ and $a_2 b_2 \nleqslant q$ where $a_2, b_2, (c' \vee c_\alpha) \in L_*$, by hypothesis ④, we have $(c' \vee c_\alpha) \leqslant q$ which implies $c_\alpha \leqslant q$. Hence by Lemma 2.4, we have $c \leqslant q$. Therefore, q is a 1-absorbing prime element of L. □

In the following result, we present Proposition 2.5 (1) of [11] in a different way.

Theorem 2.6. Let a proper element $q \in L$ be 1-absorbing prime.
①. For each $x \in L$, $x \leqslant \sqrt{q}$ if and only if $x^2 \leqslant q$.
②. If $x, y \leqslant \sqrt{q}$ where $x, y \in L$, then $xy \leqslant q$.
③. $(\sqrt{q})^2 \leqslant q$.
④. q is a semiprimary element of L.

Proof. ①. Let $h \in L_*$ be such that $h \leqslant \sqrt{q}$. Then there exists a least positive integer k such that $h^k \leqslant q$. So $hhh^{k-2} \leqslant q$ with $h^{k-2} \nleqslant q$. Since q is 1-absorbing prime, we get $hh = h^2 \leqslant q$.

Suppose $x \leqslant \sqrt{q}$ where $x \in L$. Let $a, b \in L_*$ be such that $a \leqslant x$ and $b \leqslant x$. Since $a \leqslant \sqrt{q}$ and $b \leqslant \sqrt{q}$, it follows that $a^2 \leqslant q$ and $b^2 \leqslant q$. Hence $a(a \vee b)b \leqslant q$. As q is 1-absorbing prime, we get either $a(a \vee b) \leqslant q$ or $b \leqslant q$. Consequently, $ab \leqslant q$. As $x^2 = \vee\{ab \mid a, b \in L_*, a \leqslant x, b \leqslant x\}$, it follows that $x^2 \leqslant q$. The converse is obvious.

②. Let $x, y \in L$ be such that $x \leqslant \sqrt{q}$ and $y \leqslant \sqrt{q}$. By ①, $x^2 \leqslant q$ and $y^2 \leqslant q$. Hence $x(x \vee y)y \leqslant q$. As q is 1-absorbing prime, we get either $x(x \vee y) \leqslant q$ or $y \leqslant q$. Consequently, $xy \leqslant q$.

③. It follows from ②, since $(\sqrt{q})^2 = \vee\{ab \mid a, b \in L_*, a \leqslant \sqrt{q}, b \leqslant \sqrt{q}\}$.

④. Let $a, b \leqslant L_*$ be such that $ab \leqslant \sqrt{q}$. Then by ①, $(ab)^2 \leqslant q$. Since q is 1-absorbing prime and $aab^2 \leqslant q$, we get either $aa = a^2 \leqslant q$ or $b^2 \leqslant q$. Hence either $a \leqslant \sqrt{q}$ or $b \leqslant \sqrt{q}$ and we are done. □

Corollary 2.7. *If q is a p-prime element of L, then $p^2 \leqslant q$.*

Proof. The proof is obvious. □

The following characterization of a 1-absorbing prime element in L is an outcome of the Theorem 2.6.

Theorem 2.8. A proper element $q \in L$ is a 1-absorbing prime element of L if and only if q is a semiprimary element of L and $(\sqrt{q})^2 \leqslant q$.

