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Graded S-r-ideals

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Abstract. Let G be a group, R a G-graded commutative ring with nonzero unity 1, and $S \subseteq h(R)$ a multiplicative subset of R. This article introduces the concept of graded S-r-ideal and graded S-pr-ideal. A proper graded ideal P of R is termed a *graded S-r-ideal* (resp., *graded S-pr-ideal*) if there exists an $s \in S$ such that for any nonunit elements $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$, then $sy \in P$ (resp., $sy \in Grad(P)$) with $Grad(P)$ being the graded radical of P. In this paper, we begin by providing counterexamples illustrating graded S-r-ideals and graded S-pr-ideals that do not qualify as graded S-prime ideals. Additionally, we present an example of a graded S-prime ideal that does not meet the criteria to be considered a graded S-r-ideal or a graded S-pr-ideal. Moreover, we investigate some fundamental properties of graded S-r-ideals (resp., graded S-pr-ideals), discuss their forms in finite direct products, and study their presence in Nagata's idealization ring.

Key Words: Graded S-r-ideals, Graded S-pr-ideals, Nagata's idealization.
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1 Introduction

Let G be a multiplication group with identity e. A ring R is called a G-graded ring if $R = \bigoplus_{g \in G} R_g$ with the property $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$, where R_g is an additive subgroup of R for all $g \in G$. The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\sum_{g \in G} x_g$, where x_g is the component of x in R_g. The set of all homogeneous elements of R is $h(R) = \bigcup_{g \in G} R_g$. Let P be an ideal of a G-graded ring R. Then P is called a graded ideal if $P = \bigoplus_{g \in G} P_g$, i.e, for $x \in P$, $x = \sum_{g \in G} x_g$ where $x_g \in P$ for all $g \in G$. An ideal of a G-graded ring is not necessary graded ideal (see [2]). Let P be a proper graded ideal of R. Then the graded radical of P is denoted by $Grad(P)$ and it is defined as written below:
$Grad(P) = \{x = \sum_{g \in G} x_g \in R \mid \text{for all } g \in G, \text{ there exists } n_g \in \mathbb{N} \text{ such that } x_g^{n_g} \in P\}$.
Note that $Grad(P)$ is always a graded ideal of R (see [15]). The graded radical of P is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in P$. Note that, if r is a homogeneous element of (R, G), then $r \in Grad(P)$ if and only if $r^n \in P$ for some $n \in \mathbb{N}$.

Lemma 1.1. *([14]) Let I, J be graded ideals of R. Then*

1. $I \subseteq Grad(I)$.

2. $Grad(Grad(I)) = Grad(I)$.

3. $Grad(I) = R$ iff $I = R$.

4. $Grad(IJ) = Grad(I \cap J) = Grad(I) \cap Grad(J)$.

Throughout this paper, all rings are to be commutative with $1 \neq 0$. A graded ideal P of a graded ring R is called *graded prime*, $ab \in P$ where $a, b \in h(R)$ implies $a \in P$ or $b \in P$ (see [3]). In recent years, prime ideals and graded prime ideals play a very important role in the Commutative graded rings theory. There are several ways to generalize this concept of a prime ideals graded prime ideals, for example, in [19], Yavuz and all, introduced the notion of weakly S-2-prime ideals of commutative rings. Let S a multiplicative subset of R. A proper ideal Q of R with $Q \cap S = \emptyset$ is called a weakly S-2-prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ with $0 \neq ab \in Q$, we have $sa^2 \in Q$ or $sb^2 \in Q$. In [16], H. Saber and all, further generalized this concept by introducing graded S-prime ideals. In their influential paper, a proper graded ideal I of R is termed *graded S-prime* if $ab \in I$ for some $a, b \in h(R)$ implies either $sa \in I$ or $sb \in I$ for some $s \in S$. In [12] Issoual and Ugurlu introduced the notion of graded 1-absorbing primary ideals and graded weakly 1-absorbing primary ideals. Subsequently, in 2021 the authors in ([4]) introduced the notion of graded S-primary ideals. A graded ideal I of R is called *graded S-primary* if $ab \in I$ for some $a, b \in h(R)$ implies either $sa \in I$ or $sb^n \in I$ for some $s \in S$ and some positive integer n. As usual, if R is a graded ring and $a \in R$, then $ann(a)$ denotes the set of annihilator of a in R. $ann(a) = \{r \in R \mid ra = 0\}$. In commutative algebra, graded r-ideal have an important role. There have been lots of studies on this issue (See [2]). Recall that a proper graded ideal I is called *graded r-ideal* (resp., *graded pr-ideal*), if for all $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$, then $x \in I$ (resp., $y^n \in I$ for some positive integer n). It's clear that any graded r-ideal (or graded pr-ideal) disjoint from S automatically qualifies as a graded S-r-ideal (or graded S-pr-ideal). However, Example 2.24, illustrates a counterexample showing that a graded S-r-ideal (or graded S-pr-ideal) does not necessarily retain its status as a graded r-ideal (or graded pr-ideal) in general.

The main aim of this paper is to expand upon the concept of graded ideals by introducing and investigating graded S-r-ideals (and graded S-pr-ideals) within graded rings. This approach serves to generalize essentially all the results pertaining to graded r-ideals and graded pr-ideals. Let R be a graded ring and $S \subseteq h(R)$ a multiplicative subset of R. A proper graded ideal P of a graded ring R disjoint from S is said to be a *graded S-r-ideal* (resp., *graded S-pr-ideal*) of R if whenever nonunit elements $x, y \in h(R)$ such that $xy \in P$ and $ann(a) = 0$, then $sy \in P$ (resp., $sy \in Grad(P)$). In the opening section, we commence with counterexamples illustrating graded S-r-ideals and graded S-pr-ideals that do not belong to be a graded S-prime ideals. Moreover, we offer an illustration of a graded S-prime ideal that does not satisfy the requirements to be considered a graded S-r-ideal or a graded S-pr-ideal (see Example 2.2 and 2.3).

