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On S-(m,n)-absorbing ideals and S-(m,n)-absorbing prime ideals of commutative rings

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Abstract. Let R be a commutative ring with identity, S a multiplicative subset of R, and I a proper ideal of R disjoint from S. Let m and n be positive integers with $m > n$. In this paper, we introduce the concepts of S-(m,n)-absorbing ideals, S-(m,n)-absorbing prime ideals, and S-AB-(m,n)-absorbing ideals. These notions generalize several existing concepts, including (m,n)-absorbing ideals, (m,n)-absorbing prime ideals, AB-(m,n)-absorbing ideals, and S-n-absorbing ideals. We investigate the fundamental properties of these new classes of ideals and examine their behavior under various ring constructions, such as quotient rings, homomorphic images, and localizations. Furthermore, we explore the transfer of these properties to trivial ring extensions and amalgamated rings.

Key Words:S-(m,n)-absorbing ideal, (m,n)-absorbing ideal, S-n-absorbing ideal, S-prime ideal, trivial extension, amalgamated algebra.
2010 MSC: Primary 13A15; Secondary 13B99, 13B30.

1 Introduction

Throughout this paper, all rings considered are assumed to be commutative with nonzero identity. If R is a ring, we denote by $J(R)$ and $U(R)$ the Jacobson radical of R and the set of units of R, respectively. A nonempty subset S of R is called a *multiplicative subset* if $1 \in S$, $0 \notin S$, and for all $s_1, s_2 \in S$, we have $s_1 s_2 \in S$. For a multiplicative subset S of R, the localization of R at S is denoted by $S^{-1}R = \left\{\frac{a}{s} : a \in R, s \in S\right\}$. The *saturation* of S is defined as $S^* = \left\{r \in R : \frac{r}{1} \text{ is a unit in } S^{-1}R\right\}$; note that S^* is a multiplicative subset containing S.

In recent years, various generalizations of prime ideals have been introduced and extensively studied. Among these, Abadi and Moghimi [1] introduced the concept of (m,n)-*absorbing ideals*, where m and n are positive integers with $m > n$. A proper ideal I of R is called (m,n)-absorbing if, whenever $a_1 \cdots a_m \in I$ for some nonunit elements $a_1, \ldots, a_m \in R$, there exist n of the a_i's whose product lies in I. This line of generalization was extended by Badawi, El Khalfi, and Mahdou [8], who defined (m,n)-*absorbing prime ideals*: a proper ideal I of R is called (m,n)-absorbing prime if, whenever nonunit elements $a_1, \ldots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, then either $a_1 \cdots a_n \in I$ or $a_{n+1} \cdots a_m \in I$. They also introduced the notion of AB-(m,n)-*absorbing ideals*, where a proper ideal I of R is called AB-(m,n)-absorbing if, whenever $a_1 \cdots a_m \in I$ for some $a_1, \ldots, a_m \in R$, there exist n of the a_i's whose product belongs to I.

In 2020, Hamed and Malek [18] introduced the concept of S-*prime ideals* as a generalization of prime ideals. Given a multiplicative subset S and an ideal I of R disjoint from S, the ideal I is called S-prime if there exists an $s \in S$ such that, for all $a, b \in R$, $ab \in I$ implies $sa \in I$ or $sb \in I$.

Among the many generalizations of prime ideals, the notion of n-*absorbing ideals* introduced by Anderson and Badawi [4] plays a prominent role. For a natural number n and an ideal I of R, the ideal I is said to be n-absorbing if, whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, there exist n of the x_i's whose product belongs to I. The study of n-absorbing ideals has become an active area of research, with many results published on their properties (see, for example, the survey article [6] and its references).

More recently, Baek, Choi, and Lim [5] introduced the concept of S-n-*absorbing ideals*, which generalize both S-prime ideals and n-absorbing ideals. An ideal I disjoint from S is called S-n-absorbing if there exists $s \in S$ such that for all $r_1, \ldots, r_{n+1} \in R$, the condition $r_1 \cdots r_{n+1} \in I$ implies that

$$s \prod_{\substack{1 \leq i \leq n+1 \\ i \neq j}} r_i \in I$$

for some $1 \leq j \leq n+1$.

Considering these developments, it is natural to ask whether the (m,n)-absorbing type ideals and their S-generalizations can be studied within a unified framework. In this paper, we address this question by introducing new classes of ideals. Let R be a commutative ring with identity, S a multiplicative subset of R, and I a proper ideal of R disjoint from S. Let m and n be positive integers with $m > n$. We say that I is an S-(m,n)-absorbing ideal of R if there exists $s \in S$ such that, whenever nonunit elements $a_1, \ldots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, there exist n distinct indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$.

Similarly, I is an S-(m,n)-absorbing prime ideal if there exists $s \in S$ such that, whenever $a_1, \ldots, a_m \in R$ and $a_1 \cdots a_m \in I$, then there exist n distinct indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$. An ideal I is called an S-AB-(m,n)-absorbing ideal if, for any $a_1, \ldots, a_m \in R$ with $a_1 \cdots a_m \in I$, there exist $s \in S$ and n distinct indices $\{i_1, \ldots, i_n\} \subseteq \{1, \ldots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$. In each of these cases, we say that I is associated to s.

It is easy to see that any (m,n)-absorbing prime ideal (resp., (m,n)-absorbing ideal, AB-(m,n)-absorbing ideal) disjoint from S, as well as any S-prime ideal, is an S-(m,n)-absorbing prime ideal (resp., S-(m,n)-absorbing ideal, S-AB-(m,n)-absorbing ideal) for any positive integers m, n. Furthermore, an S-n-absorbing ideal as defined in [5] is precisely an S-AB-$(n+1,n)$-absorbing ideal. We demonstrate that many interesting properties of (m,n)-absorbing type ideals and S-n-absorbing ideals continue to hold for these new classes. However, we also show that certain properties valid for the former do not extend seamlessly to the latter.

This paper is organized into three sections, including the introduction. In Section 2, we investigate the fundamental properties of S-(m,n)-absorbing prime ideals, S-(m,n)-absorbing ideals, and S-AB-(m,n)-absorbing ideals. In addition, we examine their behavior under various ring-theoretic constructions, such as homomorphic images and localizations (see Theorems 2.28 and 2.30). Moreover, Proposition 2.33 characterizes S-Noetherian rings in terms of the finiteness properties of S-(m,n)-absorbing ideals and S-AB-(m,n)-absorbing ideals.

Finally, Section 3, the concluding part of the paper, is devoted to an in-depth study of these ideals within the framework of trivial ring extensions and amalgamated algebras along an ideal (see Theorems 2.9 and 3.3). Several illustrative examples are provided throughout to clarify and support the theoretical results established in the paper.

Any undefined notation and terminology can be found in [22].

2 Main Results

The following definitions provide the foundation upon which this section is built.

Definition 2.1. Let R be a ring, S a multiplicative subset of R, and I a proper ideal of R such that $I \cap S = \emptyset$. Let m and n be positive integers with $m > n$. Then:

1. I is called an *S-(m,n)-absorbing ideal* of R if there exists $s \in S$ such that, whenever nonunit elements $a_1, \dots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, there exist n distinct indices $i_1, \dots, i_n \in \{1, \dots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$.

2. I is said to be an *S-(m,n)-absorbing prime ideal* of R if there exists $s \in S$ such that, whenever nonunit elements $a_1, \dots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, then either $sa_1 \cdots a_n \in I$ or $sa_{n+1} \cdots a_m \in I$.

3. I is defined as an *S-AB-(m,n)-absorbing ideal* of R if there exists $s \in S$ such that, whenever elements $a_1, \dots, a_m \in R$ satisfy $a_1 \cdots a_m \in I$, there exist n distinct indices $i_1, \dots, i_n \in \{1, \dots, m\}$ such that $sa_{i_1} \cdots a_{i_n} \in I$.

In all of these cases, the ideal I is said to be *associated with s*.

Remark 2.2.

