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A note on the universal property of first-order differential operators in Jacobi geometry

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Abstract. In this paper, we define first-order differential operators as an extension of derivations and we present the universal property of first-order differential operators. We use this universal property to show the existence and uniqueness of a canonical form associated with a Jacobi manifold. We recover properties analogous to those introduced by A. Lichnerowicz and we characterize the morphisms between Jacobi manifolds. Moreover, we present the relationship between a non-degenerate Jacobi structure and a locally conformal symplectic structure.

Key Words: First-order differential operators, Jacobi manifolds, Jacobi Morphisms, locally conformal symplectic structure.
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1 Introduction

Jacobi algebras were first introduced by Kirillov under the name local Lie algebras [5] and independently by Lichnerowicz as the algebraic structure on the ring of C^∞ functions on a certain kind of smooth manifolds, called Jacobi manifolds (see [7]). Jacobi algebras are abstract algebraic counterparts of Jacobi manifolds, which are generalizations of symplectic or more generally Poisson manifolds. Both Poisson and Jacobi algebras are commutative algebras endowed with a Lie bracket. However, while the Poisson bracket is a derivation of the underlying commutative algebra, the Jacobi bracket is a first-order differential operator on the commutative algebra [1, 2].

Let A be a commutative algebra with unit 1_A over a commutative field $\mathbb{K}$ with characteristic 0 and let E be an A-module [12, 13]. Recall that a first-order differential operator on A with coefficients in E is a $\mathbb{K}$-linear map $\varphi : A \longrightarrow E$ such that

$$\varphi(ab) = \varphi(a)\cdot b + a\cdot\varphi(b) - a\cdot b\cdot\varphi(1_A) \tag{1}$$

for all $a, b \in A$ (see [4]). We denote by $\mathrm{Diff}_{\mathbb{K}}(A,E)$ the A-module of all first-order differential operators on A with coefficients in E and

$$\mathrm{Der}_{\mathbb{K}}(A,E) = \{\varphi \in \mathrm{Diff}_{\mathbb{K}}(A,E) / \varphi(1_A) = 0\}$$

the A-module of all derivations on A with coefficients in E. When $E = A$, we denote by $\mathrm{Diff}_{\mathbb{K}}(A)$ the A-module of all first-order differential operators of A. The bracket $[-,-] : \mathrm{Diff}_{\mathbb{K}}(A)\times\mathrm{Diff}_{\mathbb{K}}(A) \longrightarrow \mathrm{Diff}_{\mathbb{K}}(A)$, such that for all $\varphi, \psi \in \mathrm{Diff}_{\mathbb{K}}(A)$,

$$[\varphi, \psi] = \varphi\circ\psi - \psi\circ\varphi$$

is Lie bracket. Moreover, for all $a \in A$ and for all $\varphi, \psi \in \mathrm{Diff}_{\mathbb{K}}(A)$, we have

$$[\varphi, a\cdot\psi] = [\varphi, a\cdot\psi] + [\varphi(a) - a\cdot\varphi(1_A)]\cdot\psi.$$

The purpose of this paper is to study the main features of Jacobi manifolds by using the the universal property of first-order differential operators and to give the equivalence between a non-degenerate Jacobi structure and a locally conformal symplectic structure.

The paper is organized as follows. In Section 2, we describe the universal property of first-order differential operators as well as the universal property of skew-symmetric first-order differential operators. We focus on the Schouten-Nijenhuis bracket over the module of Kähler differentials. In section 3, we give the main features of Jacobi manifolds by using the universal property of first-order differential operators and the Schouten-Nijenhuis bracket on the module of Kähler differentials. In section 4, we give necessary and sufficient conditions for any map to be Jacobi morphism. Finally, in Section 5, we introduce the Koszul bracket associated with a Kähler 2-form and we give the relation between Schouten-Nijenhuis bracket and Koszul bracket associated with a Kähler 2-form. By using the universal property of first-order differential operators, we prove that a non-degenerate Jacobi structure corresponds exactly to a locally conformal symplectic structure.

2 Universal property of first-order differential operators

Throughout this section, A is a commutative algebra with unit 1_A over a commutative field $\mathbb{K}$ with characteristic 0 and E is an A-module.

Recall that a differential module over an associative algebra A is a pair (E,δ), where E is an A-module and $\delta \in \mathrm{Der}_{\mathbb{K}}(A,E)$, such that the image $\mathrm{Im}\delta$ spans E as a module [3, 10].

In this part, the tensor products are on $\mathbb{K}$. We equip the $\mathbb{K}$-algebra $A\otimes A$ with the structure of A-module defined by the morphism $\mu : A \longrightarrow A\otimes A, a \longmapsto a\otimes 1_A$. The map $A\times A \longrightarrow A, (a,b) \longmapsto a\cdot b$ is $\mathbb{K}$-bilinear symmetric. It induces a $\mathbb{K}$-linear map $m : A\otimes A \longrightarrow A$, called multiplication, such that $m(a\otimes b) = a\cdot b$, for all a, $b \in A$. Note that the multiplication m is a morphism of $\mathbb{K}$-algebras. The kernel I of m is the A-submodule of $A\otimes A$ generated by the elements of the form $a\otimes 1_A - 1_A\otimes a$ with $a\in A$ (see [3]).

The A-module quotient $\Omega_K(A) = I/I^2$ is the $\mathbb{K}$-module of the differentials of A. For $a\in A$, we denote by $\overline{a\otimes 1_A - 1_A\otimes a}$ the class of $a\otimes 1_A - 1_A\otimes a$ in $\Omega_K(A)$. The map $d_{A/\mathbb{K}} : A \longrightarrow \Omega_{\mathbb{K}}(A)$ such that, for all $a\in A$,

$$d_{A/\mathbb{K}}(a) = \overline{a\otimes 1_A - 1_A\otimes a}$$

is a derivation called canonical derivation. Moreover, the image of $d_{A/\mathbb{K}}$ generates the A-module $\Omega_{\mathbb{K}}(A)$ (see [3, 10, 11]). An element $x\in\Omega_{\mathbb{K}}(A)$ is explicitly given by $x = \sum_{i\in J} a_i\cdot d_{A/\mathbb{K}}(b_i)$, with $a_i, b_i \in A$, $i\in J$ and J is a finite set of indices. Therefore, the pair $(\Omega_{\mathbb{K}}(A), d_{A/\mathbb{K}})$ is a differential module called the A-module of Kähler differentials of A.

Theorem 2.1. [3]Universal property of the couple $(\Omega_{\mathbb{K}}(A), d_{A/\mathbb{K}})$

For any A-module E and for any derivation $\varphi : A \longrightarrow E$, there exists a unique homomorphism $\tilde{\varphi} : \Omega_{\mathbb{K}}(A) \longrightarrow E$ of A-modules such that $\tilde{\varphi}\circ d_{A/\mathbb{K}} = \varphi$.

Thus, the map $Hom_A(\Omega_{\mathbb{K}}(A),E) \longrightarrow \mathrm{Der}_{\mathbb{K}}(A,E), \psi \longmapsto \psi\circ d_{A/\mathbb{K}}$ is an isomorphism of A-modules. In particular, $\Omega_{\mathbb{K}}(A)^* \simeq \mathrm{Der}_{\mathbb{K}}(A)$.

For $x\in E$, let us define the map $L_x : A \longrightarrow E, a \longmapsto ax$.

Proposition 2.2. *[4] A linear map $\varphi : A \longrightarrow E$ is a first-order differential operator of A into E if and only if the $\mathbb{K}$-linear map $\varphi - L_{\varphi(1_A)} : A \longrightarrow E$ is a derivation.*

Proposition 2.3. *The map $D_{A/\mathbb{K}} : A \longrightarrow A\oplus\Omega_{\mathbb{K}}(A)$ such that, for all $a\in A$,*

$$D_{A/\mathbb{K}}(a) = a + d_{A/\mathbb{K}}(a) \tag{2}$$

is a first-order differential operator.

Proof. For all $a, b \in A$, we have

$$\begin{array}{rl} & D_{A/\mathbb{K}}(ab) - aD_{A/\mathbb{K}}(b) - bD_{A/\mathbb{K}}(a) + abD_{A/\mathbb{K}}(1_A) \\ = & ab + d_{A/\mathbb{K}}(ab) - a(b + d_{A/\mathbb{K}}(b)) - b(a + d_{A/\mathbb{K}}(a)) \\ & +ab(1_A + d_{A/\mathbb{K}}(1_A)) \\ = & d_{A/\mathbb{K}}(ab) - ad_{A/\mathbb{K}}(b) - bd_{A/\mathbb{K}}(a). \end{array}$$

Since $d_{A/\mathbb{K}}(a)$ is a derivation, we get

$$D_{A/\mathbb{K}}(ab) = aD_{A/\mathbb{K}}(b) + bD_{A/\mathbb{K}}(a) - abD_{A/\mathbb{K}}(1_A).$$

Thus, $D_{A/\mathbb{K}} \in \mathrm{Diff}_{\mathbb{K}}(A, A \oplus \Omega_{\mathbb{K}}(A))$. □

For all $x \in A \oplus \Omega_{\mathbb{K}}(A)$, $x = \sum_{i\in J} a_i D_{A/\mathbb{K}}(b_i)$, with $a_i, b_i \in A$ or $x = a + \alpha$ where $\alpha = \sum_{i\in J} a_i \cdot d_{A/\mathbb{K}}(b_i)$, with $a, a_i, b_i \in A$.

