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A going-down theoretic proof that straight domains are locally divided

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Abstract. A result on going-down domains is used to give a new proof of the recent result of Spencer Secord that all straight domains are locally divided domains. Corollaries collect characterizations of straight domains (equivalently, of locally divided domains) and of prime ideals of a base domain R that are straight with respect to all overring extensions of R. For integrally closed (but not necessarily quasi-local) base domains R, it is shown how to use the "going-down to P" property and the behavior of a relatively small class of singly generated overrings of R in order to characterize the locally divided prime ideals of R, while avoiding any mention of "straight"-related concepts.

Key Words: Integral domain, prime ideal, going-down, going-down domain, divided domain, locally divided domain, torsion-free module, quotient field, overring, straight domain
2010 MSC: Primary 13B21, 13F05; Secondary 13G05, 13A15, 13C13.

1 Introduction

In this note, all rings are associative and commutative, and they are usually (integral) domains; all inclusion of rings denote subrings; and all rings, ring homomorphims, and modules are unital. If R is a domain with quotient field K, then by an *overring* of R, we mean any ring T such that $R \subseteq T \subseteq K$. For any ring A, we let $\text{Spec}(A)$ (resp., $\text{Max}(A)$) denote the set of all prime (resp., all maximal) ideals of A. In the spirit of [12], we let GD denote the going-down property of ring homomorphisms (for instance, of ring extensions). For the sake of completeness, this Introduction next gives two paragraphs of background material; then a short paragraph stating the recent result of Spencer Secord that will be reproved in Theorem 2.1; then a paragraph explaining why we believe that it would be natural to use GD-theoretic methods while seeking a new proof of Secord's result; and, finally, a short paragraph announcing some new characterizations of straight domains and locally divided prime ideals that will be proved in Corollary 2.2 and Proposition 2.4, as well as the simpler characterizations in Section 3 when the base domain is integrally closed.

Recall that if $f : A \to B$ is a (unital) ring homomorphism (for instance, a ring extension $A \hookrightarrow B$), then B can be canonically viewed as an A-module, via $a \cdot b := f(a)b$ for all $a \in A$ and $b \in B$. This fact is particularly useful when one is also given an ideal I of A, for f then induces a ring homomorphism $A/I \to B/IB$ that allows B/IB to be viewed as a module over A/I. Some of the rest of this paragraph results from editing the first and second paragraphs of the Introduction in [9]. As in [7], a ring homomorphism $R \to T$ is said to be a *prime morphism* if T/PT is torsion-free over (the domain) R/P for each $P \in \text{Spec}(R)$ (when the induced ring homomorphism $R/P \to T/PT$ is used to view T/PT as a module over R/P). It was shown, *inter alia*, in Proposition 2.3 of [9] that a ring homomorphism $R \to T$ is a prime morphism if and only if the induced map $R_M \to T_M$ $(:= T_{R\setminus M})$ is a prime morphism for each $M \in \text{Max}(R)$. As in [9], a domain R is said to be a *straight domain* if $R \hookrightarrow T$ is a prime morphism for each overring T of R. The most natural examples of straight domains are Pruefer domains (the two-fold point being that Pruefer domains are the domains all of whose overrings are flat; and any flat ring homomorphism is a prime morphism). Moreover, it follows from the proof of Proposition 3.5 (b) of [9] that if R is a domain, then: R is a straight domain $\Leftrightarrow R_P$ is a straight domain for each $P \in \mathrm{Spec}(R) \Leftrightarrow R_M$ is a straight domain for each $M \in \mathrm{Max}(R)$.

Recall from [5] and [8] that a domain R, with quotient field K, is said to be a *going-down domain* if $R \subseteq T$ satisfies GD for each overring T of R (equivalently, if $R \subseteq V$ satisfies GD for each valuation overring V of R; equivalently, if $R \subseteq T$ satisfies GD for each domain T having R as a subring; equivalently, if $R \subseteq R[u]$ satisfies GD for each $u \in K$). The most natural examples of going-down domains are domains of Krull dimension 1 and Pruefer domains. It was noted in Remark 2.7 (b) of [6] that a domain R is a going-down domain if and only if R_M is a going-down domain for all $M \in \mathrm{Max}(R)$ (equivalently, if R_Q is a going-down domain for all $Q \in \mathrm{Spec}(R)$). Next, recall from [6] that a domain R is said to be a *divided domain* if $PR_P = P$ for all $P \in \mathrm{Spec}(R)$. (Such domains were introduced by T. Akiba in [1], where they were called AV-domains.) The most natural example of a divided domain is a valuation domain. It was shown in Proposition 2.1 of [6] that each divided domain is a quasi-local going-down domain. (Example 2.9 of [6] showed that if $2 \le n \le \infty$, then there exists a quasi-local going-down domain of Krull dimension n which is not a divided domain.) Recall from Remark 2.7 (b) of [6] that a domain R is said to be a *locally divided domain* if R_M is a divided domain for all $M \in \mathrm{Max}(R)$ (equivalently, if R_Q is a divided domain for all $Q \in \mathrm{Spec}(R)$); and that any localization of a locally divided domain is itself locally divided (hence, divided domains are the same as the quasi-local locally divided domains).