1. If S consists entirely of units of R, then the notions of (m,n)-absorbing ideals and S-(m,n)-absorbing ideals coincide. The same holds for (m,n)-absorbing prime ideals and S-(m,n)-absorbing prime ideals, as well as for AB-(m,n)-absorbing ideals and S-AB-(m,n)-absorbing ideals.

2. Every S-prime ideal is both an S-(m,n)-absorbing prime ideal and an S-(m,n)-absorbing ideal for any multiplicative subset S of R and for all positive integers m and n with $m > n$.

3. For any multiplicative subset S of R, every (m,n)-absorbing ideal is an S-(m,n)-absorbing ideal. Similarly, every (m,n)-absorbing prime ideal is an S-(m,n)-absorbing prime ideal, and every AB-(m,n)-absorbing ideal is an S-AB-(m,n)-absorbing ideal.

In general, the converses of statements (2) and (3) in the previous remark do not hold, as illustrated by the following example.

Example 2.3. Let $R = \mathbb{Z}[X]$ be the ring of polynomials with integer coefficients, and let $S = \{4^n \mid n \in \mathbb{N}\}$ be a multiplicative subset of R. Consider the ideal $I := 2X^2\mathbb{Z}[X]$. Let $f, g, h \in \mathbb{Z}[X]$ such that $fgh \in I$. Since $X^2\mathbb{Z}[X]$ is $X\mathbb{Z}[X]$-primary, it follows that X^2 divides at least one of the products fg, gh, or fh. Therefore, at least one of $4fg$, $4gh$, or $4fh$ belongs to I, showing that I is an S-$(3,2)$-absorbing ideal of R.

However, I is not an S-prime ideal of R. Indeed, we have $2X \cdot X \in I$, but for every $n \in \mathbb{N}$, neither 4^nX nor $4^n(2X)$ belongs to I. Moreover, I is not a $(3,2)$-absorbing ideal of R, since $2 \cdot X \cdot X \in I$, yet neither $2X$ nor X^2 belongs to I.

The following remark follows directly from the definition of S-(m,n)-absorbing prime ideals.

Remark 2.4. Let R be a ring with a multiplicative subset $S \subseteq R$, and let I be a proper ideal of R. Let m and n be positive integers with $m > n$. Then the following statements hold:

1. I is an S-$(2,1)$-absorbing prime ideal of R if and only if I is an S-prime ideal.

2. I is an S-$(3,2)$-absorbing prime ideal of R if and only if I is an S-1-absorbing prime ideal.

3. If I is an S-(m,n)-absorbing prime ideal, then I is also an S-$(m+1,n+1)$-absorbing prime ideal.

4. I is an S-(m,n)-absorbing prime ideal if and only if it is an S-$(m,m-n)$-absorbing prime ideal.

The following example shows that the converse of Remark 2.4(3) does not necessarily hold.

Example 2.5. Let $R = \mathbb{Z}[X]$ and let $S = \{2^n \mid n \in \mathbb{N}\}$. Then S is a multiplicative subset of R, and $P = 4X\mathbb{Z}[X]$ is an S-$(2,1)$-absorbing prime ideal of R. Indeed, let $f,g \in R$ such that $fg \in P$. Since $fg \in P \subseteq X\mathbb{Z}[X]$, it follows that X divides either f or g, so $4f \in P$ or $4g \in P$. Note that $P \cap S = \emptyset$.
However, P is not an S-$(2,1)$-absorbing prime ideal of R, because $2 \cdot 2X \in P$, yet $2 \notin P$ and $2X \notin P$.

The proof of the following proposition is straightforward and is therefore omitted.

Proposition 2.6. *Let R be a ring, S a multiplicative subset of R, I a proper ideal of R, and let m and n be positive integers with $m > n$. Then the following statements hold:*

1. *If I is an S-AB-(m,n)-absorbing ideal of R, then I is also an S-(m,n)-absorbing ideal.*
2. *If I is an S-(m,n)-absorbing prime ideal of R, and $k = \max\{n, m-n\}$, then I is an S-(m,k)-absorbing ideal of R.*
3. *Every S-n-absorbing ideal is an S-AB-(m,n)-absorbing ideal for any multiplicative subset S of R.*

The following example illustrates that the converses of statements (1), (2), and (3) in Proposition 2.6 do not hold.

Example 2.7. Let $A = \mathbb{Z}_2[[X,Y,Z,W]]$ and set $H = (X^2+Y^2, Y^2+Z^2)A$. Define $R = A/H$, and let x,y,z,w denote the images of X,Y,Z,W in R, respectively. Let $S = \{w^n \mid n \in \mathbb{N}\}$ be a multiplicative subset of R. Consider the ideal

$$I = \Big(H + (XYZ, X(Y+Z), Y(X+Z), X^2)A\Big)/H.$$

Note that $I \cap S \neq \emptyset$.

1. We show that I is an S-$(4,2)$-absorbing ideal of R, but not an S-AB-$(4,2)$-absorbing ideal. Indeed, $1 \cdot x \cdot y \cdot z \in I$, but for any $s \in S$, none of sxy, syz, or sxz belongs to I. Thus, I is not S-AB-$(4,2)$-absorbing. To show I is S-$(4,2)$-absorbing, let $T = \{x,y,z\}$, and let D be the set of nonunit elements of R. An argument similar to [8, Example 2.3(1)] shows that I is a $(4,2)$-absorbing ideal of R, and hence an S-$(4,2)$-absorbing ideal.
2. Let $a_1 = x$, $a_2 = y$, $a_3 = x$, and $a_4 = z$. Then $a_1a_2a_3a_4 \in I$, but for all $s \in S$, neither $sa_1a_2 \in I$ nor $sa_3a_4 \in I$. Hence, I is not an S-$(4,2)$-absorbing prime ideal.
3. As I is an S-$(4,2)$-absorbing ideal but not an S-AB-$(4,2)$-absorbing ideal, it follows that I is not an S-n-absorbing ideal of R.

An integral domain R is said to be a *valuation domain* if, for all $0 \neq x,y \in R$, either $x \mid y$ or $y \mid x$. Every valuation domain is a Bézout domain, i.e., a domain in which every finitely generated ideal is principal. Moreover, every local Bézout domain is a valuation domain by [10, Proposition 1.5]. The following theorem provides a necessary condition under which an S-(m,n)-absorbing ideal becomes an S-n-absorbing ideal.

Theorem 2.8. Let R be a Bézout ring, and let S be a multiplicative subset of R. Then an ideal I is an S-(m,n)-absorbing ideal if and only if I is an S-n-absorbing ideal. In particular, this holds when R is a valuation domain or a principal ideal domain (PID).

Proof. Let I be an S-(m,n)-absorbing ideal of R. We show that I is also an S-$(m-1,n)$-absorbing ideal. Suppose $a_1\cdots a_{m-1}\in I$ for some nonunit elements $a_1,\ldots,a_{m-1}\in R$. Since R is a Bézout ring, the ideal $(a_1,\ldots,a_{m-1}) = xR$ for some $x\in R$. Then for each $1\le i\le m-1$, there exists $y_i\in R$ such that $a_i = xy_i$. Hence,

$$a_1\cdots a_{m-1} = x^{m-1}y_1\cdots y_{m-1}\in I.$$

By assumption, there exists $s\in S$ such that one of the following holds:

- $sx^n\in I$, or
- $sy_{i_1}\cdots y_{i_n}\in I$ for some $\{i_1,\ldots,i_n\}\subseteq\{1,\ldots,m-1\}$, or
- $sx^t y_{i_1}\cdots y_{i_{n-t}}\in I$ for some $0<t<n$ and corresponding indices.

In each case, it follows that $sa_{i_1}\cdots a_{i_n}\in I$, and thus I is an S-$(m-1,n)$-absorbing ideal. The "in particular" part follows directly. □

Remark 2.9. Let $S_1\subseteq S_2$ be multiplicative subsets of R, and let I be an ideal of R disjoint from S_2. Clearly, if I is an S_1-(m,n)-absorbing ideal of R, then I is also an S_2-(m,n)-absorbing ideal. However, the converse does not hold in general.

