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Author(s):

Shahabaddin Ebrahimi Atani

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On S-J-filters of a bounded lattice

Shahabaddin Ebrahimi Atani
Department of Pure Mathematics, Guilan University, Rasht, Iran.
e-mail: *ebrahimi@guilan.ac.ir*

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Abstract. Let $\mathcal{L}$ be a bounded lattice and S denote the join subset of $\mathcal{L}$. In this paper, we introduce the concept of S-J-filters as a new generalization of J-filters. Many of the basic properties and characterizations of this concept are investigated. We study S-J-filters under various contexts of constructions such as the stability of S-J-join subsets in some lattice-theoretic constructions, homomorphic image lattices, quotient lattices, cartesian product lattices, S-finite filters, SJ-finite filters, SJ-Noetherian lattices and filterlization lattices.

Key Words: lattice, J-filter, S-J-filter.
2020 MSC: Primary: 06B15, 06B20; Secondary 05C25, 05C40.

1 Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. Recently, the study of algebraic structures, using the properties of lattice theory, tends to a useful research topic. Associating a lattice with an algebraic structure allows us to obtain characterizations and representations of special classes of algebraic structures in terms of lattices and vice versa (see for instance [5, 6, 8-14]).

Over the years, several types of ideals have been developed in order to let us fully understand the structures of rings in general. The notion of prime ideal plays a key role in the theory of commutative algebra, and it has been widely studied (see, for example, [1]). Recall from [2], a prime ideal P of R is a proper ideal having the property that $ab \in P$ implies either $a \in P$ or $b \in P$ for each $a, b \in R$. In [19], Mohamadian defined a proper ideal I of R as an r-ideal if whenever $a, b \in R$ with $ab \in I$ and $\text{ann}(a) = 0$ imply that $b \in I$. He investigated the behavior of r-ideals and compare them with other classical ideals such as prime and maximal ideals. In [21], Tekir et al., defined and studied some subclass of r-ideals, namely, the class of n-ideals. A proper ideal I of a ring R is called an n-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{0}$, then $b \in I$. Khashan and Bani-Ata generalized the concept of n-ideals in [17] (also see [16] and [18]). A proper ideal I of a ring R is called a J-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then $b \in I$, where $J(R)$ is the Jacobson radical of R. In [4], Abouhalaka et al. defined an ideal I of a ring R disjoint from S is called an S-J-ideal, if there exists an (a fixed) $s \in S$ such that for all $a, b \in R$, if $ab \in I$, then either $sa \in J(R)$ or $sb \in I$, where S is a multiplicative subset of R. Let $\mathcal{L}$ be a bounded distributive lattice and $J(\mathcal{L})$ denote the Jacobson radical of $\mathcal{L}$. J-filters was introduced in some detail in [12]. A proper filter P of $\mathcal{L}$ is called a J-filter if whenever $x \vee y \in P$ with $x \notin J(\mathcal{L})$, then $y \in P$ for every $x, y \in \mathcal{L}$. We say that a subset $S \subseteq \mathcal{L}$ is join if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$. In this paper, we continue this approach and present S-J-filters as a generalization of J-filters. A proper filter F of $\mathcal{L}$ disjoint from S is called an S-J-filter (resp. a weakly S-J-filter) of $\mathcal{L}$, if there exists an (a fixed) $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in F$ (resp. $1 \neq x \vee y \in P$), then either $s \vee x \in J(R)$ or $s \vee y \in F$. The S-J-filter was introduced and studied in [12] in a very special case when

$S = \{0\}$.

Among many other results in this paper, the first, preliminaries section contains elementary observations needed later on. In the third section of this paper, we collect some basic properties of S-J-filters. At first, we give the definition of S-J-filters (Definition 3.1) and we provide an example of lattices for which their S-J-filters and J-ideals are the same (Proposition 3.3). Next, we provide an example (Example 3.4) of a S-J-filter of $\mathcal{L}$ that is not a J-filter (resp. is not prime filter). Many equivalent characterizations of S-J-filters for any lattice are presented in Theorem 3.7, Theorem 3.10 and Proposition 3.13. It is proved in Proposition 3.19 that an intersection of a nonempty family of S-J-filters of $\mathcal{L}$ is an S-J-filter and we provide some condition under which the converse of Proposition 3.19 is true (Theorem 3.20). Theorem 3.22 proves that S-J-filters and S-J-primary filters that contained in the Jacobson radical are the same.

Section 4 is dedicated to the investigation of the stability of S-J-join subsets in some lattice-theoretic constructions. Let X be a nonempty subset of $\mathcal{L}$ with $\mathcal{L} \setminus J(\mathcal{L}) \subseteq X$. Then X is called an S-J-join subset of $\mathcal{L}$ if there exists $s \in S$ such that for all $x, y \in \mathcal{L}$ with $s \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s \vee y \in X$ we have $s \vee x \vee y \in X$. The concept of S-J-join subset of $\mathcal{L}$ introduced in definition 4.2 is of some fundamental importance in the subject. one crucial example of the idea is $\mathcal{L} \setminus (s \vee P)$, where P is an S-J-filter of $\mathcal{L}$ for some $s \in S$ (Theorem 4.3). Similar to the prime ideals (see [20, Theorem 3.44]), it is shown in Theorem 4.5 that if P is a filter of $\mathcal{L}$ and X is an S-J-join subset of $\mathcal{L}$ such that $s \vee X \subseteq X$ and $P \cap X = \emptyset$ fo some $s \in S$, then there exists an S-J-filter K containing P such that $K \cap X = \emptyset$. Also, similar to minimal prime ideals (see [15, Theorem 2.1]), it is proved in Theorem 4.6 that if Q is an S-J-filter of $\mathcal{L}$, P is a filter of $\mathcal{L}$ with $P \subseteq s \vee Q$ for some $s \in S$ and $s \vee Q \in \min_{SJ}(P)$, then for each $x \in s \vee Q$, there is some $y \notin s \vee Q$ such that $s \vee x \vee y \in P$ (also, see Corollary 4.8 and Corollary 4.9).

In Section 5, we investigate the properties of S-J-filters similar to prime ideals. In particular, we investigate the behavior of S-J-filters under homomorphism, in factor lattices, $\vee$-lattices, cartesian product lattices, S-finite filters, SJ-finite filters, SJ-Noetherian lattices and filterlization lattices (see Theorem 5.1, Theorem 5.3, Theorem 5.5, Proposition 5.8, Theorem 5.13 and Theorem 5.14).

2 Preliminaries

A poset $(\mathcal{L}, \leq)$ is a lattice if $\sup\{a,b\} = a \vee b$ and $\inf\{a,b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the meet and $\vee$ the join). A lattice $\mathcal{L}$ is complete when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a distributive lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in $\mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\mathcal{L}$). A non-empty subset F of a lattice $\mathcal{L}$ is called a filter, if for $a \in F$, $b \in \mathcal{L}$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\mathcal{L}$ is a lattice with 0 and 1, then $1 \in F$, $\{1\}$ is a filter of £ and $0 \in F$ if and only if $F = \mathcal{L}$). A proper filter F of $\mathcal{L}$ is called prime if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter F of $\mathcal{L}$ is said to be maximal if G is a filter in $\mathcal{L}$ such that $F \subseteq G$ with $F \neq G$, then $G = \mathcal{L}$. We define the Jacobson radical of $\mathcal{L}$, denoted by $J(\mathcal{L})$, to be the intersection of all the maximal filters of $\mathcal{L}$. A proper filter P of a lattice $\mathcal{L}$ is called a J-filter if whenever $x, y \in \mathcal{L}$ with $x \vee y \in P$ and $x \notin J(\mathcal{L})$, then $y \in P$. Assume that P is a filter of $\mathcal{L}$ and let S be a join subset of $\mathcal{L}$ disjoint with P. We say that P is an S-prime filter of $\mathcal{L}$ if there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$ if $x \vee y \in P$, then $s \vee x \in P$ or $s \vee y \in P$. Let D be a subset of a lattice $\mathcal{L}$. Then the filter generated by D, denoted by $T(D)$, is the intersection of all filters that is containing D. A filter F is called finitely generated if there is a finite subset D of F such that $F = T(D)$. A lattice $\mathcal{L}$ with 1 is called $\mathcal{L}$-domain if $a \vee b = 1$ $(a, b \in \mathcal{L})$, then $a = 1$ or $b = 1$ (so $\mathcal{L}$ is $\mathcal{L}$-domain if and only if $\{1\}$ is a prime filter of $\mathcal{L}$). If $x \in \mathcal{L}$, then a complement of x in $\mathcal{L}$ is an element $y \in \mathcal{L}$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice $\mathcal{L}$ is complemented if every element of $\mathcal{L}$ has a complement in $\mathcal{L}$. If $\mathcal{L}$ and $\mathcal{L}'$ are lattices, then a lattice homomorphism $f : \mathcal{L} \to \mathcal{L}'$ is a map from $\mathcal{L}$ to $\mathcal{L}'$ satisfying $f(x \vee y) = f(x) \vee f(y)$ and

$f(x \wedge y) = f(x) \wedge f(y)$ for $x, y \in \mathcal{L}$. Let X be a nonempty subset of $\mathcal{L}$ and $a \in \mathcal{L}$. By $a \vee X$, we mean the set $\{a \vee x : x \in X\}$. Assume that $(\mathcal{L}_1, \leq_1), (\mathcal{L}_2, \leq_2)$ are lattices and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2)$ and $x \wedge_c y = (x_1 \wedge y_1, x_2 \wedge y_2)$. In this case, we say that $\mathcal{L}$ is a decomposable lattice. An element x of $\mathcal{L}$ is called identity join of a lattice $\mathcal{L}$, if there exists $1 \neq y \in \mathcal{L}$ such that $x \vee y = 1$. The set of all identity joins of a lattice $\mathcal{L}$ is denoted by $\mathrm{I}(\mathcal{L})$. A lattice $\mathcal{L}$ is called presimplifiable if $\mathrm{I}(\mathcal{L}) \subseteq J(\mathcal{L})$. First we need the following easy observation proved in [5, 6, 8, 9].