In Corollary 3.1.1 of [14], Secord recently proved that a domain is a straight domain if and only if it is a locally divided domain. As noted in [14], the "if" assertion was already known: see Corollary 3.8 of [9]. The next paragraph explains why/how Secord's result settled a natural open question; and why we have decided to devote Theorem 2.1 to giving a new proof of (the "only if" assertion in) the theorem of Secord.

A noteworthy difference between our proof in Theorem 2.1 and the exposition in [14] is that our proof features roles for the GD property of ring extensions and for the class of going-down domains. This seems appropriate to us for two reasons: all the relevant kinds of domains considered here (and many of those in [14]) are examples of going-down domains; and a technical result on quasi-local going-down domains from [6] (namely, its Corollary 2.6) will justify the only nontrivial step (apart from a displayed calculation) in our proof of Theorem 2.1. For historical and motivational purposes, we next recall the first sentence of a paragraph on page 760 of [9]: "The following observations constitute much of the motivation for the present article." That paragraph goes on to discuss the hint that was given by Kaplansky for Exercise 37 on pages 44-45 of [12]. That material details a torsion-theoretic characterization of the (commutative) ring homomorphisms that satisfy GD. The paragraphs ends as follows: "As noted in [7], it follows easily that each prime morphism satisfies GD." Since straight domains are defined as the domains R such that $R \hookrightarrow T$ is a prime morphism for each overring T of R, each straight domain is a going-down domain. As it was shown in Remark 2.7 (b) of [6] that each locally divided domain is a going-down domain, it is natural to ask what connections, if any, exist between the class of straight domains and the class of locally divided domains. In that regard, Corollary 3.8 of [9] established that each locally divided domain is a straight domain; and, after investigating a number of special contexts, the authors of [9] concluded its Remark 3.15 by stating that "we do not know an example of a straight domain that is not locally divided." The question that was implicit in that quotation was answered by Secord in Corollary 3.1.1 of [14], approximately 15 years after the publication of [9]. Frankly, one purpose of this note is to show that appropriate use of some material which was available at the time that [9] was written would have been sufficient (when conjoined with a suitable calculation) to provide a quick answer to the question that had been left open in Remark 3.15 of [9].

Corollary 2.2 collects some characterizations of straight domains. Arguably, its condition (7) goes beyond what can be found in [14]. Proposition 2.4 collects some characterizations of locally divided prime ideals of a domain. Its conditions (6) and (7) mark the first explicit appearances of sets of the form $\Omega(R,P)$ in such characterizations. (The definition of $\Omega(R,P)$ is given in Section 2.) The main point of Section 3 is that the locally divided prime ideals of an integrally closed base domain R can be characterized, without any mention of the "straight" or "prime morphism" concepts, by suitably using the "going-down to P" concept while restricting attention to overrings generated as R-algebras by single elements of $\Omega(R,P)$.

2 A new proof of a result of Secord and an application

Theorem 2.1. (Corollary 3.1.1 of [14]) Let R be a domain. Then R is a straight domain if and only if R is a locally divided domain.

Proof. The "if" assertion was proven in Corollary 3.8 of [9]. We turn to the "only if" assertion. By the material that was recalled above, we can assume, without loss of generality, that R is quasi-local, say with (unique) maximal ideal M. Suppose that the assertion fails. Then R is a quasi-local straight domain but not a divided domain. As we recalled above that any straight domain is a going-down domain, an application of Corollary 2.6 of [6] yields $w \in K$ and $P \in \mathrm{Spec}(R)$ such that $w^n \in P$ for all integers $n \geq 2$ and $w \in PR_P \setminus P$ $(= PR_P \setminus R)$. Observe that $P \neq M$ (since $MR_M = MR = M$). As $w \in PR_P$, we can write $w = p/z$ for some nonzero elements $p \in P$ and $z \in R \setminus P$. In fact, $z \in M \setminus P$, since the facts that (R,M) is quasi-local, $w \notin R$, and $p \in P$ combine to entail that z is a nonunit of R.

Now, consider the overring $T := R[w]$ of R. Since $w^2 \in R$, it follows that $T = R + Rw$. As R is a straight domain, T/PT is a torsion-free module over R/P. Consider the elements $\rho := z + P \in R/P$ and $\tau := w + PT \in T/PT$. We have $\rho \neq 0$ since $z \notin P$; and $\rho \cdot \tau = 0$, since

$$z \cdot w = z(p/z) = p \in P \subseteq PT.$$

As T/PT is a torsion-free module over R/P, we get $\tau = 0$; that is, $w \in PT = P(R + Rw) = P + Pw$. Thus, $w = p_0 + p_1 w$ for some elements $p_0, p_1 \in P$. Then

$$(1 - p_1)w = w - p_1 w = p_0 \in P.$$

Since $p_1 \in P \subseteq M$ (and M is the Jacobson radical of R), $1 - p_1$ is a unit of R. Hence, $w = (1 - p_1)^{-1} p_0 \in Rp_0 \subseteq P$, the desired contradiction. $\square$

In the spirit of the (equivalent) characterizations of going-down domains given in Theorem 1 of [8], it seems reasonable to ask if one can harmlessly focus on some naturally occurring (and possibly proper) subset of "test overrings" T in applying Theorem 2.1. In other words, is there a naturally occurring class of overrings T of a given domain R such that $R \hookrightarrow T$ being a prime morphism for all T in *that* class would ensure that R is a locally divided domain? For the special case where R is a conducive domain, it was shown in Theorem 3.19 of [9] that it is enough to use the fractional overrings of R as the test overrings T. For an arbitrary domain R with quotient field K, it is enough to use the finite-type R-subalgebras of K as the test overrings T (since it was shown in Proposition 2.4 of [9] that the direct limit of prime morphisms is a prime morphism). One can actually do better, in general. Indeed, Corollary 2.2 will use the proof of Theorem 2.1 and some globalization reasoning to achieve an analogue of the characterization of going-down domains in condition (a) of Theorem 1 of [8], by showing that if R is an arbitrary domain with quotient field K, it is enough to use the singly generated overrings of R (that is, the rings of the form $R[u]$, with $u \in K$) as the test overrings T.