To see this, consider the ideal $I=(X^2,2X)$ in the ring $R=\mathbb{Z}[X]$, and let $S_1=\{1\}$ and $S_2=\{2^n\mid n\in\mathbb{N}\}$. Note that $X^2\in I$, but $1\cdot X\notin I$, so I is not an S_1-$(2,1)$-absorbing ideal. On the other hand, $(I:2)=X\mathbb{Z}[X]$ is a prime ideal, and thus I is an S_2-prime ideal, making it an S_2-$(2,1)$-absorbing ideal.

Proposition 2.10. *Let R be a ring, and let $S_1\subseteq S_2$ be multiplicative subsets of R such that for every $s\in S_2$, there exists $t\in S_2$ with $st\in S_1$. If I is an S_2-(m,n)-absorbing ideal of R, then I is also an S_1-(m,n)-absorbing ideal of R, for some integers $1\le n<m$.*

Proof. Assume I is an S_2-(m,n)-absorbing ideal. Then there exists $s_2\in S_2$ such that, for nonunit elements $a_1,\ldots,a_m\in R$ with $a_1\cdots a_m\in I$, there are distinct indices $i_1,\ldots,i_n$ such that

$$s_2a_{i_1}\cdots a_{i_n}\in I.$$

By assumption, there exists $t\in S_2$ such that $s_1 := ts_2\in S_1$. Multiplying both sides by t, we obtain

$$s_1a_{i_1}\cdots a_{i_n}\in I,$$

showing that I is an S_1-(m,n)-absorbing ideal of R. □

Recall that if S is a multiplicative subset of a ring R with $1\in S$, then the set

$$S^* = \left\{r\in R : \frac{r}{1}\in U(S^{-1}R)\right\}$$

is called the *saturation* of S. It is easy to see that S^* is a multiplicative subset of R containing S. If $S=S^*$, then S is said to be *saturated*. Moreover, it is clear that $S^{**}=S^*$ (see [16]).

Proposition 2.11. *Let R be a ring, S a multiplicative subset of R, and I an ideal of R disjoint from S. Then I is an S-(m,n)-absorbing ideal of R if and only if I is an S^*-(m,n)-absorbing ideal of R.*

Proof. We first show that $S^*\cap I=\emptyset$. Suppose, for contradiction, that there exists $r\in S^*\cap I$. Since $\frac{r}{1}\in U(S^{-1}R)$, there exist $a\in R$ and $s\in S$ such that $\frac{r}{1}\cdot\frac{a}{s}=1$. This implies there exists $t\in S$ such that $tra=ts$, hence $tra\in I\cap S$, since $r\in I$, $a\in R$, and $t\in S$. This contradicts the assumption that $I\cap S=\emptyset$. Thus, $S^*\cap I=\emptyset$.

Now, by taking $S_1=S$ and $S_2=S^*$, the result follows immediately from Proposition 2.10. □

Theorem 2.12. Let R be a ring and S a multiplicative subset of R. Then the following assertions hold:

1. Let I be an S-(m,n)-absorbing ideal of R, and let J be an ideal of R such that $J \cap S \neq \emptyset$. Then IJ is an S-(m,n)-absorbing ideal of R.
2. Let $R \subseteq T$ be a ring extension. If J is an S-(m,n)-absorbing ideal of T, then $J \cap R$ is an S-(m,n)-absorbing ideal of R.

Proof. (1) Suppose I is an S-(m,n)-absorbing ideal of R. Let $a_1,\dots,a_m \in R$ be such that $a_1 \cdots a_m \in IJ \subseteq I$. Then, by assumption, there exists $s \in S$ and n distinct indices $i_1,\dots,i_n \in \{1,\dots,m\}$ such that $sa_{i_1}\cdots a_{i_n} \in I$. Since $J \cap S \neq \emptyset$, there exists $t \in J \cap S$, and hence $sta_{i_1}\cdots a_{i_n} \in IJ$. Therefore, IJ is an S-(m,n)-absorbing ideal of R.

(2) Suppose J is an S-(m,n)-absorbing ideal of T, and let $a_1,\dots,a_m \in R$ such that $a_1\cdots a_m \in J \cap R$. Since $a_1 \cdots a_m \in J$ and J is S-(m,n)-absorbing in T, there exists $s \in S$ and n distinct indices $i_1,\dots,i_n \in \{1,\dots,m\}$ such that $sa_{i_1}\cdots a_{i_n} \in J$. Since $a_{i_j} \in R$ for each j, it follows that $sa_{i_1}\cdots a_{i_n} \in J \cap R$. Thus, $J \cap R$ is an S-(m,n)-absorbing ideal of R. □

Theorem 2.13. Let R be a ring, S a multiplicative subset of R, and let m,n,k,ℓ be positive integers. Let I be a proper ideal of R such that $I \cap S = \emptyset$. The following statements hold:

1. I is an S-(m,n)-absorbing ideal of R if and only if there exists $s_0 \in S$ such that whenever $a_1 \cdots a_k \in I$ for nonunit elements $a_1,\dots,a_k \in R$ with $k \geq m$, there exist n distinct indices $i_1,\dots,i_n \in \{1,\dots,k\}$ such that $s_0 a_{i_1}\cdots a_{i_n} \in I$.
2. If I is an S-(m,n)-absorbing ideal of R, then I is also an S-(k,ℓ)-absorbing ideal of R for all $k \geq m$ and $\ell \geq n$. In particular, I is an S-$(m-1)$-absorbing ideal of R.
3. If each I_j is an S-(m_j,n_j)-absorbing ideal of R associated with $s_j \in S$ for $j \in \{1,\dots,p\}$, and $I = \bigcap_{j=1}^{p} I_j$ is disjoint from S, then I is an S-(m,n)-absorbing ideal of R, where $n = \sum_{j=1}^{p} n_j$ and $m = \max\{m_1,\dots,m_p,n+1\}$. Moreover, I is associated with $t = \prod_{j=1}^{p} s_j \in S$.

Proof. Parts (1) and (2) are routine and thus omitted.

(3) Assume that each I_j is an S-(m_j,n_j)-absorbing ideal associated with $s_j \in S$, for $j \in \{1,\dots,p\}$, and that $I = \bigcap_{j=1}^{p} I_j$ is disjoint from S. Set $n = \sum_{j=1}^{p} n_j$ and $m = \max\{m_1,\dots,m_p,n+1\}$. By part (2), each I_j is also an S-(m,n_j)-absorbing ideal associated with s_j.

Let $t = \prod_{j=1}^{p} s_j \in S$, and suppose $a_1,\dots,a_m \in R$ are nonunit elements such that $a_1 \cdots a_m \in I = \bigcap_{j=1}^{p} I_j$. Then $a_1 \cdots a_m \in I_j$ for each j. Since I_j is S-(m,n_j)-absorbing, there exists a subset $\Gamma_j \subseteq \{1,\dots,m\}$ with $|\Gamma_j| = n_j$ such that

$$s_j\left(\prod_{\gamma\in\Gamma_j} a_\gamma\right) \in I_j.$$

Define $\Gamma := \bigcup_{j=1}^{p} \Gamma_j$ and $P := \prod_{\gamma\in\Gamma} a_\gamma$. Then for each j, we have

$$s_j P = s_j \cdot \left(\prod_{\gamma\in\Gamma_j} a_\gamma\right) \cdot \left(\prod_{\gamma\in\Gamma\setminus\Gamma_j} a_\gamma\right) \in I_j.$$

Multiplying both sides by $\prod_{k\neq j} s_k$, we obtain $tP \in I_j$. Since this holds for all j, it follows that

$$tP \in \bigcap_{j=1}^{p} I_j = I.$$

Thus, I is an S-(m,n)-absorbing ideal associated with $t \in S$. □

Theorem 2.14. Let R be a ring, S a multiplicative subset of R, and I a proper ideal of R. Let m and n be positive integers, and suppose there exists some $i \in I$ such that $1+i \notin U(R)$. Then I is an S-AB-(m,n)-absorbing ideal of R if and only if I is an S-(m,n)-absorbing ideal of R.