In the following theorem, we give the universal property of first-order differential operators.

Theorem 2.4. Universal property of the couple $(A \oplus \Omega_{\mathbb{K}}(A), D_{A/\mathbb{K}})$

For any A-module E and for any first-order differential operator $\varphi : A \longrightarrow E$, there exists a unique homomorphism $\tilde{\varphi} : A \oplus \Omega_{\mathbb{K}}(A) \longrightarrow E$ of A-modules such that $\tilde{\varphi} \circ D_{A/\mathbb{K}} = \varphi$.

Proof. Since $\varphi : A \longrightarrow E$ is first-order differential operator, according to Proposition 2.2, the map $\varphi - L_{\varphi(1_A)} : A \longrightarrow E$ is a derivation. By the Theorem 2.1, there exists a unique homomorphism $\psi : \Omega_{\mathbb{K}}(A) \longrightarrow E$ such that $\psi \circ d_{A/\mathbb{K}} = \varphi - L_{\varphi(1_A)}$. The map $\tilde{\varphi} : A \oplus \Omega_{\mathbb{K}}(A) \longrightarrow E$ such that, for all $a \in A$ and $x \in \Omega_{\mathbb{K}}(A)$,

$$\tilde{\varphi}(a + x) = a\varphi(1_A) + \psi(x)$$

verifies

$$\begin{array}{rcl} \tilde{\varphi} \circ D_{A/\mathbb{K}}(a) & = & \tilde{\varphi}(a + d_{A/\mathbb{K}}(a)) \\ & = & a\varphi(1_A) + \psi \circ d_{A/\mathbb{K}}(a) \\ & = & a\varphi(1_A) + \left(\varphi - L_{\varphi(1_A)}\right)(a). \end{array}$$

Which implies that $\tilde{\varphi} \circ D_{A/\mathbb{K}}(a) = a\varphi(1_A) + \varphi(a) - a\varphi(1_A) = \varphi(a)$, for all $a \in A$. Thus, $\tilde{\varphi} \circ D_{A/\mathbb{K}} = \varphi$. □

Moreover, the linear mapping

$$Hom_A(A \oplus \Omega_{\mathbb{K}}(A), E) \longrightarrow \mathrm{Diff}_{\mathbb{K}}(A, E), \psi \longmapsto \psi \circ D_{A/\mathbb{K}}$$

is an isomorphism of $C^\infty(M)$-modules. In particular,

$$(A \oplus \Omega_{\mathbb{K}}(A))^* \simeq \mathrm{Diff}_{\mathbb{K}}(A).$$

Definition 2.5. Let $(\mathcal{A}, \cdot)$ be a graded commutative associative algebra with unit $1_{\mathcal{A}}$ over a commutative field $\mathbb{K}$. A linear map $\partial : \mathcal{A} \longrightarrow \mathcal{A}$ is a first-order differential operator of degree r, $r \in \mathbb{Z}$, if ∂ is of degree r and satisfying

$$\partial(x \cdot y) = \partial(x) \cdot y + (-1)^{r|x|} x \cdot \partial(y) - (-1)^{r|xy|} x \cdot y \cdot \partial(1_{\mathcal{A}}) \tag{3}$$

for all $x, y \in \mathcal{A}$.

Let us denote by $\mathrm{Diff}^r_{\mathbb{K}}(\mathcal{A})$ the set of all first-order differential operators of $\mathcal{A}$ and of degree r.

Definition 2.6. The pair $(\mathcal{A},\partial)$ is a called differential commutative associative algebra if $(\mathcal{A},\cdot)$ is a graded commutative associative algebra with unit $1_{\mathcal{A}}$ and $\partial : \mathcal{A} \longrightarrow \mathcal{A}$ is a first-order differential of degree $+1$ such that $\partial^2 = 0$.

The usual differential algebras correspond with the case $\partial(1_{\mathcal{A}}) = 0$.

In the following, M is a smooth manifold and $A = C^\infty(M)$ is a commutative associative algebra of real smooth functions on M (see [14], for more details on smooth manifolds). We denote by $\delta_M = d_{C^\infty(M)/\mathbb{R}}$ the canonical derivation and $\Delta_M = D_{C^\infty(M)/\mathbb{R}}$ the first-order differential operator defined by (2). Thus, for all $f,g \in C^\infty(M)$,

$$\delta_M(fg) = f\cdot\delta_M(g) + g\cdot\delta_M(f),$$

$$\Delta_M(fg) = f\cdot\Delta_M(g) + g\cdot\Delta_M(f) - fg\cdot\Delta_M\left(1_{C^\infty(M)}\right).$$

For $p \in \mathbb{N}$, we denote by $\Lambda^p(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$ the $C^\infty(M)$-module of skew-symmetric multilinear forms of degree p from $C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$ into $C^\infty(M)$. Let $\Lambda(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]) = \bigoplus_{p\in\mathbb{N}}\Lambda^p(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$ be the exterior algebra of the $C^\infty(M)$-module $C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$.

The elements of $\Lambda^p[C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$, with $p > 0$, are called Kähler p-forms, or simply Kähler forms. The set $\Lambda^p[C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$ is generated by elements of the for $\Delta_M(f_1)\wedge\Delta_M(f_2)\wedge\ldots\wedge\Delta_M\left(f_p\right)$ for all $f_1,\ldots,f_p \in C^\infty(M)$. The algebra $\Lambda(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$ is called algebra of Kähler forms.

Proposition 2.7. *[11] For all* $x = \sum_{i\in J} f_i\Delta_M(g_i) \in C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$, *the* $\mathbb{R}$*-linear map*

$$\Delta_M^1 : C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow \Lambda^2[C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]]$$

defined by

$$\Delta_M^1(x) = \sum_{i\in J}[\Delta_M(f_i)\wedge\Delta_M(g_i)] + x\wedge\Delta_M(1_{C^\infty(M)})$$

satisfies for all $f \in C^\infty(M)$,

$$\Delta_M^1(f\cdot x) = \Delta_M(f)\wedge x + f\cdot\Delta_M^1(x) + (f\cdot x)\wedge\Delta_M\left(1_{C^\infty(M)}\right)$$

and

$$\Delta_M^1\circ\Delta_M = 0.$$

Therefore, the pair $(\Lambda(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]),\Delta_M)$ is a differential commutative associative algebra.

Let $\varphi : C^\infty(M) \longrightarrow C^\infty(M)$ be a first-order differential operator and let $\widetilde{\varphi} : C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow C^\infty(M)$ be the'unique $C^\infty(M)$ -linear map such that $\widetilde{\varphi}\circ\Delta_M = \varphi$. The map $\sigma_\varphi : (C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])^p \longrightarrow \Lambda^{p-1}(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$ defined by

$$\sigma_\varphi(x_1,x_2,\ldots,x_p) = \sum_{i=1}^{p}(-1)^{i-1}\tilde{\varphi}(x_i)\cdot x_1\wedge\ldots\wedge\hat{x}_i\wedge\ldots\wedge x_p$$

is skew-symmetric $C^\infty(M)$-multilinear. Therefore, according to universal property of exterior algebra, there exists a unique $C^\infty(M)$-linear map

$$\tilde{\sigma}_\varphi : \Lambda^p(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]) \longrightarrow \Lambda^{p-1}(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]),$$

such that

$$\tilde{\sigma}_\varphi(x_1 \wedge x_2 \wedge \ldots \wedge x_p) = \sigma_\varphi(x_1, x_2, \ldots, x_p),$$

that is, σ_φ induces a first-order differential operator

$$i_\varphi = \tilde{\sigma}_\varphi : \Lambda\left(C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]\right) \longrightarrow \Lambda\left(C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]\right)$$

of degree -1. For $p = 1$, we have

$$i_\varphi = \tilde{\sigma}_\varphi : C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow \Lambda^0(C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]) = C^\infty(M)$$

and for all $y \in C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$i_\varphi(y) = \widetilde{\varphi}(y), \text{ that is, } i_\varphi = \widetilde{\varphi}.$$

For $p = 2$, for all $x, y \in C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$, we have

$$\sigma_\varphi(x, y) = i_\varphi(x \wedge y) = \widetilde{\varphi}(x)y - \widetilde{\varphi}(y)x.$$

