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Trivial Ring Extensions In Commutative Rings: A Survey

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Abstract. Let A be a commutative ring, and let E be an A-module. The trivial ring extension $A \propto E$ is a classical construction in commutative algebra that provides a useful framework for producing examples and counterexamples and for studying the transfer of algebraic properties between rings and modules. In this paper, we investigate several properties of the trivial ring extension $A \propto E$. We also examine conditions under which various ring-theoretic properties transfer from A and E to the extension ring $A \propto E$. Several examples and characterizations are provided to illustrate the behavior of these properties in the setting of trivial ring extensions.

Key Words: Trivial ring extensions, idealization, Prüfer rings, Arithmetical rings, valuation rings.
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1 Introduction

Throughout this survey, all rings are commutative with identity and all modules are unitary. If R is a ring and E is an R-module, then we use the following notation:

- $\mathcal{I}(R)$, the set of ideals of R;
- $\sqrt{I}$, the radical of an ideal I of R;
- $Nil(R) := \sqrt{(0)}$, the set of all nilpotent elements of R;
- $Z(R)$, the set of all zero-divisors of R;
- $Reg(R) = R \setminus Z(R)$;
- $Jac(R)$, the Jacobson radical of R;
- $Spec(R)$, the set of prime ideals of R;
- $Max(R)$, the set of maximal ideals of R;
- $Min(R)$, the set of minimal prime ideals of R;

- $T(R)$, the total quotient ring of R;
- $\overline{R}$, the integral closure of R in $T(R)$;
- $Ann_R(E) := \mathrm{Ann}(E)$, the annihilator of E;
- $Idem(R)$, the set of idempotent elements of R.

If R is an integral domain, we denote its quotient field by $qf(R)$.

Let A be a ring and let E be an A-module. The trivial ring extension of A by E, denoted by $A \propto E$, is the ring whose underlying additive group is $A \oplus E$ and whose multiplication is defined by

$$(a,e)(b,f) = (ab, af + be)$$

for all $a, b \in A$ and all $e, f \in E$. This construction is also known as the idealization of E over A. Trivial ring extensions have been extensively studied because of their usefulness in constructing examples and counterexamples in commutative algebra and in transferring several ring-theoretic properties from a ring to its extensions.

The trivial ring extension was introduced by Nagata [108] and later investigated by many authors from different perspectives [10, 22, 21, 39, 46, 52, 70, 82, 81, 80, 86, 102, 107, 124]. It provides a rich source of examples in the study of Noetherian rings, valuation rings, Pr"ufer rings, Gaussian rings, coherent rings, and homological properties. Moreover, several classical constructions may be viewed as special cases or closely related forms of trivial ring extensions. The present survey is devoted to covering most known results on trivial ring extensions.

2 Definitions and basic results

We begin by recalling the definition of trivial ring extensions and collecting the basic structural facts that will be used throughout the survey. This section is based on D. D. Anderson and M. Winders [10].

Definition 2.1. Let A be a ring and let E be an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R := A \propto E$ whose underlying additive group is $A \times E$ and whose multiplication is given by $(a,e)(a',e') = (aa', ae' + a'e)$.

We begin with a basic characterization of ideals in the idealization $A \propto E$.

Theorem 2.2. [10, Theorem 3.1] Let A be a commutative ring, I an ideal of A, E an A-module, and N a submodule of E. Then $I \propto N$ is an ideal of $A \propto E$ if and only if $IE \subseteq N$. When $I \propto N$ is an ideal, E/N is an A/I-module and $(A \propto E)/(I \propto N) \cong (A/I) \propto (E/N)$. In particular, $(A \propto E)/(0 \propto N) \cong A \propto (E/N)$ and therefore $(A \propto E)/(0 \propto E) \cong A$. Thus the ideals of $A \propto E$ containing $0 \propto E$ are of the form $J \propto E$ for some ideal J of A.

The following theorem characterizes the maximal, prime, and radical ideals of $A \propto E$.

Theorem 2.3. [10, Theorem 3.2] Let A be a commutative ring and let E be an A-module.

1. The maximal ideals of $A \propto E$ have the form $m \propto E$, where m is a maximal ideal of A. Thus $A \propto E$ is quasilocal if and only if A is quasilocal. Also, $A \propto E$ and A have the same residue fields. The Jacobson radical of $A \propto E$ is $J(A \propto E) = J(A) \propto E$.

2. The prime ideals of $A \propto E$ have the form $P \propto E$, where P is a prime ideal of A. Hence, if P is a prime ideal of A, then $\mathrm{ht}(P \propto E) = \mathrm{ht}\, P$, and so $\dim(A \propto E) = \dim A$.

3. The radical ideals of $A \propto E$ have the form $I \propto E$, where I is a radical ideal of A. If J is an ideal of $A \propto E$, then $\sqrt{J} = \sqrt{I} \propto E$, where $I = \{r \in A \mid (r,b) \in J \text{ for some } b \in E\}$ is an ideal of A. In particular, if I is an ideal of A and N is a submodule of E, then $\sqrt{I \propto N} = \sqrt{I} \propto E$; hence $Nil(A \propto E) = Nil(A) \propto E$.