Lemma 2.1. *Let $\mathcal{L}$ be a lattice.*

(1) A non-empty subset F of $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if $x \vee z \in F$ and $x \wedge y \in F$ for all $x, y \in F$, $z \in \mathcal{L}$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and F is a filter, $x \wedge y \in F$ gives $x, y \in F$ for all $x, y \in \mathcal{L}$.

(2) If F_1, F_2 are filters of $\mathcal{L}$ and $a \in \mathcal{L}$, then $F_1 \vee F_2 = \{f_1 \vee f_2 : f_1 \in F_1, f_2 \in F_2\}$ and $a \vee F_1 = \{a \vee f_1 : f_1 \in F_1\}$ are filters of $\mathcal{L}$ and $F_1 \vee F_2 = F_1 \cap F_2$.

(3) Assume that A is an arbitrary non-empty subset of $\mathcal{L}$ and let $a \in \mathcal{L}$. Then $T(A) = \{x \in \mathcal{L} :\ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x$ for some $a_i \in A$ $(1 \leq i \leq n)\}$ and $T(\{a\}) = T(a) = \{x \vee a : x \in \mathcal{L}\}$.

(4) If $\mathcal{L}$ is distributive, F, G are filters of $\mathcal{L}$ and $y \in \mathcal{L}$, then $(G :_{\mathcal{L}} F) = \{x \in \mathcal{L} : x \vee F \subseteq G\}$, $(F :_{\mathcal{L}} T(y)) = (F :_{\mathcal{L}} y) = \{a \in \mathcal{L} : a \vee y \in F\}$ and $(1 :_{\mathcal{L}} y) = \{x \in \mathcal{L} : x \vee y = 1\}$ are filters of $\mathcal{L}$.

(5) If $\{F_i\}_{i \in \Delta}$ is a chain of filters of $\mathcal{L}$, then $\bigcup_{i \in \Delta} F_i$ is a filter of $\mathcal{L}$.

(6) If $\mathcal{L}$ is distributive and F_1, F_2 are filters of $\mathcal{L}$, then $F_1 \wedge F_2 = \{f_1 \wedge f_2 : f_1 \in F_1, f_2 \in F_2\}$ is a filter of $\mathcal{L}$ and $F_1, F_2 \subseteq F_1 \wedge F_2$.

(7) If $f : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism with $f(1) = 1$, then $\mathrm{Ker}(f) = \{x \in \mathcal{L} : f(x) = 1\}$ *is a filter of $\mathcal{L}$.*

For undefined notations or terminologies in lattice theory, we refer the reader to [5, 6].

3 Properties of S-J-filters

In this paper, by $\mathcal{L}$ we mean a bounded distributive lattice, and S will denote a join subset of $\mathcal{L}$. We remind the reader of the following definition.

Definition 3.1. A filter P of $\mathcal{L}$ disjoint from S is called an S-J-filter of $\mathcal{L}$, if there exists an (a fixed) $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in P$, then either $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P$.

Remark 3.2. (1) Due to the symmetry between x and y, we can show that if P is an S-J-filter with $s \vee x, s \vee y \notin P$, then $s \vee x, s \vee y \in J(\mathcal{L})$.

(2) If $S = \{0\}$, then the J-filters and S-J-filters of $\mathcal{L}$ are the same.

(3) By definitions, it is clear that $J(\mathcal{L})$ is an S-J-filter of $\mathcal{L}$ if and only if $J(\mathcal{L})$ is S-prime.

(4) By definitions, it is clear that every J-filter is S-J-filter.

Referring to [12, Theorem 3.4] states that $\mathcal{L}$ is quasi-local if and only if every proper filter is a J-filter. The next result determines the class of lattices for which their J-filters and S-J-filters are the same.

Proposition 3.3. *If $\mathcal{L}$ is a quasi-local lattice with unique maximal filter M, then every proper filter of $\mathcal{L}$ is an S-J-filter.*

Proof. Let P be a proper filter of $\mathcal{L}$ and let $x, y \in \mathcal{L}$ such that $x \vee y \in P$ and $s \vee x \notin J(\mathcal{L}) = M$ for $s \in S$. Then $M \wedge T(s \vee x) = \mathcal{L}$ by maximality of M. If $T(s \vee x) \neq \mathcal{L}$, then $T(s \vee x) \subseteq M$ by [11, Lemma 2.1] gives $M = \mathcal{L}$, a contradiction. Therefore, $T(s \vee x) = \mathcal{L}$. So $0 = s \vee x \vee z$ for some $z \in \mathcal{L}$. Then $x = 0$ gives $y \in P$ and so $s \vee y \in P$. Thus, P is an S-J-filter. □

Example 3.4. (1) Let $\mathcal{L}_1 = \{0, a, b, c, 1\}$ be a lattice with the relations $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$. Consider $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_1$, $P = \{1, c\} \times \{1\} = \{(1,1), (c,1)\}$ and $S = \{0, c\} \times \{0, c\} = \{(0,0), (0,c), (c,0), (c,c)\}$. Then P is a filter of $\mathcal{L}$ with $P \cap S = \emptyset$, $J(\mathcal{L}_1) = \{a, c, 1\} \cap \{b, c, 1\} = \{1, c\}$ and $J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_1) = \{1, c\} \times \{1, c\} = \{(1,1), (1,c), (c,1), (c,c)\}$. Now, we show that P is an S-J-filter. Let $(x, y) \vee_c (x', y') = (x \vee x', y \vee y) \in P$ for some $(x, y), (x', y') \in \mathcal{L}$. Then by the definition of $\mathcal{L}_1$, $y \vee y' = 1$ gives either $y = 1$ or $y' = 1$. Without loss of generality, we can assume that $y = 1$. Then $(c, c) \vee_c (x, 1) = (c \vee x, 1) \in J(\mathcal{L})$ since $c \vee x = c$ or $c \vee x = 1$. Therefore, P is an S-J-filter of $\mathcal{L}$.

On the other hand, P is not a J-filter since $(a, b) \vee (b, 1) = (c, 1) \in P$ but neither $(a, b) \in P$ nor $(b, 1) \in J(\mathcal{L})$. Thus an S-J-filter need not be a J-filter.

(2) The collection of ideals of $\mathbb{Z}$, the ring of integers, form a lattice under set inclusion which we shall denote by $\mathcal{L}$ with respect to the following definitions: $m\mathbb{Z} \vee n\mathbb{Z} = (m, n)\mathbb{Z}$ and $m\mathbb{Z} \wedge n\mathbb{Z} = [m, n]\mathbb{Z}$ for all ideals $m\mathbb{Z}$ and $n\mathbb{Z}$ of $\mathbb{Z}$, where (m, n) and $[m, n]$ are greatest common divisor and least common multiple of m, n, respectively. Note that $\mathcal{L}$ is a distributive complete lattice with least element the zero ideal and the greatest element $\mathbb{Z}$. By [8, Theorem 2.9 (ii)], $\mathcal{L} \setminus \{0\}$ is the only maximal filter of $\mathcal{L}$ and so $\mathcal{L}$ is a quasi-local lattice. It follows from Proposition 3.3 that every proper filter of $\mathcal{L}$ is an S-J-filter. Consider $P = \{\mathbb{Z}, 2\mathbb{Z}, 4\mathbb{Z}, \cdots\}$. Since $5\mathbb{Z} \vee 7\mathbb{Z} = \mathbb{Z} \in P$ with $5\mathbb{Z}, 7\mathbb{Z} \notin F$, we infer that P is not a prime (maximal) filter.

Proposition 3.5. *If P is an S-J-filter of $\mathcal{L}$, then $s \vee P \subseteq J(\mathcal{L})$ for some $s \in S$. In particular, if $J(\mathcal{L}) = \{1\}$, then $s \vee P = \{1\}$ for some $s \in S$.*

Proof. We keep in mind that there exists a fixed $s \in S$ that satisfies the S-J-filter condition. On the contrary, assume that $s \vee P$ is not a subset of $J(\mathcal{L})$. Then there exists $p \in P$ such that $s \vee p \notin J(\mathcal{L})$. Now $p = p \vee 0 \in P$ and $s \vee p \notin J(\mathcal{L})$ implies that $s = s \vee 0 \in P \cap S$, a contradiction. Hence, $s \vee P \subseteq J(\mathcal{L})$. The "in particular" statement is clear. □

Corollary 3.6. *If P is a J-filter of $\mathcal{L}$, then $P \subseteq J(\mathcal{L})$. In particular, if $J(\mathcal{L}) = \{1\}$, then $P = \{1\}$.*

Proof. Take $S = \{0\}$ in Proposition 3.5. □

In the following theorem we give one other characterizations of S-J-filters.