Definition 2.8. A Lie derivative with respect to a first-order differential operator $\varphi \in \mathrm{Diff}_{\mathbb{R}}[C^\infty(M)]$ is an $\mathbb{R}$-linear map

$$\mathfrak{L}_\varphi : i_\varphi \circ \Delta^1_M + \Delta_M \circ \widetilde{\varphi} : C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)].$$

Proposition 2.9. *[11]For all $\varphi \in \mathrm{Diff}_{\mathbb{R}}[C^\infty(M)], f \in C^\infty(M)$ and for all $x = \sum_{i \in J} f_i \Delta_M(g_i) \in C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$, we have*

$$\mathfrak{L}_\varphi(x) = \sum_{i \in J} [\varphi(f_i)\Delta_M(g_i) + f_i \Delta_M(\varphi(g_i))] - \varphi(1_{C^\infty(M)}) \cdot x, \tag{4}$$

$$(\mathfrak{L}_{f \cdot \varphi})(x) = f \cdot \mathfrak{L}_\varphi(x) + \widetilde{\varphi}(x)\delta_M(f), \tag{5}$$

$$\mathfrak{L}_\varphi(f \cdot x) = \left[\varphi(f) - f \cdot \varphi(1_{C^\infty(M)})\right] x + f \cdot \mathfrak{L}_\varphi(x), \tag{6}$$

$$\mathfrak{L}_\varphi[\Delta_M(f)] = \Delta_M[\varphi(f)]. \tag{7}$$

Definition 2.10. For integer $p \geq 1$, a skew-symmetric $\mathbb{R}$-multilinear map

$$\varphi : [C^\infty(M)]^p = C^\infty(M) \times C^\infty(M) \times C^\infty(M) \times \ldots \times C^\infty(M) \longrightarrow E$$

is a skew-symmetric first-order p-differential operator if for any $i \in \{1, \ldots, p\}$, the map

$$\varphi^i = \varphi\left(f_1, \ldots, \widehat{f_i}, \ldots, f_p\right) : C^\infty(M) \longrightarrow E, f_i \longmapsto \varphi\left(f_1, f_2, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_p\right)$$

is a first-order differential operator, for all $f_1, f_2, \ldots, f_p \in C^\infty(M)$.

We denote by $\mathrm{Diff}^p_{sks}(C^\infty(M), E)$ the $C^\infty(M)$-module of all skew-symmetric first-order p-differential operators on $C^\infty(M)$ with coefficients in E . Let

$$\Delta^{(p)}_M = \Delta_M \times \Delta_M \times \ldots \times \Delta_M : [C^\infty(M)]^p \longrightarrow [C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]]^p$$

be the map such that,

$$\Delta^{(p)}_M(f_1, \ldots, f_p) = \Delta_M(f_1) \times \Delta_M(f_2) \times \ldots \times \Delta_M(f_p),$$

for all $f_1, f_2, \ldots, f_p \in C^\infty(M)$.

Theorem 2.11. For every $C^{\infty}(M)$-module E and for any skew-symmetric first-order p-differential operator $\varphi : [C^{\infty}(M)]^p \longrightarrow E$, there exists a unique skew-symmetric $C^{\infty}(M)$-multilinear map of degree p

$$\widetilde{\varphi} : [C^{\infty}(M) \oplus \Omega_{\mathbb{R}}[C^{\infty}(M)]]^p \longrightarrow E$$

such that

$$\widetilde{\varphi}\left(\Delta_M(f_1), \Delta_M(f_2), \dots, \Delta_M\left(f_p\right)\right) = \varphi\left(f_1, f_2, \dots, f_p\right) \tag{8}$$

for any $f_1, f_2, \dots, f_p \in C^{\infty}(M)$.

Proof. By definition of skew-symmetric first-order differential operator $\varphi : C^{\infty}(M)^p \longrightarrow E$ of degree p, the map $\varphi^i : C^{\infty}(M) \longrightarrow E$, such that

$$\varphi^i(f_i) = \varphi\left(f_1, f_2, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_p\right)$$

is a first-order differential operator, for all, $f_i \in C^{\infty}(M)$, $i \in \{1, \dots p\}$. According the Theorem 2.4, there exists a unique homomorphism $\widetilde{\varphi^i} : C^{\infty}(M) \oplus \Omega_{\mathbb{R}}[C^{\infty}(M)] \longrightarrow E$ of $C^{\infty}(M)$-modules such that $\widetilde{\varphi^i} \circ \Delta_M = \varphi^i$, that is,

$$\widetilde{\varphi^i}[\Delta_M(f_i)] = \varphi^i(f_i) = \varphi\left(f_1, f_2, \dots, f_{i-1}, f_i, f_{i+1}, \dots, f_p\right).$$

We deduce the existence and uniqueness of the skew-symmetric $C^{\infty}(M)$-multilinear map $\widetilde{\varphi} : [C^{\infty}(M) \oplus \Omega_{\mathbb{R}}[C^{\infty}(M$ E of degree p such that

$$\widetilde{\varphi}\left(\Delta_M(f_1), \dots, \Delta_M\left(f_p\right)\right) = \varphi\left(f_1, \dots, f_p\right)$$

for all $f_1, f_2, \dots, f_p \in C^{\infty}(M)$. □

Corollary 2.12. *For any skew-symmetric first-order p-differential operator $\varphi : [C^{\infty}(M)]^p \longrightarrow E$, there exists a unique $C^{\infty}(M)$-linear map*

$$\overline{\varphi} : \Lambda^p[C^{\infty}(M) \oplus \Omega_{\mathbb{R}}[C^{\infty}(M)]] \longrightarrow E$$

such that

$$\overline{\varphi}\left(\Delta_M(f_1) \wedge \Delta_M(f_2) \wedge \dots \wedge \Delta_M\left(f_p\right)\right) = \varphi\left(f_1, f_2, \dots, f_p\right) \tag{9}$$

for all $f_1, \dots, f_p \in C^{\infty}(M)$. Moreover, the map

$$\mathfrak{L}^p_{sks}(C^{\infty}(M) \oplus \Omega_{\mathbb{R}}[C^{\infty}(M)], E) \longrightarrow \mathrm{Diff}^p_{sks}(C^{\infty}(M), E), f \longmapsto f \circ \Delta_M^{(p)}$$

is an isomorphism of $C^{\infty}(M)$-modules.

When we denote $\widetilde{\varphi} = P \in \Lambda^p(C^{\infty}(M) \oplus \Omega_{\mathbb{R}}[C^{\infty}(M)])$ and $i_P = \bar{\varphi}$, then for all $f_1, \dots, f_p \in C^{\infty}(M)$, we have

$$\begin{aligned} i_P\left(\Delta_M(f_1) \wedge \Delta_M(f_2) \wedge \dots \wedge \Delta_M(f_p)\right) &= P\left(\Delta_M(f_1), \Delta_M(f_2), \dots, \Delta_M(f_p)\right) \\ &= \varphi\left(f_1, \dots, f_p\right). \end{aligned}$$

In particular, for all $\omega \in \Lambda^3(C^{\infty}(M) \oplus \Omega_{\mathbb{R}}[C^{\infty}(M)])$ and $f, g, h \in C^{\infty}(M)$, we have

$$\begin{aligned} & i_\omega(\Delta_M(f) \wedge \Delta_M(g) \wedge \Delta_M(h)) \\ = \; & i_\omega(\Delta_M(f) \wedge \Delta_M(g)) \cdot \Delta_M(h) - i_\omega(\Delta_M(f) \wedge \Delta_M(h)) \cdot \Delta_M(g) \\ & + i_\omega(\Delta_M(g) \wedge \Delta_M(h)) \cdot \Delta_M(f) \\ = \; & \omega(\Delta_M(f), \Delta_M(g)) \cdot \Delta_M(h) - \omega(\Delta_M(f), \Delta_M(h)) \cdot \Delta_M(g) \\ & + \omega(\Delta_M(g), \Delta_M(h)) \cdot \Delta_M(f). \end{aligned}$$

Let us give the following definition:

Definition 2.13. Let $P \in \Lambda^p(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$ and $Q \in \Lambda^q(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$. The Schouten-Nijenhuis bracket of P and Q is a mapping $[-,-]_S$ from $\Lambda^p(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]) \times \Lambda^q(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$ into $\Lambda^{p+q-1}(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]])$ such that

$$[P,Q]_S = P\circ Q - (-1)^{(p-1)(q-1)}Q\circ P \tag{10}$$

where

$$\begin{aligned}&(Q\circ P)\left(\Delta_M(f_1),\Delta_M(f_2),...,\Delta_M(f_{p+q-1})\right)\\ =& \sum_{\sigma\in S_{p,q-1}}(-1)^\sigma\,\tilde{Q}\left[\tilde{P}\left(f_{\sigma(1)},f_{\sigma(2)},...f_{\sigma(p)}\right),f_{\sigma(p+1)},...,f_{\sigma(p+q-1)}\right]\end{aligned}$$

and $\tilde{P} = \varphi \in \mathrm{Diff}^p_{\mathbb{R}}[C^\infty(M)]$ is the unique skew p-differential operator such that

$$\varphi\left(f_1,...,f_p\right) = P\left(\Delta_M(f_1),...,\Delta_M\left(f_p\right)\right).$$