We next collect some basic properties of homogeneous ideals in $A \propto E$.

Theorem 2.4. [10, Theorem 3.3] Let A be a commutative ring and let E be an A-module.

1. The homogeneous ideals of $A \propto E$ have the form $I \propto N$, where I is an ideal of A, N is a submodule of E, and $IE \subseteq N$. If J is a homogeneous ideal, then $J = I \propto N$, where $I = \{r \in A \mid (r,b) \in J \text{ for some } b \in E\}$ and $N = \{m \in E \mid (s,m) \in J \text{ for some } s \in A\}$.

2. Let $I \propto N$ and $I' \propto N'$ be two homogeneous ideals of $A \propto E$. Then $(I \propto N) \cap (I' \propto N') = (I \cap I') \propto (N \cap N')$ and $(I \propto N)(I' \propto N') = (II') \propto (IN' + I'N)$.

3. For a principal ideal $\langle (a,b) \rangle$ of $A \propto E$, the following conditions are equivalent:

   (a) $\langle (a,b) \rangle$ is homogeneous;

   (b) $\langle (a,b) \rangle = Aa \propto (Ab + aE)$;

   (c) $(a,0) \in \langle (a,b) \rangle$;

   (d) there exists $x \in A$ such that $xa = a$ and $xb \in aE$.

   In particular, if A is presimplifiable, that is, $xy = x$ implies $x = 0$ or y is a unit, then $\langle (a,b) \rangle$ is homogeneous if and only if $a = 0$ or $b \in aE$.

4. Every ideal of $A \propto E$ is homogeneous if and only if every principal ideal of $A \propto E$ is homogeneous. Hence, if A is presimplifiable, every ideal of $A \propto E$ is homogeneous if and only if $E = aE$ for each nonzero $a \in A$. Thus, if A is an integral domain, every ideal of $A \propto E$ is homogeneous if and only if E is divisible. If A is presimplifiable but not an integral domain, every ideal of $A \propto E$ is homogeneous if and only if $E = 0$.

5. Suppose that E is a finitely generated A-module. Then every ideal of $A \propto E$ is homogeneous if and only if, for each nonzero $a \in A$, there exists $x_a \in A$ such that $x_a a = a$ and $x_a E = aE$.

Corollary 2.5. *[10, Corollary 3.4] Let A be an integral domain and let E be an A-module. Then the following conditions are equivalent:*

1. *every ideal of $A \propto E$ is comparable to $0 \propto E$;*

2. *every ideal of $A \propto E$ has the form $I \propto E$ or $0 \propto N$ for some ideal I of A or some submodule N of E;*

3. *every ideal of $A \propto E$ is homogeneous;*

4. *E is divisible.*

We now characterize the regular elements and zero-divisors of $A \propto E$.

Theorem 2.6. [10, Theorem 3.5] Let A be a commutative ring and let E be an A-module. Then

$$Z(A \propto E) = \{(r,m) \mid r \in Z(A) \cup Z(E),\ m \in E\}.$$

Hence $S \propto E$ is the set of regular elements of $A \propto E$, where $S = A \setminus (Z(A) \cup Z(E))$.

We next determine when a homogeneous ideal $I \propto N$ is primary.

Theorem 2.7. [10, Theorem 3.6] Let A be a commutative ring and let E be an A-module. Let I be an ideal of A and let N be a submodule of E. Then $I \propto N$ is primary if and only if either

(a) $N = E$ and I is a primary ideal of A, or

(b) $N \subsetneq E$, $IE \subseteq N$, and both I and N are P-primary, where $P = \sqrt{I}$.

In either case, $I \propto N$ is $\sqrt{I} \propto E$-primary.

We next determine the units and idempotents of $A \propto E$.

Theorem 2.8. [10, Theorem 3.7] Let A be a commutative ring and let E be an A-module. Then the units of $A \propto E$ are $U(A \propto E) = U(A) \propto E$, and the idempotents of $A \propto E$ are $Id(A \propto E) = Id(A) \propto 0$.

The saturated multiplicatively closed subsets of $A \propto E$ are easy to determine.

Theorem 2.9. [10, Theorem 3.8] Let A be a commutative ring and let E be an A-module.

1. There is a one-to-one correspondence between the saturated multiplicatively closed subsets of A and those of $A \propto E$, given by $S \longleftrightarrow S \propto E$.
2. If S is a multiplicatively closed subset of A and N is a submodule of E, then $S \propto N$ is a multiplicatively closed subset of $A \propto E$ whose saturation is $\overline{S \propto N} = S \propto E$.

The next theorem is a regular version of part of Theorem 2.4.

Theorem 2.10. [10, Theorem 3.9] Let A be a commutative ring and let E be an A-module. Let $S = A \setminus (Z(A) \cup Z(E))$. Then the following conditions are equivalent:

1. Every regular ideal of $A \propto E$ has the form $I \propto E$, where I is an ideal of A with $I \cap S \neq \emptyset$.
2. Every regular ideal of $A \propto E$ is homogeneous.
3. For each $s \in S$ and $m \in E$, $\langle (s, m) \rangle$ is homogeneous.
4. $sE = E$ for all $s \in S$, or equivalently, $E = E_S$.