Theorem 3.7. If P is a filter of $\mathcal{L}$ with $S \cap P = \emptyset$, then the following assertions are equivalent:

(1) P is an S-J-filter of $\mathcal{L}$;

(2) There exists $s \in S$ such that for all F, G two filters of $\mathcal{L}$, if $F \vee G \subseteq P$, then either $s \vee F \subseteq J(\mathcal{L})$ or $s \vee G \subseteq P$.

Proof. (1) $\Rightarrow$ (2) Since P is an S-J-filter, we infer that there is an element $s \in S$ such that for all $a, b \in \mathcal{L}$, if $a \vee b \in P$, then $s \vee a \in J(\mathcal{L})$ or $s \vee b \in P$. On the contrary, assume that for all $t \in S$, there are F_t, G_t two filters of $\mathcal{L}$ with $F_t \vee G_t \subseteq P$, but $t \vee F_t \not\subseteq J(\mathcal{L})$ and $t \vee G_t \not\subseteq P$. So there exist F_s, G_s two filters of $\mathcal{L}$ with $F_s \vee G_s \subseteq P$, but $s \vee F_s \not\subseteq J(\mathcal{L})$ and $s \vee G_s \not\subseteq P$, as $s \in S$. This shows that there exist $a_s \in F_s$ and $b_s \in G_s$ such that $a_s \vee b_s \in P$, $s \vee a_s \notin J(\mathcal{L})$ and $s \vee b_s \notin P$ which is impossible, as P is an S-J-filter, i.e. (2) holds.

(2) $\Rightarrow$ (1) Let $a, b \in \mathcal{L}$ such that $a \vee b \in P$. Set $F = T(a)$ and $G = T(b)$. Then $F \vee G \subseteq P$ gives there exits $s \in S$ such that $s \vee a \in s \vee F \subseteq J(\mathcal{L})$ or $s \vee b \in s \vee G \subseteq P$ by (2), i.e. (1) holds. □

Corollary 3.8. *If P is a filter of $\mathcal{L}$, then P is a J-filter of $\mathcal{L}$ if and only if for all filters F, G of $\mathcal{L}$, $F \vee G \subseteq P$ implies that $F \subseteq J(\mathcal{L})$ or $G \subseteq P$.*

Proof. Take $S = \{0\}$ in Theorem 3.7. □

Proposition 3.9. *Let P be a filter of $\mathcal{L}$ disjoint from S. If there exists an $s \in S$ such that $(P :_{\mathcal{L}} s)$ is a J-filter, then P is an S-J-filter of $\mathcal{L}$.*

Proof. Let $x \vee y \in P$ for some $x, y \in \mathcal{L}$. Then $x \vee y \in (P :_{\mathcal{L}} s)$ gives $x \in J(\mathcal{L})$ or $y \in (P :_{\mathcal{L}} s)$ which implies that $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P$., as needed. □

The following theorem provides some condition under which the converse of Proposition 3.9 is true.

Theorem 3.10. Suppose that P is a filter of $\mathcal{L}$ disjoint from S and $J(\mathcal{L})$ is a J-filter of $\mathcal{L}$ such that $J(\mathcal{L}) \cap S = \emptyset$. Then P is an S-J-filter of $\mathcal{L}$ if and only if there exists $s \in S$ such that $(P :_{\mathcal{L}} s)$ is a J-filter of $\mathcal{L}$.

Proof. One side is clear by Proposition 3.9. To see the other side, assume that P is an S-J-filter of $\mathcal{L}$. Suppose that $s \in S$ satisfies S-J-filter condition for P. Let $x \vee y \in (P :_{\mathcal{L}} s)$ for some $x, y \in \mathcal{L}$ and $x \notin J(\mathcal{L})$. Then $x \vee (s \vee y) \in P$ gives either $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P$. If $s \vee x \in J(\mathcal{L})$, then $x \in J(\mathcal{L})$ since $J(\mathcal{L}) \cap S = \emptyset$, a contradiction. Therefore, $s \vee y \in P$ and so $y \in (P :_{\mathcal{L}} s)$. □

Proposition 3.11. *Let P be a filter of $\mathcal{L}$ disjoint from S. Then, P is an S-J-filter if and only if for some $s \in S$, $(P :_{\mathcal{L}} a) \subseteq (J(\mathcal{L}) :_{\mathcal{L}} s)$ for all $a \notin (P :_{\mathcal{L}} s)$.*

Proof. Suppose that P is an S-J-filter of $\mathcal{L}$. Let $s \in S$ satisfies S-J-filter condition for P. Now, let $x \in (P :_{\mathcal{L}} a)$, where $a \notin (P :_{\mathcal{L}} s)$ (so $a \vee s \notin P$). Then by assumption, $a \vee x \in P$ gives $s \vee x \in J(\mathcal{L})$, i.e. $x \in (J(\mathcal{L}) :_{\mathcal{L}} s)$. Conversely, assume that $x \vee y \in P$ for some $x, y \in \mathcal{L}$. If $s \vee y \notin P$ (i.e. $y \notin (P :_{\mathcal{L}} s)$), then by the hypothesis, $x \in (P :_{\mathcal{L}} y) \subseteq (J(\mathcal{L}) :_{\mathcal{L}} s)$. This shows that $s \vee x \in J(\mathcal{L})$. □

Corollary 3.12. *P is a J-filter of $\mathcal{L}$ if and only if $(P :_{\mathcal{L}} a) \subseteq J(\mathcal{L})$ for all $a \notin P$.*

Proof. Take $S = \{0\}$ in Proposition 3.11. □

Proposition 3.13. *Let P be a filter of $\mathcal{L}$ disjoint from S. Then, P is an S-J-filter if and only if for some $s \in S$, $(P :_{\mathcal{L}} a) \subseteq (P :_{\mathcal{L}} s)$ for all $a \notin (J(\mathcal{L}) :_{\mathcal{L}} s)$.*

Proof. Suppose that P is an S-J-filter of $\mathcal{L}$ and let $s \in S$ satisfies S-J-filter condition for P. Now, let $x \in (P :_{\mathcal{L}} a)$, where $a \notin (J(\mathcal{L}) :_{\mathcal{L}} s)$ (so $s \vee a \notin J(\mathcal{L})$). Then by assumption, $a \vee x \in P$ gives $s \vee x \in P$, i.e. $x \in (P :_{\mathcal{L}} s)$. Conversely, assume that $x \vee y \in P$ for some $x, y \in \mathcal{L}$. If $s \vee x \notin J(\mathcal{L})$ (i.e. $x \notin (J(\mathcal{L}) :_{\mathcal{L}} s)$), then by the hypothesis, $y \in (P :_{\mathcal{L}} x) \subseteq (P :_{\mathcal{L}} s)$. This shows that $s \vee y \in P$. □

Corollary 3.14. *P is a J-filter of $\mathcal{L}$ if and only if $(P :_{\mathcal{L}} a) = P$ for all $a \notin J(\mathcal{L})$.*

Proof. Take $S = \{0\}$ in Proposition 3.13. □

Corollary 3.15. *If P is a proper filter of a lattice $\mathcal{L}$, then the following statements are equivalent:*
(1) P is a J-filter of $\mathcal{L}$;
(2) $P = (P :_{\mathcal{L}} x)$ for every $x \notin J(\mathcal{L})$;
(3) For filters G and K of $\mathcal{L}$, $G \vee K \subseteq P$ and G is not subset of $J(\mathcal{L})$ gives $K \subseteq P$;
(4) $(P :_{\mathcal{L}} x) \subseteq J(\mathcal{L})$ for every $x \notin P$.

Proof. This is a direct consequence of Corollary 3.8, Corollary 3.12 and Corollary 3.14. □

Proposition 3.16. *If P is an S-J-filter of $\mathcal{L}$ and $a \notin P$, then $(P :_{\mathcal{L}} a)$ is an S-J-filter.*

Proof. Since P is an S-J-filter, we conclude that there is an element $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in P$, then $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P$. It suffices to show that there exists $u \in S$ such that for all filters F, G of $\mathcal{L}$, if $F \vee G \subseteq (P :_{\mathcal{L}} a)$, then either $u \vee F \subseteq J(\mathcal{L})$ or $u \vee G \subseteq (P :_{\mathcal{L}} a)$ by Theorem 3.7. On the contrary, assume that for all $t \in S$, there are F_t, G_t two filters of $\mathcal{L}$ with $F_t \vee G_t \subseteq P$, but $t \vee F_t$ is not a subset of $J(\mathcal{L})$ and $t \vee G_t$ is not a subset of $(P :_{\mathcal{L}} a)$. So there exist F_s, G_s two filters of $\mathcal{L}$ with $F_s \vee G_s \subseteq (P :_{\mathcal{L}} a)$, but $s \vee F_s$ is not a subset of $J(\mathcal{L})$ and $s \vee G_s$ is not a subset of $(P :_{\mathcal{L}} a)$, as $s \in S$. This shows that there exist $a_s \in F_s$ and $b_s \in G_s$ such that $a_s \vee b_s \vee a \in P$, $s \vee a_s \notin J(\mathcal{L})$ and $s \vee (b_s \vee a) \notin P$ which is impossible, as P is an S-J-filter, i.e. $(P :_{\mathcal{L}} a)$ is an S-J-filter. □

An S-J-filter P of $\mathcal{L}$ is called a maximal S-J-filter if there is no S-J-filter which contains P properly.