Proof. Suppose I is an S-AB-(m,n)-absorbing ideal of R. Then, by definition, I is also an S-(m,n)-absorbing ideal.

Conversely, assume I is an S-(m,n)-absorbing ideal of R. Suppose $a_1a_2\cdots a_m \in I$ for some elements $a_1,a_2,\ldots,a_m \in R$. Without loss of generality, assume that for some integer $k \geq n+2$, the elements $a_k,a_{k+1},\ldots,a_m$ are units in R, and $a_1,a_2,\ldots,a_{k-1}$ are nonunits.

Then $a_1\cdots a_{k-1} \in I$. Since $1+i \notin U(R)$ for some $i \in I$, set $b_k = b_{k+1} = \cdots = b_m = i+1$. Then:

$$a_1\cdots a_{k-1}b_k\cdots b_m = a_1\cdots a_{k-1}(i+1)^{m-k+1} \in I.$$

Since I is an S-(m,n)-absorbing ideal, there exists $s \in S$ and n elements among the a_i's $(1 \leq i \leq k-1)$ and b_j's $(k \leq j \leq m)$ whose product lies in I.

Note that $i \in I$, $1 \notin I$, and $(i+1)^e \notin I$ for every integer $e \geq 1$. Thus, among the n elements, at least one must come from the a_i's. Suppose there exist h such elements $a_1,\ldots,a_h$ with $1 \leq h \leq n$ such that

$$sa_1\cdots a_h(i+1)^{n-h} \in I.$$

Note that $(i+1)^{n-h} = fi+1$ for some $f \in R$ (if $h=n$, take $f=0$). Then

$$sa_1\cdots a_h(fi+1) \in I,$$

which implies $sa_1\cdots a_h \in I$ (since $fi \in I$ and I is an ideal). Hence, I is an S-AB-(m,n)-absorbing ideal. □

Corollary 2.15. *Let $m,n \geq 1$ be positive integers with $m > n$, let R be a commutative ring with a multiplicative subset S, and let I be a proper ideal of R such that $I \not\subseteq \mathrm{Jac}(R)$. Then I is an S-AB-(m,n)-absorbing ideal of R if and only if I is an S-(m,n)-absorbing ideal of R.*

Proof. Since $I \not\subseteq \mathrm{Jac}(R)$, there exists $i \in I$ such that $1+i \notin U(R)$. The result follows directly from Theorem 2.14. □

Example 2.16. Let R be a semiprimitive ring (for example, $R = \mathbb{Z}$), and let $m,n \geq 1$ be positive integers with $m > n$. Let S be a multiplicative subset of R, and let I be a proper ideal of R. Then I is an S-AB-(m,n)-absorbing ideal of R if and only if I is an S-(m,n)-absorbing ideal of R.

Proposition 2.17. *Let R_1 and R_2 be commutative rings, and let $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$ be multiplicative subsets. Let I_1 and I_2 be ideals of R_1 and R_2, respectively. If $I_1 \times I_2$ is an $(S_1\times S_2)$-(m,n)-absorbing ideal of $R_1\times R_2$, then I_1 is an S_1-(m,n)-absorbing ideal of R_1, and I_2 is an S_2-(m,n)-absorbing ideal of R_2.*

Proof. Assume that $I_1 \times I_2$ is an $(S_1\times S_2)$-(m,n)-absorbing ideal of $R_1\times R_2$. Let $a_1,\ldots,a_m \in R_1$ be such that $a_1\cdots a_m \in I_1$. Then $(a_1\cdots a_m, 0) \in I_1\times I_2$. By assumption, there exists $(s,t) \in S_1\times S_2$ such that

$$(s,t)(a_{i_1}\cdots a_{i_n}, 0) = (sa_{i_1}\cdots a_{i_n}, 0) \in I_1\times I_2,$$

for some n-tuple of indices. This implies $sa_{i_1}\cdots a_{i_n} \in I_1$. Hence, I_1 is an S_1-(m,n)-absorbing ideal of R_1. A similar argument applies to I_2. □

The converse of Proposition 2.17 does not hold in general, as shown by the following example.

Example 2.18. Let $R = \mathbb{Z} \times \mathbb{Z}$, with multiplicative subsets $S_1 = \{7^n \mid n \in \mathbb{N}\}$ and $S_2 = \{1\}$. Let $I_1 = 4\mathbb{Z}$ and $I_2 = 9\mathbb{Z}$. Then I_1 and I_2 are $(4,2)$-absorbing ideals of $\mathbb{Z}$. Hence, I_1 is S_1-$(4,2)$-absorbing and I_2 is S_2-$(4,2)$-absorbing.

Now consider $I = I_1 \times I_2 \subset R$. We claim that I is not an $(S_1 \times S_2)$-$(4,2)$-absorbing ideal of R. Let

$$x_1 = (2,1), \quad x_2 = (2,1), \quad x_3 = (1,3), \quad x_4 = (1,3).$$

Then $x_1,\dots,x_4$ are nonunit elements of R and $x_1x_2x_3x_4 \in I$. However, for every $(s,t) \in S_1 \times S_2$, the product of (s,t) and any pairwise product $x_{i_1}x_{i_2}$ is not in I. Hence, I is not an $(S_1 \times S_2)$-$(4,2)$-absorbing ideal of R.

In view of Example 2.18, we now present the following result:

Theorem 2.19. *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings, and let $S = S_1 \times S_2$ be a multiplicative subset of R, where $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$ are multiplicative subsets. Let I_1 be an ideal of R_1 and I_2 an ideal of R_2, and set $I = I_1 \times I_2$. Then the following statements are equivalent:*

1. *I is an S-(m,n)-absorbing ideal of R;*
2. *I_1 is an S_1-AB-(m,n)-absorbing ideal of R_1;*
3. *I_2 is an S_2-AB-(m,n)-absorbing ideal of R_2.*

Proof. (1) $\Rightarrow$ (2): Suppose I_1 is not an S_1-AB-(m,n)-absorbing ideal of R_1. Then there exist elements $x_1,\dots,x_m \in R_1$ such that $x_1\cdots x_m \in I_1$, but for all $s \in S_1$, no product of n of the x_i's lies in I_1 when multiplied by s.

Define $d_i = (x_i, 0) \in R$ for each $i = 1,\dots,m$. Then $d_1\cdots d_m = (x_1\cdots x_m, 0) \in I$. Since I is assumed to be S-(m,n)-absorbing, there exists $(s_1,s_2) \in S = S_1 \times S_2$ such that the product of n of the d_i's multiplied by (s_1,s_2) lies in I. This implies that some product $s_1x_{i_1}\cdots x_{i_n} \in I_1$, a contradiction. Hence, I_1 must be S_1-AB-(m,n)-absorbing.

(2) $\Rightarrow$ (3): By symmetry, the argument applies to I_2.

(3) $\Rightarrow$ (1): Since every S-AB-(m,n)-absorbing ideal is, in particular, an S-(m,n)-absorbing ideal, the conclusion follows. □

Let R be a ring with a multiplicative subset $S \subseteq R$, and let m,n be positive integers with $m > n$. In the following result, we show that if R admits an S-(m,n)-absorbing prime ideal that is not an S-$(m-1,n-1)$-absorbing prime ideal, then R must be a quasi-local ring. Moreover, if R is not quasi-local, then a proper ideal I of R is an S-(m,n)-absorbing prime ideal if and only if it is an S-prime ideal.