The following notation will be useful in this section and in Section 3. Let R be a domain with quotient field K and let $P \in \mathrm{Spec}(R)$. Put

$$\Omega(R,P) := \{u \in K \mid u \in PR_P \setminus PR_Q \text{ for some } Q \in \mathrm{Spec}(R) \text{ with } P \subseteq Q\}$$

and $\Lambda(R,P) := K \setminus (R \cup \Omega(R,P))$. Note that in the above definition of $\Omega(R,P)$, it is harmless to replace the condition "$Q \in \mathrm{Spec}(R)$" with "$Q \in \mathrm{Max}(R)$".

Corollary 2.2 collects some characterizations of straight domains (equivalently, of locally divided domains). One can make the case that the equivalence of conditions (1) and (5) in Corollary 2.2 is readily obtainable from what is explicitly in [14]. We do believe, however, that the equivalence of conditions (1) and (7) in Corollary 2.2 is new here, since the possible localization/globalization behavior of sets of the form $\Omega(R,P)$ was not addressed in [14].

Corollary 2.2. *Let R be a domain with quotient field K. Then the following conditions are equivalent:*
(1) *R is a locally divided domain;*
(2) *R is a straight domain;*
(3) *$R \hookrightarrow T$ is a prime morphism for all overrings T of R;*
(4) *T/PT is a torsion-free module over R/P for all overrings T of R and all $P \in \mathrm{Spec}(R)$;*
(5) *$R \hookrightarrow R[u]$ is a prime morphism for all $u \in K$;*
(6) *$R[u]/PR[u]$ is a torsion-free module over R/P for all $P \in \mathrm{Spec}(R)$ and all $u \in K$;*
(7) *For all $P \in \mathrm{Spec}(R)$, $R \hookrightarrow R[u]$ is a prime morphism for all $u \in \Omega(R,P)$ and $R \hookrightarrow R[u]$ satisfies* GD *for all $u \in \Lambda(R,P)$.*

Proof. (1) $\Leftrightarrow$ (2) by Corollary 3.1.1 of [14] (or by the above Theorem 2.1); (2) $\Leftrightarrow$ (3) by the definition of a straight domain; each of the equivalences (3) $\Leftrightarrow$ (4) and (5) $\Leftrightarrow$ (6) follows from the definition of a prime morphism; and each of the implications (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) is trivial. Moreover, (5) $\Rightarrow$ (7) since any prime morphism satisfies GD (cf. Exercise 37, pages 44-45 of [12]). Therefore, it will suffice to prove that (7) $\Rightarrow$ (1).

(7) $\Rightarrow$ (1): Assume (7). Fix an arbitrary $Q \in \mathrm{Spec}(R)$. Our task is to prove that R_Q is a divided domain. As it was shown in Proposition 2.3 of [9] that the "prime morphism" property is preserved by the formation of rings of fractions, (7) yields that $R_Q \hookrightarrow R[u]_{R\setminus Q}$ $(= R_Q[u])$ is a prime morphism (and hence satisfies GD) for all $u \in \Omega(R,P)$ for all $P \in \mathrm{Spec}(R)$ such that $P \subseteq Q$. On the other hand, if $u \in \Lambda(R,P)$ for some $P \in \mathrm{Spec}(R)$ such that $P \subseteq Q$, then $R \hookrightarrow R[u]$ satisfies GD by (7), and so $R_Q \hookrightarrow R[u]_{R\setminus Q}$ $(= R_Q[u])$ also satisfies GD. Thus, $R_Q \subseteq R_Q[u]$ satisfies GD for all $u \in K$. As K is (also) the quotient field of R_Q, an application of Theorem 1 of [8] shows that (the quasi-local domain) R_Q is a going-down domain. Bearing in mind that a quasi-local locally divided domain is the same thing as a divided domain, we can now revisit the proof of Theorem 2.1, beginning with its fourth sentence, with the current R_Q playing the role that R had played from the fourth sentence onward in the proof of Theorem 2.1.