Hence, if $A \propto E$ is integrally closed, then every regular ideal of $A \propto E$ has the form given in (1).

We now study the behavior of idealization with respect to localization.

Theorem 2.11. [10, Theorem 4.1] Let A be a commutative ring and let E be an A-module.

1. Let S be a multiplicatively closed subset of A and let N be a submodule of E. Then $(A \propto E)_{S \propto N} \cong A_S \propto E_S$. In the case where $N = 0$, the isomorphism is simply $(r, m)/(s, 0) \mapsto (r/s, m/s)$.
2. Let P be a prime ideal of A. Then $(A \propto E)_{P \propto E} \cong A_P \propto E_P$.
3. The total quotient ring $T(A \propto E)$ of $A \propto E$ is naturally isomorphic to $A_S \propto E_S$, where $S = A \setminus (Z(A) \cup Z(E))$.

We next determine the integral closure of $A \propto E$ in $T(A \propto E)$.

Theorem 2.12. [10, Theorem 4.2] Let A be a commutative ring and let E be an A-module. Let $S = A \setminus (Z(A) \cup Z(E))$. If $\overline{A}$ is the integral closure of A in $T(A)$, then $(\overline{A} \cap A_S) \propto E_S$ is the integral closure of $A \propto E$ in $T(A \propto E)$.

Corollary 2.13. *[10, Corollary 4.3] Let A be a commutative ring and let E be an A-module. Let $S = A \setminus (Z(A) \cup Z(E))$.*

1. *If A is integrally closed, then $A \propto E_S$ is the integral closure of $A \propto E$ in $T(A \propto E)$.*
2. *If $Z(E) \subseteq Z(A)$, then $A \propto E_S$ is integrally closed if and only if A is integrally closed.*

Suppose that A_1 and A_2 are commutative rings and that E_i is an A_i-module for $i = 1, 2$. Then $E_1 \times E_2$ is an $A_1 \times A_2$-module with action $(a_1, a_2)(e_1, e_2) = (a_1e_1, a_2e_2)$. Conversely, let $A = A_1 \times A_2$ and suppose that E is an A-module. Put $E_1 = (A_1 \times 0)E$ and $E_2 = (0 \times A_2)E$. Then E_i is an A_i-module and E is the internal direct sum of E_1 and E_2; hence $E \cong E_1 \times E_2$.

Theorem 2.14. [10, Theorem 4.4] Let A_1 and A_2 be commutative rings, and let E_i be an A_i-module for $i = 1, 2$. Then

$$(A_1 \times A_2) \propto (E_1 \times E_2) \cong (A_1 \propto E_1) \times (A_2 \propto E_2).$$

We have already remarked that $A \propto E$ is a graded ring. We next show that if A is a graded ring and E is a graded A-module, then $A \propto E$ has a natural grading.

Theorem 2.15. [10, Theorem 4.5] Let $A = A_0 \oplus A_1 \oplus \cdots$ be a graded commutative ring and let $E = E_0 \oplus E_1 \oplus \cdots$ be a graded A-module. Then $A \propto E$ is a graded ring with $(A \propto E)_n = A_n \oplus E_n$.

As a special case, we obtain the polynomial ring over $A \propto E$.

Corollary 2.16. *[10, Corollary 4.7] Let A be a commutative ring and let E be an A-module. Then $(A \propto E)[X]$ is naturally isomorphic to $A[X] \propto E[X]$.*

3 Noetherian-like conditions

This section covers results from [10], [14], and [15]. In [106], Mohamadian introduced the notion of r-ideals in a commutative ring. A proper ideal I of a ring R is said to be an r-ideal of R if, whenever $ra \in I$ for some $r \in R \setminus Z(R)$ and $a \in R$, then $a \in I$. In [88], Koç and Tekir generalized the study of r-ideals to the context of submodules as follows. A proper submodule N of an R-module M is called an r-submodule of M if $rm \in N$ with $r \in R \setminus Z_R(M)$ implies that $m \in N$ for all $r \in R$ and $m \in M$. Later, Anebri, Mahdou, and Tekir [15, 14] defined M to be an r-Noetherian, respectively r-Artinian, module if every r-submodule of M is finitely generated, respectively if M satisfies the descending chain condition on r-submodules. Also, the ring R is said to be an r-Noetherian, respectively r-Artinian, ring if R is an r-Noetherian, respectively r-Artinian, R-module.

The following theorem answers the question of when $A \propto E$ is a Noetherian, respectively Artinian, ring.

Theorem 3.1. [10, Theorem 4.8] Let A be a commutative ring and let E be an A-module. Then $A \propto E$ is Noetherian, respectively Artinian, if and only if A is Noetherian, respectively Artinian, and E is finitely generated.

We now study the transfer of the r-Noetherian property to trivial ring extensions. We say that a trivial extension $A \propto E$ satisfies $(*)$ if every r-ideal of $A \propto E$ is homogeneous.

Theorem 3.2. [15, Theorem 3.1] Let A be a ring and let E be an A-module. Then:

1. If $A \propto E$ is an r-Noetherian ring, then A is an r-Noetherian ring and E is a finitely generated r-Noetherian module.