Proof. If q is a 1-absorbing prime element of L, then by Theorem 2.6 ③ and ④, we are done. Conversely, assume that $(\sqrt{q})^2 \leqslant q$ and q is a semiprimary element of L. Let $xyz \leqslant q$ where $x, y, z \in L_*$. If either $x \leqslant q$ or $yz \leqslant q$, then we are done. So assume that $x \nleqslant q$ and $yz \nleqslant q$. Now, q being semiprimary, $\sqrt{q}$ is prime. Also, $q \leqslant \sqrt{q}$. So $xyz \leqslant \sqrt{q}$ gives $x \leqslant \sqrt{q}$ and $yz \leqslant \sqrt{q}$. Thus $x, y \leqslant \sqrt{q}$ or $x, z \leqslant \sqrt{q}$. Since $\vee\{ab \mid a, b \in L_*, a \leqslant \sqrt{q}, b \leqslant \sqrt{q}\} = (\sqrt{q})^2 \leqslant q$, it follows that either ($xy \leqslant q$ with $z \nleqslant q$) or ($xz \leqslant q$ with $y \nleqslant q$) and hence q is a 1-absorbing prime element of L. □

Theorem 2.9. Let q be a p-primary element of L. Then q is a 1-absorbing prime element of L if and only if $p^2 \leqslant q$. In particular, m^2 is a 1-absorbing prime element of L for each maximal element $m \in L$.

Proof. The proof is obvious. □

Similar to the concept of p-prime element of L, we define p-1-absorbing prime element of L.

Definition 2.10. If a proper element $q \in L$ is 1-absorbing prime, then by Theorem 2.6-④, q is semiprimary (i.e. $p = \sqrt{q}$ is a prime element of L). In this case, we say that $q \in L$ is a p-1-absorbing prime element of L.

In the following 4 successive results, we obtain results regarding the meet of 1-absorbing prime elements in L.

Theorem 2.11. If $q_1, q_2, \ldots, q_n$ are p-1-absorbing prime elements of L for some prime element $p \in L$, then $q = \bigwedge_{i=1}^{n} q_i$ is a p-2-absorbing primary element of L.

Proof. Clearly, $\bigwedge_{i=1}^{n} q_i = q \neq 1$ and by property (p4) of [12], we have $\sqrt{q} = \bigwedge_{i=1}^{n} \sqrt{q_i} = p$. Let $abc \leqslant q$, $ac \nleqslant q$ and $bc \nleqslant \sqrt{q}$ where $a,b,c \in L$. Then $ac \nleqslant q_i$ for some i $(1 \leqslant i \leqslant n)$ and $bc \nleqslant \sqrt{q_j}$ for some j $(1 \leqslant j \leqslant n)$. Without loss of generality, assume $i \neq j$. As q_i is a p-1-absorbing prime element, $(ac)b \leqslant q \leqslant q_i$ and $ac \nleqslant q_i$, we have $b \leqslant q_i \leqslant \sqrt{q_i}$ and hence $ab \leqslant \sqrt{q_i} = p = \sqrt{q}$. (Also considering q_j as a p-1-absorbing prime element, $(bc)a \leqslant q \leqslant q_j$ and $bc \nleqslant q_j$, we have $a \leqslant q_j \leqslant \sqrt{q_j}$ and hence $ab \leqslant \sqrt{q_j} = p = \sqrt{q}$). The proof is complete as p is a 2-absorbing element because every prime element of L is a 2-absorbing element of L. □

The following lemma is a statement made by C. Jayaram et. al. in [9].

Lemma 2.12. *If p_1 and p_2 are the prime elements of L, then $p_1 \wedge p_2$ is a 2-absorbing element of L.*

Theorem 2.13. If q_1 is a p_1-1-absorbing prime element of L and q_2 is a p_2-1-absorbing prime element of L for some prime elements p_1, $p_2 \in L$, then $q_1 \wedge q_2$ is a 2-absorbing primary element of L.