Among many results in paper, we investigate graded S-r-ideal (resp., graded S-pr-ideal) within the context of graded homomorphisms Proposition 2.12. Also, Theorem 2.11, proves that if P is a graded S-r-ideal of R, then $(P : a)$ is a graded S-r-ideal of R for all $a \in h(R) - P$. Additionally, Theorem 2.20 and Theorem 2.22, allows us to identify all graded S-r-ideal (resp., graded S-pr-ideal) within the Cartesian product of two cross products. Let R be a graded ring, then $zd(R)$ denotes the set of zero divisors of R and $Reg(R) = R - zd(R)$. Consider $S \subset Reg(R)$ in Proposition 2.14, we show that if I be an graded S-prime ideal of R, then I is a graded S-r-ideal of R if and only if $(I : t) \cap h(R) \subseteq zd(R)$ for some $t \in S$.

Finally, in the current section, our focus lies in investigating the graded S-r-ideal (resp., graded S-pr-ideal) and graded $S(+)M$-r-ideals (resp., graded $S(+)M$-pr-ideals) of $R(+)M$.

2 Basic results

Definition 2.1. Let R be a graded ring and $S \subseteq h(R)$ a multiplicative subset of R. A proper graded ideal P of a graded ring R disjoint from S is said to be a *graded S-r-ideal* (resp., *graded S-pr-ideal*) of R if whenever nonunit elements $x, y \in h(R)$ such that $xy \in P$ and $ann(a) = 0$, then $sy \in P$ (resp., $sy \in Grad(P)$).

Let R be a graded ring, $S \subseteq h(R)$ a multiplicative subset of R, and P a graded ideal of R disjoint from S. According to the definition of graded S-prime ideals given in [16], P is considered a *graded S-prime* ideal of R if there exists an element $s \in S$ such that for all $x, y \in h(R)$ such that $xy \in P$, then $sx \in P$ or $sy \in P$. In the following, we will provide an example of a graded S-prime ideal which is neither a graded S-r-ideal nor a graded S-pr-ideal of R.

Example 2.2. Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider the multiplicative subset $S = \{2^n \mid n \in \mathbb{N}\}$ and the graded ideal $I = 5R$ of R. To see more details to show I is graded prime ideal of R (see [4], Example 1); so S-prime ideal of R since $I \cap S = \emptyset$. But I is neither an S-r-ideal nor an S-pr-ideal of R. Indeed, we have $5 \cdot 2 \in I$ with $ann(5) = 0$, but $2^n \cdot 2 \notin Grad(I)$ for all positive integer n. This shows that I is neither an S-r-ideal nor an S-pr-ideal of R.

Now, we give an example of graded S-r-ideal which is not graded S-prime ideal.

Example 2.3. Consider $R = \mathbb{Z}_6[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}_6$ and $R_1 = i\mathbb{Z}_6$. Consider the multiplicative subset $S = \{\overline{1}, \overline{5}\}$ and the graded ideal $P = \{\overline{0}\}$ of R. Let $x, y \in h(R)$ such that $xy \in P$ and $ann(x) = \overline{0}$, then $y = \overline{0} \in P$ which implies that P is a graded r-ideal of R; so is a graded S-r-ideal of R since $P \cap S = \emptyset$. But P is not a graded S-prime ideal of R. Indeed, $\overline{2} \cdot \overline{3} = \overline{0} \in P$, but $s2 \notin P$ and $s3 \notin P$ for all $s \in S$.

Let R and S are two G-graded rings, then $R \times S$ is a G-graded ring by $(R \times S)_g = R_g \times S_g$ for all $g \in G$, (see [16]). Also, if P be an ideal of a G-graded ring R and K be an ideal of a G-graded ring S. Then $P \times K$ is a graded ideal of $R \times S$ if and only if P is a graded ideal of R and K is a graded ideal of S ([7]).

Example 2.4. Consider $R = \mathbb{Z}_2[i] \times \mathbb{Z}_2[i] \times \mathbb{Z}_2[i]$ and $G = \mathbb{Z}_2$. As $A = \mathbb{Z}_2[i]$ is a G-graded ring by $A_0 = \mathbb{Z}_2$ and $A_1 = i\mathbb{Z}_2$, R is G-graded by $R_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $R_1 = i\mathbb{Z}_2 \times i\mathbb{Z}_2 \times i\mathbb{Z}_2$. Consider the multiplicative subset $S = \{(\overline{1}, \overline{1}, \overline{0}), (\overline{1}, \overline{1}, \overline{1})\}$ and the graded ideal $P = \overline{0} \times \overline{0} \times i\mathbb{Z}_2$ of R. Let $x, y \in h(R)$ such that $xy \in P$ and $ann(x) = \overline{0}$. If $x \in R_0 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is boolean, $x^2 = x$; so $x - 1 \in ann(x) = 0$, thus $x = 1$. This implies that $y \in P$. If $x \in R_1 = i\mathbb{Z}_2 \times i\mathbb{Z}_2 \times i\mathbb{Z}_2$, then there exits $a, b, c \in \mathbb{Z}_2$ such that $x = (ia, ib, ic)$. Thus $x^2 = (-a^2, -b^2, -c^2) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is boolean, $x^4 = (-a^2, -b^2, -c^2)^2 = (-a^2, -b^2, -c^2) = x^2$. So $x^3 - x \in ann(x) = 0$, thus $x^3 = x$ which implies that $x^2 - 1 \in ann(x) = 0$; so $x^2 = 1$. As $xy \in P$, $x^2 y \in P$. This implies that $y \in P$. Hence P is a graded S-r-ideal of R. Now, we have $(\overline{0}, \overline{1}, \overline{1}), (\overline{1}, \overline{0}, \overline{1}) \in h(R)$, $(\overline{0}, \overline{1}, \overline{1})(\overline{1}, \overline{0}, \overline{1}) = (\overline{0}, \overline{0}, \overline{1}) \in P$ and $s(\overline{0}, \overline{1}, \overline{1}) \notin P$ and $s(\overline{1}, \overline{0}, \overline{1}) \notin P$ for all $s \in S$. This implies that P Does not have a graded S-prime ideal of R.

Lemma 2.5. *([11]) Let I be a proper graded ideal of a G-graded ring R. Let $c \in h(R)$, then $(I : c) = \{r \in R \mid cr \in I\}$ is a graded ideal of R.*

Theorem 2.6. Let R be a G-graded ring, $S \subseteq h(R)$ a multiplicative subset of R and P be a graded ideal of R disjoint from S. Then P is a graded S-r-ideal if and only if for all $a \in h(R)$ if $ann(a) = (0)$, then $(P : a) \subseteq (P : s)$ for some $s \in S$.

