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On J-Noetherian modules and J-coherent modules

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Abstract. This paper continues the study of the recently introduced notion of J-ideals by Khashan and Bani-Ata, extending it to a module-theoretic and homological framework. First, we introduce and investigate J-injective modules, providing a Baer-type characterization and studying their relation to J-torsion-free and J-divisible modules. We then define and examine J-Noetherian modules and rings, showing that many classical properties such as the ascending chain condition and the Cartan–Eilenberg–Bass Theorem admit J-analogues. Finally, we develop the theory of J-flat and J-coherent modules, characterizing J-coherent rings through homological conditions and examples. Our results reveal that many homological behaviors persist in this broader setting, shedding light on the structural richness of the J-ideal framework.

Key Words: J-submodule, J-injective module, J-Noetherian module, J-Noetherian ring, J-flat module, J-coherent module, J-coherent ring.
2010 MSC: 13A15, 13A18, 13F05, 13G05, 13C20.

1 Introduction

Let R be commutative ring with identity element, $J(R)$ its Jacobson radical, $Z(R)$ the set of all zero-divisors of R, and $\mathrm{Nil}(R)$ the set of all nilpotent elements of R.

Recently, Khashan and Bani-Ata [14] introduced the notion of J-ideals as follows: a proper ideal I of a ring R is called a J-ideal if whenever $a, b \in R$ with $ab \in I$ and $a \notin J(R)$, then $b \in I$. The authors study various properties and provide examples of this class of ideals. Furthermore, they explore its relationship with several well-known types of ideals, including r-ideals, prime ideals, primary ideals, and maximal ideals. They also extend the notion to J-submodules of an R-module M and investigate their properties, particularly in the case of multiplication modules.

In [18], Mimouni further investigates this concept in various contexts of commutative rings, specifically in trivial ring extensions, amalgamated constructions of rings, and pullback diagrams. Examples are provided to illustrate the scope and limitations of the results.

This notion has also been studied in other frameworks, such as the S-version, the graded version, and within multiplicative lattices [1, 2, 7, 11, 16, 22].

In contrast to the above works, the present paper approaches the theory of J-ideals from a module-theoretic and homological perspective. Our goal is to investigate this notion not only within the structure of modules but also through the lens of homological algebra, thereby revealing new insights and extending the applicability of J-ideals beyond the classical ideal-theoretic framework.

This paper aims to extend the classical theory of modules and rings by incorporating the concept of J-ideals, recently introduced by Khashan and Bani-Ata, into a broader module-theoretic and

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homological framework.

In Section 2, we introduce and investigate the notion of J-injective modules, which generalizes the classical concept of injectivity. We establish a version of Baer's Criterion in this setting and explore the relationship between J-injectivity and J-torsion-free modules via Ext-vanishing conditions. Particular attention is given to ZJ-rings, in which the zero-divisors are contained in the Jacobson radical, and under which many simplifications occur.

Section 3 is devoted to the study of J-Noetherian modules and rings. We show that J-Noetherianity satisfies analogues of the ascending chain condition and the maximal condition for J-submodules. We also present a J-version of the Cartan–Eilenberg–Bass Theorem, characterizing J-Noetherian rings through the behavior of direct sums and direct limits of J-injective modules.

In Section 4, we develop the theory of J-flat and J-coherent modules. We define J-flatness via the exactness of tensor sequences involving finitely generated J-ideals and characterize J-coherent modules as those for which every finitely generated J-submodule is finitely presented. Several equivalent conditions for a ring to be J-coherent are provided, involving vanishing Tor functors, behavior of direct products, and the existence of J-flat preenvelopes.

These developments demonstrate that many familiar homological properties can be meaningfully extended to the J-setting, offering a rich generalization of classical module theory.

2 On J-injective modules

In this section, we initiate the development of a new theory through the study of certain submodules.

Definition 2.1. Let R be a ring and M an R-module. The module M is said to be *J-torsion free* if, for every $s \in R \setminus J(R)$ and every $m \in M$, the condition $sm = 0$ implies $m = 0$.

An R-submodule N of M is called a *J-submodule* of M if either $N = 0$ or the quotient module M/N is J-torsion free; that is, for every $s \in R \setminus J(R)$ and every $m \in M$, the condition $sm \in N$ implies $m \in N$.

Remark 2.2.

1. An ideal I of R is a J-submodule if and only if either $I = 0$ or I is a J-ideal in the sense of [14].
2. When R is a local ring, it is easy to see that every submodule of an R-module is a J-submodule. In particular, every ideal of R is a J-submodule of R.