Proof. Suppose $h = q_1 \wedge q_2$. Let $abc \leqslant h$, $ac \nleqslant \sqrt{h}$ and $bc \nleqslant \sqrt{h}$ where $a,b,c \in L$. Then $a,b,c \nleqslant \sqrt{h}$. As $p_1 = \sqrt{q_1}$ and $p_2 = \sqrt{q_2}$, by property (p4) of [12], we have $\sqrt{h} = p_1 \wedge p_2$ and so by Lemma 2.12 , $\sqrt{h}$ is a 2-absorbing element of L. Hence $ab \leqslant \sqrt{h}$. Since p_1 is prime and $ab \leqslant \sqrt{h} \leqslant p_1$, we assume that $a \leqslant p_1$. Also, since p_2 is prime, $ab \leqslant \sqrt{h} \leqslant p_2$ and $a \nleqslant \sqrt{h}$, we must have $a \nleqslant p_2$ and $b \leqslant p_2$. As $b \leqslant p_2$ and $b \nleqslant \sqrt{h}$, we must have $b \nleqslant p_1$. If $a \leqslant q_1$ and $b \leqslant q_2$, then $ab \leqslant h$ and we are done. So assume that $a \nleqslant q_1$. As q_1 is 1-absorbing prime, $abc \leqslant h \leqslant q_1$ and $a \nleqslant q_1$, we have $bc \leqslant q_1 \leqslant \sqrt{q_1} = p_1$. Since $b \leqslant p_2$ and $bc \leqslant p_1$, we have $bc \leqslant \sqrt{h}$, a contradiction. Thus we must have $a \leqslant q_1$. Similarly, if we assume that $b \nleqslant q_2$, then we get a contradiction to $ac \nleqslant \sqrt{h}$ and hence we must have $b \leqslant q_2$. Therefore, $ab \leqslant h$ and the proof is complete. □

Theorem 2.14. If q_1 and q_2 are two 1-absorbing prime elements of L, then $q_1 \wedge q_2$ is a 2-absorbing element of L.

Proof. Let $abc \leqslant (q_1 \wedge q_2)$ but $ac \nleqslant (q_1 \wedge q_2)$ and $bc \nleqslant (q_1 \wedge q_2)$ where $a,b,c \in L$. Then $ac \nleqslant q_i$ for some $i \in \{1,2\}$ and $bc \nleqslant q_j$ for some $j \in \{1,2\}$. If $i = j$, then as $a(bc) \leqslant q_j$, $bc \nleqslant q_j$ and q_j is 1-absorbing prime, we have $a \leqslant q_j$ and hence $ac \leqslant q_i$, which contradicts $ac \nleqslant q_i$. So let $i \neq j$; say $ac \nleqslant q_1$ and $bc \nleqslant q_2$. As $(bc)a \leqslant q_2$, $bc \nleqslant q_2$ and q_2 is 1-absorbing prime, we have $a \leqslant q_2$ and hence $ab \leqslant q_2$. Also, as $(ac)b \leqslant q_1$, $ac \nleqslant q_1$ and q_1 is 1-absorbing prime, we have $b \leqslant q_1$ and hence $ab \leqslant q_1$. Thus $ab \leqslant (q_1 \wedge q_2)$ and we are done. □

The next result shows that the meet of a family of ascending chain of 1-absorbing prime elements of L is again 1-absorbing prime.

Theorem 2.15. Let $\{q_i \mid i \in Z_+\}$ be an ascending chain of prime elements (or 1-absorbing prime elements) in L. Then $\bigwedge\limits_{i \in Z_+} q_i$ is a 1-absorbing prime element of L.

Proof. Clearly, $(\bigwedge\limits_{i \in Z_+} q_i) \neq 1$. Let $abc \leqslant (\bigwedge\limits_{i \in Z_+} q_i)$ and $ab \nleqslant (\bigwedge\limits_{i \in Z_+} q_i)$ where $a,b,c \in L$. Then $ab \nleqslant q_j$ for some $j \in Z_+$ but $abc \leqslant q_j$ implies $c \leqslant q_j$ as $q_j \in L$ is a prime element (or a 1-absorbing prime element). Now let $q_i \neq q_j$. Then as $\{q_i\}$ is a chain, we have either $q_i < q_j$ or $q_j < q_i$. If $q_i < q_j$, then as q_i is a prime element (or a 1-absorbing prime element), $abc \leqslant q_i$ and $ab \nleqslant q_i$, we have $c \leqslant q_i$. If $q_j < q_i$, then $c \leqslant q_j \leqslant q_i$. Thus $c \leqslant (\bigwedge\limits_{i \in Z_+} q_i)$ and we are done. □

In the following result, we revisit Proposition 2.5 (2) of [11].