Theorem 2.20. *Let S be a multiplicative subset of R, I a proper ideal of R, and let m,n be positive integers with $m > n$. Then the following statements hold:*

1. *Assume that R is not a quasi-local ring. Then the following conditions are equivalent:*
   (a) *I is an S-(m,n)-absorbing prime ideal of R;*
   (b) *I is an S-$(m-1,n-1)$-absorbing prime ideal of R;*
   (c) *I is an S-$(m-n+1,1)$-absorbing prime ideal of R;*
   (d) *I is an S-prime ideal of R.*
2. *I is an S-$(n+1,n)$-absorbing prime ideal if and only if either I is an S-prime ideal, or R is quasi-local with maximal ideal M such that $SM^n \subseteq I$.*

Proof. (1) $(a) \Leftrightarrow (b)$: By Remark 2.4(3), it suffices to prove $(a) \Rightarrow (b)$. Assume I is an S-(m,n)-absorbing prime ideal but not an S-$(m-1,n-1)$-absorbing prime ideal. Then there exist nonunit elements $a_1,\dots,a_{m-1} \in R$ and $s \in S$ such that:

$$a_1 \cdots a_{m-1} \in I, \quad sa_1 \cdots a_{n-1} \notin I, \quad \text{and} \quad sa_n \cdots a_{m-1} \notin I.$$

Let d be a nonunit of R. Then $da_1 \cdots a_{m-1} \in I$. Since I is S-(m,n)-absorbing prime and $sa_n \cdots a_{m-1} \notin I$, we must have $sda_1 \cdots a_{n-1} \in I$.

Now let c be a unit of R, and suppose $d+c$ is a nonunit. Since $(d+c)a_1 \cdots a_{m-1} \in I$ and $sa_n \cdots a_{m-1} \notin I$, we get:

$$s(d+c)a_1 \cdots a_{n-1} = sda_1 \cdots a_{n-1} + sca_1 \cdots a_{n-1} \in I.$$

Because $sda_1 \cdots a_{n-1} \in I$, it follows that $sa_1 \cdots a_{n-1} \in I$, contradicting the assumption. Hence, $d+c$ must be a unit. The conclusion follows from [7, Lemma 1].

$(b) \Leftrightarrow (c)$: This follows by induction on n.

$(c) \Leftrightarrow (d)$: Assume I is S-$(m-n+1,1)$-absorbing prime. By Remark 2.4(4), this implies I is also S-$(m-n+1,m-n)$-absorbing prime. Then the implication $(a) \Rightarrow (c)$ implies that I is S-prime. The converse is obvious since every S-prime ideal is S-(m,n)-absorbing prime for any $m > n$.

(2) We prove the “only if" direction. Suppose I is an S-$(n+1,n)$-absorbing prime ideal. If R is not quasi-local, then by (1) I is an S-prime ideal. Now assume R is quasi-local with maximal ideal M and $SM^n \nsubseteq I$. We show that I is also S-$(n,n-1)$-absorbing prime.

Assume the contrary: that I is not S-$(n,n-1)$-absorbing prime. Then there exist nonunit elements $a_1,\dots,a_n \in R$ such that:

$$a_1 \cdots a_n \in I, \quad sa_1 \cdots a_{n-1} \notin I, \quad sa_n \notin I$$

for some $s \in S$. Let $x_1,\dots,x_n \in SM$. Then

$$x_1 \cdots x_n a_1 \cdots a_n \in I.$$

Since I is S-$(n+1,n)$-absorbing prime and $sa_n \notin I$, we must have

$$sx_1 \cdots x_n a_1 \cdots a_{n-1} \in I.$$

Similarly, since $sa_1 \cdots a_{n-1} \notin I$, it must be that $x_1 \cdots x_n \in I$, implying $SM^n \subseteq I$, a contradiction. Thus I is S-$(n,n-1)$-absorbing prime. Repeating this process inductively, we conclude that I is S-prime. □

As a direct consequence of Theorem 2.20, we obtain the following result.

Corollary 2.21. *Let R_1 and R_2 be commutative rings, and let $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$ be multiplicative subsets. Let m and n be positive integers with $m > n$. Suppose that $I_1 \subseteq R_1$ and $I_2 \subseteq R_2$ are ideals. Then the following statements are equivalent:*

1. *$I_1 \times I_2$ is an $S_1 \times S_2$-(m,n)-absorbing prime ideal of $R_1 \times R_2$;*
2. *$I_1 \times I_2$ is an $S_1 \times S_2$-prime ideal of $R_1 \times R_2$;*
3. *Either I_1 is an S_1-prime ideal of R_1 and $I_2 = R_2$, or $I_1 = R_1$ and I_2 is an S_2-prime ideal of R_2.*

Proof. Since $R_1 \times R_2$ is not quasi-local, the conclusion follows directly from Theorem 2.20(1). □

Corollary 2.22. *Let R be a ring with a multiplicative subset S, and let m,n be positive integers such that $m > n$. Assume that I is an S-$(n+1,n)$-absorbing prime ideal of R that is not S-prime, and that $I \subseteq J$ for some proper ideal J of R. Then J is an S-(m,n)-absorbing prime ideal of R.*

Proof. Since I is an S-$(n+1,n)$-absorbing prime ideal that is not S-prime, it follows from Theorem 2.20(2) that R is a quasi-local ring with maximal ideal M such that $SM^n \subseteq I$. Since $SM^n \subseteq I \subseteq J$, the conclusion is immediate. □

Theorem 2.23. Let R be a ring, S a multiplicative subset of R, I a proper ideal of R, and let m,n be positive integers with $m > n$. If I is an S-(m,n)-absorbing prime ideal of R associated with some $s \in S$, then $\sqrt{I}$ is an S-prime ideal of R. In particular, if I is not an S-prime ideal, then R is quasi-local with maximal ideal M, and if $m = n+1$, then $\sqrt{I} = sM$.

Proof. Assume I is an S-(m,n)-absorbing prime ideal of R. Let $x,y \in R$ such that $xy \in \sqrt{I}$. Without loss of generality, assume x and y are nonunits. Then there exists a positive integer p such that $x^p y^p \in I$. Consider the product:

$$x^{n-1} x^p y^{m-n-1} y^p = a_1 \cdots a_m \in I,$$

where we define $a_i = x$ for $1 \le i \le n-1$, $a_n = x^p$, and $a_i = y$ for $n+1 \le i \le m-1$, with $a_m = y^p$. Since I is S-(m,n)-absorbing prime, we conclude that

$$sa_1 \cdots a_n = sx^{n+p-1} \in I \quad \text{or} \quad sa_{n+1} \cdots a_m = sy^{m-n+p-1} \in I.$$

Hence, $sx \in \sqrt{I}$ or $sy \in \sqrt{I}$, which shows that $\sqrt{I}$ is an S-prime ideal.

Now assume I is not S-prime. Then by Theorem 2.20(1), R is quasi-local with maximal ideal M. If $m = n+1$, then Theorem 2.20(2) implies $sM^n \subseteq I$, hence $\sqrt{I} = sM$. □

Theorem 2.24. Let S be a multiplicative set of a ring R, and let I be an ideal of R disjoint from S. Let m,n be positive integers such that $m > n \ge 2$, and suppose that I is an S-(m,n)-absorbing prime ideal of R. Let $d \in R \setminus I$ be a nonunit element of R. Then $(I:d) = \{x \in R \mid dx \in I\}$ is an S-$(m-1,n-1)$-absorbing prime ideal of R. In particular, for every proper ideal I of R with $J \nsubseteq I$, the colon ideal $(I:J)$ is also an S-$(m-1,n-1)$-absorbing prime ideal of R.

Proof. Suppose I is an S-(m,n)-absorbing prime ideal of R. Then there exists $s \in S$ such that for any nonunit elements $a_1,\ldots,a_m \in R$ with $a_1 \cdots a_m \in I$, we have either $sa_1 \cdots a_n \in I$ or $sa_{n+1} \cdots a_m \in I$.

Now assume $a_1 \cdots a_{m-1} \in (I:d)$ for some nonunit elements $a_1,\ldots,a_{m-1} \in R$. Then $da_1 \cdots a_{m-1} \in I$. Suppose $sa_1 \cdots a_{n-1} \notin (I:d)$, i.e., $dsa_1 \cdots a_{n-1} \notin I$. Since I is S-(m,n)-absorbing prime, we must have $sa_n \cdots a_{m-1} \in (I:d)$. Hence, $(I:d)$ is an S-$(m-1,n-1)$-absorbing prime ideal of R. □

Recall that a ring R is called a *chained ring* if the set of all ideals of R is linearly ordered by inclusion. Moreover, R is called an *arithmetical ring* if R_M is a chained ring for every maximal ideal M of R. We next determine the S-$(n+1,n)$-absorbing prime ideals in a chained ring.