Proof. Let $a \in h(R)$ such that $ann(a) = (0)$ and let $x \in (P : a)$. As $(P : a)$ is a graded ideal of R, $x_g \in (P : a)$ for all $g \in G$, then $ax_g \in P$ for all $g \in G$. Since P is an S-r-ideal, $sx_g \in P$ for some $s \in S$; so $x \in (P : s)$. Conversely, let $a, b \in h(R)$ with $ab \in P$ and $ann(a) = (0)$. Then $b \in (P : a) \subseteq (P : s)$ for some $s \in S$; so $sb \in P$. Hence P is a S-r-ideal of R. $\square$

Let R be a G-graded ring, and $S \subseteq h(R)$ a multiplicatively subset of R. Then $S^* = \{a_g \in h(R) \mid \frac{a_g}{1}$ is a unit of $S^{-1}R\}$ denotes the saturation of S, $S^* \subseteq h(R)$ is a multiplicative subset of R containing S (see [17]).

Proposition 2.7. *Let R be a G-graded ring, and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then*

1. *If $S \subseteq T$ are multiplicatively subsets of R such that $P \cap T = \emptyset$ and P is a graded S-r-ideal (resp., S-pr-ideal) of R, then P is a graded T-r-ideal (resp., T-pr-ideal) of R. If for any $s \in T$, there exists $t \in T$ satisfying $st \in S$, then P is a graded T-r-ideal (resp., T-pr-ideal) of R implies P is a graded S-r-ideal (resp., S-pr-ideal) of R.*

2. *P is a graded S-r-ideal (resp., S-pr-ideal) of R if and only if P is a graded S^*-r-ideal (resp., S^*-pr-ideal) of R.*

3. *If $S \subseteq Reg(R)$, then P is a graded S-r-ideal of R if and only if $(P : s)$ is a graded r-ideal for some $s \in S$.*

4. *If $S \subseteq Reg(R)$, then P is a graded S-pr-ideal of R if and only if $(P : s)$ is a graded pr-ideal for some $s \in S$.*

Proof. (1). It is clear that if P is a graded S-r-ideal of R, then P is a graded T-r-ideal of R. Now, assume that P is graded T-r-ideal of R, there exists $s \in T$ such that if $xy \in P$ for all $x, y \in h(R)$ with $ann(x) = 0$, then $sy \in P$. By assumption $w = ts \in S$ for some $t \in T$. So $wy = tsy \in P$. Consequently, P is a graded S-r-ideal of R.

(2). Assume that P is a graded S-r-ideal of R. First, we want to show that $P \cap S^* = \emptyset$. Suppose there exists $a_g \in P \cap S^*$. Then $\frac{a_g}{1}$ is a unit of $S^{-1}R$, it follows that there exist $b_h \in h(R)$ and $s_i \in S$ such that $\frac{a_g}{1}\frac{b_h}{s_i} = 1$. Hence $t_j s_i = t_j a_g b_h$ for some $t_j \in S$. So $t_j s_i = t_j a_g b_h \in P \cap S$, a contradiction. Therefore $P \cap S^* = \emptyset$. Now, by (1) we get P is a graded S^*-r-ideal of R. Conversely, assume that P is a graded S^*-r-ideal of R. Let $xy \in P$ with $ann(x) = 0$ for some $x, y \in h(R)$. Then there exists $a_g \in S^*$ such that $a_g y \in P$. Since $\frac{a_g}{1}$ is a unit of $S^{-1}R$, there exist $b_j \in h(R)$ and $s, t \in S$ such that $ts = ta_g b_j$. It follows that $(ts)y = ta_g b_j y \in P$. Therefore, P is a graded S-r-ideal of R.

(3). Assume that P is a graded S-r-ideal of R. Then for all $x, y \in h(R)$ if $xy \in P$ with $ann(x) = 0$, then $sy \in P$ for some $s \in S$. Now, let $a, b \in h(R)$ such that $ab \in (P : s)$ with $ann(a) = 0$. Then $sab \in P$. It follows that $s^2 y \in P$; so $y \in (P : s)$ since $ann(s^2) = 0$. This implies that $(P : s)$ is a graded r-ideal of R. Conversely, assume that $(P : s)$ is a graded r-ideal of R for some $s \in S$ and let $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$. Then $xy \in (P : s)$; so $y \in (P : s)$. hence $sy \in P$.

(4). Assume that P is a graded S-pr-ideal of R. Then for all $x, y \in h(R)$ if $xy \in P$ with $ann(x) = 0$, then $sy \in Grad(P)$ for some $s \in S$. Now, let $a, b \in h(R)$ such that $ab \in (P : s)$ with $ann(a) = 0$. Then $sab \in P$. It follows that $sy \in Grad(P)$, then $s^m y^m \in P$ for some positive integer m. As $s \in Reg(R)$, $ann(s^m) = 0$; so $y^m \in (P : s)$. This implies that $(P : s)$ is a graded pr-ideal of R. Conversely, assume that $(P : s)$ is a graded pr-ideal of R for some $s \in S$ and let $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$. Then $xy \in Grad(P : s)$; so $y^n \in (P : s)$ for some positive integer n which implies that $sy^n \in P$. Hence $sy \in Grad(P)$. □

Let R be a G-graded ring, and $S \subseteq h(R)$ a multiplicatively subset of R. Note that $S^{-1}R$ is a G-graded ring with $S^{-1}R = \{\frac{r}{s} \mid r \in R, s \in S\}$. If P is a graded ideal of R and $P \cap S = \emptyset$, then $S^{-1}P \neq S^{-1}R$, $S^{-1}Grad(P) = Grad(S^{-1}P)$ and $S^{-1}P$ is a graded ideal of $S^{-1}R$ (see [6] and [13]).

