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Nil-M-Noetherian and nil-M-Artinian modules

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Abstract. Let R be a commutative ring with identity and M be an R-module. The aim of this paper is to introduce and investigate the notions of nil-M-Noetherian and nil-M-Artinian modules as generalizations of Noetherian and Artinian modules. Also, in this regard we introduce nil versions of some algebraic concepts.

Key Words: Nil-M-cyclic, nil-M-Noetherian, nil-M-Artinian, nil-M-finitely generated, nil-M-finitely cogenerated.
2020 MSC: 13C13, 13C99.

1 Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers. Also, $Nil(R)$ will denote the set of all nilpotent elements of R.

In [13], the authors introduced and investigated the notion of Ω-prime ideals as a generalization of prime ideals. A proper ideal P of R is said to be a Ω-prime ideal if $ab \in P$, for each $a, b \in R$, then either $a \in P + Nil(R)$ or $b \in P + Nil(R)$ [13]. In [7] (resp. [8]), the present author introduced the notion of nil-prime (resp. nil-primary) ideals as a generalization of prime (resp. primary) ideals and studied some basic properties of this class of ideals. A proper ideal P of R is said to be a *nil-prime ideal* if there exists $x \in Nil(R)$ such that whenever $ab \in P$, then $a \in P$ or $b \in P$ or $a + x \in P$ or $b + x \in P$ for each $a, b \in R$ (for generalizations of this notion for modules, see [9]). Also, a proper ideal P of R is said to be a *nil-primary ideal* if there exists $x \in Nil(R)$ such that whenever $ab \in P$ for some $a, b \in R$, then $a \in P$ or $b^n \in P$ or $a + x \in P$ or $b^n + x \in P$ for some $n \in \mathbb{N}$.

For a submodule N of an R-module M, a non-empty subset K of M, and a non-empty subset J of R, the residuals of N by K and J are defined as $(N :_R K) = \{a \in R : aK \subseteq N\}$ and $(N :_M J) = \{m \in M : Jm \subseteq N\}$, respectively. In particular, we use $Ann_R(M)$ to denote $(0 :_R M)$.

In algebra, some generalizations and weakenings for the concepts of Noetherian and Artinian modules have arisen. In this regard, you can refer to [11, 6]. Motivated by nil-prime ideals, the purpose of this paper is to introduce and investigate the notions of nil-M-Noetherian and nil-M-Artinian modules as generalizations of Noetherian and Artinian modules. Also, in this regard we introduce the notions of nil-M-cyclic, nil-M-finitely generated and nil-M-finitely cogenerated modules.

2 Main Results

Definition 2.1. We say that an R-module M is a *nil-M-Noetherian module* if there is $r \in \sqrt{Ann_R(M)}$ such that for each ascending chain

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$$

of submodules of M there is a positive integer t such that $N_{t+i} \subseteq N_t + rM$ for all $i \geq 0$. Moreover, we say that the ring R is a *nil-R-Noetherian ring* if R as an R-module is a nil-R-Noetherian module.

Definition 2.2. We say that an R-module M is a *nil-M-Artinian module* if there is $r \in \sqrt{Ann_R(M)}$ such that for each descending chain

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq ... \supseteq N_k \supseteq \dots$$

of submodules of M there is a positive integer t such that $N_t \cap (0 :_M r) \subseteq N_{t+i}$ for all $i \geq 0$. Also, we say that the ring R is a *nil-R-Artinian ring* if R as an R-module is a nil-R-Artinian module.

Remark 2.3. As $0 \in \sqrt{Ann_R(M)}$, we have that every Noetherian (resp. Artinian) R-module is a nil-M-Noetherian (resp. nil-M-Artinian) R-module. This motivates the Question 2.4. If $\sqrt{Ann_R(M)} = 0$, then the notions of Noetherian R-modules and nil-M-Noetherian R-modules (resp. Artinian R-modules and nil-M-Artinian R-modules) are equal. For example, for a positive integer n which is square-free (i.e., n has not a square factor), consider the ring $\mathbb{Z}_n$ of integers modulo n. Then $\sqrt{Ann_{\mathbb{Z}_n}(\mathbb{Z}_n)} = 0 = Nil(\mathbb{Z}_n)$. Also, $\sqrt{Ann_{\mathbb{Z}}(\mathbb{Z})} = 0 = Nil(\mathbb{Z})$ and $\sqrt{Ann_{F[x])}(F[x]))} = 0 = Nil(F[x])$, where F is a field.