Throughout this section, we denote $[-,-]_S$ by an unadorned bracket $[-,-]$. The description of interior product $P \in \Lambda(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$ with the Schouten bracket is similar to the interior product defined by [6, 9, 14]. Then if P and Q are two elements of the $\Lambda(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$, then

$$i_{[P,Q]} = \left[[i_P,\Delta_M],i_Q\right]. \tag{11}$$

If $P \in \Lambda^p(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$, then i_P is of degree $-p$. So

$$[i_P,\Delta_M] = i_P\circ\Delta_M - (-1)^{-p}\Delta_M\circ i_P. \tag{12}$$

Now, assuming $Q \in \Lambda^q(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$, we have

$$\begin{aligned}\left[[i_P,\Delta_M],i_Q\right] &= [i_P,\Delta_M]\circ i_Q - (-1)^{-q(1-p)}i_Q\circ[i_P,\Delta_M]\\ &= i_P\circ\Delta_M\circ i_Q + (-1)^{-p}\Delta_M\circ i_{P\wedge Q}\\ &\quad -(-1)^{-q(1-p)}i_{P\wedge Q}\circ\Delta_M + (-1)^{-q(1-p)-p}i_Q\circ\Delta_M\circ i_P.\end{aligned}$$

For all $\omega \in \Lambda^2(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$ and $\eta \in [C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]]^p$, we have

$$i_{[\omega,\omega]}\eta = 2i_\omega\Delta_M i_\omega\eta. \tag{13}$$

Proposition 2.14. *If $\omega \in \Lambda^2(C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)])$ and $f,g,h \in C^\infty(M)$, then*

$$\frac{1}{2}[\omega,\omega]_S(\Delta_M(f),\Delta_M(g),\Delta_M(h)) = \oint\omega\left(\Delta_M\left(\omega\left(\Delta_M(f),\Delta_M(g)\right)\right),\Delta_M(h)\right). \tag{14}$$

where the symbol $\oint$ means the cyclic sum in f,g,h.

Proof. For $\eta = \Delta_M(f)\wedge\Delta_M(g)\wedge\Delta_M(h)$, we have

$$\begin{aligned}i_\omega\eta &= \omega(\Delta_M(f),\Delta_M(g))\cdot\Delta_M(h) - \omega(\Delta_M(f),\Delta_M(h))\cdot\Delta_M(g)\\ &\quad+\omega(\Delta_M(g),\Delta_M(h))\cdot\Delta_M(f),\end{aligned}$$

$$\begin{aligned}\Delta_M i_\omega\eta &= \Delta_M(\omega(\Delta_M(f),\Delta_M(g)))\wedge\Delta_M(h)\\ &\quad-\Delta_M(\omega(\Delta_M(f),\Delta_M(h)))\wedge\Delta_M(g)\\ &\quad+\Delta_M(\omega(\Delta_M(g),\Delta_M(h)))\wedge\Delta_M(f)\end{aligned}$$

and

$$\begin{aligned}&i_\omega \Delta_M i_\omega \eta\\ =\ &\omega\left(\Delta_M\left(\omega\left(\Delta_M(f),\Delta_M(g)\right)\right),\Delta_M(h)\right)-\omega\left(\Delta_M\left(\omega\left(\Delta_M(h),\Delta_M(f)\right)\right),\Delta_M(g)\right)\\ &+\omega\left(\Delta_M\left(\omega\left(\Delta_M(g),\Delta_M(h)\right)\right),\Delta_M(f)\right).\end{aligned}$$

Since $i_{[\omega,\omega]}\eta = 2i_\omega \Delta_M i_\omega \eta$, we obtain (14). □

3 Jacobi manifolds

Definition 3.1. [5] A Jacobi algebra is a commutative associative $\mathbb{K}$-algebra A equipped with a Lie bracket $\{-,-\}$, called the Jacobi bracket, satisfying the Leibniz rule

$$\{a, b\cdot c\} = \{a,b\}\cdot c + b\cdot\{a,c\} - bc\{a, 1_A\}. \tag{15}$$

If $A = C^\infty(M)$ is a unital commutative associative algebra of real smooth functions on M, then the pair $(M,\{-,-\})$ is called Jacobi manifold. In this case we say that $C^\infty(M)$ is a Jacobi algebra and M is a Jacobi manifold [5, 7].

The relation (15) means that for all $f \in C^\infty(M)$, the inner derivation

$$ad(f) : C^\infty(M) \longrightarrow C^\infty(M), g \longmapsto \{f,g\}$$

is a first-order differential operator on $C^\infty(M)$. We denote $\xi = ad(1_{C^\infty(M)})$ the fundamental vector field of the Jacobi manifold M. For all $f,g \in C^\infty(M)$, we have

$$\xi\{f,g\} = \{\xi(f),g\} + \{f,\xi(g)\},\ \xi(f\cdot g) = \xi(f)\cdot g + f\cdot\xi(g). \tag{16}$$

Hence ξ is a derivation of both $(C^\infty(M),\cdot)$ and $(C^\infty(M),\{-,-\})$ and the map

$$ad : C^\infty(M) \longrightarrow \mathrm{Diff}_{\mathbb{R}}[C^\infty(M)], f \longmapsto ad(f)$$

is a first-order differential operator. Thus, according to Theorem 2.4, there exists a unique $C^\infty(M)$-linear map $\widetilde{ad} : C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow \mathrm{Diff}_{\mathbb{R}}[C^\infty(M)]$ such that

$$\widetilde{ad}\circ\Delta_M = ad. \tag{17}$$

The following theorem characterizes Jacobi manifolds:

Theorem 3.2. The following statements are equivalent:
(1) M is a Jacobi manifold.
(2) There exists a skew-symmetric 2-form

$$\omega_M : [C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]]\times[C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]] \longrightarrow C^\infty(M)$$

such that

$$\{f,g\} = \omega_M(\Delta_M(f),\Delta_M(g)) \tag{18}$$

defines a $\mathbb{R}$-Lie algebra structure on $C^\infty(M)$, for all $f,g\in C^\infty(M)$.
(3) There exists a skew-symmetric 2-form

$$\pi : \Omega_{\mathbb{R}}[C^\infty(M)]\times\Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow C^\infty(M)$$

and a vector field ξ on M such that,

$$\{f,g\} = \pi(\delta_M(f),\delta_M(g)) + f\xi(g) - g\xi(f) \tag{19}$$

defines a $\mathbb{R}$-Lie algebra structure on $C^\infty(M)$, for all $f,g\in C^\infty(M)$.

Proof. (1)⇒(2) If M is a Jacobi manifold, the bracket $\{-,-\}$ is a skew-symmetric first-order 2-differential operator. By the theorem 2.11, there exists $\omega_M \in \Lambda^2([C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)])$ such that

$$\{f,g\} = \omega_M(\Delta_M(f), \Delta_M(g))$$

for all f and g in $C^\infty(M)$.

(2)⇒(3) If $\{f,g\} = \omega_M(\Delta_M(f), \Delta_M(g))$, then

$$\begin{aligned}\{f,g\} &= \omega_M(f+\delta_M(f), g+\delta_M(g)) \\ &= f\cdot\omega_M(1_{C^\infty(M)}, \delta_M(g)) + g\cdot\omega_M(\delta_M(f), 1_{C^\infty(M)}) \\ &\quad +\omega_M(\delta_M(f), \delta_M(g)).\end{aligned}$$

Since

$$\begin{aligned}\omega_M(1_{C^\infty(M)}, f) + \omega_M(1_{C^\infty(M)}, \delta_M(f)) &= \omega_M(1_{C^\infty(M)}, \Delta_M(f)) \\ &= \{1_{C^\infty(M)}, f\} \\ &= ad(1_{C^\infty(M)})(f),\end{aligned}$$

there exists a vector field $\xi = ad(1_{C^\infty(M)})$ with

$$\xi(f) = ad(1_{C^\infty(M)})(f) = \{1_{C^\infty(M)}, f\} = \omega_M(1_{C^\infty(M)}, \delta_M(f)), \tag{20}$$

and there exists a skew-symmetric 2-form

$$\omega_{|\Omega_{\mathbb{R}}[C^\infty(M)]} = \pi : \Omega_{\mathbb{R}}[C^\infty(M)] \times \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow C^\infty(M),$$

such that

$$\begin{aligned}\{f,g\} &= \pi(\delta_M(f), \delta_M(g)) + f\cdot\omega_M(1_{C^\infty(M)}, \delta_M(g)) + g\cdot\omega_M(\delta_M(f), 1_{C^\infty(M)}) \\ &= \pi(\delta_M(f), \delta_M(g)) + f\xi(g) - g\xi(f).\end{aligned}$$