Proposition 2.8. *Let R be a graded ring $S \subseteq h(R)$ a multiplicative subset of R and P a graded ideal of R. If P is a graded S-r-ideal (resp., S-pr-ideal) of R, then $S^{-1}P$ is a graded r-ideal (resp., pr-ideal) of $S^{-1}R$.*

Proof. Assume that P is a graded S-r-ideal of R. Let $\frac{x}{\alpha}, \frac{y}{\beta}, \in h(S^{-1}R)$ such that $\frac{x}{\alpha}\frac{y}{\beta} \in S^{-1}P$ with $ann_{S^{-1}R}(\frac{x}{\alpha}) = 0$. Then there exists $s \in S$ such that $sxy \in P$ with $ann_R(x) = 0$. Since P is a graded

S-r-ideal of R, there exists $s_1 \in S$ such that $s_1 sy \in P$. Therefore, $\frac{y}{\beta} = \frac{s_1 sy}{s_1 s\beta} \in S^{-1}P$. Hence, $S^{-1}P$ is a graded r-ideal of $S^{-1}R$. □

Proposition 2.9. *Let R be a G-graded ring and $S \subseteq Reg(R)$ a multiplicative subset of R. Then P is a graded S-r-ideal if and only if $S^{-1}P$ is a graded r-ideal of $S^{-1}R$ and $(S^{-1}P) \cap h(R) = (P :_{h(R)} s)$ for some $s \in S$.*

Proof. Assume that P is a graded S-r-ideal of R. By Proposition 2.8, $S^{-1}P$ is a graded r-ideal of $S^{-1}R$. Now, we show that there exists an $s \in S$ such that $(S^{-1}P) \cap h(R) = (P :_{h(R)} s)$. As P is a graded S-r-ideal of R, there exists an $s \in S$ such that for all $x, y \in h(R)$, if $xy \in P$ with $ann(x) = 0$, then $sy \in P$. Let $x \in (P :_{h(R)} s)$. Then $sx \in P$. As $S \subseteq Reg(R)$, the graded mapping $\varphi : R \to S^{-1}R$ defined by $\varphi(r) = \frac{r}{1}$ is an injective rings homomorphism. Thus $x = \frac{x}{1} = \frac{sx}{s} \in S^{-1}P$; so $x \in (S^{-1}P) \cap h(R)$. Conversely, let $x \in (S^{-1}P) \cap h(R)$. Then $x = \frac{\alpha}{t}$ for some $\alpha \in P$ and some $t \in S$. So $xt = \alpha \in P$ with $ann(t) = 0$, then $sx \in P$ since P is a graded S-r-ideal. Hence $x \in (P :_{h(R)} s)$.

Now, assume that $S^{-1}P$ is a graded r-ideal of $S^{-1}R$ and $(S^{-1}P) \cap h(R) = (P :_{h(R)} s)$ for some $s \in S$. Let $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$. Then $\frac{x}{1}\frac{y}{1} \in S^{-1}P$ and $ann(\frac{x}{1}) = 0$; so $\frac{y}{1} \in S^{-1}P$ which implies that $y = \frac{y}{1} \in S^{-1}P \cap h(R) = (P :_{h(R)} s)$ for some $s \in S$. Hence $sy \in P$.

□

Proposition 2.10. *Let R be a G-graded ring, $S \subseteq h(R)$ a multiplicative subset of R and P a graded ideal of R disjoint from S. Then P is a graded S-r-ideal (resp., S-pr-ideal) if and only if whenever I, J are graded ideals of R such that $IJ \subseteq P$ and $I \cap Reg(h(R)) \neq \emptyset$, then $sJ \subseteq P$ (resp., $sJ \subseteq Grad(P)$) for some $s \in S$.*

Proof. Suppose that P is a graded S-r-ideal of R. Let I, J be two graded ideals of R such that $IJ \subseteq P$ and $I \cap Reg(h(R)) \neq \emptyset$. There exists $a \in I \cap Reg(h(R))$. Let $g \in G$ and $b \in Jg$. Then $ann(a) = 0$ and $ab \in IJg \subseteq IJ \subseteq P$. Since P is a graded S-r-ideal, $sb \in P$ for some $s \in S$. So $sJg \subseteq P$ for all $g \in G$ which implies that $sJ \subseteq P$. Conversely, let $a, b \in h(R)$ such that $ab \in P$ with $ann(a) = \{0\}$. Then $I = \langle a \rangle$ and $J = \langle b \rangle$ are graded ideals of R such that $IJ \subseteq P$ and $a \in I \cap Reg(h(R))$. By assumption, $sJ \subseteq P$ for some $s \in S$. So $sb \in P$. Hence, P is a graded S-r-ideal of R. □

Theorem 2.11. Let R be a graded ring, $S \subseteq h(R)$ a multiplicative subset of R and P a graded ideal of R disjoint from S. If P is a graded S-r-ideal of R, then $(P : a)$ is a graded S-r-ideal of R for all $a \in h(R) - P$.

Proof. Put $a \in h(R) - P$, then $(P : a)$ is a graded ideal of R. Let $x, y \in h(R)$ such that $xy \in (P : a)$ with $ann(x) = 0$. Then $axy \in P$; so $say \in P$ for some $s \in S$ since P is graded S-r-ideal and $ann(a) = 0$. Thus $sy \in (P : a)$. □

For G-graded rings R_1 and R_2, a G-graded ring homomorphism $f : R_1 \to R_2$ is a ring homomorphism such that $f((R_1)_g) \subseteq (R_2)_g$ for every $g \in G$ (see[11]). By [18, Remark 2.6], we have the following results.

1. If I_2 is a graded ideal of R_2 and $f : R_1 \to R_2$ is a graded ring homomorphism, then $Grad(f^{-1}(I_2)) = f^{-1}(Grad(I_2))$.
2. If I_1 is a graded ideal of R_1 such that $Ker(f) \subseteq I_1$ and f is a graded epimorphism, then $f(Grad(I_1)) = Grad(f(I_1))$.

