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Canonical solution of universal problem for the prolongation of algebra by a local algebra

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Abstract. In this paper, we introduce the notion of prolongations of algebras by a local algebra. We propose a universal problem for the prolongations of algebras by a local algebra and we describe the existence and uniqueness of the solution of this problem. We provide several propositions which illustrate the properties of functors in the category of prolongations of algebra by a local algebra.

Key Words: Prolongation of algebra, symmetric algebra, local algebra, categories of algebras.
2010 MSC: Primary 13B02, 16S10; Secondary 16P10, 08C05.

1 Introduction

The universal property is a fundamental concept in mathematics that describes the unique and universal nature of certain mathematical constructions. Universal properties define objects uniquely up to a unique isomorphism [1]. Therefore, one strategy to prove that two objects are isomorphic is to show that they satisfy the same universal property. The universal property is a powerful tool in various areas of mathematics, such as category theory, algebra, topology, and beyond [2, 3, 6].

Let K be a commutative field. A local algebra on K is a finite-dimensional commutative algebra A with unit 1_A admitting a unique maximal ideal of codimension 1 on K (see [7, 8, 11, 12] for more details). The set $\mathbb{D} = \{a+\varepsilon b \,/\, \varepsilon^2 = 0, (a,b) \in \mathbb{R}\} = \mathbb{R} \oplus \varepsilon\mathbb{R}$ is a local algebra over $\mathbb{R}$ with maximal $\mathbf{m} = \varepsilon\mathbb{R}$. The local algebra $\mathbb{D}$ is called *algebra of dual numbers* the or *Study numbers*. It is also called *truncated polynomial algebra of degree* 1 (see [4, 7, 10]). We denote by $(1^*_{\mathbb{D}}, \varepsilon^*)$ the dual basis of the basis $(1_{\mathbb{D}}, \varepsilon)$ of $\mathbb{D}$. Let $\mathbf{m}$ be the unique maximal ideal of local algebra A. Then the set $A_0 = \{\lambda \cdot 1_A / \lambda \in K\}$ is a subalgebra of A of dimension 1. Moreover, $A_0 \cap \mathbf{m} = (0)$ et $A_0 + \mathbf{m} = A$. Thus, a local algebra A can be represented by a direct sum $A = A_0 \oplus \mathbf{m}$. The map $K \longrightarrow A_0, \lambda \longmapsto \lambda \cdot 1_A$ is an isomorphism of algebras, that is, $A_0 \simeq K$. So $A = K \cdot 1_A \oplus \mathbf{m}$. When A is a local algebra on K, then the surjection $\pi_A : A \longrightarrow A/\mathbf{m} \simeq K$ is a homomorphism of algebras called *augmentation* of A (see [4, 9, 13] for more details).

In this paper, we consider an algebra F and a local algebra A. The purpose of this paper is to find an algebra X and a homomorphism γ of algebras from F into $A \otimes X$ having the following universal property: For any algebra L and any homomorphism u from algebra F into algebra $A \otimes L$, there exists a unique homomorphism $\tilde{u}$ such that $(id_A \otimes \tilde{u}) \circ \gamma = u$. The pair (X, γ) is a *solution of this universal* *problem* if the application

$$Hom_{A\lg}(X,L) \longrightarrow Hom_{A\lg}(F, A\otimes L), \varphi \longmapsto (id_A \otimes \varphi)\circ \gamma,$$

is a bijection.

The paper is organized as follows. In Section 2, we briefly recall some universal properties in commutative algebra. In Section 3, we define the prolongation of algebra by a local algebra and we construct the canonical solution to the universal problem corresponding to the algebras F and A in cases where A is a local algebra. Finally, in Section 4, we describe the covariant functor and the contravariant functor in the categories corresponding to the prolongations of algebras by a local algebra.

2 Preliminaries

In this section, we recall some universal properties in commutative algebra which will be necessary for the subsequent sections.

Throughout this section, A denotes a commutative ring with unit.

Theorem 2.1. [2] (Universal property of quotients). Let M and P be two A-modules, N a submodule of M and $\pi : M \longrightarrow M/N$ the quotient map. For every A-module homomorphism $f : M \longrightarrow P$ with $N \subseteq \ker f$, there exists a unique A-linear map $\tilde{f} : M/N \longrightarrow P$ such that $\tilde{f}\circ\pi = f$.

Let M and N be two A-modules. We assume that M is a free A-module and let $(e_i)_{i\in I}$ be its basis, where I is a set of finite indices. If $(y_i)_{i\in I}$ is a family of elements of N, then there exists a unique linear map $f : M \longrightarrow N$ such that $f(e_i) = y_i$, for all $i \in I$ (see [2, 6] for more details).

For any set X, we denote by $A^{(X)}$ the free A-module generated by X, and $(e_x)_{x\in X}$ its basis, where the map $e_x : X \longrightarrow A$ is defined by $e_x(y) = \begin{cases} 1 \text{ if } y = x \\ 0 \text{ if } y \neq x. \end{cases}$

We denote by $i_X : X \longrightarrow A^{(X)}$ the map such that $i_X(x) = e_x$, for all $x \in X$.

Given any set X and an A-module M, we look an A-module $Y = A^{(X)}$ and a map i_X of X into $A^{(X)}$ having the following universal property

Theorem 2.2. [2] (Universal property of the pair $\left(A^{(X)}, i_X\right)$). Let X be any set and M be an A-module. If $f : X \longrightarrow M$ is a map, then there exists a unique A-linear map $\tilde{f} : A^{(X)} \longrightarrow M$ such that the following diagram

$$\begin{tikzcd} A^{(X)} \arrow[dr, "\tilde{f}"] & \\ X \arrow[u, "i_X"] \arrow[r, "f"] & M \end{tikzcd}$$

commutes, that is, $\tilde{f}\circ i_X = f$. Thus, $\tilde{f}(e_x) = f(x)$, for all $x \in X$.

Definition 2.3. The pair (Y, i_X) is called *solution to the universal problem* if the map $Hom_A(Y,M) \longrightarrow Ens(X,M), \varphi \longmapsto \varphi\circ i_X$ is a bijection.

Let M and N be two A-modules and $\mathcal{H}$ be the A-submodule of $A^{(M\times N)}$, generated by the elements of the form

$$\begin{cases} e_{(x_1+x_2,y)} - e_{(x_1,y)} - e_{(x_2,y)}, \\ e_{(x,y_1+y_2)} - e_{(x,y_1)} - e_{(x,y_2)}, \\ e_{(ax,y)} - ae_{(x,y)}, \\ e_{(x,ay)} - ae_{(x,y)}, \end{cases}$$

for all $x_1, x_2, x \in M$, $y_1, y_2, y \in N$ and $a \in A$.

A *tensor product* of M and N is an A-module quotient $M \bigotimes_A N = A^{(M\times N)}/\mathcal{H}$. The class of the element $e_{(x,y)} \in A^{(M\times N)}$ in $M \bigotimes_A N$ is denoted by $\overline{e_{(x,y)}} = x \otimes y$.

The map $i_{(M,N)} = \pi \circ i_{M\times N} : M \times N \xrightarrow{\pi} A^{(M\times N)} \xrightarrow{i_{M\times N}} A^{(M\times N)}/\mathcal{H} = M \bigotimes_A N$ defined by $i_{(M,N)}(x,y) = x \otimes y$ is A-bilinear. Every element $z \in M \bigotimes_A N$ can be written as a fini sum $z = \sum_{i\in I} x_i \otimes y_i$, with $x_i \in M$ and $y_i \in N$.