Definition 2.3. A proper submodule $N \subsetneq M$ is called a *J-submodule in the sense of [14]* if, for all $a \in R$ and $m \in M$,

$$am \in N \quad \text{and} \quad a \notin \big(J(R)M : M\big) \quad \Longrightarrow \quad m \in N.$$

It is important to compare our definition of J-submodules with the one given by the authors of [14]. The following Proposition 2.4 clarifies this comparison.

Proposition 2.4. *For every R-module M and every submodule $N \subsetneq M$, if N is a J-submodule in the sense of Definition 2.1, then N is a J-submodule in the sense of Definition 2.3.*

Proof. Recall that

$$(J(R)M : M) = \{a \in R \mid aM \subseteq J(R)M\}, \quad J(R) \subseteq (J(R)M : M).$$

Hence,

$$R \setminus (J(R)M : M) \subseteq R \setminus J(R). \tag{$*$}$$

Assume N satisfies Definition 2.1. Take $a \in R \setminus (J(R)M : M)$ and $m \in M$ such that $am \in N$. By ($*$), we have $a \in R \setminus J(R)$, so the defining property of Definition 2.1 forces $m \in N$. Thus, N also satisfies Definition 2.3. □

Remark 2.5. The converse of Proposition 2.4 is *false* in general as we will show in the following Example 2.6.

Example 2.6. Let p be a prime, set $R = \mathbb{Z}$ (so $J(R) = 0$), and take

$$M = \mathbb{Z}/p\mathbb{Z}, \qquad N := 0 \subsetneq M.$$

- N **satisfies Definition 2.3.** Here $J(R)M = 0$ and $(J(R)M : M) = p\mathbb{Z}$. For $a \in \mathbb{Z} \setminus p\mathbb{Z}$ and $m \in M$, $am = 0$ implies $m = 0$ because a is invertible modulo p. Hence N is a J-submodule in the sense of Definition 2.3.
- N **fails Definition 2.1.** The quotient $M/N = M$ is not J-torsion free: choose $s = p \in R \setminus J(R)$ and $m = \overline{1} \in M$, then $sm = 0 \in N$ but $m \notin N$. Thus N is *not* a J-submodule in the sense of Definition 2.1.

Hence Definition 2.3 does not imply Definition 2.1.

As in the classical case, we introduce a new generalization of injective modules as follows.

Definition 2.7. Let R be a ring and E an R-module. We say that E is *J-injective* if, for every extension of R-modules $M \subset N$, where M is a J-submodule of N, any R-homomorphism from M to E can be extended to a homomorphism from N to E.

Remark 2.8.
1. Clearly, every injective module is J-injective. From Remark 2.2, every J-injective module over a local ring is injective.
2. Let R be a ring. Since every R-module is an epimorphic image of a free R-module, a standard verification shows that an R-module E is J-injective if and only if $\mathrm{Ext}^1_R(M,E) = 0$ for every J-torsion-free R-module M.

We now establish an analogue of the well-known Baer's Criterion to characterize J-injectivity in a manner similar to the classical case.

Theorem 2.9. Let R be a ring and E an R-module. Then E is J-injective if and only if every R-homomorphism from a J-ideal of R to E can be extended to a homomorphism from R to E.

Proof. The necessity follows directly from Remark 2.2 (1).

Conversely, let M be a J-submodule of an R-module N, and let $f : M \to E$ be an R-homomorphism. Consider the collection $\mathscr{C}$ of all pairs (C, g) such that $M \subseteq C \subseteq N$, C is a J-submodule of N, and $g : C \to E$ is an R-homomorphism satisfying $g|_M = f$. Clearly, $\mathscr{C} \neq \emptyset$ since $(M, f) \in \mathscr{C}$. Partially order $\mathscr{C}$ by declaring $(C, g) \leq (C', g')$ if $C \subseteq C'$ and $g'|_C = g$. Then $\mathscr{C}$ is inductive, and by Zorn's Lemma, it has a maximal element, say (C_0, g_0).

Suppose $C_0 \neq N$. Choose $x \in N \setminus C_0$ and define $I = \{r \in R : rx \in C_0\}$. We claim that I is a J-ideal of R: indeed, for any $a, s \in R$ with $s \notin J(R)$ and $as \in I$, it follows that $s(ax) \in C_0$, and since C_0 is a J-submodule of N, we conclude $ax \in C_0$, hence $a \in I$.

Now define $h : I \to E$ by $h(r) = g_0(rx)$. Then h is an R-homomorphism and extends to $h' : R \to E$ by assumption. Define $\bar{g} : C_0 + Rx \to E$ by $\bar{g}(c_0 + rx) = g_0(c_0) + h'(r)$.