Proposition 2.12. *Let R, R' be two G-graded rings, $f : R \to R'$ be a graded isomorphism with $f(1) = 1$ and $S \subseteq h(R)$ a multiplicatively subset of R such that $f(S)$ does not contain zero. Then*

1. *Assume that for every nonunits homogeneous elements r of $h(R)$, we have $f(r)$ is nonunit element of $h(R')$. If P' is a graded $(f(S))$-r-ideal of R', then $f^{-1}(P')$ is a graded S-r-ideal of R.*

2. *If P is a graded S-r-ideal of R, then $f(P)$ is a graded $(f(S))$-r-ideal of R'.*

Proof. (1). Assume that P' is a graded $(f(S))$-r-ideal of R'. First, we want to show that $f^{-1}(P')\cap S = \emptyset$. Suppose on the contrary that there exists $s \in f^{-1}(P')\cap S$. Then $f(s) \in P'\cap f(S)$, which is a contradiction since $P' \cap f(S) = \emptyset$. Now, let $x, y \in h(R)$ such that $xy \in f^{-1}(P')$ with $ann(x) = 0$. Since f is bijective, $ann(f(x)) = 0$. We have $f(xy) = f(x)f(y) \in P'$. Since P' is a graded $(f(S))$-r-ideal of R', there exists $s \in S$ such that $f(s)f(y) = f(sy) \in P'$. Hence $sy \in f^{-1}(P')$.

(2). First, we show that $f(P) \cap f(S) = \emptyset$. Assume not. Then there exists $u \in f(P) \cap f(S)$, which implies $u = f(x) = f(s)$ for some $x \in P$ and $s \in S$. Hence, $x - s \in Ker(f) = (0) \subseteq P$ and $s \in P \cap S$, which is a contradiction. Now, let $x', y' \in h(R')$ such that $x'y' \in f(P)$ with $ann(x') = 0$. Say $x' := f(x)$ and $y' := f(y)$ for some $x, y \in h(R)$ with $ann(x) = 0$. This yields that there exists an $s \in S$ such that $sy \in P$. Thus, clearly we have $f(s)y' \in f(P)$. □

Let R be a G-graded ring and $S \subseteq h(R)$. Then R/I is a G-graded ring by $(R/I)_g = (R_g + I)/I$ for all $g \in G$. Moreover, if P and I are graded ideals of R, then P/I is graded ideal of R/I and $\overline{S} = \{s+I \mid s \in S\}$ is a multiplicative subset of R/I (see [16]).

Proposition 2.13. *Let R be a G-graded ring and $S \subseteq h(R)$. Let I be a graded r-ideal of R disjoint from S and P a graded ideal of R such that $I \subseteq P$. Then P is a graded S-r-ideal of R if and only if P/I is a graded $\overline{S}$-r-ideal of R/I.*

Proof. Note that $P \cap S = \emptyset$ if and only if $(P/I) \cap \overline{S} = \emptyset$. Now, assume that P is a graded S-r-ideal of R. Let $a + I, b + I \in h(R/I)$ such that $(a + I)(b + I) \in P/I$ with $ann(a + I) = 0$. Then $(ab + I) \in P/I$; so $ab \in P$. As P is a graded S-r-ideal of R and $ann(a) = 0$, there exists $s \in S$ such that $sb \in P$. This implies that $(sb + I) \in P/I$. Conversely, since P/I is a graded $\overline{S}$-r-ideal of R/I, there exists $t \in S$ such that for all $a + I, b + I \in h(R)$ if $(a + I)(b + I) \in P/I$ with $ann(a + I) = \overline{0}$, then $\overline{s}(b + I) \in P/I$. Let $xy \in J$ for all $x, y \in h(R)$ with $ann(x) = 0$. Then $(x + I)(y + I) \in P/I$, since I is graded r-ideal, $ann(x + I) = \overline{0}$. So $(ty + I) = (t + I)(y + I) \in P/I$. This implies that $ty \in P$. □

Proposition 2.14. *Let R be a graded ring and $S \subseteq h(R)$ a multiplicative subset of R. Then the following are hold.*

1. *If I is a graded S-r-ideal of R, then $I \cap h(R) \subseteq zd(R)$.*

2. *Let I be an graded S-prime ideal of R. If $S \subset Reg(R)$, then I is a graded S-r-ideal of R if and only if $(I : t) \cap h(R) \subseteq zd(R)$ for some $t \in S$.*

Proof. (1). Let I be a graded S-r-ideal of R. Then for all $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$, then $sy \in I$ for some $s \in S$. Assume that $I \cap h(R) \not\subseteq zd(R)$. There exists an $a \in I \cap h(R)$ such that so $ann(a) = 0$. Since $a \in I$, $s \cdot 1 \in I$; so $s \in I \cap S$ contradiction. Hence $I \cap h(R) \subseteq zd(R)$.

(2). Assume that I is a graded S-prime ideal. There exists an $t \in S$ such that for all $x, y \in h(R)$ such that $xy \in I$, then $tx \in I$ or $ty \in I$. If I is a graded S-r-ideal, then by Proposition 2.7, $(I : t)$ is a graded r-ideal. As $I \cap S = \emptyset$, $(I : t) \cap S = \emptyset$; so $(I : t)$ is a graded S-r-ideal of R. Then by (1), $(I : t) \cap h(R) \subseteq zd(R)$. Conversely, A assume that $(I : t) \cap h(R) \subseteq zd(R)$. Let $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$. We have $tx \in I$ or $ty \in I$. If $tx \in I$, then $x \in (I : t) \cap h(R) \subseteq zd(R)$, contradiction. So $ty \in I$. This implies that I is a graded S-r-ideal of R. □

Lemma 2.15. *Let R be a graded ring, $S \subseteq h(R)$ a multiplicative subset of R and P a graded ideal of R. Then P is a graded S-pr-ideal of a graded ring R, if and only if $Grad(P)$ is a graded S-r-ideal of R.*

Proof. Assume that P is a graded S-pr-ideal. There exists $s \in S$ such that for all $a, b \in h(R)$ if $ab \in P$ with $ann(a) = 0$, then $sb \in Grad(P)$. Let $x, y \in h(R)$ such that $xy \in Grad(P)$ with $ann(x) = 0$. Then $(xy)^n \in P$. So $sx^{n-1}y^n \in P$ which implies that $s^2x^{n-2}y^n \in P$ as $ann(x) = 0$. We continue this process, we obtain $s^n y^n \in P$. This implies that $sy \in Grad(P)$. Hence $Grad(P)$ is a graded S-r-ideal of R. Conversely, Assume that $Grad(P)$ is a graded S-r-ideal. There exists $s \in S$ such that for all $a, b \in h(R)$ if $ab \in Grad(P)$ with $ann(a) = 0$, then $sb \in Grad(P)$. Let $x, y \in h(R)$ such that $xy \in P \subseteq Grad(P)$ with $ann(x) = 0$, then $sy \in Grad(P)$ for some $s \in S$. This implies that P is graded S-pr-ideal of R. □