Theorem 2.4. [2] (Universal property of the pair $(M \bigotimes_A N, i_{(M,N)})$). Let M, N, P be three A-modules. For any bilinear map $f : M \times N \longrightarrow P$, there exists a unique linear map $\tilde{f} : M \bigotimes_A N \longrightarrow P$ such that $\tilde{f} \circ i_{(M,N)} = f$, that is, $\tilde{f}(x \otimes y) = f(x,y)$, for any $x \in M$ and for any $y \in N$.

In the category of modules over a commutative ring A, the universal property of a tensor product $M \bigotimes_A N$ is the possession of a bilinear mapping $M \times N \longrightarrow M \bigotimes_A N$, that is, $M \bigotimes_A N$ is a representing object for the covariant functor which sends a module P to the set of bilinear mappings $M \times N \longrightarrow P$.

An *algebra* on A or an *A-algebra* is an A-module E which is equipped with an A-bilinear map $\mu : E \times E \longrightarrow E, (x,y) \longmapsto x \cdot y$ such as:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for all $x, y, z \in E$.

Let A be a commutative ring with unit and let M be an A-module. For $n \geq 0$, we denote $T_A^0(M) = A$, $T_A^1(M) = M$ and $T_A^n(M) = M \otimes M \otimes ... \otimes M$, (n-times). A *tensor algebra* is an A-algebra $T_A(M) = \bigoplus_{n\geq 0} T_A^n(M$ which is equipped with a multiplication $\cdot$ defined by

$$(x_1 \otimes x_2 \otimes ... \otimes x_p) \cdot (x_{p+1} \otimes x_{p+2} \otimes ... \otimes x_{p+q}) = x_1 \otimes x_2 \otimes ... \otimes x_{p+q},$$

$$a \cdot (x_1 \otimes x_2 \otimes ... \otimes x_p) = a(x_1 \otimes x_2 \otimes ... \otimes x_p),$$

for all $p \geq 0, q \geq 0, a \in A$ and $x_i \in M$.

It is immediate that the multiplication $\cdot$ defined on $T_A(M)$ is associative and admits as unit element the unit element 1_A of $A = T_A^0(M)$. We denote by $i : T^1(M) = M \longrightarrow T(M)$, the canonical injection map. The tensor algebra $T_A(M)$ has the following universal property.

Theorem 2.5. [2, 6] (Universal property of tensor algebra). For any associative A-algebra E with a unital element 1_E, and any homomorphism $f : M \longrightarrow E$ of A-modules, there exists a unique homomorphism of A-algebras $\widetilde{f} : T_A(M) \longrightarrow E$, such that $\widetilde{f}(1_A) = 1_E$ and $\widetilde{f} \circ i = f$.

The *symmetric algebra* of an A-module M, denoted by $S(M)$, is the quotient algebra over A of the tensor algebra $T_A(M)$ by the two-sided ideal $\mathcal{I}$ generated by the elements $x \otimes y - y \otimes x$ of $T_A(M)$, where $x, y \in M$. The map $i : M \longrightarrow S(M)$ is called the canonical injection

Theorem 2.6. [2, 6] (Universal property of symmetric algebras). Let M be an A-module, E be an A-algebra with and $\varphi : M \longrightarrow E$ be a linear map such that $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$, for all $x, y \in M$. There exists a unique homomorphism of A-algebras $\tilde{\varphi} : S(M) \longrightarrow E$ such that $\tilde{\varphi} \circ i = \varphi$.

3 Canonical solution of universal problem for the prolongation of algebra by a local algebra

Given an algebra F and A a local algebra. Let I be a finite set, $(a_i)_{i\in I}$ a basis of the algebra A, $(a_i^*)_{i\in I}$ the dual basis of the basis $(a_i)_{i\in I}$ and $S(A^*\otimes F)$ the symmetric algebra of the vector space $A^*\otimes F$.

Definition 3.1. The *prolongation* of the algebra F by A, denoted by F^A, is the quotient algebra of $S(A^*\otimes F)$ by the ideal $\mathcal{J}$ of $S(A^*\otimes F)$ generated by the elements of the form $\varphi\otimes 1_F - \varphi(1_A)$ and $\varphi\otimes xy - \sum\limits_{j,k}\varphi(a_ja_k)(a_j^*\otimes x)(a_k^*\otimes y)$, for all $\varphi\in A^*$ and for all $x,y\in F$.

The class of the element $\varphi\otimes x\in A^*\otimes F$ in $F^A = S(A^*\otimes F)/\mathcal{J}$ is denoted by $\overline{\varphi\otimes x}$.

Proposition 3.2. *The map $\gamma_{(A,F)}: F\longrightarrow A\otimes F^A$ such that*

$$\gamma_{(A,F)}(x) = \sum_i a_i\otimes(\overline{a_i^*\otimes x}) \tag{1}$$

is a homomorphism of algebras, for all $x\in F$. Moreover, this writing does not depend on the chosen basis.

Proof. The map $\gamma_{(A,F)}$ is by construction well-defined and is linear. By definition of $\mathcal{J}$, for all $x,y\in F$ and for all $\varphi\in A^*$, we have

$$\overline{\varphi\otimes xy - \sum_{j,k}\varphi(a_ja_k)(a_j^*\otimes x)(a_k^*\otimes y)} = 0.$$

This implies that,

$$\overline{\varphi\otimes xy} = \sum_{j,k}\varphi(a_ja_k)(\overline{a_j^*\otimes x})(\overline{a_k^*\otimes y}). \tag{2}$$

Thus, by (1) and (2), for all $x,y\in F$, we have

$$\begin{aligned}
\gamma_{(A,F)}(xy) &= \sum_i a_i\otimes\left[\sum_{j,k}a_i^*(a_ja_k)(\overline{a_j^*\otimes x})(\overline{a_k^*\otimes y})\right]\\
&= \sum_{j,k}\left[\sum_i a_i^*(a_ja_k)a_i\otimes(\overline{a_j^*\otimes x})(\overline{a_k^*\otimes y})\right]\\
&= \sum_{j,k}a_ja_k\otimes(\overline{a_j^*\otimes x})(\overline{a_k^*\otimes y}).
\end{aligned}$$

It follows that

$$\begin{aligned}
\gamma_{(A,F)}(xy) &= \sum_{j,k}\left[a_j\otimes(\overline{a_j^*\otimes x})\right]\left[a_k\otimes(\overline{a_k^*\otimes y}\right]\\
&= \left[\sum_j a_j\otimes(\overline{a_j^*\otimes x})\right]\left[\sum_k a_k\otimes(\overline{a_k^*\otimes y}\right]\\
&= \gamma_{(A,F)}(x)\gamma_{(A,F)}(y).
\end{aligned}$$

Let $(b_j)_{j\in I}$ be an other basis of A. Let $(b_j^*)_{j\in I}$ be the dual basis of the basis $(b_j)_{j\in I}$. By (2), for all x in F, we have

$$\begin{aligned}
\gamma_{(A,F)}(x) &= \sum_j b_j \otimes \overline{b_j^* \otimes x} \\
&= \sum_j b_j \otimes \left[\sum_i \overline{b_j^*(a_i)\, a_i^* \otimes x} \right] \\
&= \sum_i \left[\sum_j b_j^*(a_i)\, b_j \otimes \left(\overline{a_i^* \otimes x}\right) \right] \\
&= \sum_i a_i^*(a_i) \left(\overline{a_i^* \otimes x}\right).
\end{aligned}$$

Therefore, the homomorphism $\gamma_{(A,F)}$ does not depend on the chosen basis. □

The homomorphism $\gamma_{(A,F)}$ is called the *canonical homomorphism* of the algebra F into the algebra $A \otimes F^A$.