The following words of caution pertain to that revisiting. We will not need to use any assertions in the proof to the effect that R_Q is known to be a straight domain; we can apply Corollary 2.6 of [6] to the base ring R_Q and a supposed nondivided prime ideal $\mathfrak{P} = PR_Q$ of R_Q (with P a suitable prime ideal of R contained in Q) to find a suitable element $w \in PR_P \setminus PR_Q$, with $w = p/z$ for some $p \in P$ and $z \in Q \setminus P$, such that $w^2, w^3 \in \mathfrak{P}$; as before, we do consider $T := R_Q[w]$; most importantly, even though we do not (yet) know that R_Q is a straight domain, we *do* get from (7), since $w \in \Omega(R,P)$ (and the formation of rings of fractions preserves prime morphisms), that $R_Q \hookrightarrow T$ is a prime morphism, whence T/PT $(= T/\mathfrak{P}T)$ is a torsion-free module over $R_Q/\mathfrak{P}$: and then (with R_Q still playing the earlier role of R) the second paragraph of the proof of Theorem 2.1 carries over uneventfully, thus (obtaining the desired contradiction and) completing the proof. $\square$

Although Corollary 2.2 showed that the singly generated overrings of a domain R suffice as a family of "test overrings" (relative to appropriate torsion-free behavior) in determining whether R is a straight domain (equivalently, a locally divided domain), one may plausibly assert that such a conclusion would also be deducible from the methods of Secord in [14]. Proposition 2.4 will begin the process of sharpening Corollary 2.2 by recording that the "straight domain" (or "locally divided domain") conclusion for a given quasi-local domain R can be obtained by using a significantly smaller class of "test overrings" of a domain R. To ease the way to Proposition 2.4, we devote two paragraphs to recalling some definitions and facts about certain local/global behavior and then give Lemma 2.3, which collects some elementary but useful facts concerning such behavior. Proposition 2.4 sharpens the characterization of "sharp prime ideals" that was given in the main result of [14]. We wish to recognize that the proof of Proposition 2.4 makes explicit use of use of an argument from [14]. We wish to reiterate that for the case of integrally closed (but not necessarily quasi-local) domains, Section 3 will present an alternate GD-theoretic approach to the results and themes in the present section that will make no appeal to material from [14] (apart from using its definition of a sharp prime ideal of a domain).

By definition, a prime ideal P of a domain R is said to be a *locally divided prime ideal of* R if PR_M is a divided prime ideal of R_M for each maximal ideal M of R that contains P; equivalently, that $PR_P = PR_M$ for any such M, since

$$(PR_M)(R_M)_{PR_M} = (PR_M)R_P = P(R_M R_P) = PR_P.$$

In particular, if R is a quasi-local domain, a locally divided prime ideal of R is the same as a divided prime ideal of R.

Recall from [9] that if $R \to T$ is a ring homomorphism (for instance, an inclusion of rings) and $P \in \mathrm{Spec}(R)$, then we say that f is a *prime morphism at* P if T/PT is a torsion-free module over R/P (with respect to the module-theoretic structure induced by $R \to T$). In case f is an inclusion of domains $R \hookrightarrow T$ and $P \in \mathrm{Spec}(R)$, it will be convenient to use the terminology "P is straight with respect to $R \subseteq T$" to describe this condition. If a prime ideal P of a domain R is straight with respect to $R \subseteq T$ for all overrings T of R (contained in some preassigned quotient field of R), we will say that "P is a straight prime ideal of R". This replaces the terser (but possibly ambiguous) usage "P is straight" which occurred notably in Theorem 3.1 of [14], a result that has Corollary 3.1.1 of [14] (that is, the above Theorem 2.1) as a consequence. Note that a domain R is a straight domain if and only if each prime ideal of R is straight with respect to $R \subseteq T$ for each overring T of R, that is, if and only if each prime ideal of R is a straight prime ideal of R.

Lemma 2.3. *Let R be a domain.*

(a) *Let E be an R-module. Then E is a torsion-free R-module if and only if $E_{R\setminus M}$ is a torsion-free module over R_M for all $M \in \mathrm{Max}(R)$.*

(b) *Let $P \in \mathrm{Spec}(R)$ and let T be a domain containing R as a subring. Then P is straight with respect to $R \subseteq T$ if and only if PR_M is straight with respect to $R_M \subseteq T_{R\setminus M}$ for all $M \in \mathrm{Max}(R)$ that contain P.*

(c) *Let R be a domain with quotient field K, and let $P \in \mathrm{Spec}(R)$. Let $\mathcal{C}$ be a nonempty collection of overrings of R that are contained in K. Let $M \in \mathrm{Max}(R)$ such that $P \subseteq M$; let $\mathcal{C}_M := \{T_{R\setminus M} \mid T \in \mathcal{C}\}$. Then $\mathcal{C}_M$ is a nonempty collection of overrings of R_M that are contained in K and, moreover, each overring S of R_M that is contained in K satisfies $S = SR_M = S_{R\setminus M}$ and is an overring of R.*

(d) *Let R, K, P and $\mathcal{C}$ be as in (c). Then the following conditions are equivalent:*

(1) *P is straight with respect to $R \subseteq T$ for all $T \in \mathcal{C}$;*

(2) *For each $M \in \mathrm{Max}(R)$ that contains P, PR_M is straight with respect to $R_M \subseteq S$ for all $S \in \mathcal{C}_M$.*

(e) *Let R, K and $\mathcal{C}$ be as in (c). Then the following conditions are equivalent:*

(1) *Each prime ideal of R is straight with respect to $R \subseteq T$ for all $T \in \mathcal{C}$;*

(2) *For each $P \in \mathrm{Spec}(R)$ and each $M \in \mathrm{Max}(R)$ that contains P, PR_M is straight with respect to $R_M \subseteq S$ for all $S \in \mathcal{C}_M$.*

Proof. (a) This assertion follows easily via globalization.