If $c_0 + rx = c_0' + r'x$, then $(c_0 - c_0') = (r' - r)x \in C_0$ and hence $r' - r \in I$. It follows that

$$g_0(c_0 - c_0') = g_0((r' - r)x) = h(r' - r) = h'(r' - r),$$

so

$$g_0(c_0) + h'(r) = g_0(c_0') + h'(r').$$

Therefore, $\bar{g}$ is well-defined and agrees with f on M. Consider the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & C_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N/C_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_0 + Rx \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & N/C_0 \arrow[r] & 0 \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since N/C_0 is a J-torsion free R-module and C_0 is a J-submodule of N, we conclude that $C_0 + Rx$ is a J-submodule of Q (and hence of N). Thus, $(C_0 + Rx, \bar{g}) \in \mathscr{C}$, contradicting the maximality of (C_0, g_0). Therefore, $C_0 = N$, completing the proof. □

Corollary 2.10. *Let R be a ring and E an R-module. Then E is J-injective if and only if* $\mathrm{Ext}^1_R(R/I, E) = 0$ *for every J-ideal I of R.*

Proof. This follows immediately from Theorem 2.9. □

Corollary 2.11. *The class of J-injective modules is closed under direct products.*

Proof. This follows directly from the fact that Ext commutes with products in the second variable. □

In the remainder of this paper, we focus on rings R satisfying the condition $Z(R) \subseteq J(R)$. We refer to such rings as ZJ-rings. In particular, every local ring is a ZJ-ring.

Corollary 2.12. *Let R be a ZJ-ring. The following statements are equivalent for an R-module E:*

1. *E is J-injective;*
2. $\mathrm{Ext}^k_R(M, E) = 0$ *for every J-torsion-free R-module M and every* $k \in \mathbb{N}^*$;
3. $\mathrm{Ext}^1_R(M, E) = 0$ *for every J-torsion-free R-module M;*
4. $\mathrm{Ext}^k_R(R/I, E) = 0$ *for every J-ideal I of R and every* $k \in \mathbb{N}^*$;
5. $\mathrm{Ext}^1_R(R/I, E) = 0$ *for every J-ideal I of R.*

Proof. The implications $(2) \Rightarrow (4)$, $(4) \Rightarrow (5)$, $(2) \Rightarrow (3)$, and $(3) \Rightarrow (5)$ are all immediate.

$(5) \Rightarrow (1)$ follows directly from Corollary 2.10.

$(1) \Rightarrow (2)$: The case $k = 1$ follows from Remark 2.8 (2). Assume now that $k > 1$, and let M be a J-torsion-free R-module. Consider a projective resolution of M:

$$0 \to K \to F_{k-2} \to \cdots \to F_0 \to M \to 0,$$

where each F_i is a free R-module. Then K is a torsion-free R-module. Since R is a ZJ-ring, every torsion-free R-module is J-torsion-free. In particular, K is J-torsion-free, and thus

$$\mathrm{Ext}_R^k(M,E) \cong \mathrm{Ext}_R^1(K,E) = 0$$

by dimension shifting. Therefore, statement (2) holds for all $k \in \mathbb{N}^*$. □

It is well known that in the classical setting, injective modules are divisible. We now introduce an analogous concept for J-injectivity.

Definition 2.13. Let R be a ring, $s \in R$, and M an R-module. We say that s is a *J-regular* element if s is regular in R and the ideal (s) generated by s is a J-ideal.

An R-module M is said to be *J-divisible* if $aM = M$ for every J-regular element $a \in R$.

Remark 2.14. Clearly, when R is a local ring, J-regularity and J-divisibility coincide with classical regularity and divisibility, respectively. In particular, by [24, Theorem 2.4.5 (3)], over an integral domain, a (J)-torsion-free R-module E is J-injective if and only if E is injective.

Corollary 2.15. *Let R be a ring and E an R-module. If E is J-injective, then E is J-divisible.*

Proof. This follows immediately from Corollary 2.10 and [24, Exercise 3.3 (3)]. □

3 J-Noetherian Modules

Having introduced the notion of J-injective modules as a generalization of injectivity, we now follow the classical approach to define a new class of modules and rings that extends the concept of Noetherianity.

Definition 3.1. Let R be a ring and M an R-module. We say that M is *J-Noetherian* if M is finitely generated and every J-submodule of M is also finitely generated.

The ring R is called a *J-Noetherian ring* if it is a J-Noetherian module over itself.

Remark 3.2. Clearly, every Noetherian R-module is J-Noetherian. In particular, for a local ring R, Noetherianity and J-Noetherianity coincide by Remark 2.2 (2).

Let A be a commutative ring and let E be an A-module. The *trivial ring extension* of A by E, also called the *idealization* of E, is the set

$$A \propto E := \{(a,e) \mid a \in A,\ e \in E\},$$

with addition defined componentwise:

$$(a,e) + (a',e') := (a+a', e+e'),$$

and multiplication defined by:

$$(a,e)(a',e') := (aa', ae' + a'e).$$

It is easy to verify that $A \propto E$ is a commutative ring with unity $(1,0)$, where 0 denotes the zero of E.

