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A Counterexample Theorem for a Conjecture on Pseudo-Arithmetical Rings

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Abstract. Bakkari, Kabbaj and Mahdou conjectured that a ring is pseudo-arithmetical if and only if the zero ideal is locally irreducible. We show that this criterion fails in a broad and elementary way. Lucas proved that, over a reduced ring, a polynomial is Gaussian if and only if its content is locally principal. Hence every reduced ring is pseudo-arithmetical. On the other hand, for a reduced ring the zero ideal is locally irreducible if and only if the ring is locally a domain. Consequently every reduced ring which is not locally a domain is a counterexample to the proposed criterion. In particular, the complete local hypersurface $k[[X,Y]]/(XY)$ is pseudo-arithmetical, while its zero ideal is reducible. We also record two boundary observations: the proposed criterion holds for Artinian rings, and no Gaussian pseudo-arithmetical ring can be a counterexample.

Key Words: Gaussian polynomial, content ideal, pseudo-arithmetical ring, irreducible ideal, reduced ring, trivial extension.
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1 Introduction

All rings in this note are commutative with identity. For $f = \sum_i a_i X^i \in R[X]$, the content of f, denoted by $c(f)$, is the ideal of R generated by its coefficients. A polynomial $f \in R[X]$ is *Gaussian* if $c(fg) = c(f)c(g)$ for every $g \in R[X]$. Bakkari, Kabbaj and Mahdou introduced the following terminology.

Definition 1.1. ([1, Definition 4.1]) A ring R is *pseudo-arithmetical* if every Gaussian polynomial over R has locally principal content ideal.

In [1, Conjecture 4.5], Bakkari, Kabbaj and Mahdou conjectured that a ring R is pseudo-arithmetical if and only if the zero ideal is locally irreducible. Here an ideal I of a ring R is *irreducible* if, whenever $I = J \cap K$ for ideals J, K of R, one has $I = J$ or $I = K$. Thus the conjectural condition means that (0) is an irreducible ideal of R_M for every maximal ideal M of R.

The purpose of this note is to show that the conjecture fails not merely by an isolated example, but by an entire class of rings. The point is that Lucas established the converse to the elementary implication "locally principal content implies Gaussian" over reduced rings: if R is reduced and $f \in R[X]$, then f is Gaussian if and only if $c(f)$ is locally principal [5, Theorem 2.8]. It follows immediately that every reduced ring is pseudo-arithmetical. However, local irreducibility of the zero ideal is much more restrictive for reduced rings: it is exactly the condition of being locally a domain.

2 Reduced counterexamples

We first isolate the consequence of Lucas's theorem which will be used throughout.

Lemma 2.1. *Every reduced ring is pseudo-arithmetical.*

Proof. Let R be reduced, and let $f \in R[X]$ be Gaussian. By Lucas's theorem for reduced rings [5, Theorem 2.8.], the content ideal $c(f)$ is locally principal. Hence R is pseudo-arithmetical. □

The next elementary lemma explains exactly when the zero ideal is locally irreducible in the reduced case.

Lemma 2.2. *Let R be a reduced ring. Then the zero ideal of R is locally irreducible if and only if R is locally a domain; that is, R_M is a domain for every $M \in \operatorname{Max} R$.*

Proof. It is enough to prove the corresponding local statement. Let $(A, \mathfrak{m})$ be a reduced local ring. If A is a domain, then (0) is prime, hence irreducible.

Conversely, suppose that A is not a domain. Then there are nonzero elements $a, b \in A$ with $ab = 0$. We claim that $aA \cap bA = 0$. Indeed, if $u \in aA \cap bA$, then $u = ar = bs$ for some $r, s \in A$, and therefore $u^2 = (ar)(bs) = rsab = 0$. Since A is reduced, $u = 0$. Thus $(0) = aA \cap bA$ with $aA \neq 0$ and $bA \neq 0$, so (0) is reducible. This proves the claim. □

Combining the two lemmas gives the promised counterexample theorem.

Theorem 2.3. Let R be a reduced ring which is not locally a domain. Then R is pseudo-arithmetical, but the zero ideal of R is not locally irreducible. In particular, such a ring is a counterexample to the local irreducibility criterion proposed in [1, Conjecture 4.5].

Proof. By Lemma 2.1, R is pseudo-arithmetical. Since R is not locally a domain, there is a maximal ideal M such that R_M is not a domain. By Lemma 2.2, the zero ideal of R_M is reducible. Hence the zero ideal of R is not locally irreducible. □

Example 2.4. Let k be a field and set $R = k[[X, Y]]/(XY)$. Let x, y denote the images of X, Y in R. Since $(XY) = (X) \cap (Y)$ in the unique factorization domain $k[[X, Y]]$, the ring R is reduced. It is local and is not a domain, because $xy = 0$ while $x \neq 0$ and $y \neq 0$. Therefore R is pseudo-arithmetical by Lemma 2.1. On the other hand, $xR \cap yR = 0$, so the zero ideal of this local ring is reducible. Thus R is a one-dimensional complete Noetherian local counterexample to [1, Conjecture 4.5].