Theorem 3.17. If P is a maximal S-J-filter of $\mathcal{L}$, then P is a prime filter. If, in particular, $P = (J(\mathcal{L}) :_{\mathcal{L}} s)$ for some $s \in S$, then the converse is true.

Proof. Suppose that P is a maximal S-J-filter and let $x, y \in \mathcal{L}$ such that $x \vee y \in P$ and $x \notin P$. Then $(P :_{\mathcal{L}} x)$ is S-J-filter by Proposition 3.15 and $P \subseteq (P :_{\mathcal{L}} x)$. By Maximality of P, we have $y \in (P :_{\mathcal{L}} x) = P$ and hence P is a pime. Suppose that $P = (J(\mathcal{L}) :_{\mathcal{L}} s)$ for some $s \in S$ and let $a \vee b \in P$ for some $a, b \in \mathcal{L}$. Since P is prime, we conclude that either $s \vee a \in P$ or $s \vee b \in J(\mathcal{L})$. So P is an S-J-filter of $\mathcal{L}$. Let Q be any S-J-filer of $\mathcal{L}$. Then by Proposition 3.5, $Q \subseteq (J(\mathcal{L}) :_{\mathcal{L}} s)$, which shows the maximality of P. □

Theorem 3.18. If $J(\mathcal{L}) = \{1\}$, then the following statements are equivalent:

(1) $\mathcal{L}$ is an $\mathcal{L}$-domain;

(2) The filter $\{1\}$ is the only S-J-filter of $\mathcal{L}$;

(3) $(1 :_{\mathcal{L}} x \vee y) = (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$ for every $x, y \in \mathcal{L}$.

Proof. (1) $\Rightarrow$ (2) If P is a S-J-filter of $\mathcal{L}$, then $s \vee P = \{1\}$ for some $s \in S$ by Proposition 3.5. If $p \in P$, then $s \vee p = 1$ gives $p = 1$ and so $P = \{1\}$, as $\mathcal{L}$ is a $\mathcal{L}$-domain. Clearly, $P = \{1\}$ is an S-J-filter.

(2) $\Rightarrow$ (3) Assume that $s \in S$ satisfies S-J-filter condition for $\{1\}$. At first, we show that $(1 :_{\mathcal{L}} z)$ is a S-J-filter for each $1 \neq z \in \mathcal{L}$. Let $a, b \in \mathcal{L}$ such that $a \vee b \in (1 :_{\mathcal{L}} z)$ and $s \vee a \notin J(\mathcal{L})$. Then $\{1\}$ is an S-J-filter gives $s \vee b \vee z = 1$; hence $s \vee b \in (1 :_{\mathcal{L}} z)$. Therefore, by our hypothesis, we have $(1 :_{\mathcal{L}} z) = \{1\}$ for each $1 \neq z \in \mathcal{L}$. This immediately implies that $(1 :_{\mathcal{L}} x \vee y) = (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$ for every $x, y \in \mathcal{L}$.

(3) $\Rightarrow$ (1) Let $x \vee y = 1$, where $x, y \in \mathcal{L}$. Then $\mathcal{L} = (1 :_{\mathcal{L}} x \vee y) = (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$ implies that $0 \in (1 :_{\mathcal{L}} x) \cup (1 :_{\mathcal{L}} y)$. This means that $x = 1$ or $y = 1$, i.e. (1) holds. □

Proposition 3.19. *If $\{P_i\}_{i \in \Delta}$ is a nonempty family of S-J-filters of $\mathcal{L}$, then $P = \cap_{i \in \Delta} P_i$ is an S-J-filter of $\mathcal{L}$.*

Proof. Clearly, $P \cap S = \emptyset$. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P$ and $s \vee x \notin J(\mathcal{L})$ for some $s \in S$. Then $a \vee b \in P_i$ for each $i \in \Delta$ gives $s \vee b \in P_i$ for all $i \in \Delta$, as they are S-J-filters; hence $s \vee b \in P$ and so the result holds. □

The following theorem provides some condition under which the converse Proposition 3.19 is true.

Theorem 3.20. Let $P_1, \cdots, P_n$ be prime filters of $\mathcal{L}$ which are not comparable. Then $\cap_{i=1}^{n} P_i$ is an S-J-filter of $\mathcal{L}$ if and only if P_i is an S-J-filter for $i \in \{1, \cdots, n\}$.

Proof. One side is clear by Proposition 3.19. To see the other side, assume that $s \in S$ satisfies S-J-filter condition for $\cap_{i=1}^{n} P_i$. Let $x \vee y \in P_i$ for some $x, y \in \mathcal{L}$ and take $z \in (\cap_{i \neq j} P_j) \setminus P_i$. Therefore, $x \vee y \vee z \in \cap_{i=1}^{n} P_i$. Since $\cap_{i=1}^{n} P_i$ is an S-J-filter, we conclude that either $s \vee x \in J(\mathcal{L})$ or $s \vee y \vee z \in \cap_{i=1}^{n} P_i$ and so $s \vee y \vee z \in P_i$. This implies that $s \vee y \in P_i$, i.e. P_i is a S-J-filter of $\mathcal{L}$. □

For a filter P of a lattice $\mathcal{L}$, the Jacobson radical of P, denoted by $J(P)$, is defined as the intersection of all maximal filters of $\mathcal{L}$ containing P.

Definition 3.21. A filter P of $\mathcal{L}$ with $P \cap S = \emptyset$ is called an S-J-primary, if there exists an (a fixed) $s \in S$ such that for all $x, y \in \mathcal{L}$, if $x \vee y \in P$, then either $s \vee x \in J(P)$ or $s \vee y \in P$.

In the following result, we show that S-J-filters and S-J-primary filters that contained in $J(\mathcal{L})$ are the same.

Theorem 3.22. If P is a proper filter of a lattice $\mathcal{L}$ such that $P \subseteq J(\mathcal{L})$, then P is an S-J-filter if and only if P is S-J-primary.

Proof. Suppose P is an S-J-filter of $\mathcal{L}$. Assume that $s \in S$ satisfies S-J-filter condition and let $x \vee y \in P$ with $s \vee x \notin J(P)$. Since $J(\mathcal{L}) \subseteq J(P)$, we conclude thats $\vee x \notin J(\mathcal{L})$. Now, P is an S-J-filter gives $s \vee y \in P$ and so P is an S-J-primary filter of $\mathcal{L}$. Conversely, assume that P is a S-J-primary filter of $\mathcal{L}$ and let $a \vee b \in P$ with $s \vee a \notin J(\mathcal{L})$ for some $s \in S$. Now, $P \subseteq J(\mathcal{L})$ implies that $J(P) \subseteq J(J(\mathcal{L}) = J(\mathcal{L})$ by [12, Lemma 3.17]. Therefore, $s \vee a \notin J(P)$ and so $s \vee b \in P$, as P is S-J-primary. Hence, P is an S-J-filter. □

4 properties of S-J-join subsets

We continue this section with the investigation of the stability of S-J-join subsets in some ring-theoretic constructions. Let us begin this section with the following lemma.

Lemma 4.1. *If P is an S-J-filter of $\mathcal{L}$, then $s \vee P$ is an S-J-filter of $\mathcal{L}$ for some $s \in S$.*

Proof. If $s \vee x \in s \vee P$ for some $x \in P$, then $s \vee x \in P$, as P is a filter; hence $s \vee P \subseteq P$. Suppose that $s \in S$ satisfies S-J-filter condition for P and let $a \vee b \in s \vee P \subseteq P$ for some $a, b \in \mathcal{L}$. Since P is an S-J-filter, we conclude that either $s \vee a \in J(\mathcal{L})$ or $s \vee b \in P$ which implies that either $s \vee a \in J(\mathcal{L})$ or $s \vee b = s \vee (s \vee b) \in s \vee P$. Therefore, $s \vee P$ is an S-J-filter of $\mathcal{L}$. □

Definition 4.2. Let $\mathcal{T}$ be a nonempty subset of $\mathcal{L}$ with $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{T}$. Then $\mathcal{T}$ is called a S-J-join subset of $\mathcal{L}$ if there exists $s \in S$ such that for all $x, y \in \mathcal{L}$ with $s \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s \vee y \in \mathcal{T}$ we have $s \vee x \vee y \in \mathcal{T}$.

In the following proposition, we describe the relation between S-J-filters and S-J-join-subsets of $\mathcal{L}$.

Theorem 4.3. If P is a proper filter of $\mathcal{L}$, then the following statements are equivalent:
(1) P is an S-J-filter of $\mathcal{L}$;
(2) $\mathcal{L} \setminus (s \vee P)$ is a J-join-subset of $\mathcal{L}$ for some $s \in S$.