Proposition 3.3. *Let L be an algebra and A be a finite-dimensional algebra over K. Then, $(a_i \otimes 1_L)_{i\in I}$ is a basis of the L-module $A\otimes L$ defined by the map*

$$L \times A\otimes L \longrightarrow A\otimes L, (l, \sum_i a_i \otimes l_i) \longmapsto \sum_i a_i \otimes l l_i.$$

Proof. For $x \in A\otimes L$, there exists a family $(l_i)_{i\in I} \in L$ such that
$x = \sum\limits_i a_i \otimes l_i = \sum\limits_i l_i(a_i \otimes 1_L)$. Since x is arbitrary in $A\otimes L$, then the family $(a_i \otimes 1_L)_{i\in I}$ generates the L-module $A\otimes L$.

On the other hand, let $(l_i)_{i\in I}$ be a family of elements of L such that

$$\sum_i l_i (a_i \otimes 1_L) = \sum_i a_i \otimes l_i = 0.$$

Then, for all $j \in I$, we have $(a_j^* \otimes id_L)(\sum\limits_i a_i \otimes l_i) = 0$. Thus,

$$\sum_i a_i^*(a_i) l_i = \sum_i \delta_{ij} l_i = l_j = 0.$$

Hence, the family $(a_i \otimes 1_L)_{i\in L}$ is linearly independent. Therefore, $(a_i \otimes 1_L)_{i\in I}$ is a basis of the L-module $A\otimes L$. □

Theorem 3.4 (Universal property of the pair $(A\otimes F^A, \gamma_{(A,F)})$). Let F be an algebra, A a finite-dimensional algebra over K and $\gamma_{(A,F)}$ the canonical homomorphism of F into $A\otimes F^A$. For any algebra L and any homomorphism u from algebra F into algebra $A\otimes L$, there exists a unique homomorphism $\tilde{u}$ such that the following diagram

$$\begin{tikzcd}
A\otimes F^A \arrow[rd, "id_A \otimes \tilde{u}"] & \\
F \arrow[u, "\gamma_{(A,F)}"] \arrow[r, "u"] & A\otimes L
\end{tikzcd}$$

commutes, that is,

$$(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)} = u. \tag{3}$$

Proof. The uniqueness of $\tilde{u}$: Let $v : F^A \longrightarrow L$ be an other algebra homomorphism such that $(id_A \otimes v) \circ \gamma_{(A,F)} = u$. For all $x \in F$, we have

$$\sum_i a_i \otimes v(\overline{a_i^* \otimes x}) = \sum_i a_i \otimes \tilde{u}(\overline{a_i^* \otimes x}).$$

This implies that,

$$\begin{aligned} 0 &= \sum_i a_i \otimes v(\overline{a_i^* \otimes x}) - \sum_i a_i \otimes \tilde{u}(\overline{a_i^* \otimes x}) \\ &= \sum_i \Big[v(\overline{a_i^* \otimes x}) - \tilde{u}(\overline{a_i^* \otimes x})\Big](a_i \otimes 1_L). \end{aligned}$$

Hence, by the Proposition 3.3, $v(\overline{a_i^* \otimes x}) = \tilde{u}(\overline{a_i^* \otimes x})$, for all $i \in I$. Since the elements $\overline{a_i^* \otimes x}$ generate F^A, then $v = \tilde{u}$.

For the existence of $\tilde{u}$: For all $x \in F$, we have $u(x) = \sum_i a_i \otimes (a_i^* \otimes id_L)[u(x)]$.

Consider the map $f : A^* \times F \longrightarrow L$ such that, for all $(\varphi, x) \in A^* \times F$, $f(\varphi, x) = (\varphi \otimes id_L)[u(x)]$. This map is well-defined and bilinear by construction. According to the universal property 2.4 applied to the pair $(A^* \otimes F, i_{(A^*,F)})$, where $i_{(A^*,F)}$ is the canonical map of $A^* \times F$ into $A^* \otimes F$, there exists a unique linear map $\tilde{f} : A^* \otimes F \longrightarrow L$ such that $\tilde{f} \circ i_{(A^*,F)} = f$. Let us denote by $i_{A^*\otimes F}$ the canonical injection of the module $A^* \otimes F$ in symmetric algebra $S(A^* \otimes F)$ of $A^* \otimes F$. By virtue of the universal property of symmetric algebra (Proposition 2.6) applied to the pair $(S(A^*\otimes F), i_{A^*\otimes F})$, there is a unique homomorphism $g : S(A^* \otimes F) \longrightarrow L$ such that $g \circ i_{A^*\otimes F} = \tilde{f}$, where $i_{A^*\otimes F} : A^* \otimes F \longrightarrow S(A^* \otimes F)$. Thus, for all $\varphi \in A^*$ and $x \in F$, $(g \circ i_{A^*\otimes F})(\varphi \otimes x) = \tilde{f}(\varphi \otimes x)$, that is,

$$g(\varphi \otimes x) = \tilde{f}(\varphi \otimes x) = f(\varphi, x) = (\varphi \otimes id_L)[u(x)].$$

For all $\varphi \in A$ and for all $x, y \in F$, we have

$$g\left((\varphi \otimes xy) - \sum_{jk} \varphi(a_j a_k)(a_j^* \otimes x)(a_k^* \otimes y)\right) = 0 \tag{4}$$

Indeed, for $\varphi \in A$ and for all $x, y \in F$, we have

$$\begin{aligned} & g\left[(\varphi \otimes xy) - \sum_{jk} \varphi(a_j a_k)(a_j^* \otimes x)(a_k^* \otimes y)\right] \\ =\ & g(\varphi \otimes xy) - g[\sum_{jk} \varphi(a_j a_k)(a_j^* \otimes x)(a_k^* \otimes y)] \\ =\ & \tilde{f}(\varphi \otimes xy) - \sum_{jk} \varphi(a_j a_k) g(a_j^* \otimes x) g(a_k^* \otimes y) \\ =\ & f(\varphi, xy) - \sum_{jk} \varphi(a_j a_k) \tilde{f}(a_j^* \otimes x) \tilde{f}(a_k^* \otimes y) \\ =\ & (\varphi \otimes id_L)[u(xy)] - \sum_{jk} \varphi(a_j a_k) f(a_j^*, x) f(a_k^*, y) \\ =\ & (\varphi \otimes id_L)[u(x)u(y)] \\ & - \sum_{jk} \varphi(a_j a_k)[(a_j^* \otimes id_L)[u(x)]][(a_k^* \otimes id_L)[u(y)]]. \end{aligned}$$