Corollary 2.5. *There are counterexamples to [1, Conjecture 4.5] among one-dimensional Noetherian local rings, among complete local rings, and among reduced finitely generated algebras over a field.*

Proof. Example 2.4 gives a one-dimensional complete Noetherian local counterexample. The affine hypersurface $k[X, Y]/(XY)$ is a reduced finitely generated k-algebra which is not locally a domain, since its localization at the maximal ideal (X, Y) has two branches. The same argument applies. □

3 Boundary observations

The preceding theorem shows that the conjectural criterion fails as soon as one allows reduced local rings with more than one branch. The next observations indicate two boundaries of this failure.

Proposition 3.1. *No Gaussian pseudo-arithmetical ring can be a counterexample to [1, Conjecture 4.5]. Equivalently, if R is both Gaussian and pseudo-arithmetical, then the zero ideal of R is locally irreducible.*

Proof. Bakkari, Kabbaj and Mahdou observe that a ring is arithmetical if and only if it is both Gaussian and pseudo-arithmetical [1, Section 4]. Hence R is arithmetical. Localizing at a maximal ideal, we may assume that R is a local arithmetical ring. Then the ideals of R are linearly ordered by inclusion. If $I \cap J = 0$, then either $I \subseteq J$ or $J \subseteq I$, and so either $I = 0$ or $J = 0$. Thus (0) is irreducible locally. □

Proposition 3.2. *For Artinian rings, the pseudo-arithmetical property is equivalent to local irreducibility of the zero ideal.*

Proof. An Artinian ring is a finite product of Artinian local rings, and for a finite product Gaussianity and the locally principal condition are checked componentwise. Thus it is enough to consider an Artinian local ring $(A, \mathfrak{m}, k)$. For such a ring, (0) is irreducible if and only if the socle $\mathrm{Soc}(A) = (0 :_A \mathfrak{m})$ is one-dimensional over k. Indeed, if $\dim_k \mathrm{Soc}(A) > 1$, two independent socle elements generate nonzero ideals whose intersection is zero. Conversely, if $\dim_k \mathrm{Soc}(A) = 1$, then every nonzero ideal of A contains $\mathrm{Soc}(A)$, and hence the intersection of any two nonzero ideals is nonzero.

For an Artinian local ring $(A, \mathfrak{m}, k)$, the condition $\dim_k \mathrm{Soc}(A) = 1$ is equivalent to A being Gorenstein; see [2, Proposition 3.2.10]. Indeed, $\mathrm{Soc}(A) = (0 :_A \mathfrak{m}) \cong \mathrm{Hom}_A(k, A)$, and the Cohen–Macaulay type of A is $\dim_k \mathrm{Hom}_A(k, A)$. Heinzer and Huneke proved that locally approximately Gorenstein rings are pseudo-arithmetical [4, Theorem 1.5]; in the Artinian local case, this gives the implication from Gorenstein to pseudo-arithmetical. Conversely, the discussion recalled in [1, Remark 4.4(2)], based on [4, Remark 1.6], shows that non-Gorenstein Artinian local rings are not pseudo-arithmetical. Thus, in the Artinian local case,

$$A \text{ is pseudo-arithmetical} \iff A \text{ is Gorenstein} \iff (0) \text{ is irreducible.}$$

This proves the assertion. □

We also record a small complementary family arising from trivial extensions. Recall that, if (A, M) is a local ring and E is a vector space over A/M, then the trivial extension $A \ltimes E$ is the ring whose underlying additive group is $A \times E$ and whose multiplication is defined by $(a, e)(a', e') = (aa', ae' + a'e)$; see [3].

Proposition 3.3. *Let (A, M) be a local ring and let $0 \neq E$ be an A/M-vector space. Then $A \ltimes E$ is pseudo-arithmetical if and only if A is a field and* $\dim_{A/M} E = 1$.

Proof. If A is not a field, then $A \ltimes E$ is not pseudo-arithmetical by [1, Example 4.3]. Now assume that $A = K$ is a field. Then $K \ltimes E$ is Gaussian by [1, Theorem 3.1]. If $\dim_K E = 1$, then $K \ltimes E$ is arithmetical by [1, Theorem 3.1], hence pseudo-arithmetical. If $\dim_K E \geq 2$, then $K \ltimes E$ is Gaussian but not arithmetical by [1, Theorem 3.1]. Since a Gaussian pseudo-arithmetical ring is arithmetical by [1, Section 4], $K \ltimes E$ cannot be pseudo-arithmetical in this case. □

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