(b) Apply (a) to $E := T/PT$, noting the canonical isomorphisms

$$(R/P)_{M/P} \cong R_M/PR_M \text{ and } E_{R/P\setminus M/P} \cong T_{R\setminus M}/PR_M T_{R\setminus M}$$

for all $M \in \mathrm{Max}(R)$ that contain P.

(c) The assertion is easily verified.

(d) It suffices to apply (b) to each $T \in \mathcal{C}$.

(e) It suffices to apply (d) to each prime ideal P of R. □

Recall that one purpose of this paper is to show that some material which was available at the time that [9] was written would have been sufficient to provide a quick (and self-contained) answer to the question that had been left open in Remark 3.15 of [9]. The proof of Theorem 2.1 has done just that. However, by focusing on one prime ideal at a time in the statement of Theorem 3.1 of [14], Secord has produced a stronger result. Proposition 2.4 will sharpen that stronger result of Secord. We believe that an objective reading of the rest of this section will show that the only part of [14] that we will be using there (apart from some material which was available at the time that [9] was written) is Lemma 2.2 of [14]. For that reason, we have concluded that Lemma 2.2 of [14] is the most important contribution in [14], and we wish to express the hope that the method of proof of that result will find additional applications in the future, even though, prior to the statement of Lemma 2.2 of [14], Secord expressed the view that his Theorem 3.1 “will make this Lemma [his Lemma 2.2] redundant”.

Next, in the spirit of Corollary 2.2, the $\Omega(R,P)$ concept appears in two entries in a list of characterizations of locally divided prime ideals.

Proposition 2.4. *Let R be a domain, with quotient field K. Let $P \in \mathrm{Spec}(R)$. Then:*

(a) The following five conditions are equivalent:

(1) P is a locally divided prime ideal of R;

(2) P is straight with respect to $R \subseteq T$ for each overring T of R;

(3) T/PT is a torsion-free module over R/P for all overrings T of R;

(4) P is straight with respect to $R \subseteq R[w]$ for each $w \in K$;

(5) $R[w]/PR[w]$ is a torsion-free module over R/P for each $w \in K$.

(b) Assume also that R is quasi-local, with maximal ideal M. Then the conditions (1)-(5) in (a) are also equivalent to the following two conditions:

(6) P is straight with respect to $R \subseteq R[w]$ for each $w \in \Omega(R,P)$ and $R \subseteq R[u]$ satisfies “GD to P” for all $u \in \Lambda(R,P)$;

(7) P is straight with respect to $R \subseteq R[w]$ for each $w \in \Omega(R,P)$.

Proof. (a) Both of the equivalences (2) ⇔ (3) and (4) ⇔ (5) follow from the above definition of a prime ideal of R being straight with respect to a given domain extension of R. On the other hand, each of the implications (2) ⇒ (4) and (3) ⇒ (5) is trivial. Therefore, it will suffice to prove the two implications (1) ⇒ (3) and (5) ⇒ (1).

(1) ⇒ (3): Assume (1). As explained above, this assumption means that $PR_P = PR_M$ for each maximal ideal M of R that contains P. On the other hand, it follows from Lemma 2.3 (e), with $\mathcal{C}$ defined as the set of all overrings of R contained in K, that (3) is equivalent to PR_M being straight with respect to $R_M \subseteq T_{R\setminus M}$ for all overrings T of R (contained in K) and all maximal ideals M of R that contain P. Hence, we can assume, without loss of generality, that R is quasi-local, say with maximal ideal M. Thus $PR_P = P$, and our task is to prove that T/PT is a torsion-free module over R/P for each overring T of R. This, in turn, can be done by repeating the second paragraph of the proof of Proposition 3.7 (a) of [9] (with $A := R$ and $B := T$).

(5) ⇒ (1): Consider the analyses, in the above proof of the implication (1) ⇒ (3), of the condition (1) and of the special case of the condition (3) where the rings T are taken to be of the form $T = R[w]$, as w varies over K. The upshot is that here, in the proof of the implication (5) ⇒ (1), we can assume, without loss of generality, that R is quasi-local, say with maximal ideal M. Our task is to prove that $PR_P = P$, given that (5) holds. It will be useful to note that (4) also holds.

Suppose that the assertion fails. Choose $\xi \in PR_P \setminus P$ ($= PR_P \setminus R$). Consider the ring $T := R[\xi^{-1}]$. By Lemma 2.3 (b) and (5), T/PT is torsion-free over R/P. Also, as noted in the proof of Lemma 2.2 (i) of [14], it follows that $\xi \in PR[\xi]$. (For the sake of completeness, we recall its proof: as $\xi = p/z$ for some $p \in P$ and $z \in R \setminus P$, one need only combine the fact that $z\xi = p \in PR[\xi]$ with the hypothesis that P is straight with respect to $R \subseteq R[\xi]$.) Next, as (4) gives that P is straight with respect to $R \subseteq T$, it follows from the hint given by Kaplansky for Exercise 37 on pages 44-45 of [12] that $R \subseteq T$ satisfies "GD to P". That fragment of GD-like behavior is enough to allow us to adapt the first paragraph of the proof of Lemma 2.4 (a) of [6] in order to conclude that ξ is integral over R. In particular, $S := R[\xi]$ is a finitely generated R-module. To get the desired contradiction (that $S = R$), it will suffice (by Nakayama's Lemma) to show that $M(S/R) = S/R$, that is, to show that $S \subseteq R + MS$. As S is the subring of K generated by $R \cup \{\xi\}$, we need only note that $R \subseteq R + MS$ and $\xi \in PR[\xi] = PS \subseteq MS \subseteq R + MS$.