2. Suppose that $A \propto E$ satisfies $(*)$ and $Z(A) = Z_A(E)$. Then $A \propto E$ is an r-Noetherian ring if and only if A is an r-Noetherian ring and E is a finitely generated r-Noetherian module.

An A-module E is called torsion-free if $r \in A$, $m \in E$, and $rm = 0$ imply that either $r = 0$ or $m = 0$.

Corollary 3.3. *[15, Corollary 3.2] Let A be a domain and let E be a torsion-free A-module. Then $A \propto E$ is an r-Noetherian ring if and only if E is a finitely generated r-Noetherian module.*

Corollary 3.4. *[15, Corollary 3.3] Let A be a domain and let E be a finitely generated r-Noetherian A-module such that $T_A(E) \neq E$. Then $A \propto (E/T_A(E))$ is an r-Noetherian ring.*

Remark 3.5. [15, Remark 3.4] An r-ideal of $A \propto E$ need not be homogeneous. For example, $J = \langle(\bar{2}, \bar{1})\rangle$ is an r-ideal of $\mathbb{Z}_4 \propto \mathbb{Z}_2$, but J does not have the form $I \propto N$. Also, if $I \propto M$ is an r-ideal of $A \propto E$, then I is not necessarily an r-ideal of A. Indeed, $J = 2\mathbb{Z} \propto \mathbb{Z}_2$ is a prime ideal of $\mathbb{Z} \propto \mathbb{Z}_2$ consisting entirely of zero-divisors of $\mathbb{Z} \propto \mathbb{Z}_2$, and so J is an r-ideal. However, $2\mathbb{Z}$ is not an r-ideal of $\mathbb{Z}$.

Example 3.6. [15, Example 3.5] Let A be a non-Noetherian domain, for example $A = \mathbb{Z} + X\mathbb{Q}[X]$. Then $A \propto A$ is an r-Noetherian ring that is not Noetherian. Also, $A \propto A$ is not a weakly Noetherian ring because it contains a regular element that is neither a unit nor a zero-divisor.

The following theorem studies the transfer of the r-Artinian property to $A \propto E$. Recall from [115] that an A-module E is said to be an S-Artinian module, for some multiplicatively closed subset S of A, if for any descending chain of submodules $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$ of E, there exist $s \in S$ and $k \in \mathbb{N}$ such that $sN_k \subseteq N_n$ for each $n \geq k$.

Theorem 3.7. [14, Theorem 2.18] Let A be a ring and let E be an A-module. Then the following statements hold:

1. If $A \propto E$ is an r-Artinian ring, then A is an r-Artinian ring and E is an r-Artinian A-module.

2. Assume that $Z_A(E) \subseteq Z(A)$. If A is an r-Artinian ring and E is an S-finite module, where $S = A \setminus Z(A)$, then $A \propto E$ is an r-Artinian ring.

3. Suppose that A is an S-Artinian ring and E is an S-finite module, where $S = A \setminus (Z(A) \cup Z_A(E))$. Then $A \propto E$ is an r-Artinian ring.

Remark 3.8. [14, Remark 2.19] It is interesting that the assertion in Theorem 3.7(2) fails if $Z_A(E) \not\subseteq Z(A)$. In fact, $A := \mathbb{Z} \propto \mathbb{Z}_2$ is not an r-Artinian ring since the following descending chain of r-ideals is not stationary:

$$2\mathbb{Z} \propto \mathbb{Z}_2 \supseteq 2^2\mathbb{Z} \propto \mathbb{Z}_2 \supseteq \cdots \supseteq 2^n\mathbb{Z} \propto \mathbb{Z}_2 \supseteq \cdots.$$

The following example gives an r-Artinian ring that is not an S-Artinian ring.

Example 3.9. [14, Example 2.20] Consider the ring of integers $A := \mathbb{Z}$ and let $E := \mathbb{Z}^n$. Then $A \propto E$ is an r-Artinian ring which is not an S-Artinian ring, for every multiplicatively closed subset S of $A \propto E$.

4 Coherent-like conditions

We next collect results concerning coherent-type finiteness conditions for trivial ring extensions, including v-coherence, n-coherence, weak coherence, and Nil_*-coherence.

4.1 v-Coherent rings with zero-divisors

This subsection is based on Kabbaj and Mahdou [81]. A ring R is quasi-coherent, respectively finite conductor, if $(0:a)$ and $a_1R\cap\cdots\cap a_nR$, respectively $bR\cap cR$, are finitely generated ideals of R for any finite set of elements a and $a_1,\ldots,a_n$, respectively b,c, of R [27, 67, 125]. Also, R is called a G-GCD ring if every principal ideal of R is projective and the intersection of any two finitely generated flat ideals of R is a finitely generated flat ideal of R [6, 67]. By an ideal of R, we mean an integral ideal of R. Let I and J be two nonzero fractional ideals of R. We define the fractional ideal $(I:J)=\{x\in T(R)\mid xJ\subseteq I\}$. We denote $(R:I)$ by I^{-1} and $(I^{-1})^{-1}$ by I_v, called the v-closure of I. A nonzero fractional ideal I is said to be invertible if $II^{-1}=R$, divisorial, or a v-ideal, if $I_v=I$, and v-finite if $I_v=J_v$, or equivalently, if $I^{-1}=J^{-1}$, for some finitely generated fractional ideal J of R. The ring R is said to be v-coherent if $(0:a)$ and $\bigcap_{1\leq i\leq n}Ra_i$ are v-finite ideals of R for any finite set of elements a and $a_1,\ldots,a_n$ of R [81].