Theorem 2.16. If a proper element $q \in L$ is 1-absorbing prime, then for every $x \in L$ such that $x \nleqslant q$, $(q : x)$ is a prime element of L.

Proof. Let $x \in L$ be such that $x \nleqslant q$. Then $(q:x) \neq 1$. Let $ab \leqslant (q:x)$ and $a \nleqslant (q:x)$ where $a,b \in L$. Since q is 1-absorbing prime, $axb \leqslant q$, $ax \nleqslant q$ gives $b \leqslant q \leqslant (q:x)$. Therefore, $(q:x)$ is a prime element of L. □

Corollary 2.17. *Let q be a 1-absorbing prime element of L. Then the following statements hold true:*
① *If $x,y \in L$ such that $x \nleqslant q$ and $xy \nleqslant q$, then $(q:xy) = (q:x) = q$.*
② *If $x,y \in L$ such that $x \nleqslant q$ and $(q:x) < (q:y)$, then $(q:(ax \vee by)) = (q:x)$, $\forall\ a,b \in L$ such that $ab \nleqslant (q:x)$. In particular, $(q:(x \vee y)) = (q:x)$.*

Proof. ①. Let $x,y \in L$ be such that $x \nleqslant q$ and $xy \nleqslant q$. To show $(q:xy) \leqslant (q:x)$, consider $z \in L$ such that $z \leqslant (q:xy)$. Then $yz \leqslant (q:x)$ and $y \nleqslant (q:x)$ which implies $z \leqslant (q:x)$, because as $x \nleqslant q$, by Theorem 2.16, $(q:x)$ is prime. Hence $(q:xy) \leqslant (q:x)$. Further, as $xy \leqslant x$, we have $(q:x) \leqslant (q:xy)$ and hence $(q:xy) = (q:x)$. By Theorem 2.5③, as $xy \nleqslant q$, we have $(q:xy) = q$ and we are done.
②. Let $x,y \in L$ be such that $x \nleqslant q$ and $(q:x) < (q:y)$. Let $a,b \in L$ be such that $ab \nleqslant (q:x)$. Clearly, $(q:x) \leqslant (q:(ax \vee by))$. Assume that $(q:(ax \vee by)) \neq (q:x)$. Then there exists $z \leqslant (q:(ax \vee by))$ such that $z \nleqslant (q:x)$ for some $z \in L$. So $zax \leqslant q$ and hence $za \leqslant (q:x)$. As $x \nleqslant q$, by Theorem 2.16, $(q:x)$ is prime. It follows that either $z \leqslant (q:x)$ or $a \leqslant (q:x)$, a contradiction. Therefore, $(q:(ax \vee by)) = (q:x)$. For the particular case, put $a = b = 1$ and we are done. □

We will investigate under which condition(s), the converse of Theorem 2.16 holds true.

Theorem 2.18. Let q be an element of L such that $q \neq \sqrt{q}$ and $\sqrt{q}$ is a prime element of L. If $(q:x)$ is a prime element of L for each $x \leqslant \sqrt{q}$ with $x \nleqslant q$ such that $(q:x) < (q:a)\ \forall\ a \in L$ where $x \neq a$ and $a \neq 1$, then q is a 1-absorbing prime element of L.