This implies that,

$$
\begin{aligned}
& g[(\varphi\otimes xy)-\sum_{jk}\varphi(a_ja_k)(a_j^*\otimes x)(a_k^*\otimes y)] \\
= \;& (\varphi\otimes id_L)(\sum_{jk} a_i\otimes(a_j^*\otimes id_L)[u(x)]\sum_{jk} a_l\otimes(a_l^*\otimes id_L)[u(y)]) \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& (\varphi\otimes id_L)(\sum_{jk} a_i\otimes(a_j^*\otimes id_L)[u(x)]a_l\otimes(a_l^*\otimes id_L)[u(y)]) \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& (\varphi\otimes id_L)(\sum_{jk} a_ia_l\otimes[(a_j^*\otimes id_L)[u(x)]][(a_l^*\otimes id_L)[u(y)]]) \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& (\varphi\otimes id_L)(\sum_{jk} a_ia_l\otimes[(a_j^*\otimes id_L)[u(x)]][(a_l^*\otimes id_L)[u(y)]]) \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& \sum_{il}\varphi(a_ia_l)[(a_i^*\otimes id_L)[u(x)]][(a_l^*\otimes id_L)[u(y)]] \\
& -\sum_{jk}\varphi(a_ja_k)[(a_j^*\otimes id_L)[u(x)]][(a_k^*\otimes id_L)[u(y)]] \\
= \;& 0.
\end{aligned}
$$

For all $\varphi\in A^*$, we have

$$
g(\varphi\otimes 1_F-\varphi(1_A))=0. \tag{5}
$$

Indeed, for all $\varphi\in A^*$, we have

$$
\begin{aligned}
g(\varphi\otimes 1_F-\varphi(1_A)) &= \tilde{f}(\varphi\otimes 1_F)-\varphi(1_A)g(1_K) \\
&= f(\varphi,1_F)-\varphi(1_A)\cdot 1_L \\
&= (\varphi\otimes id_L)[u(1_F)]-\varphi(1_A)\cdot 1_L \\
&= (\varphi\otimes id_L)(1_A\otimes 1_L)-\varphi(1_A)\cdot 1_L \\
&= \varphi(1_A)\cdot id_L(1_L)-\varphi(1_A)\cdot 1_L \\
&= 0.
\end{aligned}
$$

By (4) and (5), the map g vanishes on $\mathcal{J}$. Therefore, according to Theorem 2.1, there exists a unique linear map $\tilde{u}: F^A\longrightarrow L$ such that the following diagram

$$
\begin{tikzcd}
S(A^*\otimes F)/\mathcal{J} \arrow[dr, "\tilde{u}"] & \\
S(A^*\otimes F) \arrow[u, "p"] \arrow[r, "g"] & L
\end{tikzcd}
$$

commutes, that is, $\tilde{u} \circ p = g$, where $p : S(A^* \otimes F) \longrightarrow F^A = S(A^* \otimes F)/\mathcal{J}$ is the canonical surjection. Furthermore, for all $\varphi \in A^*$and $x \in F$,

$$(\tilde{u} \circ p)(\varphi \otimes x) = g(\varphi \otimes x),$$

that is, $\tilde{u}\overline{(\varphi \otimes x)} = g(\varphi \otimes x)$. Hence, for all $\varphi_1, \varphi_2 \in A^*$ and for all $x_1, x_2 \in F$,

$$\begin{aligned}
\tilde{u}\overline{[(\varphi_1 \otimes x_1)(\varphi_2 \otimes x_2)]} &= g[(\varphi_1 \otimes x_1)(\varphi_2 \otimes x_2)] \\
&= g(\varphi_1 \otimes x_1) g(\varphi_2 \otimes x_2) \\
&= \tilde{u}\overline{(\varphi_1 \otimes x_1)}\tilde{u}\overline{(\varphi_2 \otimes x_2)}.
\end{aligned}$$

Since the elements $\overline{\varphi \otimes x}$ generate F^A, then $\tilde{u}$ is a homomorphism of algebras.

On the other hand, for all $x \in F$,

$$\begin{aligned}
\Big[(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}\Big](x) &= (id_A \otimes \tilde{u})[\sum_i a_i \otimes (\overline{a_i^* \otimes x})] \\
&= \sum_i a_i \otimes \tilde{u}(\overline{a_i^* \otimes x}) \\
&= \sum_i a_i \otimes (a_i^* \otimes id_L)[u(x)] \\
&= u(x).
\end{aligned}$$

Since the element x is arbitrary in F, we conclude $(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)} = u$. □

The pair $(F^A, \gamma_{(A,F)})$ is called *solution of the universal problem* for homomorphisms of an algebra F into an algebra $A \otimes F^A$.

Remark 3.5. The existence and uniqueness of homomorphism $\tilde{u} : F^A \longrightarrow L$ such that $(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)} = u$ for every homomorphism $u : F \longrightarrow A \otimes L$ proves that the map

$$Hom(F^A, L) \longrightarrow Hom(F, A \otimes L), \tilde{u} \longmapsto (id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}$$

is a bijection.

Example 3.6. For $L = K$, there exists a canonical bijection of $Hom(F^A, K)$ into $Hom(F, A)$. In particular, if M is a paracompact smooth manifold and if A is a local algebra on $\mathbb{R}$, then the manifold $M^A = Hom_{Alg}(C^\infty(M), A)$ of infinitely near points is canonically in bijection with $Hom_{Alg}(C^\infty(M)^A, \mathbb{R})$ (see [7, 9, 13] for more details).

Proposition 3.7. *Let G be a commutative algebra with unit over K and β a homomorphism of the algebra G into the algebra $A \otimes L$. If the pair (G, β) is another solution of the same universal problem, then there exists a unique isomorphism θ of the algebra F^A into the algebra G such that the following diagram*

$$\begin{tikzcd}
A \otimes F^A \arrow[dr, "id_A \otimes \theta"] & \\
F \arrow[u, "\gamma_{(A,F)}"] \arrow[r, "\beta"'] & A \otimes G
\end{tikzcd}$$

commutes, that is,

$$(id_A \otimes \theta) \circ \gamma_{(A,F)} = \beta. \tag{6}$$

Proof. The existence and uniqueness of homomorphism θ follows from Theorem 3.4 applied to algebra G and to homomorphism β of algebra F into the algebra $A \otimes G$. Moreover, let θ' be the unique homomorphism of G into F^A such that

$$(id_A \otimes \theta') \circ \gamma_{(A,F)} = \beta. \tag{7}$$

We deduce from relations (6) and (7) that

$$[id_A \otimes (\theta' \circ \theta)] \circ \gamma_{(A,F)} = \gamma_{(A,F)} \tag{8}$$

and

$$[id_A \otimes (\theta \circ \theta')] \circ \beta = \beta. \tag{9}$$

Since the pairs $(F^A, \gamma_{(A,F)})$ and (G, β) are two solutions of the universal problem for homomorphisms of an algebra F into an algebra $A \otimes L$, then the map $\theta' \circ \theta$ is the unique endomorphism of the algebra F^A satisfying the relation (8) and the map $\theta \circ \theta'$ is the unique endomorphism of the algebra G satisfying the relation (9).

Since $id_A \otimes id_{F^A} = id_{A \otimes F^A}$ and $id_A \otimes id_G = id_{A \otimes G}$, then the endomorphisms id_{F^A} and id_G respectively verify relations (8) and (9). It follows from the uniqueness that $\theta' \circ \theta = id_{F^A}$ and $\theta \circ \theta' = id_G$. Therefore, the homomorphism θ is an isomorphism. □

4 Functorials properties

In this section, we describe the covariant functor and the contravariant functor in the categories corresponding to the prolongations of algebras by a local algebra.