(3)⇒(1) If the bracket

$$\{f,g\} = \pi(\delta_M(f), \delta_M(g)) + f\xi(g) - g\xi(f) = \omega_M(\Delta_M(f), \Delta_M(g))$$

defines a $\mathbb{R}$-Lie algebra structure on $C^\infty(M)$, then for all $f,g,h \in C^\infty(M)$,

$$\{f, g\cdot h\} = \omega_M(\Delta_M(f), \Delta_M(g\cdot h)).$$

Since Δ_M is a first-order differential operator, for all $f,g,h \in C^\infty(M)$,

$$\begin{aligned}\{f, g\cdot h\} &= g\cdot\omega_A(\Delta_M(f), \Delta_M(h)) + h\cdot\omega_A(\Delta_M(f), \Delta_M(g)) \\ &\quad -g\cdot h\cdot\omega_A(\Delta_M(f), \Delta_M(1_{C^\infty(M)})) \\ &= g\cdot\{f,h\} + \{f,g\}\cdot h - g\cdot h\cdot\{f, 1_{C^\infty(M)}\}.\end{aligned}$$

That is, $ad(f)$ is a first-order differential operator. Therefore, M is a Jacobi manifold. □

In this case, we say that the pair (π,ξ) defines a Jacobi structure on M and (M,π,ξ) is a Jacobi manifold. The skew-symmetric 2-form ω_M on $C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$ is called Jacobi 2-form of the Jacobi manifold M and the pair (M,ω_M) is called Jacobi manifold.

Consider the Jacobiator $J(.,.,.)$ defined as

$$J(f,g,h) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}$$

for all $f,g,h \in C^\infty(M)$ and denote by $[-,-]$ the restriction of Schouten-Nijenhuis bracket to $\Lambda(\Omega_{\mathbb{R}}[C^\infty(M)])$.

Lemma 3.3. *If $\pi \in \Lambda^2(\Omega_{\mathbb{R}}[C^\infty(M)])$, $\xi \in \Lambda^1(\Omega_{\mathbb{R}}[C^\infty(M)])$ and $f,g \in C^\infty(M)$, then*

$$\begin{aligned}[\xi,\pi](\delta_M(f),\delta_M(g)) &= \xi\cdot\pi(\delta_M(f),\delta_M(g)) \\ &\quad -\pi(\delta_M(\xi(f)),\delta_M(g)) - \pi(\delta_M(f),\delta_M(\xi(g)))\end{aligned} \tag{21}$$

Proof.

$$\begin{aligned}[\xi,\pi](\delta_M(f),\delta_M(g)) &= \left(\mathfrak{L}_\xi\pi\right)(\delta_M(f),\delta_M(g)) \\ &= \xi\cdot\pi(\delta_M(f),\delta_M(g)) - \pi\left(\mathfrak{L}_\xi\delta_M(f)),\delta_M(g)\right) \\ &\quad -\pi\left(\delta_M(f),\mathfrak{L}_\xi\delta_M(g)))\right) \\ &= \xi\cdot\pi(\delta_M(f),\delta_M(g)) \\ &\quad -\pi(\delta_M(\xi(f)),\delta_M(g)) - \pi(\delta_M(f),\delta_M(\xi(g))).\end{aligned}$$

□

Lemma 3.4. *Let (M,π,ξ) be a Jacobi manifold. Then, for all $f,g,h \in C^\infty(M)$,*

$$[\xi,\pi](\delta_M(f),\delta_M(g)) = J\left(f,g,1_{C^\infty(M)}\right); \tag{22}$$

$$\begin{aligned}J(f,g,h) &= \left(\frac{1}{2}[\pi,\pi]-\xi\wedge\pi\right)(\delta_M(f),\delta_M(g),\delta_M(h)) \\ &\quad -(\oint f\cdot[\xi,\pi](\delta_M(g),\delta_M(h)))),\end{aligned} \tag{23}$$

where $[-,-]$ is the Schouten-Nijenhuis bracket defined on $\Lambda(\Omega_{\mathbb{R}}[C^\infty(M)])$.

Proof. From (21) and (20), for all $f,g,h \in C^\infty(M)$, we have

$$\begin{aligned}&[\xi,\pi](\delta_M(f),\delta_M(g)) \\ =\ & \xi\cdot\pi(\delta_M(f),\delta_M(g)) - \pi(\delta_M(\xi(f)),\delta_M(g)) - \pi(\delta_M(f),\delta_M(\xi(g))) \\ =\ & -\{\pi(\delta_M(f),\delta_M(g)),1_{C^\infty(M)}\} + \pi(\delta_M\left(\{f,1_{C^\infty(M)}\}\right),\delta_M(g)) \\ =\ & \{f,\{g,1_{C^\infty(M)}\}\} + \{g,\{1_{C^\infty(M)},f\}\} + \{1_{C^\infty(M)},\{f,g\}\}.\end{aligned}$$

Therefore, $[\xi,\pi](\delta_M(f),\delta_M(g)) = J\left(f,g,1_{C^\infty(M)}\right)$, for all $f,g,h \in C^\infty(M)$.

Using the equation (19), for all $f,g,h \in C^\infty(M)$, we have

$$\{\{f,g\},h\} = \pi(\delta_M(\{f,g\}),\delta_M(h)) + \{f,g\}\cdot\xi(h) - h\cdot\xi(\{f,g\}).$$

From (16) and (19), we get,

$$\begin{aligned}&\{\{f,g\},h\} \\ =\ & \pi(\delta_M(\pi(\delta_M(f),\delta_M(g)),\delta_M(h)) + f\cdot\pi(\delta_M(\xi(g)),\delta_M(h)) \\ & +\xi(g)\cdot\pi(\delta_M(f),\delta_M(h)) - g\cdot\pi(\delta_M(\xi(f)),\delta_M(h)) - h\cdot\xi(\pi(\delta_M(f),\delta_M(g)) \\ & -\xi(f)\cdot\pi(\delta_M(g),\delta_M(h)) + \xi(h)\cdot\pi(\delta_M(f),\delta_M(g)) + f\cdot\xi(g)\xi(h) \\ & -g\cdot\xi(f)\xi(h) - hf\cdot\xi(\xi(g)) + hg\cdot\xi(\xi(f)).\end{aligned}$$

Using the skew-symmetry of π and rearranging the terms, we obtain

$$\begin{aligned} & J(f,g,h) \\ = \; & \oint \pi(\delta_M(\pi(\delta_M(f),\delta_M(g))),\delta_M(h)) - \oint \xi(f)\cdot\pi(\delta_M(g),\delta_M(h)) \\ & -\oint f\cdot[\xi(\pi(\delta_M(g),\delta_M(h))) - \pi(\delta_M(\xi(g)),\delta_M(h)) - \pi(\delta_M(g),\delta_M(\xi(h)))] \\ = \; & \left(\frac{1}{2}[\pi,\pi] - \xi\wedge\pi\right)(\delta_M(f),\delta_M(g),\delta_M(h)) - \oint f\cdot[\xi,\pi](\delta_M(g),\delta_M(h)). \end{aligned}$$

□

Theorem 3.5. For all $f,g\in C^\infty(M)$, the bracket

$$\{f,g\} = \pi(\delta_M(f),\delta_M(g)) + f\xi(g) - g\xi(f)$$

satisfies the Jacobi identity if and only if

$$[\xi,\pi] = 0 \text{ and } [\pi,\pi] = 2\xi\wedge\pi. \tag{24}$$

Proof. Assume that the bracket $\{,\}$ satisfies the Jacobi identity, then from 3.4, $[\xi,\pi]=0$ and $[\pi,\pi] = 2\xi\wedge\pi$.

Conversely, assume the equations (24), then from 3.4, we have,

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

□

In this case, the pair (π,ξ) such that $[\pi,\pi] = 2\xi\wedge\pi$ and $[\xi,\pi]=0$ is called Jacobi pair. If $\xi=0$, we have $[\pi,\pi]=0$ and the pair (M,π) is called Poisson manifold [10].

4 Jacobi Morphisms

Definition 4.1. Let $(M,\{-,-\}_M)$ and $(N,\{-,-\}_N)$ be two Jacobi manifolds. A smooth map $f: M\longrightarrow N$ is called a Jacobi morphism or a Jacobi map if the dual morphism

$$f^*: C^\infty(N)\longrightarrow C^\infty(M), \varphi\longmapsto f^*(\varphi) = \varphi\circ f$$

is a morphism of Lie algebras, that is,

$$f^*(\{\varphi,\psi\}_N) = \{f^*(\varphi), f^*(\psi)\}_M, \tag{25}$$

for all $\varphi,\psi\in C^\infty(N)$.