Proof. (1) $\Rightarrow$ (2) We keep in mind that there exists a fixed $s \in S$ that satisfies the S-J-filter condition for P. Since $s \vee P \subseteq J(\mathcal{L})$ by Proposition 3.4, we conclude that $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{L} \setminus (s \vee P)$. Let $s \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s \vee y \in \mathcal{L} \setminus (s \vee P)$. We show that $s \vee x \vee y \in \mathcal{L} \setminus (s \vee P)$. On the contrary, assume that $s \vee x \vee y \in s \vee P$. Since $s \vee P$ is an S-J-filter by Lemma 4.1 and $s \vee x \notin J(\mathcal{L})$, we infer that $s \vee y \in s \vee P$ which is a contradiction. Thus, $s \vee x \vee y \in \mathcal{L} \setminus (s \vee P)$ and so $\mathcal{L} \setminus (s \vee P)$ is an S-J-join subset of $\mathcal{L}$.

(2) $\Rightarrow$ (1) Let $x, y \in \mathcal{L}$ and $x \vee y \in P$ (so $s \vee x \vee y \in s \vee P$) with $s \vee x \notin J(\mathcal{L})$. Then we have $s \vee y \in s \vee P \subseteq P$ since otherwise we would have $s \vee x \vee y \in \mathcal{L} \setminus (s \vee P)$ which is impossible by (2), i.e. (1) holds. □

Proposition 4.4. *For S-J-filters $P_1, \cdots, P_n$ of $\mathcal{L}$, let $s_i \in S$ satisfies S-J-filter condition for P_i, where $i \in \{1, \cdots, n\}$. Then $\mathcal{T} = \cap_{i=1}^{n} (\mathcal{L} \setminus (s_i \vee P_i))$ is an S-J-join subset of $\mathcal{L}$.*

Proof. It is clear that $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{T}$. Set $s = \vee_{i=1}^{n} s_i$. Let $x, y \in \mathcal{L}$ such that $s \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s \vee y \in \mathcal{T}$. Then for each $i \in \{1, \cdots, n\}$, $s_i \vee x \in \mathcal{L} \setminus J(\mathcal{L})$ and $s_i \vee y \in \mathcal{L} \setminus (s_i \vee P_i)$, as $J(\mathcal{L})$, $s_i \vee P_i$ are filters of $\mathcal{L}$. This shows that $s_i \vee x \vee y \in \mathcal{L} \setminus (s_i \vee P_i)$ for each $i \in \{1, \cdots, n\}$ by Proposition 4.3. We claim that $s \vee x \vee y = ((\vee_{i \neq j}^{n} s_j) \vee x) \vee (s_i \vee y) \in \mathcal{L} \setminus (s_i \vee P_i)$ for all $i \in \{1, \cdots, n\}$. Otherwise, $s_i \vee P_i$ is an S-J-filter gives either $s \vee x \in J(\mathcal{L})$ or $s_i \vee y \in s_i \vee P_i$ which is impossible. This implies that $s \vee x \vee y \in \mathcal{T}$, as needed. □

The following Theorem is a lattice counterpart of Theorem 3.44 in [20] describing the structure of S-J-filters.

Theorem 4.5. Suppose P is a filter of $\mathcal{L}$ and let $\mathcal{T}$ be a S-J-join subset of $\mathcal{L}$ such that $s \in S$ satisfies S-J-join subset condition for $\mathcal{T}$, $s \vee \mathcal{T} \subseteq \mathcal{T}$ and $P \cap \mathcal{T} = \emptyset$. Then there exists an S-J-filter Q containing P such that $Q \cap \mathcal{T} = \emptyset$.

Proof. Let Δ be the set of all filters G of $\mathcal{L}$ such that $P \subseteq G$ and $G \cap \mathcal{T} = \emptyset$. Since $P \in \Delta$, we conclude that $\Delta \neq \emptyset$. Clearly, $(\Delta, \subseteq)$ is a partial order. Let $\{P_i\}_{i \in \Lambda}$ be a chain of elements of Δ and set $H = \bigcup_{i \in \Lambda} P_i$. It is clear that H is a filter of $\mathcal{L}$ and $H \in \Delta$. Thus H is an upper bound of the chain $\{P_i\}_{i \in \Lambda}$; hence by Zorn's Lemma, Δ has a maximal element Q containing P such that $Q \cap \mathcal{T} = \emptyset$. It remains to show that Q is an S-J-filter of $\mathcal{L}$. Assume that $s \in S$ satisfies S-J-join subset condition for $\mathcal{T}$. On the contrary, let $x \vee y \in Q$ such that $s \vee x \notin J(\mathcal{L})$ and $s \vee y \notin Q$ (so $y \notin Q$). Since $Q \subsetneq (Q :_{\mathcal{L}} x)$, we infer that $(Q :_{\mathcal{L}} x) \cap \mathcal{T} \neq \emptyset$ by maximality of Q and so there exists $t \in \mathcal{T}$ with $s \vee t \in s \vee \mathcal{T} \subseteq \mathcal{T}$ such that $t \vee x \in Q$. Since $s \vee x \notin J(\mathcal{L})$ and $s \vee t \in \mathcal{T}$, we conclude that $s \vee t \vee x \in Q \cap \mathcal{T}$ which is a contradiction. Therefore, Q is an S-J-filter of $\mathcal{L}$ with $Q \cap \mathcal{T} = \emptyset$ and $P \subseteq Q$. □

If P is a filter in $\mathcal{L}$, the set of all minimal S-J-filters (resp. the set of all minimal J-filters) over P will be denoted by $\min_{SJ}(P)$ (resp. $\min_J(P)$) (set $\min_{SJ}(\{1\}) = \min_{SJ}(\mathcal{L})$ and $\min_J(\{1\}) = \min_J(\mathcal{L})$)).

The following theorem is a weakly S-J-filters counterpart of Theorem 2.1 in [15] describing the structure of such filters.

Theorem 4.6. Suppose that Q is an S-J-filter of $\mathcal{L}$ such that $s \in S$ satisfies S-J-filter condition for Q and let P be a filter of $\mathcal{L}$ with $P \subseteq s \vee Q$. If $s \vee Q \in \min_{SJ}(P)$, then for each $x \in s \vee Q$, there is some $y \notin s \vee Q$ such that $s \vee x \vee y \in P$.

Proof. Set $X = \mathcal{L} \setminus (s \vee Q)$. At first, we show that $s \vee X \subseteq X$. Let $s \vee x \in s \vee X$ for some $x \in X$ but $s \vee x \notin X$ (so $s \vee x \in s \vee Q$). Since $\mathcal{L} \setminus J(\mathcal{L}) \subseteq X$ by Proposition 3.4, then $s \vee x \notin J(\mathcal{L})$. Now, $(s \vee x) \vee 0 \in s \vee Q$ and $s \vee (s \vee x) = s \vee x \notin J(\mathcal{L})$ gives $s = s \vee 0 \in (s \vee Q) \cap S$, a contradiction, as $s \vee Q$ is an S-J-filter by Lemma 4.1. So $s \vee X \subseteq X$.

Next, we prove that $\mathcal{L} \setminus (s \vee Q)$ is a S-J-join subset that is maximal with $(\mathcal{L} \setminus (s \vee Q)) \cap P = \emptyset$. Since $(\mathcal{L} \setminus (s \vee Q)) \cap P = \emptyset$, the set Δ of all S-J-join subsets, say $\mathcal{T}'$, with $\mathcal{T}' \cap P = \emptyset$ is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ. Now Δ is easily seen to be inductive under inclusion, so by Zorn's Lemma Δ has a maximal element $\mathcal{T}$ with $P \cap \mathcal{T} = \emptyset$ and $s \vee \mathcal{T} \subseteq \mathcal{T}$. By Theorem 4.5, there is a filter K of $\mathcal{L}$ containing P that is maximal with respect to being disjoint from $\mathcal{T}$ which is S-J-filter. Note that $K \cap (\mathcal{L} \setminus (s \vee Q)) = \emptyset$ which implies that $K = Q$. Hence, $\mathcal{T} = \mathcal{L} \setminus (s \vee Q)$. Therefore, $\mathcal{L} \setminus (s \vee Q)$ has the stated property. Let $1 \neq a \in Q$ and set

$$\mathcal{T} = \{s \vee b) \vee (\wedge_{j=1}^{n} a) : s \vee b \in \mathcal{L} \setminus (s \vee Q),\ n = 0, 1, \cdots\}$$

(note that $\wedge_{i=1}^{0} a$ is interpreted as 0, and clearly, $\wedge_{i=1}^{n} a = a$). Since Q is a S-J-filter of $\mathcal{L}$, we conclude that $\mathcal{L} \setminus (s \vee Q)$ is a S-J-join subset of $\mathcal{L}$ by Proposition 3.7; so $\mathcal{L} \setminus J(\mathcal{L}) \subseteq \mathcal{L} \setminus (s \vee Q) \subseteq \mathcal{T}$. Let $u, v \in \mathcal{L}$ such that $s \vee u \notin J(\mathcal{L})$ and $s \vee v \in \mathcal{T}$. We show that $s \vee u \vee v \in \mathcal{T}$. By the hypothesis, $s \vee v = (s \vee b) \vee a$ for some $s \vee b \notin s \vee Q$ which implies that $s \vee u \vee v = (s \vee b \vee u) \vee a$. Then $s \vee b \vee u \notin s \vee Q$ (otherwise, either $s \vee u \in J(\mathcal{L})$ or $s \vee b \in s \vee Q$ by Lemma 4.1, a contradiction) and so $s \vee u \vee v \in \mathcal{T}$. Then $\mathcal{T}$ is an S-J-join subset that properly contains $\mathcal{L} \setminus (s \vee Q)$; so $P \cap \mathcal{T} \neq \emptyset$ by maximality of $\mathcal{L} \setminus s \vee Q$. So there is some $s \vee b \in \mathcal{L} \setminus (s \vee Q)$ such that $s \vee a \vee b \in P$. □