The main result is the following.

Theorem 4.1. [81, Theorem 2.6] Let (A,M) be a local ring and let E be an A-module with $ME=0$. Let $R:=A\propto E$ be the trivial ring extension of A by E. Then:

1. R is a v-coherent ring that is not G-GCD.
2. R is coherent, respectively quasi-coherent or finite conductor, if and only if A is coherent, respectively quasi-coherent or finite conductor, M is finitely generated, and E is an (A/M)-vector space of finite rank.

Lemma 4.2. *[81, Lemma 2.7] Under the hypotheses of Theorem 4.1, $(0:c)$ is a finitely generated ideal of R for each $c\in R$ if and only if $(0:a)$ is a finitely generated ideal of A for each $a\in A$, M is finitely generated, and E is an (A/M)-vector space of finite rank.*

Next, we explore a different context, namely the trivial ring extension of a domain by its quotient field.

Theorem 4.3. [81, Theorem 2.8] Let A be a domain that is not a field, let $K=qf(A)$, and let $R:=A\propto K$ be the trivial ring extension of A by K. Then:

1. R is not a finite conductor ring. In particular, R is neither quasi-coherent nor coherent.
2. R is a v-coherent ring if and only if A is v-coherent.

New examples of coherent-like rings with zero-divisors and arbitrary Krull dimension may be obtained from Theorems 4.1 and 4.3, as shown by the following constructions.

Example 4.4. [81, Example 2.9] Let K be a field and let $X_1,X_2,\ldots$ be indeterminates over K. Let n be an integer with $n\geq 1$, let $A=K[[X_1,\ldots,X_n]]$ be the power series ring in n variables over K, and let $R:=A\propto K$. Then R is an n-dimensional coherent ring that is not G-GCD.

Example 4.5. [81, Example 2.10] Let A be as in the preceding example and let $R:=A\propto K[Y]$, where Y is another indeterminate over K. Then R is an n-dimensional v-coherent ring that is not finite conductor.

Example 4.6. [81, Example 2.11] Let $R:=\mathbb{Z}[X_1,\ldots,X_{n-1}]\propto\mathbb{Q}(X_1,\ldots,X_{n-1})$, where $n\geq 1$. Then R is an n-dimensional v-coherent ring that is not finite conductor.

4.2 n-Coherence and strong n-coherence

This subsection is based on Kabbaj and Mahdou [81]. Let R be a ring. For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0,$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules. Given a nonnegative integer d, R is called an (n,d)-ring if every n-presented R-module has projective dimension at most d, and a weak (n,d)-ring if every n-presented cyclic R-module has projective dimension at most d, equivalently, if every $(n-1)$-presented ideal of R has projective dimension at most $d-1$. For instance, the $(0,1)$-domains are the Dedekind domains, the $(1,1)$-domains are the Prüfer domains, and the $(1,0)$-rings are the von Neumann regular rings [40]. The ring R is said to be n-coherent if each $(n-1)$-presented ideal of R is n-presented, and R is said to be strong n-coherent if each n-presented R-module is $(n+1)$-presented [47, 48]. This terminology is not the same as that of Costa [40]; more precisely, Costa's n-coherence is our strong n-coherence. In particular, 1-coherence coincides with coherence, and 0-coherence coincides with Noetherianity. Any strong n-coherent ring is n-coherent, and the converse holds for $n = 1$ or for coherent rings. Strong n-coherence arose naturally in Costa's study [40] of (n,d)-rings. In fact, every (n,d)-ring is strong $\max(n,d)$-coherent [40, Theorem 2.2], and an (n,d)-ring is strong r-coherent, with $r < n$, only if it is an (r,d)-ring [40, Theorem 2.4].

The main theorem examines the transfer of strong n-coherence and n-coherence, for $n \geq 2$, to the trivial ring extension of a domain by its quotient field.

Theorem 4.7. [81, Theorem 3.1] Let A be a domain that is not a field, let $K = qf(A)$, and let $R := A \propto K$ be the trivial ring extension of A by K. Let $n \geq 2$ and $d \geq 1$ be integers. Then:

1. R is not coherent.
2. R is strong n-coherent, respectively n-coherent, if and only if so is A.
3. R is an (n,d)-ring, respectively a weak (n,d)-ring, if and only if so is A.

Let us fix the notation for the next two results. Let R be as in Theorem 4.7, and let H be an R-submodule of R^m, where m is a positive integer. Set $U = \{x \in A^m \mid (x,e) \in H \text{ for some } e \in K^m\}$ and $E = \{e \in K^m \mid (x,e) \in H \text{ for some } x \in A^m\}$.