The ring $A \propto E$ contains a natural ideal $0 \propto E := \{(0,e) \mid e \in E\}$, which is isomorphic to E as an A-module. Also, $A \propto E/(0 \propto E) \cong A$, so the projection map onto the first component gives a ring homomorphism $A \propto E \to A$.

This construction, introduced by D. D. Anderson and M. Winders [3], is often used to transfer or study ring-theoretic and module-theoretic properties in a new context (for instance, [13]), particularly for investigating properties like coherence, clean rings, or Gorenstein dimensions.

It is important to illustrate that the J-Noetherian property does not imply classical Noetherianity. The following example provides such a case.

Example 3.3. Let $\pi : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ be the projection map defined by $\pi(a,b) = a$. It is easy to verify that π is a ring homomorphism. Let $A = \mathbb{Z}\times\mathbb{Z}$. Then every abelian group can be viewed as an A-module via the structure induced by π. Let p be a prime integer, and set $R = A \propto \mathbb{Z}_{p^\infty}$. Then R is a J-Noetherian ring that is not Noetherian.

Proof. First, observe that $\mathbb{Z}_{p^\infty}$ is not a finitely generated abelian group. Consequently, it is not a finitely generated A-module, and thus R is not a Noetherian ring.

On the other hand, as noted in [4], the group $\mathbb{Z}_{p^\infty}$ is *almost Noetherian*, meaning that every proper subgroup is finitely generated. Therefore, every proper abelian subgroup of $\mathbb{Z}_{p^\infty}$ is a finitely generated R-module.

Next, observe that $J(R) = \mathrm{Nil}(R) = 0 \propto \mathbb{Z}_{p^\infty}$, which is not a J-ideal of R. Indeed, we have:

$$((1,0),0)\cdot((0,1),0) = ((0,0),0) \in J(R),$$

yet neither $((1,0),0)$ nor $((0,1),0)$ belongs to $J(R)$.

Suppose, for contradiction, that there exists a J-ideal $I \subsetneq J(R)$ that is not finitely generated. Then I has an infinite generating set of the form $\{(0,0),e_i\}_i$. It is easy to see that $R((0,0),e_i) = (0,0) \propto Ae_i$ for each i, and so

$$I = (0,0) \propto \sum_i Ae_i \subsetneq J(R).$$

Hence, $H = \sum_i Ae_i$ is a proper A-submodule of $\mathbb{Z}_{p^\infty}$. Since H is a finitely generated abelian subgroup of $\mathbb{Z}_{p^\infty}$, it is also a finitely generated A-module — a contradiction.

Therefore, every J-ideal of R contained in $J(R)$ must be finitely generated. By [14, Proposition 2.2], we conclude that R is a J-Noetherian ring. □

We now show that J-Noetherian modules satisfy the ascending chain condition and the maximal condition, in a form analogous to the classical Noetherian case.

Theorem 3.4. Let R be a ring. The following statements are equivalent for an R-module M:

1. M is a J-Noetherian module;
2. M satisfies the ascending chain condition (ACC) on J-submodules, that is, for any ascending chain
$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots,$$
where each M_i is a J-submodule of M, there exists a positive integer m such that $M_n = M_m$ for all $n \geq m$;
3. M satisfies the J-maximal condition, that is, every nonempty set of J-submodules of M contains a maximal element.

Proof. (1) ⇒ (2): Let $\{M_n\}$ be an ascending chain of J-submodules of M, and define $N := \bigcup_n M_n$. By assumption, N is a J-submodule of M, and hence finitely generated. Write $N = Rx_1 + \cdots + Rx_k$. Then, for some m, all x_i belong to M_m, so $N \subseteq M_m$. Since $M_m \subseteq N$, we have $N = M_m$. Therefore, the chain stabilizes at M_m.

(2) ⇒ (3): Let Γ be a nonempty set of J-submodules of M with no maximal element. Choose any $M_1 \in \Gamma$. Since M_1 is not maximal, there exists $M_2 \in \Gamma$ such that $M_1 \subset M_2$. Repeating this process, we obtain an infinite strictly ascending chain

$$M_1 \subset M_2 \subset \cdots \subset M_n \subset \cdots,$$

contradicting the assumption that every ascending chain of J-submodules stabilizes.