Corollary 4.7. *Suppose that Q is an S-J-filter of $\mathcal{L}$ such that $s \in S$ satisfies S-J-filter condition for Q. If $s \vee Q \in \min_{SJ}(P)$, then for each $x \in s \vee Q$, there is some $y \notin s \vee Q$ such that $s \vee x \vee y = 1$.*

Proof. Take $P = \{1\}$ in Theorem 4.6. □

Corollary 4.8. *Suppose that Q is a J-filter of $\mathcal{L}$ and let P be a filter of $\mathcal{L}$ with $P \subseteq s \vee Q$. If $Q \in \min_J(P)$, then for each $x \in Q$, there is some $y \notin Q$ such that $x \vee y \in P$.*

Proof. Take $S = \{0\}$ in Theorem 4.6. □

Corollary 4.9. *Suppose that Q is a J-filter of $\mathcal{L}$. If $Q \in \min_J(\mathcal{L})$, then for each $x \in Q$, there is some $y \notin Q$ such that $x \vee y = 1$.*

Proof. Take $S = \{0\}$ in Corollary 4.7. □

Theorem 4.10. Suppose P is an S-J-filter of a presimplifiable lattice $\mathcal{L}$ and let $s \in S$ satisfies S-J-filter condition for P. If P is prime, $s \vee P \in \min_{SJ}(\mathcal{L})$ and $c \in \mathcal{L}$ has a complement in $\mathcal{L}$, then $Q = (s \vee P) \wedge (1 :_{\mathcal{L}} c)$ is an S-J-filter.

Proof. Let $x,y \in \mathcal{L}$ such that $x \vee y \in Q$ with $s \vee x \notin J(\mathcal{L})$. Then $x \vee y = p \wedge b$ for some $p \in s \vee P$ and $b \vee c = 1$. By Corollary 4.7, there exists $q \notin s \vee P$ such that $s \vee p \vee q = 1$. Therefore, $(s \vee x) \vee (c \vee q \vee y) = (c \vee s \vee q) \vee (x \vee y) = (c \vee s \vee q) \vee (p \wedge b) = (c \vee s \vee q \vee p) \wedge (c \vee s \vee q \vee b) = 1 \vee 1 = 1$. Since $s \vee x \notin J(\mathcal{L})$, we conclude that $s \vee x \notin \mathrm{I}(\mathcal{L})$; so $c \vee q \vee y = 1 \in P$ which implies that $c \vee y \in P$, as P is prime. By assumption, there is an element $c' \in \mathcal{L}$ such that $c \vee c' = 1$ and $c \wedge c' = 0$. Now, $s \vee y = (s \vee y) \vee (c \wedge c') = (s \vee y \vee c) \wedge (s \vee y \vee c') \in Q$, and therefore Q is an S-J-filter. □

5 S-J-filters in related lattices

We begin this section with the investigation of the stability of S-J-filters in various lattice-theoretic constructions.

Theorem 5.1. Let $\mathcal{L}$ be a complemented lattice. If $f : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism such that $f(0) = 0$ and $f(1) = 1$, then the following hold:

(1) If f is epimorphism and P is an S-J-filter of $\mathcal{L}$ with $\ker(f) \subseteq P$, then $f(P)$ is an $f(S)$-J-filter of $\mathcal{L}'$.

(2) If K is an $f(S)$-J-filter of $\mathcal{L}'$ with $\ker(f) \subseteq J(\mathcal{L})$, then $f^{-1}(K)$ is an S-J-filter.

Proof. At first, we need to prove that $f(S) \cap f(P) = \emptyset$. Otherwise, there are elements $p \in P$ and $t \in S$ such that $f(p) = f(t)$. By assumption, $p \vee p' = 1$ and $p \wedge p' = 0$ for some $p' \in \mathcal{L}$. Since $f(t \vee p') = f(t) \vee f(p') = f(p \vee p') = 1$, we conclude that $t \vee p' \in \ker(f) \subseteq P$ which implies that $t = t \vee (p \wedge p') = (t \vee p) \wedge (t \vee p') \in P \cap S$, as P is a filter, a contradiction. Also, it is clear that $f(S)$ is a join subset of $\mathcal{L}'$.

(1) Let $x,y \in \mathcal{L}'$ such that $x \vee y \in f(P)$. Since f is an epimorphism, there exist $a,b \in \mathcal{L}$ such that $x = f(a)$ and $y = f(b)$. Then $x \vee y = f(a \vee b) \in f(P)$ and so $f(a \vee b) = f(c)$ for some $c \in P$. By the hypothesis, $c \vee c' = 1$ and $c \wedge c' = 0$ for some $c' \in \mathcal{L}$. Since $f(a \vee b \vee c') = f(a \vee b) \vee f(c') = 1$, we conclude that $a \vee b \vee c' \in \ker(f) \subseteq P$; hence $a \vee b = (a \vee b) \vee (c \wedge c') = (a \vee b \vee c) \wedge (a \vee b \vee c') \in P$, as P is a filter. Then there exists $s \in S$ such that either $s \vee a \in J(\mathcal{L})$ or $s \vee b \in P$, as P is an S-J-filter. Therefore, either $f(s) \vee x = f(s \vee a) \in f(J(\mathcal{L})) \subseteq J(\mathcal{L}')$ or $f(s) \vee y = f(s \vee b) \in f(P)$, as required.

(2) Since K is an $f(S)$-J-filter of $\mathcal{L}'$, we infer that there exists $s \in S$ such that for all $x', y' \in \mathcal{L}'$ with $x' \vee y' \in K$ we have $f(s) \vee x' \in J(\mathcal{L}')$ or $f(s) \vee y' \in K$. Now, let $a,b \in \mathcal{L}$ with $a \vee b \in f^{-1}(K)$ and $s \vee a \notin J(\mathcal{L})$. Then $f(a) \vee f(b) = f(a \vee b) \in K$. First, we show that $f(s \vee a) \notin J(\mathcal{L}')$. On the contrary, assume that $f(s \vee a) \in J(\mathcal{L}')$. Let M be any maximal filter of $\mathcal{L}$. Then $f(M)$ is a maximal filter of $\mathcal{L}'$. Since $f(s \vee a) \in J(\mathcal{L}') \subseteq f(M)$, we conclude that $f(s \vee a) = f(m)$ for some $m \in M$. By assumption, $m \vee m' = 1$ and $m \wedge m' = 0$ for some $m' \in \mathcal{L}$. As $f(s \vee a \vee m') = f(m) \vee f(m') = 1$, we conclude that $s \vee a \vee m' \in \ker(f) \subseteq M$; hence $s \vee a = (s \vee a) \vee (m \wedge m') = (s \vee a \vee m) \wedge (s \vee a \vee m') \in M$, as M is a filter. Therefore, $s \vee a \in J(\mathcal{L})$ which is a contradiction. Since K is an $f(S)$-J-filter, $f(a) \vee f(b) \in K$ and $f(s) \vee f(a) \notin J(\mathcal{L}')$, we get $f(s) \vee f(b) = f(s \vee b) \in K$ and so $s \vee b \in f^{-1}(K)$. □

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If P is a filter of a lattice $(\mathcal{L}, \leq)$, we define a relation on $\mathcal{L}$, given by $x \sim y$ if and only if there exist $a,b \in P$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $\mathcal{L}$, and we denote the equivalence class of a by $a \wedge P$ and these collection of all equivalence classes by $\mathcal{L}/P$. We set up a partial order $\leq_Q$ on $\mathcal{L}/P$ as follows: for each $a \wedge P, b \wedge P \in \mathcal{L}/P$, we write $a \wedge P \leq_Q b \wedge P$ if and only if $a \leq b$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}/P, \leq_Q)$ is a lattice with $(a \wedge P) \vee_Q (b \wedge P) = (a \vee b) \wedge P$ and $(a \wedge P) \wedge_Q (b \wedge P) = (a \wedge b) \wedge P$ for all elements $a \wedge P, b \wedge P \in \mathcal{L}/P$. Note that $p \wedge P = P$ if and only if $p \in P$ (see [10, Remark 4.2 and Lemma 4.3]). An inspection will show that if P is a filter of $\mathcal{L}$, then $S_Q = \{s \wedge P : s \in S\}$ is a join subset of $\mathcal{L}/P$.

Corollary 5.2. *Suppose $\mathcal{L}$ is a complemented lattice and let P, Q be two filters of $\mathcal{L}$ with $P \subseteq Q$. Then the followings hold:*

(1) If Q is an S-J-filter of $\mathcal{L}$, then Q/P is an S_Q-J-filter of $\mathcal{L}/P$;
(2) If Q/P is an S_Q-J-filter of $\mathcal{L}/P$ and $P \subseteq J(\mathcal{L})$, then Q is an S-J-filter of $\mathcal{L}$;
(3) If Q/P is an S_Q-J-filter of $\mathcal{L}/P$ and P is a J-filter of $\mathcal{L}$, then Q is an S-J-filter.