Proof. Consider $xyz \leqslant q$ where $x,y,z \in L$. Since, $\sqrt{q}$ is prime, at least one of the elements x,y,z is less than or equal to $\sqrt{q}$. Without loss of generality, assume that $x \leqslant \sqrt{q}$. If $x \leqslant q$, then we are done. So suppose $x \nleqslant q$. As $(q:x)$ is a prime element of L and $yz \leqslant (q:x)$, we have either $y \leqslant (q:x)$ or $z \leqslant (q:x)$. It follows that either $y \leqslant (q:z)$ or $z \leqslant (q:y)$, since $(q:x) < (q:a)\ \forall\ a \in L$ where $x \neq a$ and $a \neq 1$. Therefore in any case, $yz \leqslant q$ and we are done. □

Theorem 2.19. Let q be a 1-absorbing prime element of L and let p and p' be prime elements of L. If $\sqrt{q} = (p \wedge p')$, then $(q:x)$ is a prime element of L and $\sqrt{(q:x)} = \sqrt{q}$, for all $x \nleqslant (p \vee p')$.

Proof. Let $x \nleqslant (p \vee p')$. Then $x \nleqslant \sqrt{q} = (p \wedge p')$ and hence $x \nleqslant q$. So by Theorem 2.16, $(q:x)$ is a prime element of L. Now to show $\sqrt{(q:x)} \leqslant \sqrt{q}$, let $y \leqslant \sqrt{(q:x)}$ where $y \in L_*$. Then there exists $k \in Z_+$ such that $y^k \leqslant (q:x)$. As $(q:x)$ is prime, it follows that $y \leqslant (q:x)$ and hence $xy \leqslant \sqrt{q}$. Since, by Theorem 2.6 ④, $\sqrt{q}$ is prime and $x \nleqslant \sqrt{q}$, we have $y \leqslant \sqrt{q}$ and thus $\sqrt{(q:x)} \leqslant \sqrt{q}$. Clearly, $\sqrt{q} \leqslant \sqrt{(q:x)} \leqslant \sqrt{q}$ and we are done. □

Lemma 2.20. *Let q be a 1-absorbing prime element of L and p_1, p_2 be two distinct minimal primes over q. If $x_1, x_2 \in L_*$ are such that $x_1 \leqslant p_1$, $x_1 \nleqslant p_2$, $x_2 \leqslant p_2$ and $x_2 \nleqslant p_1$, then $x_1x_2 \leqslant q$.*

Proof. Let $x_1, x_2 \in L_*$ be such that $x_1 \leqslant p_1$, $x_1 \nleqslant p_2$, $x_2 \leqslant p_2$ and $x_2 \nleqslant p_1$. By Lemma 3.5 in [1], there exist $c_1, c_2 \in L_*$, $c_2 \nleqslant p_1$, and $c_1 \nleqslant p_2$ such that $c_2x_1^n \leqslant q$ and $c_1x_2^m \leqslant q$, for some $m,n \in Z_+$. As q is 1-absorbing prime and $(c_2x_1^{n-1})x_1 \leqslant q$, it follows that either $c_2x_1 \leqslant q$ or $x_1 \leqslant q$. If $x_1 \leqslant q$, then $x_1 \leqslant p_2$, a contradiction. So the only possibility is $c_2x_1 \leqslant q$. Similarly, $c_1x_2^m \leqslant q$ will lead to $c_1x_2 \leqslant q$. Clearly, $(c_1 \vee c_2)x_1x_2 \leqslant q$. Since $(c_1 \vee c_2) \nleqslant p_1$, $(c_1 \vee c_2) \nleqslant p_2$ and p_1, p_2 are primes, we conclude that $(c_1 \vee c_2)x_1 \nleqslant p_2$, $(c_1 \vee c_2)x_2 \nleqslant p_1$. Hence $(c_1 \vee c_2)x_1 \nleqslant q$ and $(c_1 \vee c_2)x_2 \nleqslant q$ because $q \leqslant p_1$ and $q \leqslant p_2$ as p_1, p_2 are minimal primes over q. Moreover, $(c_1 \vee c_2) \nleqslant q$ because if $(c_1 \vee c_2) \leqslant q$, then $c_2 \leqslant (c_1 \vee c_2) \leqslant p_1$, a contradiction to $c_2 \nleqslant p_1$. Considering $((c_1 \vee c_2)x_1)x_2 \leqslant q$, $(c_1 \vee c_2)x_1 \nleqslant q$ and q is 1-absorbing prime, we get $x_2 \leqslant q$ which contradicts $x_2 \nleqslant p_1$. Considering $((c_1 \vee c_2)x_2)x_1 \leqslant q$, $(c_1 \vee c_2)x_2 \nleqslant q$ and q is 1-absorbing prime, we get $x_1 \leqslant q$ which contradicts $x_1 \nleqslant p_2$. Considering $(c_1 \vee c_2)(x_1x_2) \leqslant q$, $(c_1 \vee c_2) \nleqslant q$ and q is 1-absorbing prime, we get $x_1x_2 \leqslant q$. □