Question 2.4. Let M be an R-module. Is every nil-M-Noetherian (resp. nil-M-Artinian) module a Noetherian (resp. Artinian) module?

Definition 2.5. We say that a submodule N of an R-module M is a *nil-M-cyclic* if there exist $r \in \sqrt{Ann_R(M)}$ and $x \in M$ such that $N = Rx + rM$.

Let M be an R-module. Since $0 \in \sqrt{Ann_R(M)}$, clearly each cyclic submodule of M is a nil-M-cyclic submodule of M. But Example 2.6 shows that the converse is not true in general.

Example 2.6. The submodule $N = \mathbb{Z}x + 2\mathbb{Z}_4[x,y]$ of the $\mathbb{Z}$-module $M = \mathbb{Z}_4[x,y]$ is a nil-M-cyclic since $2 \in \sqrt{Ann_{\mathbb{Z}}(M)}$ and $x \in \mathbb{Z}_4[x,y]$. But N is not a cyclic submodule of M.

Definition 2.7. We say that a submodule N of an R-module M is a *nil-M-finitely generated* (resp. *weakly nil-M-finitely generated*) if there exist $r \in \sqrt{Ann_R(M)}$ and $x_1, x_2, \dots, x_t \in M$ such that $N = Rx_1 + Rx_2 + \cdots + Rx_t + rM$ (resp. $N + rM = Rx_1 + Rx_2 + \cdots + Rx_t + rM$).

Example 2.8. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \oplus \cdots$. Then $6 \in \sqrt{Ann_{\mathbb{Z}}(M)}$. For each finitely generated submodule N of M we have $N + 6M$ is a nil-M-finitely generated submodule of M which is not a finitely generated submodule of M.

Remark 2.9. Let M be an R-module. Clearly, every nil-M-finitely generated submodule of M is a weakly nil-M-finitely generated submodule of M. If N is a weakly nil-M-finitely generated submodule of M such that $(N :_R M)$ is prime ideal of R, then $\sqrt{Ann_R(M)} \subseteq \sqrt{(N :_R M)} = (N :_R M)$. This in turn implies that N is a nil-M-finitely generated submodule of M.

Proposition 2.10. *Let S be a multiplicatively subset of R and let M be an R-module. If N is a nil-M-finitely generated submodule of M, then $S^{-1}N$ is a nil-$S^{-1}M$-finitely generated submodule of $S^{-1}M$.*

Proof. Assume that N is a nil-M-finitely generated submodule of M. Then there exist $r \in \sqrt{Ann_R(M)}$ and $x_1, x_2, \dots, x_t \in M$ such that $N = Rx_1 + Rx_2 + \cdots + Rx_t + rM$. By using the proof of [14, Lemma 9.12(ii)] and [14, Lemma 5.31(iv)], we have $S^{-1}(\sqrt{Ann_R(M)}) \subseteq \sqrt{Ann_{S^{-1}R}(S^{-1}M)}$. Therefore, $r/1 \in \sqrt{Ann_{S^{-1}R}(S^{-1}M)}$ and $x_1/1, x_2/1, \dots, x_t/1 \in S^{-1}M$ such that

$$S^{-1}N = (S^{-1}R)(x_1/1) + (S^{-1}R)(x_2/1) + \cdots + (S^{-1}R)(x_t/1) + r/1(S^{-1}M).$$

□

A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. By [10], every submodule of M is an intersection of completely irreducible submodules of M. Thus the intersection of all completely irreducible submodules of M is zero [10].

Definition 2.11. We say that a submodule N of an R-module M is a *nil-M-completely irreducible submodule* if there exist $r \in \sqrt{Ann_R(M)}$ and completely irreducible submodule L of M such that $N = L \cap (0 :_M r)$.