Proof. (1) Let $v : \mathcal{L} \to \mathcal{L}/P$ be the natural epimorphism defined by $v(x) = x \wedge P$ for $x \in \mathcal{L}$. Then $\ker(v) = \{x \in \mathcal{L} : x \wedge P = 1 \wedge P\} = P \subseteq Q$ by [10, Lemma 4.3] and so by Theorem 4.1 (1) and [10, Lemma 4.3] that $v(Q) = \{x \wedge P : x \in Q\} = Q/P$ is an S_Q-J-filter of $\mathcal{L}/P$, where $v(S) = S_Q$.

(2) Consider the natural epimorphism $v : \mathcal{L} \to \mathcal{L}/P$ defined by $v(x) = x \wedge P$ for $x \in \mathcal{L}$. Since $P \subseteq J(\mathcal{L})$, we conclude that $v^{-1}(Q/P) = Q$ is an S-J-filter of $\mathcal{L}$ by Theorem 5.1 (2).

(3) Since P is a J-filter of $\mathcal{L}$, we conclude that $P \subseteq J(\mathcal{L})$ by Corollary 3.5. Now the assertion follows from (2). □

If F is a proper filter of $\mathcal{L}$, then by a $\vee$-factorization of F we mean an expression of F as an intersection $\bigcap_{i=1}^{n} F_i$ of S-J-filters. We call $\mathcal{L}$ a $\vee$-lattice if every proper filter has a $\vee$-factorization.

Theorem 5.3. Let P be a proper filter of a complemented lattice $\mathcal{L}$. If $\mathcal{L}$ is an $\vee$-lattice, then $\mathcal{L}/P$ is an $\vee$-lattice.

Proof. Let Q be a proper filter of $\mathcal{L}/P$. Then $Q = K/P$ for some proper filter K of $\mathcal{L}$ by [10, Lemma 4.3 (5)]. Let $K = \cap_{i=1}^{n} P_i$ be a $\vee$-factorization of K. Then $Q = (\cap_{i=1}^{n} P_i)/P = \cap_{i=1}^{n} P_i/P$ by [10, Lemma 4.3 (7)]. By Corollary 5.2 (1), we conclude that P_i/P is an Q_S-J-filter of $\mathcal{L}/P$ for each $i \in \{1, 2, \cdots, n\}$. It means that $\mathcal{L}/P$ is an $\vee$-lattice. □

Lemma 5.4. *Suppose $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice. Then $J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$.*

Proof. If P is a proper filter of $\mathcal{L}$, then P is a maximal filter of $\mathcal{L}$ if and only if $P = P_1 \times \mathcal{L}_2$ for some maximal filter P_1 of $\mathcal{L}_1$ or $P = \mathcal{L}_1 \times P_2$ for some maximal filter P_2 of $\mathcal{L}_2$ by [12, Lemma 4.16]. Since $(P_i \times \mathcal{L}_2) \cap (\mathcal{L}_1 \times P_j) = P_i \times P_j$ and $(P_i \times P_j) \cap (P_i' \times P_j') = (P_i \cap P_i') \times (P_j \cap P_j')$, we infer that $J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$. □

We next show that S-J-filters are really only of interest in indecomposable lattices.

Theorem 5.5. Assume that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a decomposable lattice and let P_1, P_2 be filters of $\mathcal{L}_1$, $\mathcal{L}_2$, respectively. If S_i is a join subset of $\mathcal{L}_i$, $i = 1, 2$, then the following hold:
(1) $P_1 \times \mathcal{L}_2$ is an $(S_1 \times S_2)$-J-filter of $\mathcal{L}$ if and only if P_1 is an S_1-J-filter of $\mathcal{L}_1$ and $J(\mathcal{L}_2) \cap S_2 \neq \emptyset$.
(2) $\mathcal{L}_1 \times P_2$ is an $(S_1 \times S_2)$-J-filter of $\mathcal{L}$ if and only if P_2 is an S_2-J-filter of $\mathcal{L}_2$ and $J(\mathcal{L}_1) \cap S_1 \neq \emptyset$.

Proof. (1) It is easy to see that $(P_1 \times \mathcal{L}_2) \cap S = \emptyset$ if and only if $P_1 \cap S_1 = \emptyset$ and $S = S_1 \times S_2$ is a join subset of $\mathcal{L}$. Suppose that $P = P_1 \times \mathcal{L}_2$ is an S-J-filter of $\mathcal{L}$ and let $(s_1, s_2) \in S$ satisfies S-J-filter condition for P. Since $(1, 0) \vee_c (0, 1) = (1, 1) \in P$, we conclude that either $(s_1, s_2) \vee_c (0, 1) = (s_1, 1) \in P$ or $(s_1, s_2) \vee_c (1, 0) = (1, s_2) \in J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$ by Lemma 5.4. This shows that either $s_2 \in J(\mathcal{L}_2)$ (so $J(\mathcal{L}_2) \cap S_2 \neq \emptyset$) or $s_1 \in P_1 \cap S_1$ which is impossible. Let $p \vee p' \in P_1$ for some $p, p' \in \mathcal{L}_1$. Then $(p, 0) \vee_c (p', 0) = (p \vee p', 0) \in P$ gives $(s_1, s_2) \vee_c (p, 0) = (s_1 \vee p, s_2) \in J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$ or $(s_1, s_2) \vee_c (p', 0) = (s_1 \vee p', s_2) \in P_1 \times \mathcal{L}_2$; hence either $s_1 \vee p \in J(\mathcal{L})$ or $s_1 \vee p' \in P_1$. Therefore, P_1 is an S_1-J-filter.

Conversely, suppose P_1 is an S_1-J-filter of $\mathcal{L}_1$ and $J(\mathcal{L}_2) \cap S_2 \neq \emptyset$. Assume $s_1 \in S_1$ satisfies S_1-J-filter condition for P_1 and let $(p, q), (p', q') \in \mathcal{L}$ such that $(p, q) \vee_c (p', q') = (p \vee p', q \vee q') \in P$. Then $s_1 \vee p \in J(\mathcal{L}_1)$ or $s_1 \vee p' \in P_1$, as $p \vee p' \in P_1$. By the hypothesis, there exists $s_2 \in S_2$ such that $s_2 \in J(\mathcal{L}_2)$. This shows that $(p, q) \vee_c (s_1, s_2) = (s_1 \vee p, s_2 \vee q) \in J(\mathcal{L}) = J(\mathcal{L}_1) \times J(\mathcal{L}_2)$ or $(s_1, s_2) \vee_c (p', q') = (s_1 \vee p', s_2 \vee q') \in P$. Hence, P is an S-J-filter of $\mathcal{L}$.

(2) The proof is similar to that in case (1) and we omit it. □

Lemma 5.6. *If P is a finitely generated filter of $\mathcal{L}$ and $a \in \mathcal{L} \setminus P$, then $a \vee P$ is a finitely generated filter.*

Proof. Let $P = T(A)$, where $A = \{p_1, \cdots, p_n\} \subseteq P$. An inspection will show that $a \vee P = T(B)$, where $B = \{a \vee p_1, \cdots, a \vee p_n\} \subseteq a \vee P$. □

We say that a filter P of $\mathcal{L}$ is S-finite if $s \vee P \subseteq F \subseteq P$ for some finitely generated filter F of $\mathcal{L}$ and some $s \in S$.

Proposition 5.7. *For the lattice $\mathcal{L}$, the following hold:*

(1) Let P be an intersection of maximal filters of $\mathcal{L}$. If P is a finitely generated S-J-filter, then $J(\mathcal{L})$ is S-finite.

(2) If $J(\mathcal{L})$ is finitely generated, then it is S-finite. In particular, if P is a finitely generated maximal S-J-filter, then $J(\mathcal{L})$ is S-finite.

Proof. (1) Since P is S-J-filter, we conclude that $J(\mathcal{L}) \subseteq P \subseteq (J(\mathcal{L}) :_{\mathcal{L}} s)$ for some $s \in S$ by Proposition 3.4. Therefore, $s \vee J(\mathcal{L}) \subseteq s \vee P \subseteq J(\mathcal{L})$ and since $s \vee P$ is finitely generated by Lemma 5.6, we infer that $J(\mathcal{L})$ is S-finite.

(2) This is a direct consequence of (1). □

Proposition 5.8. *Let P, Q and K be filters of $\mathcal{L}$ such that $s \vee K \not\subseteq J(\mathcal{L})$ for all $s \in S$. Then the following hold:*

(1) Let P and Q be S-J-filters of $\mathcal{L}$. If P is a finitely generated with $K \vee P = K \vee Q$, then Q is a S-finite filter.

(2) Let P and Q be S-J-filters of $\mathcal{L}$. If Q a finitely generated with $K \vee P = K \vee Q$, then P is a S-finite filter.

(3) If $K \vee P$ is a finitely generated S-J-filter, then P is S-finite.