Lemma 2.21. *If q is a 1-absorbing prime element of L, then there are at most two prime elements of L that are minimal over q.*

Proof. Let $A = \{p_i \mid p_i$ is a prime element of L that is minimal over $q\}$. Assume that A has at least three elements. Let p_1, $p_2 \in A$ be two distinct prime elements. Then there exists x_1, $x_2 \in L_*$ such that $x_1 \leqslant p_1$, $x_1 \nleqslant p_2$ and $x_2 \leqslant p_2$, $x_2 \nleqslant p_1$. By Lemma 2.20, we have $x_1x_2 \leqslant q$. Now assume that there exists $p_3 \in A$ distinct from p_1 and p_2. Then we can choose $y_i \in L_*$ such that $y_i \leqslant p_i$, $y_i \nleqslant p_j$ for $i \neq j$ where $i, j = 1, 2, 3$. By Lemma 2.20, we have $y_1y_2 \leqslant q \leqslant p_3$, a contradiction. Therefore, A has at most two elements. □

Theorem 2.22. If q is a 1-absorbing prime element of L, then one of the following statements hold true:

①. $\sqrt{q} = p$ is a prime element of L such that $p^2 \leqslant q$.

②. $\sqrt{q} = p_1 \wedge p_2$ and $p_1p_2 \leqslant q$, where p_1, p_2 are the only distinct prime elements of L that are minimal over q.

Proof. By Lemma 2.21, we have either $\sqrt{q} = p$ is a prime element of L or $\sqrt{q} = p_1 \wedge p_2$ where p_1, p_2 are the only distinct prime elements of L that are minimal over q. If $\sqrt{q} = p$ is a prime element of L, then by Theorem 2.6, $p^2 \leqslant q$, so the condition ① holds.

Now assume that $\sqrt{q} = p_1 \wedge p_2$ where p_1, p_2 are the only distinct prime elements of L that are minimal over q. We show that $p_1p_2 \leqslant q$. Observe that $p_1p_2 = \vee\{xy \mid x, y \in L_*, x \leqslant p_1, y \leqslant p_2\}$. Following 3 cases arise:

Case 1. If x, $y \leqslant p_1 \wedge p_2 = \sqrt{q}$, then by Theorem 2.6-②, $xy \leqslant q$.

Case 2. If x, $y \in L_*$ are such that $x \leqslant p_1$, $x \nleqslant p_2$, and $y \leqslant p_2$, $y \nleqslant p_1$, then by Lemma 2.20, $xy \leqslant q$.

Case 3. Let x, $y \in L_*$ be such that $x \leqslant \sqrt{q}$, and $y \leqslant p_2$, $y \nleqslant p_1$. Choose $y_1 \in L_*$ such that $y_1 \leqslant p_1$ and $y_1 \nleqslant p_2$. Then by Lemma 2.20, $y_1y \leqslant q$. Clearly, $(x \vee y_1) \leqslant p_1$ but $(x \vee y_1) \nleqslant p_2$. So by Lemma 2.20, $(x \vee y_1)y \leqslant q$ and hence $xy \leqslant q$. Similarly, one can show that if $y \leqslant \sqrt{q}$, $x \leqslant p_1$ and $x \nleqslant p_2$, then $xy \leqslant q$.