Definition 2.12. For a submodule N of an R-module M, we say that M/N is a *nil-M-finitely cogenerated* (resp. *weakly nil-M-finitely cogenerated*) if there exists $r \in \sqrt{Ann_R(M)}$ such that $N = \cap_{i\in I} N_i$ implies that $N = \cap_{i\in J} N_i \cap (0 :_M r)$ (resp. $\cap_{i\in J} N_i \cap (0 :_M r) = N \cap (0 :_M r)$) for some finite subset J of I.

Proposition 2.13. *If for a submodule N of an R-module M we have M/N is a nil-M-finitely cogenerated (resp. weakly nil-M-finitely cogenerated) module, then there exist $r \in \sqrt{Ann_R(M)}$ and completely irreducible submodules $L_1, L_2, \dots, L_t$ of M such that $N = L_1 \cap L_2 \cap \cdots \cap L_t \cap (0 :_M r)$ (resp. $L_1 \cap L_2 \cap \cdots \cap L_t \cap (0 :_M r) = N \cap (0 :_M r)$).*

Proof. This follows from the fact that every submodule of M is an intersection of completely irreducible submodules of M. □

Remark 2.14. Let M be an R-module. Clearly, every nil-M-finitely cogenerated homomorphic image of M is a weakly nil-M-finitely cogenerated homomorphic image of M. It is easy to see that, if M/N is a weakly nil-M-finitely cogenerated homomorphic image of M such that $Ann_R(N)$ is prime ideal of R, then M/N is a nil-M-finitely cogenerated homomorphic image of M.

Proposition 2.15. *Let M be an R-module. Then we have the following.*

(a) If for a submodule N of M and $r \in \sqrt{Ann_R(M)}$ we have $N + rM = M$, then $N = M$.

(b) If for a submodule N of M and $r \in \sqrt{Ann_R(M)}$ we have $N \cap (0 :_M r) = 0$, then $N = 0$.

Proof. (a) Suppose that for a submodule N of M and $r \in \sqrt{Ann_R(M)}$ we have $N + rM = M$. Then there exists $t \in \mathbb{N}$ such that $r^t M = 0$. We have $rN + r^2M = rM$. This implies that $rN + r^2M + N = rM + N$ and so $N + r^2M = M$. Continuing this way, we get that $M = N + r^tM = N + 0 = N$.

(b) Suppose that for a submodule N of M and $r \in \sqrt{Ann_R(M)}$ we have $N \cap (0 :_M r) = 0$. Then there exists $t \in \mathbb{N}$ such that $r^t M = 0$. Clearly, $N \cap (0 :_M r) = (0 :_N r)$. Hence, $(0 :_N r) = 0$. Therefore,

$$(0 :_N r^2) = ((0 :_N r) :_N r) = (0 :_N r) = N \cap (0 :_M r) = 0.$$

Continuing this way, we have $0 = (0 :_N r^t) = N \cap (0 :_M r^t) = N \cap M = N$. □

Corollary 2.16. *Let M be an R-module. Then we have the following.*

(a) M is a nil-M-cyclic module if and only if it is a cyclic module.

(b) M is a weakly nil-M-finitely generated module if and only if it is a finitely generated module.

(c) The submodule 0 of M is a nil-M-completely irreducible submodule if and only if it is a completely irreducible submodule.

(d) M is a weakly nil-M-finitely cogenerated module if and only if it is a finitely cogenerated module.

Proof. (a) Let M be a nil-M-cyclic R-module. Then there exist $r \in \sqrt{Ann_R(M)}$ and $x \in M$ such that $M = Rx + rM$. This implies that $M = Rx$ by Proposition 2.15 (a). Thus M is a cyclic module. Since $0 \in \sqrt{Ann_R(M)}$, the converse is clear.

(b) Let M be a weakly nil-M-finitely generated R-module. Then there exist $r \in \sqrt{Ann_R(M)}$ and $x_1, x_2, \dots, x_t \in M$ such that $M = M + rM = Rx_1 + Rx_2 + \cdots + Rx_t + rM$. Thus $M = Rx_1 + Rx_2 + \cdots + Rx_t$ by Proposition 2.15 (a). So, M is a finitely generated module. Since $0 \in \sqrt{Ann_R(M)}$, the converse is clear.