Proposition 4.1. *Let F and G be two algebras over K, let A be a local algebra over K and let $\Psi : F \longrightarrow G$ be a homomorphism of algebras. Then, there exists a unique homomorphism of algebras Ψ^A from the algebra F^A into the algebra G^A such that the following diagram*

$$\begin{tikzcd}
G \arrow[r, "\gamma_{(A,G)}"] & A \otimes G^A \\
F \arrow[u, "\Psi"] \arrow[r, "\gamma_{(A,F)}"] & A \otimes F^A \arrow[u, "id_A \otimes \Psi^A"']
\end{tikzcd}$$

commutes, that is, $(id_A \otimes \Psi^A) \circ \gamma_{(A,F)} = \gamma_{(A,G)} \circ \Psi$.

Moreover, for all $\varphi \in A^*$ and $x \in F$, we have

$$\Psi^A \overline{(\varphi \otimes x)} = \overline{\varphi \otimes \Psi(x)}.$$

Proof. The existence and uniqueness of the homomorphism Ψ^A results from the Theorem 3.4, applied to the algebra G^A and to homomorphism $\gamma_{(A,G)} \circ \Psi : F \to A \otimes G^A$. For all $x \in F$, the homomorphism Ψ^A is such that

$$[(id_A \otimes \Psi^A) \circ \gamma_{(A,F)}](x) = (\gamma_{(A,F)} \circ \Psi)(x).$$

Thus, by (1), we have

$$(id_A \otimes \Psi^A)\left[\sum_i a_i \otimes (\overline{a_i^* \otimes x})\right] = \gamma_{(A,F)}[\Psi(x)].$$

This implies that,

$$\sum_i a_i \otimes \Psi^A(\overline{a_i^* \otimes x}) = \sum_i a_i \otimes (\overline{a_i^* \otimes \Psi(x)},$$

that is

$$\sum_i \Big[a_i \otimes \Psi^A(\overline{a_i^* \otimes x}) - a_i \otimes (\overline{a_i^* \otimes \Psi(x)}\Big] = 0,$$

hence

$$\sum_i a_i \otimes \Big[\Psi^A(\overline{a_i^* \otimes x}) - (\overline{a_i^* \otimes \Psi(x)}\Big] = 0.$$

According to Proposition 3.3, we conclude that $\Psi^A(\overline{a_i^* \otimes x}) = (\overline{a_i^* \otimes \Psi(x)}$, for any $i \in I$. It follows that, $\Psi^A\overline{(\varphi \otimes x)} = \overline{\varphi \otimes \Psi(x)}$, for all $\varphi \in A^*$ and $x \in F$, since the elements $\overline{a_i^* \otimes x}$ generate F^A. □

Proposition 4.2. *For any local algebra A, the map $F \longrightarrow F^A$ is a covariant functor of the category of algebras over K in itself.*

Proof. Consider the composition $\Phi \circ \Psi : F \longrightarrow G \longrightarrow H$ of two homomorphisms of algebras. According to Proposition 4.1, we have

$$\begin{aligned}
\Big[id_A \otimes (\Phi \circ \Psi)^A\Big] \circ \gamma_{(A,F)} &= \gamma_{(A,H)} \circ (\Phi \circ \Psi) \\
&= (\gamma_{(A,H)} \circ \Phi) \circ \Psi \\
&= (id_A \otimes \Phi^A) \circ \gamma_{(A,G)} \circ \Psi \\
&= (id_A \otimes \Phi^A) \circ (id_A \otimes \Psi^A) \circ \gamma_{(A,F)} \\
&= [id_A \otimes (\Phi^A \circ \Psi^A)] \circ \gamma_{(A,F)}.
\end{aligned}$$

Since $(\Phi \circ \Psi)^A$ is the unique homomorphism verifying the relation

$$\Big[id_A \otimes (\Phi \circ \Psi)^A\Big] \circ \gamma_{(A,F)} = \gamma_{(A,H)} \circ (\Phi \circ \Psi).$$

We conclude that $(\Phi \circ \Psi)^A = \Phi^A \circ \Psi^A$.

On the other hand, since the elements $\overline{\varphi \otimes x}$, where $\varphi \in A^*$ and $x \in F$, generate F^A, then $(id_F)^A(\overline{\varphi \otimes x}) = \overline{\varphi \otimes id_F(x)} = \overline{\varphi \otimes x}$, for all $\varphi \in A^*$ and $x \in F$. It follows, according to the uniqueness of the homomorphism $(id_F)^A$, we deduce that $(id_F)^A = id_{F^A}$. □

Proposition 4.3. *Let A and B be two finite-dimensional algebras and $u : A \longrightarrow B$ be a homomorphism of algebras. For any algebra F, there exists a unique homomorphism $F^u : F^B \longrightarrow F^A$ such that*

$$(id_B \otimes F^u) \circ \gamma_{(B,F)} = (u \otimes id_{F^A}) \circ \gamma_{(A,F)}.$$

Furthermore, the homomorphism F^u is such that $F^u(\overline{\varphi \otimes x}) = \overline{{}^t u(\varphi) \otimes x}$, for all $\varphi \in B^*$ and $x \in F$.

Proof. The existence and uniqueness of the homomorphism F^u result from the Theorem 3.4 applied to the algebra F^A and to the homomorphism $(u \otimes id_{F^A}) \circ \gamma_{(A,F)} : F \overset{\gamma_{(A,F)}}{\longrightarrow} A \otimes F^A \overset{u \otimes id_{F^A}}{\longrightarrow} B \otimes F^A$. Then, we have

$$(id_B \otimes F^u) \circ \gamma_{(B,F)} = (u \otimes id_{F^A}) \circ \gamma_{(A,F)},$$

that is, for all $x \in F$,

$$[(id_B \otimes F^u) \circ \gamma_{(B,F)}](x) = [(u \otimes id_{F^A}) \circ \gamma_{(A,F)}](x).$$

Hence,

$$(id_B \otimes F^u)\left[\sum_j b_j \otimes (\overline{b_j^* \otimes x})\right] = (u \otimes id_{F^A})\left[\sum_i a_i \otimes (\overline{a_i^* \otimes x})\right]$$

implies that

$$\begin{aligned}\sum_j b_j \otimes F^u[(\overline{b_j^* \otimes x})] &= \sum_i u(a_i) \otimes (\overline{a_i^* \otimes x}) \\ &= \sum_i \left(\sum_j b_j^*[u(a_i)] \right) b_j \otimes (\overline{a_i^* \otimes x}) \\ &= \sum_i \left(\sum_j [{}^t u(b_j^*)](a_i) b_j \otimes (\overline{a_i^* \otimes x}) \right),\end{aligned}$$

that is,

$$\begin{aligned}\sum_j b_j \otimes F^u[(\overline{b_j^* \otimes x})] &= \sum_i \sum_j b_{j\otimes} \left(\overline{[{}^t u(b_j^*)](a_i) a_i^* \otimes x} \right) \\ &= \sum_j b_j \otimes \left(\sum_i \overline{[{}^t u(b_j^*)](a_i) a_i^* \otimes x} \right) \\ &= \sum_j b_j \otimes \overline{{}^t u(b_j^*) \otimes x}.\end{aligned}$$