(b) (4) $\Rightarrow$ (6): As $\Omega(R,P) \subseteq K$, it suffices to appeal once again to Exercise 37 on pages 44-45 of [12].

(6) $\Rightarrow$ (1): Since R is assumed quasi-local, it suffices to rework the second paragraph of the above proof that (5) $\Rightarrow$ (1). Observe that the element ξ in that paragraph is assumed to be in $\Omega(R,P)$. In reworking the earlier argument, it is also necessary to observe that $\xi^{-1} \in \Lambda(R,P)$. Since $\xi \notin R$, this comes down to showing that $\xi^{-1} \notin \Omega(R,P)$. This, in turn, holds, since the proper ideal PR_P of R_P would otherwise contain both ξ and ξ^{-1}, whence $\xi\xi^{-1} \in PR_P$, a patent contradiction.

(6) $\Rightarrow$ (7): Trivial.

(7) $\Rightarrow$ (1): In our opinion, this is the main application in [14]. For the sake of completeness, we summarize Secord's argument here. As in the above proof that (5) $\Rightarrow$ (1), one uses Nakayama's Lemma to show that R is integrally closed in $R + PR_P$; and as noted in the proof of Lemma 2.2 (i) of [14], one shows that each $\xi \in \Omega(R,P)$ satisfies $\xi \in PR[\xi]$. Thus, since R is quasi-local, we get that ξ is the root of a polynomial in $R[X]$ having a unit coefficient. So, since $\xi \notin R$, it follows from a well known generalization of Seidenberg's "u, u^{-1} Lemma" (cf. Lemma 3.8 of [10]) that $\xi^{-1} \in R$. This would give a contradiction, since ξ would be an element of the proper ideal PR_P of the larger ring $R + PR_P$. Hence, no such ξ can exist. This completes the proof. $\square$

We close this section with a multi-faceted remark/*mélange*.

Remark 2.5. (a) The equivalent conditions (4) and (5) in the statement of Proposition 2.4 gave the sharpest "straight-theoretic" or "torsion-theoretic" characterizations of a locally divided prime ideal of a domain that I was able to develop by using mostly only "old" material involving GD and going-down domains. About "mostly": I humbly acknowledge that the final two sentences of the proof that (5) $\Rightarrow$ (1) in the proof of Proposition 2.4 were a mild adaptation of an innovative argument using Nakayama's Lemma that was part of Secord's proof of Lemma 2.2 (ii) in [14]. Thus, apart from that appeal to Nakayama's Lemma (and the use of the definition of a prime ideal being straight with respect to a ring extension), our GD-theoretic proof that (4) and (5) each characterize locally divided prime ideals of a domain is, in effect, independent of [14].

As the property of being a locally divided prime ideal of a domain is a local property, it seemed reasonable to include a part (b) of Proposition 2.4 to address the context of a quasi-local base domain. For that context, the proof that condition (6) characterizes a divided prime ideal (of a quasi-local domain) featured roles (that we found pleasant) for the "$\Omega(R,P)$" and "GD to P" concepts, with proofs that were also, in effect, independent of [14] (apart from Secord's above-mentioned innovative use of Nakayama's Lemma). *That* explains our interest in including condition (6) in the statement of Proposition 2.4. I am glad to acknowledge and appreciate the superiority of the condition (7) (*vis-à-vis* condition (6)) in Proposition 2.4 as a characterization of a divided prime ideal of a quasi-local domain. In that regard, we note again that the above proof that (7) $\Rightarrow$ (1) in Proposition 2.4 is merely a reworking of some of Secord's arguments.

(b) By way of partial summary of (b), I am pleased to acknowledge that Secord has given a sharper characterization of a divided prime ideal of a quasi-local domain than I could while using only GD-theoretic methods. One may ask whether the characterization of (not necessarily quasi-local) straight domains (equivalently, locally divided domains) given by condition (7) in Corollary 2.2 surpasses anything that could reasonably be attributed to [14]. Because of the content of the next paragraph, I would caution against rushing to such a rash conclusion.

By combining Lemma 2.3 (b) and Proposition 2.4 (b), one gets the following result. Let R be a (not necessarily quasi-local) domain and let $P \in \mathrm{Spec}(R)$. Then P is a locally divided prime ideal of R if and only if, for each $M \in \mathrm{Max}(R)$ that contains P, PR_M is straight with respect to $R_M \subseteq R_M[w]$ for each $w \in \Omega(R,P)$. It seems reasonable to conclude that this result would be noticed by many readers of [14].

(c) If R is any domain, we can obtain another characterization of locally divided domains R (equivalently, straight domains R) by applying universal quantification over $P \in \mathrm{Spec}(R)$ to the result in the second paragraph of (b). That characterization deserves to be added to the list of equivalent conditions in the statement of Corollary 2.2.

(d) In view of the roles played by sets of the form $\Omega(R,P)$ and the prevalence of globalization considerations in this section, it is tempting to ask about the relationship, given an arbitrary domain R and prime ideals $P \subseteq M$ of R, between $\Omega(R_M, PR_M)$ and $\Omega(R,P)_{R\setminus M}$. In our opinion, this question merits further scrutiny.