Thus, in any case, $xy \leqslant q$. Consequently, we get $p_1p_2 \leqslant q$, so the condition ② holds. □

Lemma 2.23. *Let q be a 1-absorbing prime element of L such that $\sqrt{q} = p$ is a prime element of L and assume that $q \neq p$. Then the following statements hold true:*

①. For each $x \leqslant p$ and $x \nleqslant q$, $(q : x)$ is a prime element of L such that $p \leqslant (q : x)$.

②. Either $(q : x) \leqslant (q : y)$ or $(q : y) \leqslant (q : x)$, for all x, $y \leqslant p$ and x, $y \nleqslant q$.

Proof. ①. Let $x \leqslant p$ and $x \nleqslant q$. By Theorem 2.16, $(q : x)$ is prime. Now, by Theorem 2.6 ③, $p^2 \leqslant q$. Clearly, $xp \leqslant p^2$ and hence $p \leqslant (q : x)$.

②. Let x, $y \leqslant p$ and x, $y \nleqslant q$. Choose any $z \in L_*$ such that $z \leqslant (q : x)$ and $z \nleqslant (q : y)$. As $x \leqslant p$ and $x \nleqslant q$, by ①, we have $p \leqslant (q : x)$. As $y \leqslant p$ and $y \nleqslant q$, by ①, we have $p \leqslant (q : y)$. So $z \nleqslant p$ and hence $z \nleqslant q$. To show $(q : y) \leqslant (q : x)$, consider $w \in L_*$ such that $w \leqslant (q : y)$. If $w \leqslant p$, then as $p \leqslant (q : x)$, we have $w \leqslant (q : x)$ and we are done. So assume that $w \nleqslant p$. Clearly, $(x \vee y)zw \leqslant q$ and $zw \nleqslant q$. Moreover, if $(x \vee y)z \leqslant q$, then $yz \leqslant q$ which contradicts $z \nleqslant (q : y)$. Considering $(x \vee y)(zw) \leqslant q$, $zw \nleqslant q$ and q is 1-absorbing prime, we get $(x \vee y) \leqslant q$ which is a contradiction, since $x, y \nleqslant q$. Considering $((x \vee y)z)w \leqslant q$, $(x \vee y)z \nleqslant q$ and q is 1-absorbing prime, we get $w \leqslant q$ which implies $w \leqslant (q : x)$, since $q \leqslant (q : x)$ and we are done. Considering $((x \vee y)w)z \leqslant q$, $z \nleqslant q$ and q is 1-absorbing prime, we get $((x \vee y)w) \leqslant q$ which implies $xw \leqslant q$ and hence $w \leqslant (q : x)$. □

Lemma 2.24. *Let q be a 1-absorbing prime element of L such that $q \neq \sqrt{q} = p_1 \wedge p_2$ where p_1 and p_2 are the only non-zero distinct prime elements of L that are minimal over q. Then the following statements hold true:*

①. *For each $x \leqslant \sqrt{q}$ and $x \nleqslant q$, $(q:x)$ is a prime element of L such that $p_1 \leqslant (q:x)$ and $p_2 \leqslant (q:x)$.*

②. *Either $(q:x) \leqslant (q:y)$ or $(q:y) \leqslant (q:x)$, for all $x, y \leqslant \sqrt{q}$ and $x, y \nleqslant q$.*

Proof. ①. Let $x \leqslant \sqrt{q}$ and $x \nleqslant q$. By Theorem 2.16, $(q:x)$ is prime. Further, by Theorem 2.22, $p_1 p_2 \leqslant q$. So we have $xp_1 \leqslant q$ and $xp_2 \leqslant q$. Thus $p_1 \leqslant (q:x)$ and $p_2 \leqslant (q:x)$.