Proposition 2.16. *Let R be a graded ring and $S \subseteq h(R)$ a multiplicative subset of R. Let P be an graded S-primary ideal of R. If $S \subset Reg(R)$, then P is a graded S-pr-ideal of R if and only if $(P : s) \cap h(R) \subseteq zd(R)$ for some $s \in S$.*

Proof. Let P be a graded S-primary ideal of R. Then for all $x, y \in h(R)$ if $xy \in P$, then $sx \in P$ or $sy \in Grad(P)$ for some $s \in S$. Assume that P is a graded S-pr-ideal of R, then $Grad(P)$ is a graded S-r-ideal of R. So $(Grad(P) : s) \cap h(R) \subseteq zd(R)$. As $P \subseteq Grad(P)$, $(P : s) \cap h(R) \subseteq (Grad(P) : s) \cap h(R) \subseteq zd(R)$. Conversely, assume that $(P : s) \cap h(R) \subseteq zd(R)$. Let $x, y \in h(R)$ such that $xy \in P$ with $ann(x) = 0$, then $sx \in P$ or $sy \in Grad(P)$. If $sa \in P$, then $a \in (P : s) \cap h(R) \subseteq zd(R)$ contradiction. Then $sb \in Grad(P)$. This implies that P is a graded S-pr-ideal of R. □

Note that in R a G-graded ring and I, J are graded ideals of R, then $I \cap J$ is a graded ideal of R and $Grad(I \cap J) = Grad(I) \cap Grad(J)$ (see [9] and [14]).

Theorem 2.17. Let R be a graded ring, $S \subseteq h(R)$ a multiplicative subset of R and I, J are a graded S-r-ideals (resp., S-pr-ideals) of R disjoint from S. Then $I \cap J$ is graded S-r-ideal (resp., S-pr-ideal) of R.

Proof. First, since $I \cap S = \emptyset$ and $J \cap S = \emptyset$, $I \cap J \cap S = \emptyset$. Let $x, y \in h(R)$ such that $xy \in I \cap J$ with $ann(x) = 0$. Then $xy \in I$. Since I is a graded S-r-ideal, $sy \in I$ for some $s \in S$. Similarly, $ty \in J$ for some $t \in S$ and hence $sty \in I \cap J$. Therefore, $I \cap J$ is a graded S-r-ideal of R. □

Theorem 2.18. Let I be a graded S-r-ideal (resp., S-pr-ideal) of R. For a graded ideal J of R with $J \cap S \neq \emptyset$, $I \cap J$ is graded S-r-ideal (resp., S-pr-ideal) of R.

Proof. Since $I \cap S = \emptyset$, clearly we have $(I \cap J) \cap S = \emptyset$. Suppose that I is a graded S-r-ideal of R. There exists an $s \in S$ such that for all $x, y \in h(R)$, if $xy \in I$ with $ann(x) = 0$, then $sy \in I$. Let $xy \in I \cap J \subseteq I$ with $ann(x) = 0$. Then $sy \in I$. Choose $t \in J \cap S$. This means $w = st \in S$. Thus $wy \in I$ which implies that $wy \in I \cap J$. So $I \cap J$ is a graded S-r-ideal of R. □

Theorem 2.19. Let R be a G-graded ring, $S \subseteq h(R)$ a multiplicative subset of R and I, J be graded prime ideals of R which are not comparable disjoint from S. If $I \cap J$ is a graded S-r-ideal of R, then I and J are graded S-r-ideals of R.

Proof. Let $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$. Suppose that $a \in J - I$. Then $xya \in I \cap J$. Since $I \cap J$ is graded S-r-ideal, $sya \in I \cap J$ for some $s \in S$. Then $sya \in I$. Since I is a graded prime ideal of R and $a \notin I$, $sy \in I$. Hence, I is a graded S-r-ideal of R. Similarly, J is a graded S-r-ideal of R. □

Theorem 2.20. Let $R = R_1 \times R_2$ where each R_i is a G-graded ring for all $i \in \{1, 2\}$, $P = P_1 \times P_2$ a graded ideal of R where each P_i is a graded ideal of R_i and $S = S_1 \times S_2 \subseteq h(R)$ a multiplicative subset of R such that $S_1 \cap P_1 = \emptyset$ and $S_2 \cap P_2 = \emptyset$. Then the following are equivalents.

1. $P = P_1 \times P_2$ is a graded S-r-ideal of R.

2. $P_1 = R_1$ and P_2 is a graded S_2-r-ideal of R_2 or $P_2 = R_2$ and P_1 is a graded S_1-r-ideal of R_1 or either P_1 is a graded S_1-r-ideal of R_1 and P_2 is a graded S_2-r-ideal of R_2.

Proof. (1)$\Rightarrow$(2). Suppose that $P = P_1 \times P_2$ is a graded S-r-ideal of R and $P_2 = R_2$. We will show that P_1 is a graded S_1-r-ideal of R_1. Let $x, y \in h(R_1)$ such that $xy \in P_1$ with $ann(x) = 0$. Note that $ann(a, 1) = (0, 0)$ and $(a, 1)(b, 0) = (ab, 0) \in P$. Since P is a graded S-r-ideal of R, there exists an $s = (s_1, s_2) \in S$ such that $s(b, 0) \in P$; so $s_1 b \in P_1$. Hence P_1 is a graded S_1-r-ideal of R_1. Similarly, one can easily show that P_2 is a graded S_2-r-ideal of R_2 and $P_1 = R_1$. Assume that P_1, I_2 are proper ideals, similarly, P_1, P_2 be graded S_i-r-ideals of R_i respectively.