(3) $\Rightarrow$ (1): Let N be a J-submodule of M, and consider the set

$$\Gamma := \{A \subseteq N \mid A \text{ is a finitely generated } J\text{-submodule of } N\}.$$

Clearly, Γ is nonempty since $0 \in \Gamma$. By assumption, Γ has a maximal element, say A. If $A \neq N$, then there exists $x \in N \setminus A$, and $A_1 := A + Rx$ is a finitely generated submodule of N strictly containing A. As in the proof of Theorem 2.9, one can verify that A_1 is a J-submodule of M, contradicting the maximality of A. Hence, $N = A$ is finitely generated, and M is J-Noetherian. □

Corollary 3.5. *Let R be a ring. The following statements are equivalent:*

1. *R is a J-Noetherian ring;*
2. *R satisfies the ascending chain condition (ACC) on J-ideals, that is, for any ascending chain*
$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots,$$
there exists a positive integer m such that $I_n = I_m$ for all $n \geq m$;
3. *R satisfies the J-maximal condition, that is, every nonempty set of J-ideals of R has a maximal element.*

Proof. This follows immediately from Theorem 3.4. □

We now establish the relationship between exact sequences and J-Noetherianity, in analogy with the classical theory.

Theorem 3.6. Let R be a ring, and let

$$0 \to M' \to M \to M'' \xrightarrow{\pi} 0$$

be a short exact sequence of R-modules, where M' is a J-submodule of M. Then M is a J-Noetherian module if and only if both M' and M'' are J-Noetherian modules.

Proof. Assume that M is J-Noetherian. Let X be a J-submodule of M'. To show that X is finitely generated, it suffices to prove that X is a J-submodule of M. Let $x \in M$ and $s \in R \setminus J(R)$ such that $sx \in X$. Then $s\pi(x) = 0$ in M''. Since M'' is J-torsion free, it follows that $\pi(x) = 0$, and hence $x \in M'$. So $x \in X$ as $sx \in X$ and X is a J-submodule of M'. Thus, X is a J-submodule of M, and therefore finitely generated. Hence M' is J-Noetherian.

Now let Y be a J-submodule of M'', and consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

Since $M''/Y \cong P/M$, and M''/Y is J-torsion free, we conclude that P is a J-submodule of M. By assumption, M is J-Noetherian, so P is finitely generated. It follows that Y is finitely generated, and hence M'' is J-Noetherian.

Conversely, suppose M' and M'' are J-Noetherian. Let

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$$

be an ascending chain of J-submodules of M. Then

$$M' \cap M_1 \subseteq M' \cap M_2 \subseteq \cdots$$

is an ascending chain of J-submodules of M', and

$$\frac{M_1 + M'}{M'} \subseteq \frac{M_2 + M'}{M'} \subseteq \cdots$$

is an ascending chain of J-submodules of M/M': for each i, we have

$$\frac{M_i + M'}{M'} \cong \frac{M_i}{M_i \cap M'}.$$

Let $s \in R \setminus J(R)$ and suppose $\bar{x} \in M/M'$ satisfies $s\bar{x} \in M_i/(M_i \cap M')$. Then $sx - m_i \in M_i \cap M'$ for some $m_i \in M_i$, so $sx \in M_i$ and since M_i is a J-submodule, it follows that $x \in M_i$, hence $\bar{x} \in M_i/(M_i \cap M')$. Thus, $M_i/(M_i \cap M')$ is a J-submodule of M/M'.

Since M' and M/M' are J-Noetherian, the above chains stabilize at some $m \in \mathbb{N}$. That is,

$$M' \cap M_n = M' \cap M_m \quad \text{and} \quad M' + M_n = M' + M_m \quad \text{for all } n \geq m.$$

It follows that $M_n = M_m$ for all $n \geq m$, so M is J-Noetherian by Theorem 3.4. □

Corollary 3.7. *Let R be a ring and $\{M_i\}_{i\in I}$ a finite collection of J-torsion-free R-modules. Then $\bigoplus_{i\in I} M_i$ is a J-Noetherian R-module if and only if each M_i is J-Noetherian.*

Proof. We proceed by induction on the number of summands, applying Theorem 3.6. □

Corollary 3.8. *Let R be a ring and I a J-ideal of R. Then R is a J-Noetherian ring if and only if both I and R/I are J-Noetherian R-modules.*

Proof. This follows immediately from Theorem 3.6. □

Corollary 3.9. *Let R be a ZJ-ring that is J-Noetherian, and let M be a J-torsion-free R-module. Then M is J-Noetherian if and only if it is a finitely generated R-module.*

Proof. The forward direction is immediate, since J-Noetherian modules are, in particular, finitely generated. Conversely, suppose M is finitely generated. Then M is a quotient of a finite direct sum $R^{(n)}$. Since $Z(R) \subseteq J(R)$, R is J-torsion-free, and by Corollary 3.7, $R^{(n)}$ is J-Noetherian. Therefore, M is J-Noetherian by Theorem 3.6. □

Theorem 3.10 (J-version of Cartan–Eilenberg–Bass)**.** The following statements are equivalent for a ring R:

1. R is J-Noetherian;
2. every direct limit of J-injective R-modules is J-injective;
3. every direct sum of J-injective R-modules is J-injective.