Lemma 4.2. *Let M and N be two smooth manifolds and $f: M\longrightarrow N$ a smooth map. Then, there exists a unique $C^\infty(M)$-linear map*

$$\Omega(f^*): C^\infty(N)\oplus\Omega_{\mathbb{R}}[C^\infty(N)] \longrightarrow C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$$

such that

$$\Omega(f^*)\circ\Delta_N = \Delta_M\circ f^*. \tag{26}$$

Moreover, for $\varphi\in C^\infty(N)$ and $x\in C^\infty(N)\oplus\Omega_{\mathbb{R}}[C^\infty(N)]$, we have

$$\Omega(f^*)(\varphi x) = f^*(\varphi)\Omega(f^*)(x).$$

Proof. For $\varphi \in C^\infty(N)$ and $y \in C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$, the map

$$* : C^\infty(N) \times (C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]) \longrightarrow C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$$

such that

$$\varphi * y = f^*(\varphi) \cdot y$$

defines the $C^\infty(N)$-module structure on $C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$ and the map

$$\Delta_M \circ f^* : C^\infty(N) \longrightarrow C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)]$$

is a first-order differential operator, that is, for all $\varphi_1, \varphi_2 \in C^\infty(N)$,

$$\begin{aligned} (\Delta_M \circ f^*)(\varphi_1 \cdot \varphi_2) &= [(\Delta_M \circ f^*)(\varphi_1)] * \varphi_2 + \varphi_1 * [(\Delta_M \circ f^*)(\varphi_2)] \\ &\quad - (\varphi_1 \cdot \varphi_2) * \Delta_M \left(1_{C^\infty(M)}\right). \end{aligned}$$

Thus, according to Theorem 2.4, there exists a unique $C^\infty(N)$-linear map

$$\widetilde{\Delta_M \circ f^*} = \Omega(f^*) : (C^\infty(N) \oplus \Omega_{\mathbb{R}}[C^\infty(N)], \cdot) \longrightarrow (C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)], *)$$

such that

$$\Omega(f^*) \circ \Delta_N = \Delta_M \circ f^*.$$

By direct computations we obtain

$$\Omega(f^*)(\varphi x) = f^*(\varphi)\Omega(f^*)(x),$$

for all $\varphi \in C^\infty(N)$ and $x \in C^\infty(N) \oplus \Omega_{\mathbb{R}}[C^\infty(N)]$. □

Theorem 4.3. Let (M, ω_M) and (N, ω_N) be two Jacobi manifolds. Then, a smooth map $f : M \longrightarrow N$ is a Jacobi morphism if and only if

$$\omega_M \circ ((\Omega(f^*) \times \Omega(f^*)) = f^* \circ \omega_N. \tag{27}$$

Proof. Assume that $f : M \longrightarrow N$ verifies the relation (25) For all x and $y \in C^\infty(N) \oplus \Omega_{\mathbb{R}}[C^\infty(N)]$,

$x = \sum_i \varphi_i \Delta_N(\varphi_i')$, $y = \sum_j \psi_j \Delta_N(\psi_j')$ with $\varphi_i, \varphi_i', \psi_j, \psi_j' \in C^\infty(N)$

$$
\begin{aligned}
(f^* \circ \omega_N)(x,y) &= (f^* \circ \omega_N)(\sum_i \varphi_i \Delta_N(\varphi_i'), \sum_j \psi_j \Delta_N(\psi_j')) \\
&= \sum_{i,j} f^*(\varphi_i \psi_j \omega_N(\Delta_N(\varphi_i'), \Delta_N(\psi_i'))) \\
&= \sum_{i,j} f^*(\varphi_i) f^*(\psi_j) f^*(\omega_N(\Delta_N(\varphi_i'), \Delta_N(\psi_i'))) \\
&= \sum_{i,j} f^*(\varphi_i) f^*(\psi_j) [(f^* \omega_N)(\Delta_N(\varphi_i'), \Delta_N(\psi_j'))] \\
&= \sum_{i,j} f^*(\varphi_i) f^*(\psi_j) [\omega_M(\Delta_M(f^*(\varphi_i')), \Delta_M(f^*(\psi_j')))] \\
&= \sum_{i,j} f^*(\varphi_i) f^*(\psi_j) [\omega_M((\Delta_M \circ f^*)(\varphi_i')), (\Delta_M \circ f^*)(\psi_j'))] \\
&= \sum_{i,j} f^*(\varphi_i) f^*(\psi_j) [\omega_M((\Omega(f^*) \circ \Delta_N)(\varphi_i')), (\Omega(f^*) \circ \Delta_N)(\psi_j'))] \\
&= \sum_{i,j} f^*(\varphi_i) f^*(\psi_j) [\omega_M(\Omega(f^*)(\Delta_N(\varphi_i')), \Omega(f^*)(\Delta_N(\psi_j')))] \\
&= \sum_{i,j} f^*(\varphi_i) f^*(\psi_j) \omega_M(\Omega(f^*)(\Delta_N(\varphi_i')), \Omega(f^*)(\Delta_N(\psi_j'))) \\
&= \omega_M(\sum_i f^*(\varphi_i) \Omega(f^*)(\Delta_N(\varphi_i')), \sum_j f^*(\psi_j) \Omega(f^*)(\Delta_N(\psi_j'))) \\
&= \omega_M(\Omega(f^*)(x), \Omega(f^*)(y)) \\
&= \omega_M(\Omega(f^*) \times \Omega(f^*))(x,y).
\end{aligned}
$$

Hence, $f^* \circ \omega_N = \omega_M \circ (\Omega(f^*) \times \Omega(f^*))$.

Conversely, assume the relation (27). Then, for all φ, $\psi \in C^\infty(N)$, we have

$$
\begin{aligned}
f^*(\{\varphi, \psi\}_N) &= f^*(\omega_N(\Delta_N(\varphi), \Delta_N(\psi))) \\
&= (f^* \circ \omega_N)(\Delta_N(\varphi), \Delta_N(\psi)) \\
&= (\omega_M \circ (\Omega(f^*) \times \Omega(f^*))(\Delta_N(\varphi), \Delta_N(\psi)) \\
&= \omega_M[\Omega(f^*)(\Delta_N(\varphi)), \Omega(f^*)(\Delta_N(\psi))] \\
&= \omega_M[(\Omega(f^*) \circ \Delta_N)(\varphi), (\Omega(f^*) \circ \Delta_N)(\psi)] \\
&= \omega_M[(\Delta_M \circ f^*)(\varphi), (\Delta_M \circ f^*)(\psi)] \\
&= \omega_M[\Delta_M(f^*(\varphi)), \Delta_M(f^*(\psi))] \\
&= \{f^*(\varphi), f^*(\psi)\}_M.
\end{aligned}
$$

□

Corollary 4.4. *Let (M, ω_M), (N, ω_N) and (P, ω_P) be three Jacobi manifolds, let $f : M \longrightarrow N$ and $g : N \longrightarrow P$ be Jacobi morphisms. Then, $g \circ f$ is a Jacobi morphism.*

Proof. By direct calculations and using (27), we find

$$(g \circ f)^* \circ \omega_P = \omega_M \circ ((\Omega((g \circ f)^*) \times \Omega(f^* \circ g^*))).$$

□

5 Non-degenerate Jacobi structure

If (M,ω_M) is a Jacobi manifold, for $f\in C^\infty(M)$, the map $X_f : C^\infty(M)\longrightarrow C^\infty(M)$ such that

$$X_f(g)=\omega_M(\Delta_M(f),\delta_M(g))=\{f,g\}-g\{f,1_{C^\infty(M)}\}$$

is a vector field on M. Moreover, the map $\Phi : C^\infty(M)\longrightarrow \mathrm{Der}_{\mathbb{R}}[C^\infty(M)], f\longmapsto X_f$ is a morphism of Lie algebras and is a first-order differential operator. That is,

$$\Phi(\{f,g\})=[\Phi(f),\Phi(g)]$$

and

$$\Phi(fg)=f\cdot\Phi(g)+g\cdot\Phi(f)-fg\cdot\Phi\left(1_{C^\infty(M)}\right),$$

for all $f,g\in C^\infty(M)$. The vector field X_f is called hamiltonian vector field associated with f, and f is called the hamiltonian function of X_f.

Since $\Phi : C^\infty(M)\longrightarrow \mathrm{Der}_{\mathbb{R}}[C^\infty(M)]$ is a first-order differential operator, according to Theorem 2.4, there exists a unique $C^\infty(M)$-linear map $\tilde{\Phi} : C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]\longrightarrow \mathrm{Der}_{\mathbb{R}}[C^\infty(M)]$ such that $\tilde{\Phi}\circ\Delta_M=\Phi$.