Lemma 4.8. *[81, Lemma 3.2] Under the above notation, the following statements are equivalent:*

1. *H is finitely generated and E is a K-vector space.*
2. *U is finitely generated and $H = U \propto KU$.*

Lemma 4.9. *[81, Lemma 3.3] Let n be an integer with $n \geq 1$. Under the above notation, the following statements are equivalent:*

1. *H is n-presented.*
2. *U is n-presented and $H = U \propto KU$.*

For $n \leq 1$ or $d = 0$, the (n,d)-property may not survive, in general, in the trivial extension R, even under strong assumptions on A. This is illustrated by the next example.

Example 4.10. [81, Example 3.4] Let A be an arbitrary Prüfer domain, and let R be the trivial ring extension of A by its quotient field. Then R is a $(2,1)$-ring which is neither a semihereditary ring, that is, a $(1,1)$-ring, nor a 2-von Neumann regular ring, that is, a $(2,0)$-ring.

The Bézout property, however, transfers reciprocally from A to R, as shown by the next result.

Proposition 4.11. *[81, Proposition 3.5] Let R be as in Theorem 4.7. Then R is a Bézout ring if and only if A is a Bézout domain.*

It is worth noting that new families of non-semihereditary Bézout rings arise from the combination of Example 4.10 and Proposition 4.11.

At this point, for the convenience of the reader, we recall from [82] the main result that establishes the transfer of the (n,d)-property to trivial ring extensions of local rings by their residue fields.

Theorem 4.12. [82, Theorem 1.1] Let (A,M) be a local ring and let $R := A \propto A/M$ be the trivial ring extension of A by A/M. Then:

1. R is a $(3,0)$-ring provided M is not finitely generated.
2. R is not a $(2,d)$-ring, for each integer $d \geq 0$, provided M contains a regular element.

Clearly, Theorems 4.7 and 4.12 generate new examples of n-coherent rings which, moreover, reflect no obvious correlation between strong n-coherence and the large class of finite conductor rings.

Example 4.13. [81, Example 3.7] Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Q} = qf(\mathbb{Z})$. Then $R := \mathbb{Z} \propto \mathbb{Q}$ is a strong 2-coherent ring which is not a finite conductor ring.

Example 4.14. [81, Example 3.8] Let (V,M) be a nondiscrete valuation domain. Then $R := V \propto V/M$ is a 3-coherent ring which is neither 2-coherent nor a finite conductor ring.

4.3 On weakly coherent rings

This subsection is based on Bakkari and Mahdou [23]. A ring R is called a weakly coherent ring if every finitely generated ideal of R contained in a finitely presented proper ideal of R is itself finitely presented. If R is coherent, then R is naturally weakly coherent [23].

We begin this subsection by giving an example of a non-coherent weakly coherent ring.

Example 4.15. [23, Example 2.1] Let (A,M) be a local ring with maximal ideal M, let E be an A/M-vector space of infinite rank, and let $R := A \propto E$ be the trivial ring extension of A by E. Then:

1. R is a weakly coherent ring.
2. R is not a coherent ring.

The main theorem examines the transfer of the weakly coherent property to trivial ring extensions arising from local rings.

Theorem 4.16. [23, Theorem 2.6] Let (A,M) be a local ring with maximal ideal M, let E be an A-module such that $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of A by E. Then R is a weakly coherent ring if and only if one of the following two properties holds:

1. E is an A/M-vector space of infinite rank.
2. E is an A/M-vector space of finite rank and A is weakly coherent.

We now give a new class of non-coherent weakly coherent rings.

Example 4.17. [23, Example 2.7] Let (A,M) be a local coherent ring with non-finitely generated maximal ideal M, let E be an A-module such that $ME = 0$, let $R := A \propto E$ be the trivial ring extension of A by E, and set $I := 0 \propto E$. Then:

1. R is a weakly coherent ring by Theorem 4.16.
2. R is not a coherent ring since M is not finitely generated.

The localization of a weakly coherent ring is not always weakly coherent, as the following example shows.

Example 4.18. [23, Example 2.8] Let $A := \mathbb{Z}_{(2)} + X\mathbb{R}[[X]]$ be a local ring with maximal ideal $M = 2\mathbb{Z}_{(2)} + X\mathbb{R}[[X]]$, let E be an A/M-vector space of infinite rank, and let $R := A \propto E$ be the trivial ring extension of A by E. Let S be the multiplicative subset of R given by $S := \{(2,0)^n \mid n \in \mathbb{N}\}$, and let S_0 be the multiplicative subset of A given by $S_0 := \{2^n \mid n \in \mathbb{N}\}$. Then:

1. R is a weakly coherent ring.
2. $S^{-1}R$ is not a weakly coherent ring.

We know that a coherent ring is weakly coherent and strong 2-coherent. The following two examples show that the class of weakly coherent rings and the class of strong 2-coherent rings are not comparable.

Example 4.19. [23, Example 2.9] Let R be a non-coherent strong 2-coherent domain. Then R is not a weakly coherent domain since R is not coherent.