(c) Assume that the submodule 0 of M is a nil-M-completely irreducible submodule. Then there exist $r \in \sqrt{Ann_R(M)}$ and completely irreducible submodule L of M such that $0 = L \cap (0 :_M r)$. This implies that $L = 0$ by Proposition 2.15 (b). So, 0 is a completely irreducible submodule of M. Since $0 \in \sqrt{Ann_R(M)}$, the converse is clear.

(d) Suppose that M is a weakly nil-M-finitely cogenerated R-module and $0 = \cap_{i \in I} N_i$. Then there exist $r \in \sqrt{Ann_R(M)}$ and finite subset J of I such that $\cap_{i \in J} N_i \cap (0 :_M r) = 0 \cap (0 :_M r) = 0$. It follows that $\cap_{i \in J} N_i = 0$ by Proposition 2.15 (b). Therefore, M is a finitely cogenerated module. Since $0 \in \sqrt{Ann_R(M)}$, the converse is clear. □

Theorem 2.17. Let M be an R-module. Consider the following statements:

(a) Every submodule of M is nil-M-finitely generated;

(b) M is a nil-M-Noetherian module;

(c) There exists $r \in \sqrt{Ann_R(M)}$ such that any non-empty collection Σ of submodules of M with the form $K + rM$ has a maximal element $N + rM$;

(d) Every submodule of M is weakly nil-M-finitely generated.

Then $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, and $(c) \Rightarrow (d)$.

Proof. $(a) \Rightarrow (b)$ Assume that the following is an increasing sequence of submodules of M

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots.$$

Consider $N := \bigcup_{i \geq 1} N_i$. Then N is a submodule of M, hence it is nil-M-finitely generated by part (a). Thus there exist $r \in \sqrt{Ann_R(M)}$ and $u_1, \dots, u_t \in M$ such that $N = Ru_1 + \cdots + Ru_t + rM$. Then we can find s such that $u_j \in N_s$ for all $j = 1, 2, \dots, t$. In this case we have $N + rM \subseteq N_s + rM$. Therefore, for all $i \geq 0$ we have $N_{s+i} \subseteq N \subseteq N + rM \subseteq N_s + rM$, as needed.

$(b) \Rightarrow (c)$ First note that by part (b), there is $r \in \sqrt{Ann_R(M)}$ such that for each ascending chain

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$$

of submodules of M there is a positive integer t such that $N_{t+i} \subseteq N_t + rM$ for all $i \geq 0$. Suppose that there exists a non-empty collection Σ of submodules of M with the form $N + rM$ such that it has no maximal element. Let us choose $N_1 + rM \in \Sigma$. Since this is not maximal, there is $N_2 + rM \in \Sigma$ such that $N_1 + rM \subset N_2 + rM$, and we continue in this way to construct the following infinite strictly increasing sequence of submodules of M:

$$N_1 + rM \subset N_2 + rM \subset N_3 + rM \subset \ldots.$$

By part (b), there is a positive integer t such that

$$N_{t+1} + rM \subseteq N_t + rM + rM = N_t + rM,$$

which is a contradiction.

$(c) \Rightarrow (d)$ Suppose that there exists $r \in \sqrt{Ann_R(M)}$ such that any non-empty collection Σ of submodules of M with the form $K+rM$ has a maximal element. Let N be a submodule of M and consider the family Σ of all submodules of M with the form $K+rM$, where $K = Rx_1 + \cdots + Rx_t$ for some $x_i \in N$. Clearly, Σ is non-empty. By part (c), Σ has a maximal element $\acute{N} + rM$, where $\acute{N} = Rx_1 + \cdots + Rx_n$ for some $x_i \in N$. If $\acute{N} + rM \neq N + rM$, then there is $u \in N + rM \setminus \acute{N} + rM$ and the submodule $Ru + \acute{N} + rM$ strictly containing $\acute{N} + rM$, a contradiction. Thus $\acute{N} + rM = N + rM$, as needed. □

Definition 2.18. (a) We say that a submodule N of an R-module M is a *nil-M-Noetherian submodule* if there is $r \in \sqrt{Ann_R(M)}$ such that for each ascending chain

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \ldots \subseteq N_k \subseteq \ldots$$

of submodules of N there is a positive integer t such that $N_{t+i} \subseteq N_t + rM$ for all $i \geq 0$.