For all $f\in C^\infty(M)$ and for all $x,y\in C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$, we find

$$[\tilde{\Phi}(x)](f)=\omega_M(x,\delta_M(f));$$

$$[\widetilde{ad}(x)](f)=\omega_M(x,\Delta_M(f));$$

$$[\widetilde{\widetilde{ad}(x)}](y)=\omega_M(x,y). \tag{28}$$

Let $\overline{ad}=\tilde{\Phi}_{|\Omega_{\mathbb{R}}[C^\infty(M)]} : \Omega_{\mathbb{R}}[C^\infty(M)]\longrightarrow \mathfrak{X}(M)$ be the restriction of $\tilde{\Phi}$ to $\Omega_{\mathbb{R}}[C^\infty(M)]$. We have, for all $f\in C^\infty(M)$ and $\alpha,\beta\in\Omega[C^\infty(M)]$

$$\left[\overline{ad}(\alpha)\right](f)=\pi(\alpha,\delta_M(f)), \tag{29}$$

$$\overline{\left[\overline{ad}(\alpha)\right]}(\beta)=\pi(\alpha,\beta). \tag{30}$$

If (M,π,ξ) is a Jacobi manifold, then we obtain

$$\tilde{\Phi}(f+\alpha)=\overline{ad}(\alpha)-f\cdot\xi$$

for all $f\in C^\infty(M)$ and $\alpha\in\Omega_{\mathbb{R}}[C^\infty(M)]$.

By the isomorphism $\mathrm{Der}_{\mathbb{R}}[C^\infty(M)]\simeq\Omega_{\mathbb{R}}[C^\infty(M)]^*$, $\xi\in\Omega_{\mathbb{R}}[C^\infty(M)]^*$ can also be written as

$$\xi(\alpha)=\omega_M(1_{C^\infty(M)},\alpha)$$

where $\alpha\in\Omega_{\mathbb{R}}[C^\infty(M)]$.

Let M be a manifold. The 2-form ω_M induces on $C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$ a bracket $[-,-]_{\omega_M}$ called the Koszul bracket :

$$[x,y]_{\omega_M}=\mathfrak{L}_{\widetilde{ad}(x)}y-\mathfrak{L}_{\widetilde{ad}(y)}x-\Delta_M(\omega_M(x,y))$$

for all $x,y\in C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$. Let $[-,-]_\pi=[-,-]_{\omega_{M|\Omega_{\mathbb{R}}[C^\infty(M)]\times\Omega_{\mathbb{R}}[C^\infty(M)]}}$ be the restriction of $[-,-]_{\omega_M}$ to $\Omega_{\mathbb{R}}[C^\infty(M)]\times\Omega_{\mathbb{R}}[C^\infty(M)]$. The Koszul bracket is given by

$$[\alpha,\beta]_\pi=\mathfrak{L}_{\overline{ad}(\alpha)}\beta-\mathfrak{L}_{\overline{ad}(\beta)}\alpha-\delta_M(\pi(\alpha,\beta)),$$

for all $\alpha,\beta\in\Omega_{\mathbb{R}}[C^\infty(M)]$. Let

$$d_{\overline{ad}} : \Lambda^k(\Omega_{\mathbb{R}}[C^\infty(M)])\longrightarrow\Lambda^{k+1}(\Omega_{\mathbb{R}}[C^\infty(M)]),\quad Q\longmapsto d_{\overline{ad}}Q$$

be the cohomology operator associated to the representation

$$\overline{ad} = \widetilde{\Phi}_{|\Omega_{\mathbb{R}}[C^\infty(M)]} : \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow \mathfrak{X}(M), \alpha \longmapsto \overline{ad}(\alpha),$$

such that, for any $\alpha_1, ..., \alpha_{k+1} \in \Omega_{\mathbb{R}}[C^\infty(M)$,

$$\begin{aligned} d_{\overline{ad}} Q(\alpha_1, ..., \alpha_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} \overline{ad}(\alpha_i) \cdot Q(\alpha_1, ..., \hat{\alpha}_i, ..., \alpha_{k+1}) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} Q([\alpha_i, \alpha_j]_\pi, \alpha_1, ..., \hat{\alpha}_i, ..., \hat{\alpha}_j, ..., \alpha_{k+1}). \end{aligned}$$

For all $\pi \in \Lambda^2(\Omega_{\mathbb{R}}[C^\infty(M)])$ and $Q \in \Lambda^k(\Omega_{\mathbb{R}}[C^\infty(M)])$, then

$$d_{\overline{ad}} Q = -[\pi, Q]. \tag{31}$$

In particular, if $Q = \pi$, then for any $\alpha, \beta, \gamma \in \Omega_{\mathbb{R}}[C^\infty(M)]$,

$$[\pi, \pi](\alpha, \beta, \gamma) = -\oint \overline{ad}(\alpha) \cdot \pi(\beta, \gamma) + \oint \pi([\alpha, \beta]_\pi, \gamma) \tag{32}$$

where the symbol $\oint$ means the cyclic sum in α, β, γ.

For all $\alpha, \beta, \gamma \in \Omega_{\mathbb{R}}[C^\infty(M)]$, we have

$$\pi([\alpha, \beta]_\pi, \gamma) - \left(\overline{[\overline{ad}(\alpha), \overline{ad}(\beta)]}\right)(\gamma) = \frac{1}{2}[\pi, \pi](\alpha, \beta, \gamma). \tag{33}$$

Definition 5.1. [8] A locally conformal symplectic structure on M is a pair (η, θ) of a differential closed 1-form θ and a non-degenerate differential 2-form η on M such that

$$d\eta = -\theta \wedge \eta. \tag{34}$$

The 1-form θ is known as the Lee form. When $\theta = 0$, then M is a symplectic manifold (see [7, 14]).

Definition 5.2. A Jacobi structure is non-degenerate if the Jacobi 2-form ω_M is non-degenerate, that is, the map

$$\omega_M^\flat : C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow (C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)])^*, x \longmapsto i_x \omega_M$$

is an isomorphism of $C^\infty(M)$-modules, where $(i_x \omega_M)(y) = \omega_M(x, y)$.

If the Jacobi 2-form ω_M is non-degenerate, then the map $\widetilde{ad}$ is an isomorphism of $C^\infty(M)$-modules. indeed, the map

$$\widetilde{ad} = K \circ \omega_M^\flat : C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)] \longrightarrow Diff_{\mathbb{R}}[C^\infty(M)]$$

is an isomorphism of $C^\infty(M)$-modules, where

$$K : (C^\infty(M) \oplus \Omega_{\mathbb{R}}[C^\infty(M)])^* \longrightarrow Diff_{\mathbb{R}}[C^\infty(M)], \varphi \longmapsto \varphi \circ \Delta_M$$

is an isomorphism of $C^\infty(M)$-modules defined by the Theorem 2.4, and for any $f, g \in C^\infty(M)$,

$$\begin{aligned} \left(K \circ \omega_M^\flat\right)(\Delta_M(f))(g) &= \left[i_{\Delta_M(f)} \omega_M \circ \Delta_M\right](g) \\ &= \widetilde{ad}(\Delta_M(f))(g). \end{aligned}$$

Proposition 5.3. *If (M,π,ξ) is a Jacobi manifold and if the restriction $\pi = \omega_{M|\Omega_{\mathbb{R}}[C^\infty(M)]\times\Omega_{\mathbb{R}}[C^\infty(M)]}$ of ω_M to $\Omega_{\mathbb{R}}[C^\infty(M)]\times\Omega_{\mathbb{R}}[C^\infty(M)]$ is non-degenerate, then there exists $\eta\in\Lambda^2(M)$ and $\theta\in\Lambda^1(M)$ such that for any $X,Y\in\mathfrak{X}(M)$ and $\alpha,\beta\in\Omega_{\mathbb{R}}[C^\infty(M)]$,*

$$\eta(X,Y)=\pi(\alpha,\beta), \tag{35}$$

$$\theta(X)=-\xi(\alpha). \tag{36}$$

Proof. If $\pi:\Omega_{\mathbb{R}}[C^\infty(M)]\times\Omega_{\mathbb{R}}[C^\infty(M)]\longrightarrow C^\infty(M)$ is non-degenerate, then the map

$$\overline{ad}:\Omega_{\mathbb{R}}[C^\infty(M)]\longrightarrow\Omega_{\mathbb{R}}[C^\infty(M)]^*\cong \mathrm{Der}_{\mathbb{R}}[C^\infty(M)]$$

is an isomorphism. Let $\overline{ad}^{-1}$ be the inverse isomorphism of $\overline{ad}$. Thus, for any $X,Y\in\mathfrak{X}(M)$, there exists $\alpha,\beta\in\Omega_{\mathbb{R}}[C^\infty(M)]$ such that $\alpha=\overline{ad}^{-1}(X)$, $\beta=\overline{ad}^{-1}(Y)$. The map

$$\eta=\pi\circ(\overline{ad}^{-1}\times\overline{ad}^{-1}):\mathrm{Der}_{\mathbb{R}}[C^\infty(M)]\times\mathrm{Der}_{\mathbb{R}}[C^\infty(M)]\longrightarrow C^\infty(M)$$

satisfies

$$\begin{aligned}\eta(X,Y) &= \pi\circ(\overline{ad}^{-1}\times\overline{ad}^{-1})(X,Y)\\ &= \pi(\overline{ad}^{-1}(X),\overline{ad}^{-1}(Y))\\ &= \pi(\alpha,\beta).\end{aligned}$$