Proof. (1) ⇒ (2): Let $E = \varinjlim E_j$, where each E_j is a J-injective R-module, and let I be a J-ideal of R. Then

$$\operatorname{Ext}^1_R(R/I, E) = \varinjlim \operatorname{Ext}^1_R(R/I, E_j)$$

by [24, Theorem 3.9.4 (1)]. Since each E_j is J-injective, each term vanishes, so $\operatorname{Ext}^1_R(R/I, E) = 0$. Thus, E is J-injective by Corollary 2.10.

(2) $\Rightarrow$ (3): This follows from [24, Example 2.5.30 (3)], since direct sums are special cases of direct limits.

(3) $\Rightarrow$ (1): Suppose R is not J-Noetherian. Then, by Theorem 3.5, there exists a strictly ascending chain $I_1 \subset I_2 \subset \cdots$ of J-ideals that does not stabilize. Let $I = \bigcup_{i=1}^{\infty} I_i$. Then $I/I_i \neq 0$ for all i.

Define f_i as the composition of the natural map $\tau_i : I \to I/I_i$ with the inclusion into its injective envelope $E(I/I_i)$. Define $f : I \to \bigoplus_{i=1}^{\infty} E(I/I_i)$ by $f(a) = (f_i(a))_i$. Since each $a \in I$ belongs to only finitely many I_i, $f(I)$ is well-defined.

If $\bigoplus_{i=1}^{\infty} E(I/I_i)$ were J-injective, then f would extend to an R-homomorphism $g : R \to \bigoplus_{i=1}^{\infty} E(I/I_i)$ by Theorem 2.9. Let π_j be the projection onto the j-th component. Then, for sufficiently large i, we have $\pi_i \circ g(1) = 0$, so for $a \in I$,

$$f_i(a) = \pi_i \circ f(a) = \pi_i \circ g(a) = a \cdot \pi_i(g(1)) = 0.$$

This implies that each f_i is eventually zero, contradicting the assumption that $I/I_i \neq 0$ for all i. Hence, the direct sum is not J-injective, and the result follows. $\square$

Remark 3.11. Let $R = A \propto \mathbb{Z}_{p^\infty}$, where $A = \mathbb{Z} \times \mathbb{Z}$, and $\mathbb{Z}_{p^\infty}$ is the Prüfer p-group viewed as an A-module via the projection map $\pi((a,b)) = a$. Over this ring (see Example 3.3), there exists a J-injective R-module which is not injective. Consider the R-module

$$E := \frac{R}{J(R)} \cong A = \mathbb{Z} \times \mathbb{Z}.$$

(i) E **is** J**-injective.** By Theorem 2.9, an R-module E is J-injective if every R-homomorphism $\varphi : I \to E$, where I is a J-ideal of R, extends to a homomorphism $R \to E$. In this case, the only proper J-ideal of R is $J(R) = 0 \propto \mathbb{Z}_{p^\infty}$. Since $J(R) \cdot E = 0$, any such homomorphism must be the zero map, which clearly extends to R. Thus, E is J-injective.
(ii) E **is not injective.** Consider the short exact sequence

$$0 \longrightarrow J(R) \xrightarrow{\iota} R \xrightarrow{\pi} E \longrightarrow 0.$$

If E were injective, this sequence would split, implying that $R \cong E \oplus J(R)$ as R-modules. Hence, $J(R)$ would be isomorphic to eR for some idempotent $e \in R$. However, since $J(R) \subseteq \mathrm{Nil}(R)$, and the Jacobson radical contains no nontrivial idempotents, we must have $e = 0$, contradicting the assumption $J(R) \cong eR \neq 0$. Therefore, the sequence does not split and E is not injective.

4 On J-flat Modules and J-coherent Modules

We now generalize the concept of flat modules in the context of the J-setting.

Definition 4.1. Let R be a ring and F an R-module. We say that F is *J-flat* if, for every finitely generated J-ideal I of R, the sequence

$$0 \to I \otimes_R F \to R \otimes_R F$$

is exact.

Remark 4.2. Clearly, every flat R-module is J-flat. Moreover, by Remark 2.2, J-flatness coincides with classical flatness in certain cases.

It is straightforward to verify that the class of J-flat modules is closed under direct sums.