Example 4.20. [23, Example 2.9] Let (A,M) be a local coherent domain with non-finitely generated maximal ideal M, and let $R := A \propto (A/M)$ be the trivial ring extension of A by A/M. Then:

1. R is a weakly coherent ring by Theorem 4.16.
2. R is not a strong 2-coherent ring.

4.4 Nil_*-coherence and special Nil_*-coherence

This subsection is based on Alaoui, Dobbs, and Mahdou [3]. Let R be a ring and let M be an R-module. Then M is called a Nil_*-coherent R-module if every finitely generated R-submodule of $Nil(R)M$ is a finitely presented R-module. A ring R is said to be a Nil_*-coherent ring if it is Nil_*-coherent as an R-module, that is, if every finitely generated ideal of R contained in $Nil(R)$ is finitely presented. Also, an R-module M is said to be a special Nil_*-coherent R-module if $Nil(R)M$ is a coherent R-module; equivalently, if $Nil(R)M$ is a finitely generated R-module and every finitely generated R-submodule of $Nil(R)M$ is finitely presented. A ring R is said to be a special Nil_*-coherent ring if it is special Nil_*-coherent as an R-module; equivalently, if $Nil(R)$ is a coherent R-module; equivalently, if $Nil(R)$ is a finitely generated ideal of R and each finitely generated ideal of R contained in $Nil(R)$ is finitely presented.

Lemma 4.21. *[3, Lemma 4.1] Let (A,M) be a quasi-local ring and let E be an A-module such that $ME = 0$. Then:*

(a) $(r,e)^n = (r^n, 0)$ for all $r \in M$, $e \in E$, and all integers $n > 1$.

(b) $Nil(A \propto E) = Nil(A) \propto E$.

Theorem 4.22. [3, Theorem 4.2] Let (A,M) be a quasi-local ring and set $k := A/M$. Let E be an A-module such that $ME = 0$, that is, a vector space over k. Set $R := A \propto E$. Then:

(a) $Nil(R)$ is a finitely generated ideal of R if and only if $Nil(A)$ is a finitely generated ideal of A and E is a finite-dimensional vector space over k.

(b) R is a Nil_*-coherent ring, respectively a special Nil_*-coherent ring, if and only if A is a Nil_*-coherent ring, respectively a special Nil_*-coherent ring, and E is a finite-dimensional vector space over k.

The next corollary produces a class of coherent rings that are also special Nil_*-coherent rings.

Corollary 4.23. *[3, Corollary 4.3]*

(a) Let (A, M) be a quasi-local coherent ring such that M is a finitely generated ideal of A, and set $k := A/M$. Let E be a finitely generated A-module such that $ME = 0$, that is, a finite-dimensional vector space over k. Then $R := A \propto E$ is a coherent ring.

(b) Let k be a field and let E be a nonzero finite-dimensional vector space over k. Then $A := k \propto E$ is a coherent, in fact Noetherian, ring, and hence a special Nil_*-coherent ring, which is not reduced.

We next construct new examples of special Nil_*-coherent rings that are not coherent rings.

Corollary 4.24. *[3, Corollary 4.4] Let (A, M) be a quasi-local special Nil_*-coherent ring that is not coherent, and let E be a finite-dimensional vector space over A/M. Then $A \propto E$ is a special Nil_*-coherent ring that is not coherent.*

Corollary 4.25. *[3, Corollary 4.5] Let (A, M) be a quasi-local integral domain that is not coherent. Then $A \propto A/M$ is a special Nil_*-coherent ring that is not coherent.*

The next corollary gives another, not necessarily Noetherian, example illustrating Corollary 4.23(a).

Corollary 4.26. *[3, Corollary 4.6] Let $k \subset K$ be distinct fields and let (V, M) be a valuation domain of the form $V = K + M$. Put $A := k + M$ and $R := A \propto k$. Then:*

(a) R is a special Nil_*-coherent ring.

(b) R is not a coherent ring if and only if either $[K : k] = \infty$ or $M = M^2$, or both.

To close this subsection, we give an example of a coherent ring that is not a special Nil_*-coherent ring.

Example 4.27. [3, Corollary 4.4] Let k be a field. Inductively define a sequence of rings as follows. Let $R_0 := k$ and, for each nonnegative integer n, let $R_{n+1} := R_n \propto R_n$. Put $R := \varinjlim R_n$. Then R is a coherent ring, and hence a Nil_*-coherent ring, but not a special Nil_*-coherent ring, since $Nil(R)$ is not a finitely generated ideal of R.

4.5 P-Coherent rings

This subsection is due to Mahdou [99]. A ring R is called P-coherent if every finitely generated prime ideal of R is finitely presented. A coherent ring is naturally a P-coherent ring.

Theorem 4.28. [99, Theorem 3.1] Let A be a domain but not a field, $K = qf(A)$, and $R := A \propto K$ be the trivial ring extension of A by K. Then:

1. R is a P-coherent ring if and only if A is a P-coherent ring.
2. R is not coherent.

If A is a coherent domain, we obtain by Theorem 4.28:

Corollary 4.29. *[99, Corollary 3.2] Let A be a coherent domain which is not a field, $K = qf(D)$, and $R := A \propto K$ be the trivial ring extension of A by K. Then R is P-coherent which is not coherent.*

Next, we explore a different context, namely, the trivial ring extension of a local ring (A, M) by an A-module E such that $ME = 0$.