Theorem 2.25. Let R be a chained ring with maximal ideal M, S a multiplicative set of R, and I an S-$(n+1,n)$-absorbing prime ideal of R disjoint from S, for some integer $n \ge 2$. If I is not an S-prime ideal of R, then $I = SM^k$ for some integer k with $2 \le k \le n$. Furthermore, if $k < n$, then I is also an S-$(k+1,k)$-absorbing prime ideal of R.

Proof. Assume I is an S-$(n+1,n)$-absorbing prime ideal of R that is not S-prime. Since R is a quasi-local ring with maximal ideal M, Theorem 2.20(2) implies that $SM^n \subseteq I$.

Let $k = \min\{i \mid M^i \subseteq I\}$. We claim that $I = SM^k$. Suppose for contradiction that $SM^k \subsetneq I$. Then there exist elements $a \in SM^{k-1} \setminus I$ and $b \in I \setminus SM^k$. Since R is chained, $b \in aR$, so $b = ar$ for some $r \in R$, and hence $r \in M$. Then $b = ar \in SM^k$, contradicting the choice of b. Thus, $I = SM^k$.

It is evident that SM^k is an S-$(k+1,k)$-absorbing prime ideal. □

Corollary 2.26. *Let R be an arithmetical ring with Jacobson radical M, and let I be an S-$(n+1,n)$-absorbing prime ideal of R for some integer $n \ge 2$. If I is not an S-prime ideal, then $I = SM^k$ for some k with $2 \le k \le n$. Furthermore, if $k < n$, then I is also an S-$(k+1,k)$-absorbing prime ideal of R.*

Proof. If R is not quasi-local, then I is S-prime by Theorem 2.20(1). Assume instead that R is quasi-local with maximal ideal M. Since R is an arithmetical quasi-local ring, it follows that R is a chained ring. Thus, by Theorem 2.25, $I = SM^k$ for some k with $2 \leq k \leq n$. □

Theorem 2.27. Let S be a multiplicative subset of R, and let I be a proper ideal of R. Then I is an S-(m,n)-absorbing prime ideal of R associated to some $s \in S$ if and only if, for any proper ideals $I_1, \dots, I_m$ of R with $I_1 \cdots I_m \subseteq I$, we have $sI_1 \cdots I_n \subseteq I$ or $sI_{n+1} \cdots I_m \subseteq I$.

Proof. It suffices to prove the “if" direction. Suppose that I is an S-(m,n)-absorbing prime ideal associated to some $s \in S$, and let $I_1, \dots, I_m$ be proper ideals of R such that $I_1 \cdots I_m \subseteq I$ and $sI_{n+1} \cdots I_m \nsubseteq I$. Then there exist elements $a_{n+1} \in I_{n+1}, \dots, a_m \in I_m$ such that $sa_{n+1} \cdots a_m \notin I$. For arbitrary $b_1 \in I_1, \dots, b_n \in I_n$, we have $b_1 \cdots b_n a_{n+1} \cdots a_m \in I$. Since I is S-(m,n)-absorbing and $sa_{n+1} \cdots a_m \notin I$, it follows that $sb_1 \cdots b_n \in I$. Thus, $sI_1 \cdots I_n \subseteq I$. □

Let R and T be commutative rings with identity, and let S be a multiplicative subset of R. Let $f : R \to T$ be a ring homomorphism such that $f(S)$ does not contain zero. Then $f(S)$ is a multiplicative subset of $f(R)$.

Theorem 2.28. Let R and T be commutative rings with identity, and let S be a multiplicative subset of R. Suppose $f : R \to T$ is a ring homomorphism such that $f(S)$ contains no zero divisors. Then the following statements hold:

1. If J is an $f(S)$-(m,n)-absorbing ideal of T, then $f^{-1}(J)$ is an S-(m,n)-absorbing ideal of R.
2. If f is surjective and I is a proper ideal of R containing $\mathrm{Ker}(f)$, then I is an S-(m,n)-absorbing ideal of R if and only if $f(I)$ is an $f(S)$-(m,n)-absorbing ideal of T.

Proof. (1) Assume that $a_1 \cdots a_m \in f^{-1}(J)$ for some nonunit elements $a_1, \dots, a_m \in R$. Then $f(a_1) \cdots f(a_m) \in J$. Since J is an $f(S)$-(m,n)-absorbing ideal, there exists $s \in S$ and n distinct indices $i_1, \dots, i_n \in \{1, \dots, m\}$ such that

$$f(s) f(a_{i_1}) \cdots f(a_{i_n}) \in J.$$

Thus, $sa_{i_1} \cdots a_{i_n} \in f^{-1}(J)$, and $f^{-1}(J)$ is an S-(m,n)-absorbing ideal of R.

(2) Suppose $f(I)$ is an $f(S)$-(m,n)-absorbing ideal of T. Since $I = f^{-1}(f(I))$ and $\mathrm{Ker}(f) \subseteq I$, it follows from part (1) that I is an S-(m,n)-absorbing ideal of R.

Conversely, assume that I is an S-(m,n)-absorbing ideal of R. Let $x_1 \cdots x_m \in f(I)$ for nonunit elements $x_1, \dots, x_m \in T$. Then there exist $a_1, \dots, a_m \in R$ such that $f(a_i) = x_i$ for $i = 1, \dots, m$, and $a_1 \cdots a_m \in I$ since $\mathrm{Ker}(f) \subseteq I$. Since I is S-(m,n)-absorbing, there exists $s \in S$ and n distinct indices $i_1, \dots, i_n$ such that $sa_{i_1} \cdots a_{i_n} \in I$, hence

$$f(s) x_{i_1} \cdots x_{i_n} = f(sa_{i_1} \cdots a_{i_n}) \in f(I).$$

Therefore, $f(I)$ is an $f(S)$-(m,n)-absorbing ideal of T. □

Corollary 2.29. *Let R be a ring, S a multiplicative subset of R, and $I \subseteq J$ proper ideals of R. Let $f : R \to R/I$ be the canonical surjection. Assume that $a + I$ is a nonunit in R/I for every nonunit element $a \in R$. Then J is an S-(m,n)-absorbing ideal of R if and only if J/I is an $f(S)$-(m,n)-absorbing ideal of R/I.*

Let R be a ring and I an ideal of R. The set

$$Z_I(R) := \{x \in R : xy \in I \text{ for some } y \notin I\}$$

is known as the set of zero-divisors modulo I (see [21]).

Theorem 2.30. Let R be a ring, and let $S, S' \subseteq R$ be multiplicative subsets. Let I be an ideal of R disjoint from S. If I is an S'-(m,n)-absorbing ideal of R with $I \cap S = \emptyset$, then $S^{-1}I$ is an $S^{-1}S'$-(m,n)-absorbing ideal of $S^{-1}R$. The converse holds if $Z_I(R) \cap S = \emptyset$.

Proof. If S' is a multiplicative subset of R, then $S^{-1}S'$ is a multiplicative subset of $S^{-1}R$. Since $I \cap S = \emptyset$, $S^{-1}I$ is a proper ideal of $S^{-1}R$ and $S^{-1}I \cap S^{-1}S' = \emptyset$.

Assume $\frac{a_1}{s_1} \cdots \frac{a_m}{s_m} \in S^{-1}I$ for nonunit elements $a_1, \dots, a_m \in R$ and $s_1, \dots, s_m \in S$. Then $ta_1 \cdots a_m \in I$ for some $t \in S$. Since I is S'-(m,n)-absorbing, there exists $s' \in S'$ and distinct indices $i_1, \dots, i_n \in \{1, \dots, m\}$ such that $s'ta_{i_1} \cdots a_{i_n} \in I$. Hence,

$$\frac{s'}{1} \cdot \frac{a_{i_1}}{s_{i_1}} \cdots \frac{a_{i_n}}{s_{i_n}} \in S^{-1}I,$$

proving $S^{-1}I$ is $S^{-1}S'$-(m,n)-absorbing.