Proof. (1) Assume that $s \in S$ satisfies S-J-filter condition for Q. Since $K \vee P = K \vee Q \subseteq Q$ and $s \vee K \not\subseteq J(\mathcal{L})$, we conclude that $s \vee P \subseteq Q$ by Theorem 3.6, as Q is an S-J-filter. Similarly, There exists $t \in S$ such that $t \vee Q \subseteq P$. Therefore, $(s \vee t) \vee Q \subseteq s \vee P \subseteq Q$. For $s' = s \vee t \in S$, $s' \vee Q \subseteq s \vee P \subseteq Q$, where $s \vee P$ is finitely generated by Lemma 5.6. Hence, Q is S-finite.

(2) Similar to the proof of the part (1), one can show that P is S-finite.

(3) Since $s \vee K \not\subseteq J(\mathcal{L})$ and $K \vee P$ is S-J-filter, we infer that $s \vee P \subseteq K \vee P \subseteq P$ and so P is S-finite. □

Corollary 5.9. *Let P, Q and K be filters of $\mathcal{L}$ such that $K \not\subseteq J(\mathcal{L})$. Then the following hold:*

(1) Let P and Q be J-filters of $\mathcal{L}$. If P is a finitely generated with $K \vee P = K \vee Q$, then Q is a S-finite filter.

(2) Let P and Q be J-filters of $\mathcal{L}$. If Q a finitely generated with $K \vee P = K \vee Q$, then P is a S-finite filter.

(3) If $K \vee P$ is a finitely generated J-filter, then P is S-finite.

Proof. Take $S = \{0\}$ in Proposition 5.8. □

Definition 5.10. (1) A proper filter P of $\mathcal{L}$ is called SJ-finite if $s \vee P \subseteq G \subseteq P \subseteq J(\mathcal{L})$ for some finitely generated filter K of $\mathcal{L}$ and some $s \in S$.

(2) We say that $\mathcal{L}$ is SJ-Noetherian if each proper filter of $\mathcal{L}$ is SJ-finite.

Lemma 5.11. *If $P \subseteq J(\mathcal{L})$ is a filter of $\mathcal{L}$ which is maximal among all non-SJ-finite filters of $\mathcal{L}$, then P is a J-filter of $\mathcal{L}$.*

Proof. Let $a, b \in \mathcal{L}$ such that $a \vee b \in P$ and $a \notin J(\mathcal{L})$ (so $a \notin P$). We show that $b \in P$. On the contrary, assume that $b \notin P$. By an argument like that in [13, Lemma 2.20], there exist $s, t \in S$, $p_1, \cdots, p_n \in P$ and $b_1, \cdots, b_k \in (P :_{\mathcal{L}} a)$ such that $(s \vee t) \vee P \subseteq T(A) \subseteq P$, where $A = \{p_1 \vee t, \cdots, p_n \vee t, a \vee b_1, \cdots, a \vee b_k\} \subseteq P$; hence P is S-finite, a contradiction. Thus P is a J-filter of $\mathcal{L}$. □

Proposition 5.12. *$\mathcal{L}$ is SJ-Noetherian if and only if every J-filter of $\mathcal{L}$ (disjoint from S) is S-finite.*

Proof. One side is clear. To see the other side, suppose that P is SJ-finite for each J-filter P of $\mathcal{L}$ disjoint from S. On the contrary, Suppose that $\mathcal{L}$ is not SJ-Noetherian. Then the set Δ of all non-SJ-finite filters of $\mathcal{L}$ is inductively ordered under inclusion. By Zorn's Lemma, choose Q maximal in Δ. Then Lemma 5.11 implies that Q is a J-filter of $\mathcal{L}$. If $Q \cap S \neq \emptyset$, then $s \vee Q \subseteq T(s) \subseteq Q$ for every $s \in Q \cap S$ gives Q is SJ-finite, a contradiction. Thus $Q \cap S = \emptyset$. Now, by the hypothesis, Q is SJ-finite which is impossible since $Q \in \Delta$. Thus $\mathcal{L}$ is SJ-Noetherian. □

In the following theorem we give an SJ-version of Cohen's Theorem (Theorem 2 in [7])

Theorem 5.13. For a lattice $\mathcal{L}$, the following assertions are equivalent:
(1) $\mathcal{L}$ is SJ-Noetherian;
(2) Every S-J-filter of $\mathcal{L}$ is SJ-finite;
(3) Every J-filter of $\mathcal{L}$ is SJ-finite.

Proof. (1) $\Rightarrow$ (2) By assumption, each proper filter of $\mathcal{L}$ is SJ-finite.
(2) $\Rightarrow$ (3) Let P be a J-filter of $\mathcal{L}$. If $P \cap S \neq \emptyset$, then $s \vee P \subseteq T(s) \subseteq P$ for every $s \in P \cap S$ gives P is SJ-finite. If $P \cap S = \emptyset$, then P is an S-J-filter of $\mathcal{L}$; hence by the hypothesis, P is S-finite.
(3) $\Rightarrow$ (1) Follows from Proposition 5.12. □

Similar to the idealization of a module over a ring (Huckaba in [14]), the remaining part of this section mainly devoted to investigation the stability of S-J-filters in the filterlization of a lattice and a filter. Let $\mathcal{M}$ be a filter of lattice $\mathcal{L}$. The filterlization $\mathcal{L}(+)\mathcal{M} = \{(a,m) : a \in \mathcal{L}, m \in \mathcal{M}\}$ of $\mathcal{M}$ in $\mathcal{L}$ is a lattice with respect to the following definitions: (1) $(a,m) = (a',m')$ if $a = a'$ and $m = m'$, (2) $(a,m_1) \wedge_F (b,m_2) = (a \wedge b, m_1 \wedge m_2)$ and (3) $(a,m_1) \vee_F (b,m_2) = (a \vee b, (a \vee m_2) \wedge (b \vee m_1))$. Under these definitions $\mathcal{L}(+)\mathcal{M}$ becomes a lattice. An easy inspection will show that $S_{\mathcal{M}} = S(+)\mathcal{M}$ is a join subset of $\mathcal{L}(+)\mathcal{M}$ and $J(\mathcal{L}(+)\mathcal{M}) = J(\mathcal{L})(+)\mathcal{M}$. In addition, if F is a filter of $\mathcal{L}$, then for a subfilter N of $\mathcal{M}$, $F(+)N$ is a filter of $\mathcal{L}(+)\mathcal{M}$ if and only if $F \vee \mathcal{M} \subseteq N$. Here we clarify the relationships between S-J-filters in a lattice $\mathcal{L}$ and in a filterlization lattice $\mathcal{L}(+)\mathcal{M}$.

Theorem 5.14. Let P be a filter of a lattice $\mathcal{L}$ and K be a subfilter of $\mathcal{M}$. Then the following hold:
(1) If $P(+)K$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$, then P is an S-J-filter of $\mathcal{L}$.
(2) $P(+)\mathcal{M}$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$ if and only if P is an S-J-filter of $\mathcal{L}$.

Proof. (1) Let $P(+)K$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$. So $S_{\mathcal{M}} \cap (P(+)K) = \emptyset$ gives $S \cap P = \emptyset$. Let $x, y \in \mathcal{L}$ such that $x \vee y \in P$. Then $(x,1) \vee_F (y,1) = (x \vee y, 1) \in P(+)K$ implies that there exists $(s,m) \in S_{\mathcal{M}}$ such that either $(s,m) \vee_F (x,1) = (s \vee x, x \vee m) \in J(\mathcal{L}(+)\mathcal{M}) = J(\mathcal{L})(+)\mathcal{M}$ or $(s,m) \vee_F (y,1) = (s \vee y, y \vee m) \in P(+)K$. Hence, either $s \vee x \in J(\mathcal{L})$ or $s \vee y \in P$ and so P is an S-J-filter of $\mathcal{L}$.
(2) Suppose that $P(+)\mathcal{M}$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$. Then P is an S-J-filter of $\mathcal{L}$ by (1). To see the other side, suppose P is an S-J-filter of $\mathcal{L}$ and let $s \in S$ satisfies S-J-filter condition for P. Note that $S_{\mathcal{M}} \cap (P(+)\mathcal{M}) = \emptyset$, as $S \cap P = \emptyset$. Let $(p,m) \vee_F (p',m') = (p \vee p', (p \vee m') \wedge (p' \vee m) \in P(+)\mathcal{M}$ for some $(p,m), (p',m') \in \mathcal{L}(+)\mathcal{M}$. By the hypothesis, since $p \vee p' \in P$, we conclude that either $s \vee p \in J(\mathcal{L})$ or $s \vee p' \in P$ which implies either $(s,m) \vee_F (p,m) \in J(\mathcal{L})(+)\mathcal{M} = J(\mathcal{L}(+)\mathcal{M})$ or $(s,m) \vee_F (p',m') \in P(+)\mathcal{M}$. This shows that $P(+)\mathcal{M}$ is an $S_{\mathcal{M}}$-J-filter of $\mathcal{L}(+)\mathcal{M}$. □

Corollary 5.15. *Let P be a filter of a lattice $\mathcal{L}$ and K be a subfilter of $\mathcal{M}$. Then the following hold:*
(1) If $P(+)K$ is a J-filter of $\mathcal{L}(+)\mathcal{M}$, then P is an J-filter of $\mathcal{L}$.
(2) $P(+)\mathcal{M}$ is an J-filter of $\mathcal{L}(+)\mathcal{M}$ if and only if P is an J-filter of $\mathcal{L}$.

Proof. Take $S = \{0\}$ in Theorem 5.14. □

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