Proposition 4.3. *Let R be a ring and F an R-module. Then F is J-flat if and only if $\mathrm{Tor}_1^R(R/I,F)=0$ for every finitely generated J-ideal I of R. Moreover, if R is a ZJ-ring, then F is J-flat if and only if $\mathrm{Tor}_k^R(R/I,F)=0$ for every $k\in\mathbb{N}^*$ and every finitely generated J-ideal I of R.*

Proof. The first assertion follows by adapting the classical flatness criterion via Tor. The second assertion follows similarly to the argument used in the proof of Corollary 2.12. □

In what follows, we denote by $\mathscr{F}^J$ the class of all finitely presented cyclic J-torsion-free R-modules.

Definition 4.4. Let R be a ring and M an R-module. We say that M is *J-coherent* if M is finitely generated and every finitely generated J-submodule of M is finitely presented.

We say that R is a *J-coherent ring* if R is a J-coherent module over itself.

Remark 4.5. A ring R is J-coherent if and only if it is $\mathscr{F}^J$-coherent in the sense of [27], where $\mathscr{C}=\mathscr{F}^J$.

Example 4.6. Following [6], a ring R is said to be *left J-coherent* if its Jacobson radical $J(R)$ is a coherent left R-module.

It is easy to see that J-coherence in the sense of [6] implies our notion of J-coherence. In particular, by [6, Example 2.8], let R be a non-coherent commutative domain and let G be a free abelian group of infinite rank. Then the group ring RG is not coherent, yet it is J-coherent in our sense.

Theorem 4.7. Let R be a ZJ-ring. If R is a J-Noetherian ring, then R is a J-coherent ring.

Proof. Let I be a J-ideal of R. Since R is J-Noetherian, I is finitely generated. Consider the following pullback diagram of R-modules:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R^{(n-1)} \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R^{(n-1)} \arrow[r] & R^{(n)} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
 & & R/I \arrow[r, equal] \arrow[d] & R/I \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since R is a ZJ-ring, Corollary 3.7 implies that $R^{(n)}$ is J-Noetherian. Because P is a J-submodule of $R^{(n)}$, Theorem 3.6 shows that P is also J-Noetherian. Therefore, as a J-submodule of itself, P is finitely presented. By [10, Theorem 2.1.2 (2)], it follows that I is a finitely presented ideal. Hence, R is J-coherent. □

Here is another example of a J-coherent ring in our sense, which is not J-coherent in the sense of [6], constructed via a trivial ring extension.

Example 4.8. Let $R=\mathbb{Z}\propto\mathbb{Q}$, the trivial ring extension of $\mathbb{Z}$ by $\mathbb{Q}$. Then R is a J-coherent ring in our sense, since the only finitely generated J-ideal of R is the zero ideal.

Indeed, by [3, Corollary 3.4] and [14, Proposition 2.2], any nonzero J-ideal I of R is of the form $I=0\propto G$ where G is a subgroup of $\mathbb{Q}$.

If $G=\mathbb{Q}$, then I is a prime ideal but not finitely generated. Suppose instead that G is a proper subgroup of $\mathbb{Q}$. Then there exists $\frac{p}{q}\in\mathbb{Q}\setminus G$.

Assume $\mathbb{Z} \subset G$. Then we have:

$$(q,0)(0,\tfrac{p}{q}) = (0,p) \in 0 \propto G,$$

but $(0,\frac{p}{q}) \notin 0 \propto G$, contradicting the definition of a J-ideal. Thus, $\mathbb{Z} \not\subset G$, and there exists $\alpha \in \mathbb{Z} \setminus G$.

Now choose $\beta \in \mathbb{Z}^* \cap G$. Such β exists: if $\frac{x}{s} \in G \setminus \{0\}$, then $x \in sG \subset G$ (since G is a group), and we may take $\beta = x$. Then $\frac{\alpha}{\beta} \notin G$, but:

$$(\beta^2,0)(0,\tfrac{\alpha}{\beta}) = (0,\beta\alpha) \in 0 \propto G.$$

Hence, $0 \propto G$ is not a J-ideal of R, a contradiction. It follows that the only J-ideal of R is zero, which is trivially finitely generated. Thus, R is J-coherent in our sense, but not J-coherent in the sense of [6], as it is not coherent.

The following result is an analogue of the well-known behavior described in [10, Theorem 2.2.1].

Theorem 4.9. Let R be a ring and consider a short exact sequence of R-modules

$$0 \to P \xrightarrow{f} N \xrightarrow{g} M \to 0$$

such that P is a J-submodule of N. Then:

1. If N is a J-coherent R-module and P is finitely generated, then M is a J-coherent R-module.
2. If M is a coherent R-module and P is a J-coherent R-module, then N is a J-coherent R-module.
3. If both N and M are J-coherent R-modules, then so is P.