②. The proof is similar to the proof of Lemma 2.23 ② and hence omitted. □

Lemma 2.25. *Let q be a 1-absorbing prime element of L such that $q \neq \sqrt{q} = p_1 \wedge p_2$ where p_1 and p_2 are the only non-zero distinct prime elements of L that are minimal over q. Then the following statements hold true:*

①. *For each $x \leqslant \sqrt{q}$ and $x \nleqslant q$, $(q:x)$ is a prime element of L such that $p_1 p_2 \leqslant q$.*

②. *If $(q:x)$ is proper and either $x \leqslant p_1$ or $x \leqslant p_2$, then $(q:x)$ is a prime element of L.*

Proof. ①. Let $x \leqslant \sqrt{q}$ and $x \nleqslant q$. By Theorem 2.16, $(q:x)$ is prime. Further, by Theorem 2.22, $p_1 p_2 \leqslant q$.

②. Let $x \leqslant p_1$ and $x \nleqslant p_2$. Since by Theorem 2.22, $p_1 p_2 \leqslant q$, we have $xp_2 \leqslant q$. Hence $(q:x) = p_2$ is a prime element of L. Similarly, if $x \leqslant p_2$ and $x \nleqslant p_1$, then $(q:x) = p_1$ is a prime element of L. If $x \leqslant p_1$ and $x \leqslant p_2$ and $x \nleqslant q$, then by Theorem 2.16, $(q:x)$ is a prime element of L. □

Theorem 2.26. Let $q \in L$ such that $q \neq \sqrt{q} = p_1 \wedge p_2$ where p_1 and p_2 are the only non-zero distinct prime elements of L that are minimal over q. Let $(q:x)$ be proper element of L such that either $x \leqslant p_1$ or $x \leqslant p_2$. If $(q:x) < (q:a)$ $\forall a \in L$ where $x \neq a$ and $a \neq 1$, then q is a 1-absorbing prime element of L.

Proof. Consider $xyz \leqslant q$ where $x, y, z \in L$. Without loss of generality, assume that $x \leqslant p_1$. If $x \leqslant q$, then we are done. So suppose $x \nleqslant q$. By Lemma 2.25, $(q:x)$ is a prime element of L. So as $yz \leqslant (q:x)$, we have either $y \leqslant (q:x)$ or $z \leqslant (q:x)$. It follows that either $y \leqslant (q:z)$ or $z \leqslant (q:y)$, since $(q:x) < (q:a)$ $\forall$ $a \in L$ where $x \neq a$ and $a \neq 1$. Therefore in any case, $yz \leqslant q$ and we are done. □

3 Future scope

In an attempt to push ahead and give future scope to the concept of "1-absorbing prime element" in L, we define the following notions and conclude the paper.

Definition 3.1. A proper element $q \in L$ is called a weakly 1-absorbing prime element if for all proper elements $a, b, c \in L$ such that the product of a, b, c need not collapse to one of the element among them, $0 \neq abc \leqslant q$ imply either $ab \leqslant q$ or $c \leqslant q$.

Clearly, every 1-absorbing prime element of L is weakly 1-absorbing prime.

Definition 3.2. Let $n \geqslant 1$ in Z_+. A proper element $q \in L$ is said to be n-1-absorbing prime if for all proper elements $x_1, \cdots, x_{n+1} \in L$ such that the product of $x_1, \cdots, x_{n+1}$ need not collapse to one of the element among them, $x_1 \cdots x_{n+1} \leqslant q$ imply either $x_1 \cdots x_n \leqslant q$ or $x_{n+1} \leqslant q$.

Clearly, "2-1-absorbing prime element" of L is "1-absorbing prime element" of L.

Obviously, "1-1-absorbing prime element" of L is "prime element" of L.

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