If ω_M is non-degenerate, the map $\tilde{\Phi}:C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]\longrightarrow\mathrm{Der}_{\mathbb{R}}[C^\infty(M)]$ is an isomorphism. Let $\tilde{\Phi}^{-1}$ be the inverse isomorphism of $\tilde{\Phi}$, that is, for all $f+\alpha\in C^\infty(M)\oplus\Omega_{\mathbb{R}}[C^\infty(M)]$, there exists $X\in\mathrm{Der}_{\mathbb{R}}[C^\infty(M)]$ such that $\tilde{\Phi}^{-1}(X)=f+\alpha$. The map

$$\theta=\left(-i_{1_{C^\infty(M)}}\omega_M\right)\circ\tilde{\Phi}^{-1}:Der_{\mathbb{R}}[C^\infty(M)]\longrightarrow C^\infty(M)$$

satisfies

$$\begin{aligned}\theta(X) &= (-i_{1_{C^\infty(M)}}\omega_M)[\tilde{\Phi}^{-1}(X)]\\ &= (-i_{1_{C^\infty(M)}}\omega_M)(f+\alpha)\\ &= -\omega_M(1_{C^\infty(M)},\alpha)\\ &= -\xi(\alpha).\end{aligned}$$

□

Lemma 5.4. *If (M,π,ξ) is a non-degenerate Jacobi manifold, then*

i)

$$\eta([X,Y],Z)=-\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma)+\pi([\alpha,\beta]_\pi,\gamma); \tag{37}$$

ii)

$$\eta(X,\mathfrak{L}_\xi Y)=-\pi(\mathfrak{L}_\xi\alpha,\beta)+\xi[\pi(\alpha,\beta)]; \tag{38}$$

iii)

$$(d\eta+\theta\wedge\eta)(X,Y,Z)=\left(\frac{1}{2}[\pi,\pi]-\xi\wedge\pi\right)(\alpha,\beta,\gamma); \tag{39}$$

iv)

$$\mathfrak{L}_\xi\eta(X,Y)=-\mathfrak{L}_\xi\pi(\alpha,\beta), \tag{40}$$

for all $X,Y,Z\in\mathfrak{X}(M)$ and for all $\alpha,\beta,\gamma\in\Omega_{\mathbb{R}}[C^\infty(M)]$.

Proof. i) For all $X, Y, Z \in \mathfrak{X}(M)$ and for all $\alpha, \beta, \gamma \in \Omega_{\mathbb{R}}[C^\infty(M)]$, we have

$$\begin{aligned}\eta([X,Y],Z) &= \pi \circ (\overline{ad}^{-1} \times \overline{ad}^{-1})([X,Y],Z) \\ &= \pi(\overline{ad}^{-1}([\overline{ad}(\alpha),\overline{ad}(\beta)]),\gamma)\end{aligned}$$

From (30), we have

$$\begin{aligned}\eta([X,Y],Z) &= \overline{\left(\overline{ad}\left[\overline{ad}^{-1}([\overline{ad}(\alpha),\overline{ad}(\beta)])\right]\right)}(\gamma) \\ &= \left(\overline{[\overline{ad}(\alpha),\overline{ad}(\beta)]}\right)(\gamma).\end{aligned}$$

By (33), it follows that

$$\eta([X,Y],Z) = -\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) + \pi([\alpha,\beta]_\pi,\gamma).$$

ii) For $X = \overline{ad}(\alpha)$ and $Y = \overline{ad}(\beta)$, we have

$$\pi(\mathfrak{L}_\xi \alpha, \beta) = -\pi(\beta, \mathfrak{L}_\xi \alpha) = -(\mathfrak{L}_\xi \alpha)(\overline{ad}(\beta)),$$

that is,

$$\begin{aligned}\pi(\mathfrak{L}_\xi \alpha,\beta) &= -(\mathfrak{L}_\xi \alpha)(Y) \\ &= -\xi\alpha(Y) + \alpha(\mathfrak{L}_\xi Y) \\ &= \xi[\eta(X,Y)] - \eta(X,\mathfrak{L}_\xi Y).\end{aligned}$$

Thus,

$$\eta(X,\mathfrak{L}_\xi Y) = -\pi(\mathfrak{L}_\xi \alpha,\beta) + \xi[\pi(\alpha,\beta)].$$

iii) For all $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}d\eta(X,Y,Z) &= X\cdot[\eta(Y,Z)] - Y\cdot[\eta(X,Z)] + Z\cdot[\eta(X,Y)] \\ &\quad -\eta([X,Y],Z) + \eta([X,Z],Y) - \eta([Y,Z],X).\end{aligned}$$

From expressions (35) and (37), we have

$$\begin{aligned}d\eta(X,Y,Z) &= \overline{ad}(\alpha)\cdot\pi(\beta,\gamma) - \overline{ad}(\beta)\cdot\pi(\alpha,\gamma) + \overline{ad}(\gamma)\cdot\pi(\alpha,\beta) \\ &\quad +\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) - \pi([\alpha,\beta]_\pi,\gamma) \\ &\quad -\frac{1}{2}[\pi,\pi](\alpha,\gamma,\beta) + \pi([\alpha,\gamma]_\pi,\beta) \\ &\quad -\frac{1}{2}[\pi,\pi](\alpha,\gamma,\beta) + \pi([\alpha,\gamma]_\pi,\beta),\end{aligned}$$

that is,

$$\begin{aligned}d\eta(X,Y,Z) &= \oint \overline{ad}(\alpha)\cdot\pi(\beta,\gamma) - \oint \pi([\alpha,\beta]_\pi,\gamma) \\ &\quad +\frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma) - \frac{1}{2}[\pi,\pi](\alpha,\gamma,\beta) + \frac{1}{2}[\pi,\pi](\beta,\gamma,\alpha).\end{aligned}$$

From (32), we get

$$d\eta(X,Y,Z) = \frac{1}{2}[\pi,\pi](\alpha,\beta,\gamma). \tag{41}$$

Since

$$(\theta \wedge \eta)(X,Y,Z) = \theta(X)\cdot\eta(Y,Z) - \theta(Y)\cdot\eta(X,Z) + \theta(Z)\cdot\eta(X,Y)$$

and from (36), we have

$$\begin{aligned}(\theta \wedge \eta)(X,Y,Z) &= -\xi(\alpha)\cdot\pi(\beta,\gamma) + \xi(\beta)\cdot\pi(\alpha,\gamma) - \xi(\gamma)\cdot\pi(\alpha,\beta) \\ &= -(\xi \wedge \pi)(\alpha,\beta,\gamma).\end{aligned} \quad (42)$$

From (41) and (42), we deduce that

$$(d\eta + \theta \wedge \eta)(X,Y,Z) = \left(\frac{1}{2}[\pi,\pi] - \xi \wedge \pi\right)(\alpha,\beta,\gamma).$$

iv) By the relation

$$\mathfrak{L}_\xi \eta(X,Y) = \xi[\eta(X,Y)] - \eta(\mathfrak{L}_\xi X, Y) - \eta(X, \mathfrak{L}_\xi Y)$$

and from (38), we have

$$\begin{aligned}\mathfrak{L}_\xi \eta(X,Y) &= -\xi[\pi(\alpha,\beta)] + \pi(\mathfrak{L}_\xi \alpha,\beta) + \pi(\alpha,\mathfrak{L}_\xi \beta) \\ &= -\mathfrak{L}_\xi \pi(\alpha,\beta).\end{aligned}$$

□

Theorem 5.5. The pair (π,ξ) is a non-degenerate Jacobi structure if and only if the pair (η,θ) is a locally conformal symplectic structure.

Proof. Using the Lemma 5.4 and the Cartan formula, we get

$$\mathfrak{L}_\xi \eta = di_\xi \eta + i_\xi d\eta = -d\theta - d(\theta \wedge \eta) = -d\theta.$$

Then, $[\pi,\xi] = \mathfrak{L}_\xi \pi = 0$ if and only if $d\theta = 0$, where $\mathfrak{L}_\xi \pi := [\pi,\xi]$ is the Lie derivative with respect to ξ.

From (39), we deduce that the identity $d\eta + \theta \wedge \eta = 0$ is satisfied if and only if the identity $[\pi,\pi] = 2\xi \wedge \pi$. □

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