Proof. (1) Since N is finitely generated, so is M. Let F be a finitely generated J-submodule of M. Then $F \cong A/P$, where A is a finitely generated J-submodule of N and P is a submodule of A. Since N is J-coherent, A is finitely presented. Therefore, by [15, (4.54) Lemma], the quotient F is finitely presented. Thus, M is a J-coherent module.

(2) Since M is coherent and P is finitely generated, N is finitely generated by [10, Theorem 2.1.2 (1)]. Let X be a finitely generated J-submodule of N, and consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(g_{|X}) \arrow[r] \arrow[d] & X \arrow[r, "g"] \arrow[d] & g(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r, "f"] & N \arrow[r, "g"] & M \arrow[r] & 0
\end{tikzcd}$$

Since X is finitely generated, $g(X)$ is also finitely generated. As M is coherent, $g(X)$ is finitely presented. Moreover, $\ker(g_{|X}) \subseteq P$ is a J-submodule and hence finitely presented by the J-coherence of P. It follows that X is finitely presented, and by [10, Theorem 2.1.2], N is J-coherent.

(3) By [10, Theorem 2.1.2 (3)], P is finitely generated. Let G be a finitely generated J-submodule of P. Then G is also a J-submodule of N. Since N is J-coherent, G is finitely presented. Hence P is J-coherent. □

Theorem 4.10. Let R be a ring and let I be a finitely generated ideal contained in $J(R)$. Then an R/I-module M is J-coherent if and only if it is J-coherent as an R-module. In particular:

1. If R is a J-coherent ring and $I \subseteq J(R)$ is a finitely generated ideal, then R/I is a J-coherent ring.

2. If R/I is a coherent ring and I is a J-coherent R-module, then R is a J-coherent ring.

Proof. We use the result from [10, Theorem 2.1.8], which states that an R-module M is finitely presented if and only if M/IM is a finitely presented R/I-module.

Now assume that M is a J-coherent R-module. We claim that M is also a J-coherent R/I-module. Let N be a finitely generated J-submodule of M, considered as an R/I-module. Then N is also a finitely generated J-submodule of M as an R-module. Since M is assumed J-coherent over R, it follows that N is finitely presented as an R-module. Hence, by the above result, N is finitely presented over R/I, and thus M is a J-coherent R/I-module.

Conversely, suppose M is a J-coherent R/I-module. Let N be a finitely generated J-submodule of M as an R-module. Then N is also a finitely generated J-submodule of M considered as an R/I-module. Since M is J-coherent over R/I, N is finitely presented over R/I, and therefore, by [10, Theorem 2.1.8], N is also finitely presented over R. Thus, M is J-coherent over R.

For part (2), assume that R/I is a coherent ring and I is a J-coherent R-module. Then by part (1) and Theorem 4.9 (2), it follows that R is a J-coherent ring. □

To characterize J-coherent rings, we introduce a J-analogue of the well-known concept of FP-injective modules.

Definition 4.11. An R-module M is said to be *J-FP-injective* if $\mathrm{Ext}^1_R(R/I, M) = 0$ for every finitely generated J-ideal I of R.

Theorem 4.12. The following statements are equivalent for a ring R:

1. R is J-coherent;
2. For any finitely generated J-ideal I of R and any direct system $(M_\alpha)_{\alpha\in A}$ of R-modules, we have
$$\varinjlim \mathrm{Ext}^1_R(R/I, M_\alpha) \cong \mathrm{Ext}^1_R(R/I, \varinjlim M_\alpha);$$
3. For any family $\{N_\alpha\}$ of R-modules and any finitely generated J-ideal I of R, we have
$$\mathrm{Tor}^R_1\left(\prod N_\alpha, R/I\right) \cong \prod \mathrm{Tor}^R_1(N_\alpha, R/I);$$
4. Any direct product of copies of R is J-flat;
5. Any direct product of J-flat R-modules is J-flat;
6. Any direct limit of J-FP-injective R-modules is J-FP-injective;
7. Any direct limit of injective R-modules is J-FP-injective;
8. An R-module M is J-FP-injective if and only if M^+ is J-flat;
9. An R-module M is J-FP-injective if and only if M^{++} is J-FP-injective;
10. An R-module M is J-flat if and only if M^{++} is J-flat;
11. For any ring S, finitely generated J-ideal I of R, (R,S)-bimodule B, and injective S-module E, we have
$$\mathrm{Tor}^R_1(\mathrm{Hom}_S(B,E), R/I) \cong \mathrm{Hom}_S\left(\mathrm{Ext}^1_R(R/I,B), E\right);$$
12. Every R-module has a J-flat preenvelope.

Proof. This follows immediately from Remark 4.5 and [27, Theorem 3.3]. □

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