Conversely, suppose $S^{-1}I$ is $S^{-1}S'$-(m,n)-absorbing and $Z_I(R) \cap S = \emptyset$. Let $a_1 \cdots a_m \in I$ for nonunit elements $a_1, \dots, a_m \in R$. Then $\frac{a_1}{1} \cdots \frac{a_m}{1} \in S^{-1}I$. By assumption, there exists $\frac{s'}{s} \in S^{-1}S'$ and indices $i_1, \dots, i_n$ such that

$$\frac{s'}{s} \cdot \frac{a_{i_1}}{1} \cdots \frac{a_{i_n}}{1} \in S^{-1}I.$$

Hence, for some $v \in S$, we get $vs'a_{i_1} \cdots a_{i_n} \in I$. Since $v \notin Z_I(R)$, it follows that $s'a_{i_1} \cdots a_{i_n} \in I$. Therefore, I is S'-(m,n)-absorbing. □

Theorem 2.31. Let S be a multiplicative subset of R, I an ideal of R, and m, n positive integers with $m > n$. Then there exists an S-(m,n)-absorbing ideal of R that is minimal among all S-(m,n)-absorbing ideals of R containing I. In particular, R has a minimal S-(m,n)-absorbing ideal.

Proof. Let V denote the set of all S-(m,n)-absorbing ideals of R that contain I. Since any maximal ideal containing I is S-(m,n)-absorbing, V is non-empty. Order V by reverse inclusion: for $I_1, I_2 \in V$, define $I_1 \leq I_2$ if and only if $I_1 \supseteq I_2$.

Let $C = \{I_\lambda\}_{\lambda \in \Lambda}$ be a non-empty chain in V and let $J = \bigcap_{\lambda \in \Lambda} I_\lambda$. Clearly $J \neq R$ and $J \in V$. Let $a_1, \dots, a_m \in R$ be nonunit elements with $a_1 \cdots a_m \in J$, and suppose that for every $s \in S$, no n-subproduct of the a_i's multiplied by s lies in J. Since C is a chain, there exists some I_λ such that this also fails in I_λ, contradicting that I_λ is S-(m,n)-absorbing. Hence J is S-(m,n)-absorbing.

By Zorn's Lemma, V has a maximal element with respect to inclusion, which is a minimal S-(m,n)-absorbing ideal of R containing I. □

Corollary 2.32. *Let S be a multiplicative subset of R, I an ideal of R disjoint from S, and n a positive integer. Then there exists a minimal S-n-absorbing ideal of R containing I.*

As a generalization of Noetherian rings, Anderson and Dumitrescu introduced the concept of S-Noetherian rings. Recall from [3] that a ring R is called an S-Noetherian ring if every ideal I of R is S-finite; that is, there exists $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$. The following result characterizes the S-Noetherian property in terms of S-AB-(m,n)-absorbing ideals and S-(m,n)-absorbing ideals.

Proposition 2.33. *Let S be a multiplicative subset of a ring R. Then the following statements are equivalent:*

1. *R is an S-Noetherian ring;*

2. *Every S-AB-(m,n)-absorbing ideal is S-finite;*

3. *Every S-(m,n)-absorbing ideal is S-finite.*

Proof. (1) $\Rightarrow$ (2): Suppose that R is S-Noetherian. Then every ideal of R is S-finite. In particular, every S-AB-(m,n)-absorbing ideal is S-finite.

(2) $\Rightarrow$ (3): Assume that every S-AB-(m,n)-absorbing ideal is S-finite. Let I be an S-(m,n)-absorbing ideal of R. Then I is also an S-AB-(m,n)-absorbing ideal, so by assumption, I is S-finite.

(3) $\Rightarrow$ (1): Assume that every S-(m,n)-absorbing ideal is S-finite. In particular, every S-prime ideal is S-finite. Thus, by [18, Theorem 3], R is an S-Noetherian ring. □

3 S-(m,n)-absorbing Ideals in Trivial Extension Rings and Amalgamated Algebras

Let E be a unitary A-module. The idealization of E in A, denoted by $A \propto E = A \oplus E$, is a commutative ring with componentwise addition and multiplication defined by

$$(a_1, e_1)(a_2, e_2) = (a_1 a_2, a_1 e_2 + a_2 e_1)$$

for all $a_1, a_2 \in A$ and $e_1, e_2 \in E$ [2]. For an ideal I of A and a submodule F of E, the set $I \propto F = I \oplus F$ is not necessarily an ideal of $A \propto E$; it is an ideal if and only if $IE \subseteq F$ [2, Theorem 3.1]. If S is a multiplicative subset of A, then the set $S \propto E = \{(s,e) \mid s \in S,\ e \in E\}$ is clearly a multiplicative subset of $A \propto E$. For background on trivial ring extensions, see [14, 20]. We now present some properties of S-(m,n)-absorbing ideals in the context of the trivial extension.

Theorem 3.1. Let A be a ring, E an A-module, S a multiplicative subset of A, I an ideal of A, and F a submodule of E such that $IE \subseteq F$. Then the following statements hold:

1. If $I \propto F$ is an $S \propto E$-(m,n)-absorbing ideal of $A \propto E$, then I is an S-(m,n)-absorbing ideal of A.

2. $I \propto E$ is an $S \propto E$-(m,n)-absorbing ideal of $A \propto E$ if and only if I is an S-(m,n)-absorbing ideal of A.

Proof. (1) Suppose $I \propto F$ is an $S \propto E$-(m,n)-absorbing ideal of $A \propto E$, and let $a_1, \ldots, a_m$ be nonunit elements of A such that $a_1 \cdots a_m \in I$. Then,

$$(a_1, 0) \cdots (a_m, 0) = (a_1 \cdots a_m, 0) \in I \propto F.$$

Hence, there exists $(s,t) \in S \propto E$ and n distinct indices $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that

$$(s,t)(a_{i_1}, 0) \cdots (a_{i_n}, 0) \in I \propto F.$$

Therefore, $sa_{i_1} \cdots a_{i_n} \in I$, proving the assertion.

(2) By part (1), it suffices to prove the "if" direction. Let $(a_1, e_1), \ldots, (a_m, e_m)$ be nonunit elements of $A \propto E$ such that

$$(a_1, e_1) \cdots (a_m, e_m) \in I \propto E.$$

Then clearly $a_1 \cdots a_m \in I$. Since I is S-(m,n)-absorbing, there exists $s \in S$ and n distinct indices $i_1, \ldots, i_n$ such that

$$sa_{i_1} \cdots a_{i_n} \in I.$$

Note that

$$(s,0)(a_{i_1}, e_{i_1}) \cdots (a_{i_n}, e_{i_n}) = (sa_{i_1} \cdots a_{i_n}, c)$$

for some $c \in E$. Hence, $(s,0)(a_{i_1}, e_{i_1}) \cdots (a_{i_n}, e_{i_n}) \in I \propto E$, proving the claim. □

The following example illustrates Theorem 3.1.

Example 3.2. Let A be a ring, E an A-module, S a multiplicative subset of A, and F a submodule of E such that $IE \subseteq F$. Let m,n be positive integers. If I is an S-prime ideal of A (e.g., $A = \mathbb{Z}[X]$, $E = F = 4X\mathbb{Z}[X]$, $I = 8X\mathbb{Z}[X]$, and $S = \{2^k \mid k \in \mathbb{Z}\}$), then by Remark 2.2, I is an S-(m,n)-absorbing ideal of A. Hence, by Theorem 3.1, $I \propto F$ is an $S \propto E$-(m,n)-absorbing ideal of $A \propto E$.