Thus,

$$\sum_j b_j \otimes F^u[(\overline{b_j^* \otimes x})] = \sum_j b_j \otimes \overline{{}^t u(b_j^*) \otimes x},$$

that is

$$\sum_j b_j \otimes [F^u(\overline{b_j^* \otimes x}) - \overline{{}^t u(b_j^*) \otimes x}] = 0.$$

It follows that,

$$\sum_j [F^u(\overline{b_j^* \otimes x}) - \overline{{}^t u(b_j^*) \otimes x}](b_j \otimes 1_F) = 0.$$

According to Proposition 3.3, we have $F^u(\overline{b_j^* \otimes x}) - \overline{{}^t u(b_j^*) \otimes x} = 0$, for all $j \in J$, that is, $F^u(\overline{b_j^* \otimes x}) = \overline{u^t(b_j^*) \otimes x}$, for all $j \in J$. By linearity of F^u and ${}^t u$, we conclude that $F^u(\overline{\varphi \otimes x}) = \overline{{}^t u(\varphi) \otimes x}$, for all $\varphi \in B^*$ and $x \in F$. □

Lemma 4.4. *For any algebra F, the map $A \longrightarrow F^A$ is a contravariant functor from the category of finite-dimensional algebras over K in the category of algebras over K.*

Proof. Let $u : A \longrightarrow B$ and $v : B \longrightarrow C$ be two algebras homomorphisms. According to Proposition 4.3, we have

$$(id_B \otimes F^u) \circ \gamma_{(B,F)} = (u \otimes id_{F^A}) \circ \gamma_{(A,F)};$$

$$(id_C \otimes F^v) \circ \gamma_{(C,F)} = (v \otimes id_{F^B}) \circ \gamma_{(B,F)}$$

and

$$(id_C \otimes F^{v\circ u}) \circ \gamma_{(C,F)} = ((v \circ u) \otimes id_{F^A}) \circ \gamma_{(A,F)}.$$

Thus,

$$\begin{aligned}(id_C\otimes(F^u\circ F^v)\circ\gamma_{(C,F)} &= (id_C\otimes F^u)\circ(id_C\otimes F^v)\circ\gamma_{(C,F)}\\ &= (id_C\otimes F^u)\circ(v\otimes id_{F^B})\circ\gamma_{(B,F)}\\ &= [(id_C\circ v)\otimes(F^u\otimes id_{F^B})\circ\gamma_{(B,F)}\\ &= (v\otimes F^u)\circ\gamma_{(B,F)}\\ &= (v\otimes id_{F^A})\circ(id_B\otimes F^u)\circ\gamma_{(B,F)}\\ &= (v\otimes id_{F^A})\circ(u\otimes id_{F^A})\circ\gamma_{(A,F)}\\ &= ((v\circ u)\otimes id_{F^A})\circ\gamma_{(A,F).}\end{aligned}$$

By uniqueness, we conclude that $F^{v\circ u}=F^u\circ F^v$. In particular, for $B=A$ and $u=id_A$, we have $(id_A\otimes F^{id_A})\circ\gamma_{(A,F)}=(id_A\otimes id_{F^A})\circ\gamma_{(A,F)}$. We conclude by uniqueness that $F^{id_A}=id_{F^A}$. □

Given an algebra L and a homomorphism $\tilde{u}$ from F^A into L, we know how to determine the homomorphism from F into $A\otimes F$ corresponding to $\tilde{u}$: it is the homomorphism $(id_A\otimes\tilde{u})\circ\gamma_{(A,F)}$. It is useful to determine the homomorphism from F^A onto L corresponding to F in $A\otimes L$.

Lemma 4.5. *Let L be an algebra and A be a finite-dimensional algebra. There exists a unique homomorphism β from $(A\otimes L)^A$ into L such that*

$$(id_A\otimes\beta)\circ\gamma_{(A,A\otimes L)}=id_{A\otimes L}. \tag{10}$$

Moreover, the homomorphism β is such that

$$\beta(\overline{\varphi\otimes l\otimes a})=\varphi(a)\cdot l,$$

for all $\varphi\in A^*$, $a\in A$ and $l\in L$,

Proof. The existence and uniqueness of the homomorphism $\beta:(A\otimes L)^A\longrightarrow L$ result from the Theorem 3.4 applied to the algebra L and to the homomorphism $id_{A\otimes L}$. Thus, $(id_A\otimes\beta)\circ\gamma_{(A,A\otimes L)}=id_{A\otimes L}$ and for all $a\in A$ and $l\in L$, we have

$$[(id_A\otimes\beta)\circ\gamma_{(A,A\otimes L)}](a\otimes l)=a\otimes l.$$

It follows that,

$$(id_A\otimes\beta)[\sum_i a_i\otimes(\overline{a_i^*\otimes a\otimes l})]=\sum_i a_i^*(a)a_i\otimes l.$$

Hence,

$$\sum_i a_i\otimes\beta(\overline{a_i^*\otimes a\otimes l})=\sum_i a_i\otimes a_i^*(a)\cdot l$$

implies that

$$\begin{aligned}0 &= \sum_i a_i\otimes[\beta(\overline{a_i^*\otimes a\otimes l})-a_i^*(a)\cdot l]\\ &= \sum_i\Big[\beta(\overline{a_i^*\otimes a\otimes l})-a_i^*(a)\cdot l\Big](a_i\otimes 1_{A\otimes L}).\end{aligned}$$

By the Proposition 3.3, we have $\beta(\overline{a_i^*\otimes a\otimes l})-a_i^*(a)\cdot l=0$, for all $i\in I$ that is, $\beta(\overline{a_i^*\otimes a\otimes l})=a_i^*(a)\cdot l$, for all $i\in I$. By linearity, we conclude that $\beta(\overline{\varphi\otimes l\otimes a})=\varphi(a)\cdot l$, for all $\varphi\in A^*, a\in A$ and $l\in L$. □

Proposition 4.6. *Let $\Phi : Hom(F^A, L) \longrightarrow Hom(F, A \otimes L), \tilde{u} \longmapsto (id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}$ be the bijection from $Hom(F^A, L)$ into $Hom(F, A \otimes L)$. Then the map $\Psi : Hom(F, A \otimes L) \longrightarrow Hom(F^A, L), u \longmapsto \beta \circ u^A$ is the inverse bijection of Φ.*

Proof. For all $u \in Hom(F, A \otimes L)$, we have

$$\begin{aligned}(\Phi \circ \Psi)(u) &= \Phi(\beta \circ u^A) \\ &= \big(id_A \otimes (\beta \circ u^A)\big) \circ \gamma_{(A,F)} \\ &= (id_A \otimes \beta) \circ (id_A \circ u^A) \circ \gamma_{(A,F)}\end{aligned}$$

According to Proposition 4.1 and by (10), we have

$$\begin{aligned}(\Phi \circ \Psi)(u) &= (id_A \otimes \beta) \circ \gamma_{(A,A \otimes L)} \circ u \\ &= id_{A \otimes L} \circ u = u.\end{aligned}$$

Thus, $\Phi \circ \Psi = id_{Hom(F, A \otimes F)}$.