(b) We say that a quotient module M/N of an R-module M is a *nil-M-Noetherian quotient module* if there is $r \in \sqrt{Ann_R(M)}$ such that for each ascending chain

$$N_1/N \subseteq N_2/N \subseteq N_3/N \subseteq \ldots \subseteq N_k/N \subseteq \ldots$$

of submodules of M/N there is a positive integer t such that $N_{t+i}/N \subseteq (N_t + rM)/N$ for all $i \geq 0$.

Recall that a submodule N of an R-module M is said to be *pure* in M if $rM \cap N = rN$ for each $r \in R$ [12]. M is said to be *fully pure* if every submodule of M is pure [2].

Proposition 2.19. *Let M be a nil-M-Noetherian R-module. Then we have the following.*

(a) Any pure submodule N of M is a nil-N-Noetherian R-module.

(b) Any quotient module M/N of M is a nil-M/N-Noetherian R-module.

(c) Any submodule N of M is a nil-M-Noetherian R-module.

(d) Any quotient module M/N of M is a nil-M-Noetherian R-module.

Proof. (a) Let N be a pure submodule of M and

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_t \subseteq \ldots$$

be an ascending chain of submodules of N. Then since this chain is also an ascending chain of submodules of M, there is a positive integer s such that $K_{s+i} \subseteq K_s + rM$ for all $i \geq 0$, where $r \in \sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(N)}$. As N is pure, $N \cap rM = rN$. Therefore, for $r \in \sqrt{Ann_R(N)}$ we get that

$$K_{s+i} = K_{s+i} \cap N \subseteq (K_s + rM) \cap N = (K_s \cap N) + (rM \cap N) = K_s + rN,$$

for all $i \geq 0$. Thus N is a nil-N-Noetherian R-module.

(b) Let N be a submodule of M and

$$H_1/N \subseteq H_2/N \subseteq H_3/N \subseteq \ldots \subseteq H_t/N \subseteq \ldots$$

be an ascending chain of submodules of M/N. Then

$$H_1 \subseteq H_2 \subseteq H_3 \subseteq \ldots \subseteq H_t \subseteq \ldots$$

is an ascending chain of submodules of M. Thus there is a positive integer t such that $H_{t+i} \subseteq H_t + rM$ for all $i \geq 0$, where $r \in \sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(M/N)}$. This implies that for all $i \geq 0$, $H_{t+i}/N \subseteq H_t/N + r(M/N)$, where $r \in \sqrt{Ann_R(M/N)}$. Hence, M/N is a nil-M/N-Noetherian R-module.

(c) This follows from the fact that any increasing sequence of submodules of N is also an increasing sequence of submodules of M.

(d) This follows from the fact that any increasing sequence of submodules of M/N corresponds to a sequence of submodules of M containing N, So M/N is a nil-M-Noetherian module. □

Theorem 2.20. Let M be an R-module. Consider the following statements:

(a) Every homomorphic image of M is nil-M-finitely cogenerated R-module;

(b) M is a nil-M-Artinian module;

(c) There exists $r \in \sqrt{Ann_R(M)}$ such that any non-empty collection Σ of submodules of M with the form $K \cap (0 :_M r)$ has a minimal element $N \cap (0 :_M r)$.

(d) Every homomorphic image of M is weakly nil-M-finitely cogenerated.

Then $(a) \Rightarrow (b)$, $(b) \Rightarrow (c)$, and $(c) \Rightarrow (d)$.

Proof. $(a) \Rightarrow (b)$ Suppose that the following is a descending chain of submodules of M

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \ldots \supseteq N_k \supseteq \ldots.$$

Consider $N := \bigcap_{i \geq 1} N_i$. Then N is a submodule of M, hence M/N is nil-M-finitely generated by part (a). Thus there exist $r \in \sqrt{Ann_R(M)}$ and $N_{j_1}, \ldots, N_{j_t}$ of the above chain such that $N = N_{j_1} \cap \ldots \cap N_{j_t} \cap (0 :_M r)$. Then we can find s such that $N_{j_s} \subseteq N_{j_i}$ for all $i = 1, \ldots, t$. Therefore, $N = N_{j_s} \cap (0 :_M r)$. Now, for all $i \geq 0$ we have $N_{j_s} \cap (0 :_M r) = N \subseteq N_{j_{s+i}}$, as needed.