Let A and B be commutative rings, J an ideal of B, and $f : A \to B$ a ring homomorphism. Define the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A,\ j \in J\}.$$

The ring $A \bowtie^f J$ is called the *amalgamation of A with B along J with respect to f*. This construction generalizes the amalgamated duplication of a ring along an ideal (cf. [11, 13, 12, 15]).

Let I be an ideal of A, S a multiplicative subset of A, and K an ideal of $f(A)+J$. Define:

$$\begin{aligned} S' &:= \{(s, f(s)) \mid s \in S\}, \\ I \bowtie^f J &:= \{(i, f(i)+j) \mid i \in I,\ j \in J\}, \\ S^{\bowtie^f} &:= \{(s, f(s)+j) \mid s \in S,\ j \in J\}, \\ \overline{K}^f &:= \{(a, f(a)+j) \mid a \in A,\ j \in J,\ f(a)+j \in K\}. \end{aligned}$$

Clearly, S' and $S^{\bowtie^f}$ are multiplicatively closed subsets of $A \bowtie^f J$, and both $I \bowtie^f J$ and $\overline{K}^f$ are ideals of $A \bowtie^f J$. Our next result characterizes when these ideals are S-(m,n)-absorbing in $A \bowtie^f J$.

Theorem 3.3. Let A and B be commutative rings, $f : A \to B$ a ring homomorphism, J an ideal of B, I an ideal of A such that $S \cap I = \emptyset$, and K an ideal of $f(A)+J$ disjoint from $f(S)$. Let S be a multiplicative subset of A, and let m,n be positive integers such that $m > n$. Then the following statements hold:

1. I is an S-(m,n)-absorbing ideal of A if and only if $I \bowtie^f J$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$.

2. K is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$ if and only if $\overline{K}^f$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$.

Proof. (1) Define $\varphi : A \to (A \bowtie^f J)/(\{0\} \bowtie^f J)$ by $\varphi(a) = (a, f(a)) + (\{0\} \bowtie^f J)$. Then φ is a ring isomorphism and $\varphi(S) = S^{\bowtie^f}/(\{0\} \bowtie^f J)$. By Theorem 2.28(1), I is an S-(m,n)-absorbing ideal of A if and only if $\varphi(I) = (I \bowtie^f J)/(\{0\} \bowtie^f J)$ is an $\varphi(S)$-(m,n)-absorbing ideal of $(A \bowtie^f J)/(\{0\} \bowtie^f J)$. Applying Corollary 2.29, this is equivalent to $I \bowtie^f J$ being an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$.

(2) Define $\psi : A \bowtie^f J \to f(A)+J$ by $\psi(a, f(a)+j) = f(a)+j$. Then ψ is an epimorphism with $\mathrm{Ker}(\psi) = f^{-1}(J) \times \{0\}$, $\psi(\overline{K}^f) = K$, and $\psi(S^{\bowtie^f}) = f(S)$.

Suppose K is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$. Let $x_1, \dots, x_m$ be nonunit elements of $A \bowtie^f J$ such that $x_1 \cdots x_m \in \overline{K}^f$. Then $\psi(x_1 \cdots x_m) = \psi(x_1) \cdots \psi(x_m) \in K$, and each $\psi(x_i)$ is a nonunit in $f(A)+J$. By assumption, there exists $f(s) \in f(S)$ and distinct indices $i_1, \dots, i_n$ such that $f(s)\psi(x_{i_1}) \cdots \psi(x_{i_n}) \in K$. Thus,

$$\psi\Big((s, f(s)) x_{i_1} \cdots x_{i_n}\Big) \in K,$$

which implies $(s, f(s)) x_{i_1} \cdots x_{i_n} \in \overline{K}^f$. Hence, $\overline{K}^f$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal.

Conversely, assume $\overline{K}^f$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$, and let $y_1, \dots, y_m$ be nonunit elements of $f(A)+J$ such that $y_1 \cdots y_m \in K$. Since ψ is surjective, there exist $x_1, \dots, x_m \in A \bowtie^f J$ with

$\psi(x_j) = y_j$ for all j. As x_j are preimages of nonunits, they are also nonunits. Then $x_1 \cdots x_m \in \overline{K}^f$. By assumption, there exists $(s, f(s)) \in S^{\bowtie^f}$ and distinct indices $i_1, \ldots, i_n$ such that

$$(s, f(s))x_{i_1} \cdots x_{i_n} \in \overline{K}^f.$$

Applying ψ yields $f(s)y_{i_1} \cdots y_{i_n} \in K$. Hence, K is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$. □

Example 3.4. Let $A = \mathbb{Z}[X]$, $B = k[X]$ with k a field, J an ideal of B, and let $S = \{5^k : k \in \mathbb{N}\}$ be a multiplicative subset of A. Let m, n be positive integers. Consider the ideal $I = 5X\mathbb{Z}[X]$. It is easy to verify that I is an S-prime ideal of A, and hence I is an S-(m,n)-absorbing ideal of A. Therefore, by Theorem 3.3(1), $I \bowtie^f J$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$.

Let I be a proper ideal of A. The (amalgamated) duplication of A along I (see [9]) is a special case of amalgamation and is defined as

$$A \bowtie I := \{(a, a+i) \mid a \in A,\ i \in I\}.$$

The next corollary is an immediate consequence of Theorem 3.3 regarding the transfer of the S-(m,n)-absorbing ideal property to duplications.

Corollary 3.5. *Let S be a multiplicative subset of a ring A, J an ideal of A, and let m,n be positive integers with $m > n$. Define $S^{\bowtie} = \{(s,s) \mid s \in S\}$, which is a multiplicative subset of $A \bowtie J$. Then:*

1. *I is an S-(m,n)-absorbing ideal of A if and only if $I \bowtie J$ is an $S^{\bowtie}$-(m,n)-absorbing ideal of $A \bowtie J$.*
2. *K is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$ if and only if $\overline{K}$ is an $S^{\bowtie}$-(m,n)-absorbing ideal of $A \bowtie J$.*

Corollary 3.6. *Let A and B be rings, J an ideal of B, and $f : A \to B$ a ring homomorphism. Consider the amalgamation of A and B along J with respect to f. Let S be a multiplicative subset of A, and let I be an ideal of A such that $S \cap I = \emptyset$. Let K be an ideal of $f(A)+J$ disjoint from $f(S)$, and let m,n be positive integers with $m > n$. Then:*

1. *Every $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$ containing $\{0\} \times J$ is of the form $I \bowtie^f J$, where I is an S-(m,n)-absorbing ideal of A.*
2. *Every $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$ is of the form $\overline{K}$, where K is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$.*

Proof. Let L be an ideal of $A \bowtie^f J$ disjoint from $S^{\bowtie^f}$. Define two ring epimorphisms:

$$\pi_1 : A \bowtie^f J \to A, \quad \pi_1(a, f(a)+j) = a,$$

$$\pi_2 : A \bowtie^f J \to f(A)+J, \quad \pi_2(a, f(a)+j) = f(a)+j.$$

(1) By Theorem 3.3(1), $I \bowtie^f J$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal for any S-(m,n)-absorbing ideal I of A. Conversely, assume L is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$ containing $\{0\} \times J$. Then it is straightforward to verify that $L = \pi_1(L) \bowtie^f J$. Since $\mathrm{Ker}(\pi_1) = \{0\} \times J$, it follows from Theorem 2.28 that $\pi_1(L)$ is an S-(m,n)-absorbing ideal of A.
(2) Similarly, by Theorem 3.3(2), $\overline{K}$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$ for any $f(S)$-(m,n)-absorbing ideal K of $f(A)+J$. Now suppose L is an $S^{\bowtie^f}$-(m,n)-absorbing ideal containing $f^{-1}(J) \times \{0\}$. Then one can check that $L = \overline{\pi_2(L)}^f$. Since $\mathrm{Ker}(\pi_2) = f^{-1}(J) \times \{0\}$, it follows again from Theorem 2.28 that $\pi_2(L)$ is an $f(S)$-(m,n)-absorbing ideal of $f(A)+J$, as desired. □

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