Conversely, for all $\tilde{u} \in Hom(F^A, L)$, we have

$$\begin{aligned}(\Psi \circ \Phi)(\tilde{u}) &= \Psi[(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}] \\ &= \beta \circ [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A.\end{aligned}$$

Putting $\beta \circ [(id_A \otimes \tilde{u}) \circ \gamma]^A = v$, we have

$$\begin{aligned}\Phi(v) &= \Phi\Big[\beta \circ [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A\Big] \\ &= \Big(id_A \otimes \Big[\beta \circ [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A\Big]\Big) \circ \gamma_{(A,F)} \\ &= (id_A \otimes \beta) \circ \Big[id_A \otimes [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A\Big] \circ \gamma_{(A,F)}.\end{aligned}$$

According to Proposition 4.1, we have

$$\Phi(v) = (id_A \otimes \beta) \circ \gamma_{(A,A \otimes L)} \circ (id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}.$$

By (10), we get $\Phi(v) = (id_A \otimes \tilde{u}) \circ \gamma_{(A,F)} = \Phi(\tilde{u})$. Since Φ is a bijection, we have

$$\beta \circ [(id_A \otimes \tilde{u}) \circ \gamma_{(A,F)}]^A = \tilde{u}.$$

Therefore,

$$\Psi \circ \Phi = id_{Hom(F^A, L)}.$$

□

For $L = F^A$ and for $u = \gamma_{(A,F)} : F \longrightarrow A \otimes F^A$, the corresponding homomorphism is $id_{F^A} \in Hom(F^A, F^A)$. Indeed, for all $x \in F$,

$$\begin{aligned}[(id_A \otimes id_{F^A}) \circ \gamma_{(A,F)}](x) &= (id_A \otimes id_{F^A})\left[\sum_i a_i \otimes (\overline{a_i^* \otimes x})\right] \\ &= \sum_i a_i \otimes (\overline{a_i^* \otimes x}).\end{aligned}$$

That is, $(id_A \otimes id_{F^A}) \circ \gamma_{(A,F)} = \gamma_{(A,F)}$. By the uniqueness, we conclude that $\tilde{u} = \widetilde{\gamma_{(A,F)}} = id_{F^A}$.

Proposition 4.7. *Let A be a finite-dimensional algebra over K. Then the homomorphism*

$$\varphi : K \longrightarrow K^A, \lambda \longmapsto \overline{1_A^* \otimes \lambda}$$

is an isomorphism of algebras. Its inverse homomorphism $\tilde{u} : K^A \longrightarrow K$ is such that

$$(id_A \otimes \tilde{u}) \circ \gamma_{(A,K)} = u,$$

where $u : K \longrightarrow A = A \otimes K, \lambda \longmapsto \lambda \cdot 1_A = 1_A \otimes \lambda$ is the canonical homomorphism.

Proof. The map φ is well-defined by construction. Since, for all $\lambda_1, \lambda_2 \in K$, $\overline{[1_A^* \otimes (\lambda_1 + \lambda_2)]} = \left(\overline{1_A^* \otimes \lambda_1}\right) + \left(\overline{1_A^* \otimes \lambda_2}\right)$ and $\overline{1_A^* \otimes \lambda_1 \lambda_2} = \lambda_1 \left(\overline{1_A^* \otimes \lambda_2}\right)$, we have $\varphi(\lambda_1 + \lambda_2) = \varphi(\lambda_1) + \varphi(\lambda_2)$ and $\varphi(\lambda_1 \lambda_2) = \lambda_1 \varphi(\lambda_2)$. On the other hand, for all $\lambda \in K$, we have $\varphi(\lambda) = \lambda(\overline{1_A^* \otimes 1_K}) = \lambda \overline{1_A^*(1_A)} = \lambda 1_{K^A}$. So, since for all $\lambda_1, \lambda_2 \in K$, $\overline{1_A^* \otimes \lambda_1 \lambda_2} = \lambda_1 \lambda_2 \left(\overline{1_A^* \otimes 1_K}\right) = (\lambda_1 \lambda_2) \overline{1_A^*(1_A)}$, we $\varphi(\lambda_1 \lambda_2) = (\lambda_1 \lambda_2) 1_{K^A} = (\lambda_1 \cdot 1_{K^A})(\lambda_2 \cdot 1_{K^A}) = \varphi(\lambda_1)\varphi(\lambda_2)$. The map φ is therefore a homomorphism of algebras.

Let $u : K \longrightarrow A = A \otimes K, \lambda \longmapsto \lambda \cdot 1_A = 1_A \otimes \lambda$ be the canonical homomorphism and let $\tilde{u} : K^A \longrightarrow K$ be the unique homomorphism of algebras such that $id_A \otimes \tilde{u}) \circ \gamma_{(A,K)} = u$. Since $\gamma_{(A,K)}$ does not depend of a choisen basis in A, consider the basis $(e_i)_{i \in I}$ of A. By the Proposition 3.2, we have for all $\lambda \in K$,

$$\gamma_{(A,K)}(\lambda) = \sum_i e_i \otimes (\overline{e_i^* \otimes \lambda}) = \sum_i e_i \otimes \lambda(\overline{e_i^* \otimes 1_K}) = \sum_i e_i \otimes \lambda e_i^*(1_A),$$

implies that

$$\gamma_{(A,K)}(\lambda) = 1_A \otimes \cdot \overline{1_A^*(\lambda \cdot 1_A)} = 1_A \otimes \lambda \cdot \overline{1_A^*(1_A)} = 1_A \otimes \overline{(1_A^* \otimes \lambda)}.$$

Thus, for all $\lambda \in K$,

$$[(id_A \otimes \varphi) \circ u](\lambda) = (id_A \otimes \varphi)[u(\lambda)] = (id_A \otimes \varphi)(1_A \otimes \lambda)$$

That is

$$[(id_A \otimes \varphi) \circ u](\lambda) = 1_A \otimes \overline{(1_A^* \otimes \lambda)} = \gamma_{(A,K)}(\lambda),$$

for all $\lambda \in K$. Hence, $(id_A \otimes \varphi) \circ u = \gamma_{(A,K)}$. Since $\gamma_{(A,K)} : K \longrightarrow A \otimes K^A$ is a homomorphism of algebras, the map $\varphi \otimes \tilde{u} : K^A \longrightarrow K \longrightarrow K^A$ is the unique homomorphism of algebras such that

$$[id_A \otimes (\varphi \circ \tilde{u})] \circ \gamma_{(A,K)} = \gamma_{(A,K)}. \tag{11}$$

Consider the map $(id_A \otimes id_{K^A}) \circ \gamma_{(A,K)} : K \longrightarrow A \otimes K^A \longrightarrow A \otimes K^A$. For all $\lambda \in K$, we have

$$\begin{aligned} [(id_A \otimes id_{K^A}) \circ \gamma_{(A,K)}](\lambda) &= (id_A \otimes id_{K^A})[\gamma_{(A,K)}(\lambda)] \\ &= (id_A \otimes id_{K^A})[1_A \otimes \overline{(1_A^* \otimes \lambda)}] \\ &= \gamma_{(A,K)}(\lambda). \end{aligned}$$

We deduce that

$$(id_A \otimes id_{K^A}) \circ \gamma_{(A,K)} = \gamma_{(A,K)}. \tag{12}$$

By (11) and (12), It follows by uniqueness that $\varphi \circ \tilde{u} = id_{K^A}$.