$(b) \Rightarrow (c)$ First note that by part (b), there is $r \in \sqrt{Ann_R(M)}$ such that for each descending chain

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq ... \supseteq N_k \supseteq \ldots$$

of submodules of M there is a positive integer t such that $N_t \cap (0 :_M r) \subseteq N_{t+i}$ for all $i \geq 0$. Suppose that there exists a non-empty collection Σ of submodules of M with the form $N \cap (0 :_M r)$ such that it has no minimal element. Let us choose $N_1 \cap (0 :_M r) \in \Sigma$. Since this is not minimal, there is $N_2 \cap (0 :_M r) \in \Sigma$ such that $N_2 \cap (0 :_M r) \subset N_1 \cap (0 :_M r)$, and we continue in this way to construct the following infinite strictly descending chains of submodules of M:

$$N_1 \cap (0 :_M r) \supset N_2 \cap (0 :_M r) \supset N_3 \cap (0 :_M r) \supset \ldots.$$

By part (b), there is a positive integer t such that

$$N_t \cap (0 :_M r) = N_t \cap (0 :_M r) \cap (0 :_M r) \subseteq N_{t+1} \cap (0 :_M r),$$

which is a contradiction.

$(c) \Rightarrow (d)$ Suppose that there exists $r \in \sqrt{Ann_R(M)}$ such that any non-empty collection Σ of submodules of M with the form $K \cap (0 :_M r)$ has a minimal element. Let N be a submodule of M and $N = \cap_{i \in I} N_i$. Consider the family Σ of all submodules of M with the form $\cap_{i \in J} N_i \cap (0 :_M r)$, where J is a finite subset of I. Clearly, Σ is non-empty. By part (c), Σ has a minimal element $\cap_{i \in H} N_i \cap (0 :_M r)$, where H is a finite subset of I. By the minimality, we get that $N \cap (0 :_M r) = \cap_{i \in I} N_i \cap (0 :_M r) = \cap_{i \in H} N_i \cap (0 :_M r)$, as needed. □

Corollary 2.21. *Let M be an R-module. Then we have the following.*

(a) If M is a nil-M-Noetherian R-module, then M is a finitely generated R-module.

(b) If M is a nil-M-Artinian R-module, then M is a finitely cogenerated R-module.

Proof. (a) Assume that M is a nil-M-Noetherian R-module. Then M is a weakly nil-M-finitely generated module by Theorem 2.17. Now the result follows from Corollary 2.16 (b).

(b) Suppose that M is a nil-M-Artinian R-module. Then M is a weakly nil-M-finitely cogenerated module by Theorem 2.20. Now the result follows from Corollary 2.16 (d) □

Definition 2.22. (a) We say that a submodule N of an R-module M is a *nil-M-Artinian submodule* if there is $r \in \sqrt{Ann_R(M)}$ such that for each descending chain

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \ldots \supseteq N_k \supseteq \ldots$$

of submodules of N there is a positive integer t such that $N_t \cap (0 :_M r) \subseteq N_{t+i}$ for all $i \geq 0$.

(b) We say that a quotient module M/N of an R-module M is a *nil-M-Artinian quotient module* if there is $r \in \sqrt{Ann_R(M)}$ such that for each descending chain

$$N_1/N \supseteq N_2/N \supseteq N_3/N \supseteq \ldots \supseteq N_k/N \supseteq \ldots$$

of submodules of M/N there is a positive integer t such that $(N_t \cap (0 :_M r))/N \subseteq N_{t+i}/N$ for all $i \geq 0$.

A submodule N of an R-module M is said to be *copure* in M if $(N :_M I) = N + (0 :_M I)$ for each ideal I of R [3]. M is said to be *fully copure* if every submodule of M is copure [2].