(e) Although the concept of a prime ideal being straight with respect to a ring extension was innovative and has been fruitful, we believe that more can/should be learned about it. One possible source of questions is provided by known results on prime morphisms. The next four paragraphs discuss the use of one such source, namely, Proposition 2.6 of [9], which gave some general results on how the above-mentioned concept may be preserved or reflected under composition of (for our applications, injective) ring homomorphisms.

Suppose that $R \subset S$ are (distinct) domains, but not fields, such that $\mathrm{Spec}(R) = \mathrm{Spec}(S)$ (as sets). The first paper studying the "$\mathrm{Spec}(R) = \mathrm{Spec}(S)$" condition was [2]. The most familiar example of a ring extension satisfying this condition is given by an arbitrary pseudo-valuation domain R (in the sense of Hedstrom and Houston [11]) which is not a field and its canonically associated valuation overring S. By adapting the proof of Proposition 2.16 of [9], one can obtain the following conclusion (for $R \subset S$ distinct domains, but not fields, with the same prime ideals). Let $P \in \mathrm{Spec}(R)$ $(= \mathrm{Spec}(S))$ and let T be an overring of S (that is, an overring of R that contains S). Then: if P is straight with respect to $S \subseteq T$, then P is straight with respect to $R \subseteq T$.

The next paragraph will comment briefly on the proof of the above conclusion. Before that, we will make two points that help to explain the tractability of data satisfying the "$\mathrm{Spec}(R) = \mathrm{Spec}(S)$" condition. The first of these was noted in Proposition 2.1 (a) of [9]: because every vector space over a field K is a torsion-free K-module, if follows that if A is a (commutative) ring and $\mathcal{M} \in \mathrm{Max}(A)$, then $\mathcal{M}$ is a straight ideal of A (that is, $\mathcal{M}$ is straight with respect to $A \subseteq B$ for any commutative ring B containing A as a subring). Second, if $R \subset S$ are domains, but not fields, such that $\mathrm{Spec}(R) = \mathrm{Spec}(S)$, then R and S are each quasi-local, they have the same maximal ideal (say M), and S is an overring of R (this follows from Proposition 3.3 of [2]) and if in addition $P \in \mathrm{Spec}(R) \setminus \mathrm{Max}(R)$, then $R_P = S_{S\setminus P}$ (so, the stalks of the canonical structure sheaves of the affine schemes $\mathrm{Spec}(R)$ and $\mathrm{Spec}(S)$ differ *only* at the point M).

The proof of the above conclusion is an easy consequence of the proof of Proposition 2.6 (c) of [9], which uses the fact that the "$\mathrm{Spec}(R) = \mathrm{Spec}(S)$" hypothesis ensures that the inclusion map $R \hookrightarrow S$ is a prime-producing homomorphism that satisfies the quasi-lying-over property.

It seems natural to ask for additional hypotheses under which the converse of the above result would be valid. In other words: if $\mathrm{Spec}(R) = \mathrm{Spec}(S)$ (as sets) for domains $R \subset S$, T is an overring of S, and $P \in \mathrm{Spec}(R)$ is straight with respect to $R \subseteq T$, under what additional conditions (if any) must P be straight with respect to $S \subseteq T$? It seems that none of the *other* parts of Proposition 2.6 of [9] would be helpful in settling this question. Only some of the homological properties used in the proof of the just-cited result, such as flatness, have relevance in the study of overrings of domains. Some other homological properties that were involved in that proof, such as faithfullyflatness, would be irrelevant for the kind of focused study that I am proposing. (After all, no domain has a proper faithfully flat overring.) I suggest that it may be necessary to develop a deeper study of some ideal-theoretic properties in order to obtain a satisfactory answer to this question. Such an answer will surely call for something that would go beyond a facile restatement involving conductors.

3 Integrally closed domains admit a simpler characterization of locally divided prime ideals

Many of the objects of study in this paper (certain kinds of domains or prime ideals) are defined, and often characterized, via formulations involving universal quantifications. Proposition 2.4 pursued Secord's approach of removing the universal quantification over the prime spectrum of a given domain R (which had been involved in the definition of a straight domain in [9]) by focusing on whether a given prime ideal P of R is a locally divided prime ideal of R (equivalently, a straight prime ideal of R) while still using universal quantification over the overrings of that domain R that can be generated over R by a single element of $\Omega(R,P)$. As Proposition 2.4 (b) recorded, that approach is successful when the base domain R is quasi-local. It is natural to ask if one can remove (or, at least, relax) that hypothesis of quasi-locality. An initial attempt along such lines was made in the second paragraph of Remark 2.5 (b), at the cost of replacing consideration of a triple (P,R,T) with consideration of the set of triples $\{(PR_M, R_M, T_{R\setminus M}) : M \in \mathrm{Max}(R), P \subseteq M\}$. In this brief section, we show that one *can*, for the special case where R is integrally closed, remove the "quasi-local" hypothesis *and* also avoid consideration of the above-mentioned maximal ideals M: see Corollary 3.5.