On the other hand, $\varphi \circ \tilde{u}$ is the unique homomorphism of algebras such that

$$[id_A \otimes (\tilde{u} \circ \varphi)] \circ u = u. \tag{13}$$

Since for all $\lambda \in K$,

$$[(id_A \otimes id_K) \circ u](\lambda) = (id_A \otimes id_K)[u(\lambda)] = (id_A \otimes id_K)(1_A \otimes \lambda),$$

that is $[(id_A \otimes id_K) \circ u](\lambda) = 1_A \otimes \lambda = u(\lambda)$, for any $\lambda \in K$. Hence

$$(id_A \otimes id_K) \circ u = u \tag{14}$$

By (13) and (14), we deduce by uniqueness that $\tilde{u} \circ \varphi = id_K$. Therefore φ is an isomorphism of algebras. This isomorphism allows us to identify K^A with K. □

Let $u : F \longrightarrow F = K \otimes F, x \longmapsto 1_K \otimes x$ be the canonical isomorphism.

Proposition 4.8. *Let F be an algebra over K and $u : F \longrightarrow F = K \otimes F$ be the canonical isomorphism. Then, the canonical homomorphism $\gamma_{(K,F)} : F \longrightarrow K \otimes F^K = F^K$ is an inverse isomorphism of the homomorphism $\tilde{u} : F^K \longrightarrow F$ such that $(id_K \otimes \tilde{u}) \circ \gamma_{(K,F)} = u$.*

Proof. Let $\gamma_{(K,F)} : F \longrightarrow K \otimes F^K$ be the canonical homomorphism and let $\gamma_{(K,F)} \otimes \tilde{u}$ be the unique homomorphism of algebras such that

$$[id_K \otimes (\gamma_{(K,F)} \circ \tilde{u})] \circ \gamma_{(K,F)} = \gamma_{(K,F)}. \tag{15}$$

Since for all $x \in F$, $\gamma_{(K,F)}(x) = 1_K \otimes \overline{(1_K^* \otimes x)}$ and

$$\begin{aligned} 1_K \otimes \overline{(1_K^* \otimes x)} &= id_K(1_K) \otimes id_{F^K}\left(\overline{1_K^* \otimes x}\right) \\ &= (id_K \otimes id_{F^K})[1_K \otimes \overline{(1_K^* \otimes x)}], \end{aligned}$$

then $\gamma_{(K,F)}(x) = (id_K \otimes id_{F^K})[\gamma_{(K,F)}(x)] = [(id_K \otimes id_{F^K}) \circ \gamma_{(K,F)}](x)$.Which implies

$$(id_K \otimes id_{F^K}) \circ \gamma_{(K,F)} = \gamma_{(K,F)}. \tag{16}$$

By (15) and (16), we conclude by uniqueness that $\gamma_{(K,F)} \circ \tilde{u} = id_{F^K}$.
On the other hand, since for all x in F,

$$\begin{aligned} [(id_K \otimes \gamma_{(K,F)}) \circ u](x) &= (id_K \otimes \gamma)(1_K \otimes x) \\ &= 1_K \otimes [1_K \otimes \overline{(1_K^* \otimes x)}], \end{aligned}$$

that is,

$$[(id_K \otimes \gamma_{(K,F)}) \circ u](x) = 1_K \cdot 1_K \otimes \overline{(1_K^* \otimes x)} = \gamma_{(K,F)}(x),$$

for all x in F. Hence, $\left(id_K \otimes \gamma_{(K,F)}\right) \circ u = \gamma_{(K,F)}$.
The homomorphism $\tilde{u} \circ \gamma_{(K,F)}$ is the unique homomorphism of algebras such that

$$[(id_K \otimes (\tilde{u} \circ \gamma_{(K,F)})] \circ u = u. \tag{17}$$

As $(id_K \otimes id_F) \circ u = u$ and since for all $x \in F$,

$$\begin{aligned} [(id_K \otimes id_F) \circ u](x) &= (id_K \otimes id_F)[u(x)] \\ &= (id_K \otimes id_F)(1_K \otimes x), \end{aligned}$$

implies that

$$[(id_K \otimes id_F) \circ u](x) = (1_K \otimes x) = u(x),$$

for all $x \in F$. Hence,

$$(id_K \otimes id_F) \circ u = u. \tag{18}$$

By (17) and (18), we deduce by uniqueness that $\tilde{u} \circ \gamma_{(K,F)} = id_F$.

It follows that $\gamma_{(K,F)}$ is an isomorphism of algebras with inverse $\tilde{u}$. This canonical isomorphism allows us to identify F with F^K. □

Proposition 4.9. *Let A and B be two algebras of finite dimensions. Then, there exists a unique isomorphism $\theta : F^{A\otimes B} \longrightarrow (F^A)^B$ such that*

$$(id_{A\otimes B} \otimes \theta) \circ \gamma_{(A\otimes B,F)} = \left(id_A \otimes \gamma_{(B,F^A)}\right) \circ \gamma_{(A,F)}.$$

Proof. By applying the Proposition 3.3 to the algebra $(F^A)^B$ and to the homomorphism $u = \left(id_A \otimes \gamma_{(B,F^A)}\right) \circ \gamma_{(A,F)} : F \longrightarrow A \otimes F^A \longrightarrow (A \otimes B) \otimes (F^A)^B$, we prove the existence and uniqueness of the homomorphism $\theta : F^{A\otimes B} \longrightarrow (F^A)^B$ such that

$$(id_{A\otimes B} \otimes \theta) \circ \gamma_{(A\otimes B,F)} = \left(id_A \otimes \gamma_{(B,F^A)}\right) \circ \gamma_{(A,F)}.$$

Furthermore, let L be an algebra and $\alpha : F \longrightarrow A \otimes B \otimes L$ be a homomorphism of algebras. Let $\bar{\alpha} : F^A \longrightarrow B \otimes L$ be the unique homomorphism of algebras such that

$$(id_A \otimes \bar{\alpha}) \circ \gamma_{(A,F)} = \alpha, \tag{19}$$

and let $\tilde{\alpha} : (F^A)^B \longrightarrow F^A$ be the unique homomorphism such that

$$(id_B \otimes \tilde{\alpha}) \circ \gamma_{(B,F^A)} = \bar{\alpha}. \tag{20}$$

By composing by tensor product on the left of the relation (20) by id_A, we have

$$id_A \otimes [(id_B \otimes \tilde{\alpha}) \circ \gamma_{(B,F^A)}] = id_A \otimes \bar{\alpha}.$$

It follows that,

$$(id_A \otimes [(id_B \otimes \tilde{\alpha}) \circ \gamma_{(B,F^A)}]) \circ \gamma_{(A,F)} = (id_A \otimes \bar{\alpha}) \circ \gamma_{(A,F)}$$

that is

$$(id_A \otimes id_B \otimes \tilde{\alpha}) \circ (id_A \otimes \gamma_{(B,F^A)}) \circ \gamma_{(A,F)} = [(id_A \otimes \bar{\alpha})] \circ \gamma_{(A,F)}. \tag{21}$$

Since $id_A \otimes id_B = id_{A\otimes B}$, $\left(id_A \otimes \gamma_{(B,F^A)}\right) \circ \gamma_{(A,F)} = u$ and by (19), the relation (21) becomes $(id_A \otimes id_B \otimes \tilde{\alpha}) \circ u = \alpha$. By uniqueness of the homomorphism $\tilde{\alpha}$, the pair $\left((F^A)^B, u\right)$ is therefore a solution of the universal problem for the algebras F and $A \otimes B \otimes L$, for any algebra L.
It follows that the homomorphism $\theta : F^{A\otimes B} \longrightarrow (F^A)^B$ is an isomorphism of algebras. □

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