Proposition 2.23. *Let M be a nil-M-Artinian R-module. Then we have the following.*

(a) Any submodule N of M is a nil-N-Artinian R-module.

(b) Any quotient module M/N of M is a nil-M/N-Artinian R-module, where N is a copure submodule of M.

(c) Any submodule N of M is a nil-M-Artinian R-module.

(d) Any quotient module M/N of M is a nil-M-Artinian R-module.

Proof. (a) Let N be a submodule of M and

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots \supseteq K_t \supseteq \ldots$$

be a descending chain of submodules of N. Then since this chain is also a descending chain of submodules of M, there is a positive integer s such that $K_s \cap (0 :_M r) \subseteq K_{s+i}$ for all $i \geq 0$, where $r \in \sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(N)}$. This implies that $K_s \cap (0 :_N r) = N \cap K_s \cap (0 :_M r) \subseteq N \cap K_{s+i} = K_{s+i}$ for $r \in \sqrt{Ann_R(N)}$.

(b) Let N be a copure submodule of M and

$$H_1/N \supseteq H_2/N \supseteq H_3/N \supseteq \ldots \supseteq H_t/N \supseteq \ldots$$

be a descending chain of submodules of M/N. Then we have the following descending chain of submodules of M.

$$H_1 \supseteq H_2 \supseteq H_3 \supseteq \ldots \supseteq H_t \supseteq \ldots.$$

Thus there is a positive integer t such that $H_t \cap (0 :_M r) \subseteq H_{t+i}$ for all $i \geq 0$, where $r \in \sqrt{Ann_R(M)} \subseteq \sqrt{Ann_R(M/N)}$. Since N is copure, $(N :_M r) = N + (0 :_M r)$. Thus we have

$$H_t \cap (N :_M r) = H_t \cap (N + (0 :_M r)) = (H_t \cap (0 :_M r)) + (N \cap H_t)$$
$$= (H_t \cap (0 :_M r)) + N \subseteq H_{t+i} + N = H_{t+i}.$$

This implies that for all $i \geq 0$, $H_t/N \cap (0 :_{M/N} r) \subseteq H_{t+i}/N$, where $r \in \sqrt{Ann_R(M/N)}$. Hence, M/N is a nil-M/N-Artinian R-module.

(c) This follows from the fact that any descending chain of submodules of N is also a descending chain of submodules of M.

(d) This follows from the fact that any descending chain of submodules of M/N corresponds to a descending chain of submodules of M containing N, So M/N is a nil-M-Artinian module. □

Theorem 2.24. Let M be an R-module. Then we have the following.

(a) If M is a fully pure nil-M-Noetherian R-module, then M is a Noetherian R-module.

(b) If M is a fully copure nil-M-Artinian R-module, then M is an Artinian R-module.

Proof. (a) Assume that M is a fully pure nil-M-Noetherian R-module. Then by Proposition 2.19 (a), every submodule N of M is a nil-N-Noetherian R-module. Now, by Corollary 2.21 (a), every submodule N of M is a finitely generated. Thus M is a Noetherian R-module.

(b) Assume that M is a fully copure nil-M-Artinian R-module. Then by Proposition 2.23 (b), for every submodule N of M we have M/N is a nil-M/N-Artinian R-module. Now, by Corollary 2.21 (b), M/N is a finitely generated. Hence, M is an Artinian R-module. □

Recall that an R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [5]. An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [1]. It is easy to see that M is a multiplication (resp. comultiplication) module if and only if $N = (N :_R M)M$ (resp. $N = (0 :_M Ann_R(N))$) for each submodule N of M.

Proposition 2.25. *Let M be an R-module. Then we have the following.*

(a) If R is a nil-R-Noetherian ring and M is a multiplication R-module, then M is a nil-M-Noetherian module.

(b) If R is a nil-R-Artinian ring and M is a faithful comultiplication R-module, then M is a nil-M-Noetherian module.