(2)$\Rightarrow$(1). Assume that P_1, P_2 be S_i-r-ideals of R_i for every $i = 1, 2$. Now, let $(a_1, a_2), (b_1, b_2) \in h(R)$ such that $(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2) \in P_1 \times P_2$ with $ann(a_1, a_2) = (0, 0)$. Then $ann(a_1) = ann(a_2) = 0$. Note that $a_i b_i \in P_i$ with $ann(a_i) = 0$. So $s_i b_i \in P_i$ for some $s_i \in S$ since P_1, P_2 be graded S_i-r-ideals of R_i for every $i = 1, 2$. Put $s = (s_1, s_2) \in S$. Then $(s_1, s_2)(b_1, b_2) = (s_1 b_1, s_2 b_2) \in P$. This implies that P is a graded S-r-ideal of R. In other cases, one can similarly prove that P is a graded S-r-ideal of R. □

Let R be a G-graded ring. Then the graded radical of R is denoted by $Grad(R)$ and equal R (see [14]).

Lemma 2.21. *([5]) Let $R = R_1 \times R_2$ where each R_i is a G-graded ring for all $i \in \{1, 2\}$. Then the following hold.*

1. *P_1 is a graded ideal of R_1, if and only if $Grad(P_1 \times R_2) = Grad(P_1) \times R_2$.*
2. *P_2 is a graded ideal of R_2, if and only if $Grad(R_1 \times P_2) = R_1 \times Grad(P_2)$.*
3. *P_1 is a graded ideal of R_1 and P_2 is a graded ideal of R_2, if and only if $Grad(P_1 \times P_2) = Grad(P_1) \times Grad(P_2)$.*

By analogy, in Theorem 2.20, it is straightforward to demonstrate the preceding theorem.

Theorem 2.22. Let $R = R_1 \times R_2$ where each R_i is a G-graded ring for all $i \in \{1, 2\}$, $P = P_1 \times P_2$ a graded ideal of R where each P_i is a graded ideal of R_i and $S = S_1 \times S_2 \subseteq h(R)$ a multiplicative subset of R such that $S_1 \cap P_1 = \emptyset$ and $S_2 \cap P_2 = \emptyset$. Then the following are equivalents.

1. $P = P_1 \times P_2$ is a graded S-pr-ideal of R.
2. $P_1 = R_1$ and P_2 is a graded S_2-pr-ideal of R_2 or $P_2 = R_2$ and P_1 is a graded S_1-pr-ideal of R_1 or either P_1 is a graded S_1-pr-ideal of R_1 and P_2 is a graded S_2-pr-ideal of R_2.

Corollary 2.23. *Let $R = R_1 \times R_2$ where each R_i is a G-graded ring for all $i \in \{1, 2\}$ and $P = P_1 \times P_2$ a graded ideal of R where each P_i is a graded ideal of R_i. Then the following are equivalents.*

1. *$P = P_1 \times P_2$ is a graded r-ideal (resp., pr-ideal) of R.*
2. *$P_1 = R_1$ and P_2 is a graded r-ideal (resp., pr-ideal) of R_2 or $P_2 = R_2$ and P_1 is a graded r-ideal (resp., pr-ideal) of R_1 or either P_1 is a graded r-ideal of R_1 and P_2 is a graded r-ideal (resp., pr-ideal) of R_2.*

In the case $S = \{1\}$, a proper graded ideal P of a graded ring R is said to be a *graded r-ideal* (resp., *pr-ideal*) of R if whenever nonunit elements $x, y \in h(R)$ such that $xy \in P$ and $ann(x) = 0$, then $y \in P$ (resp., $y \in Grad(P)$) [2]. It is recognizable that a graded r-ideal (resp., pr-ideal) of R disjoint with S is a graded S-r-ideal (resp., S-pr-ideal) of R. However, a graded S-r-ideal (resp., S-pr-ideal) of R needs not be a graded r-ideal (resp., pr-ideal) of R, see the following example.

Example 2.24. Let R_1, R_2 be two graded rings. Let P_1 be a proper non graded pr-ideal ring and P_2 be a proper graded S_2-r-ideal of R_2, where S_2 a multiplicative subset of R_2. We consider $R = R_1 \times R_2$ and a multiplicative subset $S := (S_1 \cup \{0\} \times S_2)$, where S_1 be a multiplicative subset of R_1. Then S is a multiplicative subset of R. By Corollary 2.23, $P_1 \times P_2$ is neither an graded r-ideal nor graded

pr-ideal of R. Now, we will show that $P_1 \times P_2$ is an graded S-r-ideal of R, and so graded S-pr-ideal of R. Since P_2 is a graded S_2-r-ideal, there exists an $s_2 \in S_2$ such that for any $x, y \in h(R_2)$ such that $xy \in P_2$ with $ann(x) = 0$, then $s_2 y \in P_2$. Let $(a,x),(b,y) \in h(R_1 \times R_2)$ such that $(a,x),(b,y) \in P_1 \times P_2$ with $ann((a,x)) = (0,0)$, then $(xy \in P_2$ with $ann(x) = 0$. So $s_2 y \in P_2$. Take $s := (0, s_2) \in S$. Then for any $(0,s_2)(b,y) = (0, s_2 y) \in \{0\} \times P_2 \subseteq P_1 \times P_2$. This implies that $P_1 \times P_2$ is a graded S-r-ideal.

Assume that M is an R-module. Then M is said to be G-graded if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$ where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. It is clear that M_g is an R_e-submodule of M for all $g \in G$. We assume that $h(M) \bigcup_{g \in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be graded R-submodule if $N = \bigoplus_{g \in G} (N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g \in G} x_g$ where $x_g \in N$ for all $g \in G$. It is known that an R-submodule of a graded R-module need not be graded. For more terminology (see [10] and [13]). Let G be an abelian group and M be a G-graded R-module. Then $X = R(+)M$ is G-graded by $X_g = R_g(+)M_g$ for all $g \in G$. Note that, X_g is an additive subgroup of X for all $g \in G$. Also, for $g, h \in G$, $X_g X_h = (R_g(+)M_g)(R_h(+)M_h) = (R_g R_h, R_g M_h + R_h M_g) \subseteq (R_{gh}, M_{gh} + M_{hg}) \subseteq (R_{gh}, M_{gh}) = X_{gh}$ as G is abelian [18]. Moreover, if P is an ideal of R and N is an R-submodule of M such that $PM \subseteq N$, then $P(+)N$ is a graded ideal of $R(+)M$ if and only if P is a graded ideal of R and N is a graded R-submodule of M [18, Proposition 3.3]. Consequently, $h(R(+)M) = \{(a,x); a \in h(R), x \in h(M)\}$. Clearly, if S is a multiplicative subset of $h(R)$, then $S(+)\{0\}$ and $S(+)h(M)$ are multiplicative subset of $h(R(+)M)$.