It seems natural to focus on something like the universe of integrally closed domains in studying the "straight" and "locally divided" concepts. Indeed, in Corollary 11 of [13], McAdam showed (using other terminology) that an integrally closed domain is a going-down domain if and only if it is a locally divided domain. (The statement of the just-cited result included the extra hypothesis that the given domain is quasi-local, but that stated result easily globalizes.) This was generalized by replacing "integrally closed" with "root closed" in Corollary 2.8 of [5], whose method of proof yields the further generalization in which "root closed" is replaced by "seminormal". In short, for a seminormal domain R: R is a straight domain $\Leftrightarrow$ R is a going-down domain $\Leftrightarrow$ R is a locally divided domain. Note, however, that a going-down domain need not be a straight domain (and hence need not be a locally divided domain). This was shown in Example 4.4 of [9] by further analyzing a certain two-dimensional quasi-local domain which had been constructed for other purposes by Boisen and Sheldon in [4] (and which had been shown to be a quasi-local going-down domain that is not a divided domain in Example 2.9 of [6]).

The first result in this section extracts part of what was actually shown in a result in [5]. This is then used in proving the main result of this section, Theorem 3.2 (b), which gives a companion for the results in the earlier sections by having a statement and proof featuring the "going-down to P" concept while ignoring the "P is straight with respect to $R \subseteq T$" concept. The subsequent corollaries specialize matters to integrally closed based domains and feature statements replacing the occurrences of "going-down to P" with their less general "straight" analogues.

Lemma 3.1. *Let R be a domain with quotient field K and let $u \in K \setminus \{0\}$, such that R is integrally closed in both $R[u]$ and $R[u^{-1}]$. Let $P \in \mathrm{Spec}(R)$ such that $R \subseteq R[u]$ satisfies* GD *to P. Then $R \subseteq R[u^{-1}]$ satisfies* GD *to P.*

Proof. The assertion can be extracted from the proof of Theorem 2.6 of [5]. □

Theorem 3.2. (a) Let R be a quasi-local domain, $P \in \operatorname{Spec}(R)$ and $u \in PR_P \setminus \{0\}$. Suppose that R is integrally closed in both $R[u]$ and $R[u^{-1}]$ and that $R \subseteq R[u]$ satisfies GD to P. Then $u \in P$.

(b) Let R be a domain, $P \in \operatorname{Spec}(R)$ and $u \in PR_P \setminus \{0\}$. Suppose that R is integrally closed in both $R[u]$ and $R[u^{-1}]$ and that $R \subseteq R[u]$ satisfies GD to P. Then $u \in PR_M$ for each maximal ideal M of R that contains P.

Proof. (a) By Lemma 3.1, $R \subseteq R[u^{-1}]$ also satisfies "GD to P". Therefore, it follows, as in the proof of Lemma 2.4 (a) of [5], that $1 \in PR[u^{-1}]$. Hence, $1 = p_0 + \sum_{i=1}^{n} p_i u^{-i}$ for some finite list $p_0, \dots, p_n$ of elements of P (and some $n \geq 1$). Multiplying this equation through by u^n reveals that u is integral over R. As R is integrally closed in $R[u]$, we get $u \in R \cap PR_P = P$, as asserted.

(b) Fix a maximal ideal M of R that contains P. Then $u \in PR_P = (PR_M)(R_M)_{PR_M}$. Since $R \subseteq R[u]$ satisfies GD to P, it is easy to use the lore of localizations to check that $R_M \subseteq R[u]_{R\setminus M}$ $(= R_M[u])$ satisfies GD to PR_M. Also, the lore of integrality (cf. Proposition 5.12 of [3]) leads to R_M being integrally closed in both $R[u]_{R\setminus M}$ $(= R_M[u])$ and $R[u^{-1}]_{R\setminus M}$ $(= R_M[u^{-1}])$. Hence, by applying (a) to the base ring R_M and its prime ideal PR_M, we get $u \in PR_M$. □

Corollary 3.3. *Let R be a domain, $P \in \operatorname{Spec}(R)$ and $u \in PR_P \setminus \{0\}$. Suppose that R is integrally closed in both $R[u]$ and $R[u^{-1}]$ and that P is straight with respect to $R \subseteq R[u]$. Then $u \in PR_M$ for each maximal ideal M of R that contains P.*

Proof. Since P is straight with respect to $R \subseteq R[u]$, it follows from the oft-cited exercise of Kaplansky [12] that $R \subseteq R[u]$ satisfies "GD to P". So, the assertion follows from Theorem 3.2 (b). □

Corollary 3.4. *Let R be an integrally closed domain, $P \in \operatorname{Spec}(R)$ and $u \in PR_P \setminus \{0\}$ such that P is straight with respect to $R \subseteq R[u]$. Then $u \in PR_M$ for each maximal ideal M of R that contains P.*

Proof. Apply Corollary 3.3. □

We close by showing that if the base domain R is integrally closed (but not necessarily quasi-local), it suffices to use the overrings generated by a single element of $\Omega(R,P)$ to determine whether a prime ideal P of R is locally divided.

Corollary 3.5. *Let R be an integrally closed domain and $P \in \operatorname{Spec}(R)$, such that P is straight with respect to $R \subseteq R[u]$ for all $u \in \Omega(R,P)$. Then P is a locally divided prime ideal of R.*

Proof. It suffices to show that $PR_P = PR_M$ for each maximal ideal M of R that contains P. One has, on general principles, that $PR_M \subseteq PR_P$. If the assertion fails, there exists $u \in PR_P \setminus PR_M$ for some maximal ideal M of R that contains P. Then $u \in \Omega(R,P)$, and so an application of Corollary 3.4 gives the desired contradiction, thus completing the proof. □

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