Proof. (a) Let

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_t \subseteq \ldots$$

be an ascending chain of submodules of M. Then we have

$$(K_1 :_R M) \subseteq (K_2 :_R M) \subseteq (K_3 :_R M) \subseteq \ldots \subseteq (K_t :_R M) \subseteq \ldots$$

is an ascending chain of ideals of R. Thus by assumption, there exist $r \in \sqrt{0} \subseteq \sqrt{Ann_R(M)}$ and a positive integer t such that $(N_{t+i} :_R M) \subseteq (N_t :_R M) + rR$ for all $i \geq 0$. This implies that

$$N_{t+i} = (N_{t+i} :_R M)M \subseteq (N_t :_R M)M + rM = N_t + rM$$

for all $i \geq 0$, as needed.

(b) Let

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K_t \subseteq \ldots$$

be an ascending chain of submodules of M. Then we have

$$Ann_R(K_1) \supseteq Ann_R(K_2) \supseteq Ann_R(K_3) \supseteq \ldots \supseteq Ann_R(K_t) \supseteq \ldots$$

is a descending chain of ideals of R. Thus by assumption, there exist $r \in \sqrt{0} \subseteq \sqrt{Ann_R(M)}$ and a positive integer t such that $Ann_R(N_t) \cap (0 :_R r) \subseteq Ann_R(N_{t+i})$ for all $i \geq 0$. As M is faithful, $Ann_R(r) = Ann_R(rM)$. Thus by using [4, Proposition 3.3], we have

$$N_{t+i} = (0 :_M Ann_R(N_{t+i})) \subseteq (0 :_M Ann_R(N_t) \cap (0 :_R r)) =$$

$$(0 :_M Ann_R(N_t) \cap Ann_R(rM)) = N_t + rM.$$

for all $i \geq 0$, as needed. □

Definition 2.26. Let M be an R-module and f be an endomorphism of M. We say that f is *nil-epic* if there exists $r \in \sqrt{Ann_R(M)}$ such that $(0 :_M r) \subseteq Imf$. Also, we say that f is *nil-monic* if there exists $r \in \sqrt{Ann_R(M)}$ such that $Kerf \subseteq rM$.

Theorem 2.27. Let M be an R-module and f be an endomorphism of M.

(a) If M is a nil-Artinian module, then there exists $r \in \sqrt{Ann_R(M)}$ such that $(0 :_M r) \subseteq Imf^n + Kerf^n$ for some n, whence if f is monic, then f is nil-epic.

(b) If M is a nil-Noetherian module, then there exists $r \in \sqrt{Ann_R(M)}$ such that $Imf^n \cap Kerf^n \subseteq rM$ for some n, whence if f is epic, then f is nil-monic.

Proof. (a) Clearly, we have

$$Imf \supseteq Imf^2 \supseteq \cdots.$$

Thus by assumption, there exists $r \in \sqrt{Ann_R(M)}$ such that $Imf^n \cap (0 :_M r) \subseteq Imf^{2n}$ for some positive integer n. Now, let $x \in (0 :_M r)$. Then $rx = 0$ and so $rf(x) = 0$. Hence, $f(x) \in (0 :_M r)$. This implies that $f^n(x) \in (0 :_M r)$. Therefore, $f^n(x) \in Imf^n \cap (0 :_M r) \subseteq Imf^{2n}$. Thus $f^n(x) = f^{2n}(y)$. Clearly,

$$x = f^n(y) + (x - f^n(y)) \in Imf^n + Kerf^n.$$

Finally, if f is monic, then $Kerf^n = 0$. This implies that $(0 :_M r) \subseteq Imf^n \subseteq Imf$.

(b) Since

$$Kerf \subseteq Kerf^2 \subseteq \cdots,$$

by assumption, there exists $r \in \sqrt{Ann_R(M)}$ such that $Kerf^{2n} \subseteq Kerf^n + rM$ for some positive integer n. Now, let $x \in Imf^n \cap Kerf^n$. Then $x = f^n(y)$ and $f^n(x) = 0$. Thus $f^{2n}(y) = 0$. Therefore, $y \in Kerf^{2n} \subseteq Kerf^n + rM$. Thus

$$x = f^n(y) \in f^n(Kerf^n) + f^n(rM) = rf^n(M) \subseteq rM.$$

Finally, if f is epic, then $Imf^n = M$. It follows that $Kerf \subseteq Kerf^n \subseteq rM$, as needed. □

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