Lemma 2.25. *([8], [11] and [18]) Let $R = \bigoplus_{g \in G} R_g$ be G-graded ring and $M = \bigoplus_{g \in G} M_g$ be a G-graded R-module. Let I be a graded ideal of R and N a graded submodule of M such that $IM \subseteq N$. Then the following hold:*

1. *$I(+)N$ is a graded ideal of $R(+)M$.*
2. *$Grad(I(+)N) = Grad(I)(+)M$.*

Theorem 2.26. Let G be an abelian group, R be a G-graded ring, M be a G-graded R-module, S be a multiplicative subset of $h(R)$ and I be a graded ideal of R disjoint with S. Then the following are hold.

1. I is a graded S-r-ideal of R, then $I(+)M$ is an $(S(+)M)$-r-ideal of $R(+)M$.
2. If $Z(M) = zd(R)$ and $I(+)M$ an $S(+)M$-r-ideal, then I is an S-r-ideal of R.

Proof. (1). First, $I(+)M \cap (S(+)M) = \emptyset$, as $I \cap S = \emptyset$. Now, let $(x,\alpha),(y,\beta) \in h(R(+)M)$ such that $(x,\alpha)(y,\beta) = (xy, x\beta + y\alpha) \in I(+)M$ with $ann(x,\alpha) = (0,0)$. Then $xy \in I$ with $ann(x) = 0$; so $s\alpha \in I$ for some $s \in S$ since I is a graded S-r-ideal of R. This implies that $(s,0)(y,\beta) = (sy, s\alpha) \in I(+)M =$. Hence $I(+)M$ is a graded $(S(+)\{0\})$-r-ideal of $R(+)M$. As $(S(+)\{0\}) \subseteq S(+)h(M)$, then by Proposition 2.7, $I(+)M$ is a graded $(S(+)M)$-r-ideal of $R(+)M$.

(2). Let $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$. Then $(x,0),(y,0) \in h(R(+)M)$ with $(x,0)(y,0) = (xy,0) \in I(+)M$. As $Z(M) = zd(R)$, $ann(x,0) = (0,0)$. This implies that $(s,m)(y,0) = (sy, ym) \in I(+)M$ for some $(s,m) \in S(+)h(M)$. Thus $sy \in I$; so I is a graded S-r-ideal of R.

$\square$

Example 2.27. Consider the $G = \mathbb{Z}_2$, $R = \mathbb{Z}[i]$ a G-graded ring and $M = R$ a G-graded R-module, by $R_0 = M_0 = \mathbb{Z}$ and $R_1 = M_1 = i\mathbb{Z}$. Let $S = \{1, -1\}$ a multiplicative subset of R, $I = \{0\}$ a graded ideal of R and $N = 6R$ a graded submodule of M. Note that I is a graded S-r-ideal of R. But $I(+)N$ note have a graded $(S(+)M)$-r-ideal of $R(+)M$. Indeed, we have $(1,2)(0,3) \in h(R(+)M)$, $(1,2)(0,3) = (0,6) \in I(+)N$ and $ann(1,2) = (0,0)$ but $s(0,3) \notin I(+)N$ for all $s \in S(+)M$. This shows that $I(+)N$ does not have a graded $(S(+)M)$-r-ideal of R.

Theorem 2.28. Let G be an abelian group, R be a G-graded ring, M be a G-graded R-module, S be a multiplicative subset of $h(R)$, I be a graded ideal of R disjoint with S and N be a graded submodule of M with $IM \subseteq N$. Then the following are hold.

1. I is a graded S-pr-ideal of R, then $I(+)N$ is an $(S(+)M)$-pr-ideal of $R(+)M$.
2. If $Z(M) = zd(R)$ and $I(+)N$ an $S(+)M$-pr-ideal, then I is an S-pr-ideal of R.

Proof. (1). First, $I(+)M \cap (S(+)M) = \emptyset$, as $I \cap S = \emptyset$. Now, let $(x,\alpha),(y,\beta) \in h(R(+)M)$ such that $(x,\alpha)(y,\beta) = (xy, x\beta + y\alpha) \in I(+)N$ with $ann(x,\alpha) = (0,0)$. Then $xy \in I$ with $ann(x) = 0$; so $s\alpha \in Grad(I)$ for some $s \in S$ since I is a graded S-pr-ideal of R. This implies that $(s,0)(y,\beta) = (sy, s\alpha) \in Grad(I)(+)M = Grad(I(+)N)$. Hence $I(+)N$ is a graded $(S(+)\{0\})$-r-ideal of $R(+)M$. As $(S(+)\{0\}) \subseteq S(+)h(M)$, then by Proposition 2.7, $I(+)M$ is a graded $(S(+)M)$-pr-ideal of $R(+)M$.

(2). Let $x, y \in h(R)$ such that $xy \in I$ with $ann(x) = 0$. Then $(x,0),(y,0) \in h(R(+)M)$ with $(x,0)(y,0) = (xy,0) \in I(+)N$. As $Z(M) = zd(R)$, $ann(x,0) = (0,0)$. This implies that $(s,m)(y,0) = (sy, ym) \in Grad(I(+)N) = Grad(I)(+)M$ for some $(s,m) \in S(+)h(M)$. Thus $sy \in Grad(I)$; so I is a graded S-pr-ideal of R. $\square$

Example 2.29. Consider the $G = \mathbb{Z}_2$, $R = \mathbb{Z}[i]$ a G-graded ring and $M = R$ a G-graded R-module, by $R_0 = M_0 = \mathbb{Z}$ and $R_1 = M_1 = i\mathbb{Z}$. Let $S = \{1, -1\}$ a multiplicative subset of R, $I = \{0\}$ a graded ideal of R and $N = 6R$ a graded submodule of M. Note that P is a graded S-pr-ideal of R. Then by Theorem 2.28, $I(+)N$ is a graded $(S(+)M)$-pr-ideal of $R(+)M$. But by Example 2.27, $I(+)N$ does not have a graded $(S(+)M)$-r-ideal of R.

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