Theorem 4.30. [99, Theorem 3.3] Let (A, M) be a local ring and E an A-module with $ME = 0$. Let $R := A \propto E$ be the trivial ring extension of A by E. Then R is P-coherent if and only if one of the following conditions holds:

1. E is an (A/M)-vector space of infinite rank;
2. E is an (A/M)-vector space of finite rank, A is P-coherent, and M is a finitely generated ideal of A.

Before proving Theorem 4.30, we establish the following Lemma.

Lemma 4.31. *[99, Lemma 3.4] Under the hypotheses of Theorem 4.30, assume that E is an (A/M)-vector space of infinite rank. Then each prime ideal of R is not finitely generated. In particular, R is P-coherent.*

Now, we are able to give a class of P-coherent rings with zero-divisors which are not coherent.

Corollary 4.32. *[99, Corollary 3.5] Under the hypotheses of Theorem 4.30, assume that E is an (A/M)-vector space of infinite rank. Then:*

1. *R is P-coherent by Lemma 4.31.*
2. *R is not coherent since E is an (A/M)-vector space of infinite rank.*

Next, we give an example of non P-coherent ring.

Example 4.33. [99, Example 3.6] Let $A := K[[X_1, \ldots]] = K + M$ be the formal power series over a field K, where $(X_i)_{i=1,\ldots,\infty}$ is an indeterminates over K and M is the maximal ideal of A. Then the trivial ring extension of A by A/M is not P-coherent by Theorem 4.30 since M is not finitely generated.

Let A be a ring, E an A-module, and $R := A \propto E$ the trivial ring extension of A by E. In general, the implication that R is P-coherent $\Rightarrow$ A is P-coherent fails as the following example shown:

Example 4.34. [99, Example 3.7] Let (A, M) be a local non P-coherent ring (see Example 4.33) and let E be an (A/M)-vector space of infinite rank. Then, the trivial ring extension $R := A \propto E$ of A by E is P-coherent by Theorem 4.30.

5 Valuation-like properties

In [74], Hedstrom and Houston introduced a class of integral domains closely related to the class of valuation domains. An integral domain A with quotient field K is called a pseudo-valuation domain (PVD) if each prime ideal P of A is strongly prime, in the sense that for every $x, y \in K$, if $xy \in P$, then $x \in P$ or $y \in P$. An interesting survey article on pseudo-valuation domains is [18]. In [20], the study of pseudo-valuation domains was generalized to arbitrary rings with zero-divisors. Recall from [20] that a prime ideal P of a ring A is said to be strongly prime if aP and bA are comparable for all $a, b \in A$. A ring A is called a pseudo-valuation ring (PV-ring) if each prime ideal of A is strongly prime. A ring A is a PV-ring if and only if it is local with its maximal ideal strongly prime [20, Lemma 3]. Also, an integral domain is a PV-ring if and only if it is a PVD by [4, Proposition 3.1], [5, Proposition 4.2], and [19, Proposition 3].

In [11], D. D. Anderson and M. Zafrullah introduced and studied the notion of almost valuation domains. An integral domain A is called an almost valuation domain (AVD) if, for every nonzero $x \in K$, there exists an integer $n \geq 1$ such that either $x^n \in A$ or $x^{-n} \in A$. In [78] and [101], a generalization of almost valuation domains to arbitrary commutative rings with zero-divisors is considered as follows: A is called an almost valuation ring (AV-ring) if, for any two elements a and b in A, there exists a positive integer $n \geq 1$ such that a^n divides b^n or b^n divides a^n. An AV-ring is necessarily local [78, Proposition 2.2].

In [17], Badawi introduced a new class of integral domains as follows. A prime ideal P of an integral domain A is called a pseudo-strongly prime ideal if, whenever $x, y \in K$ and $xyP \subseteq P$, there is an integer $n \geq 1$ such that either $x^n \in A$ or $y^n P \subseteq P$. If each prime ideal of A is a pseudo-strongly prime ideal, then A is called a pseudo-almost valuation domain (PAVD). Also, an integral domain A is a PAVD if and only if, for every nonzero $x \in K$, there is a positive integer $n \geq 1$ such that either $x^n \in A$ or $ax^{-n} \in A$ for every nonunit $a \in A$. In [77], a generalization of pseudo-almost valuation domains to arbitrary commutative rings with zero-divisors is considered as follows: a prime ideal P of a ring A is said to be pseudo-strongly prime if, for every $a, b \in A$, there is an integer $n \geq 1$ such that either $a^n A \subseteq b^n A$ or $b^n P \subseteq a^n P$. A ring A is called a pseudo-almost valuation ring (PAV-ring) if each maximal ideal of A is pseudo-strongly prime. A PAV-ring is necessarily local. Also, an integral domain A is a PAV-ring if and only if A is a PAVD [77, Proposition 2.7].

5.1 Valuation rings

This subsection is based on Kabbour and Mahdou [83].

The main result is the following.

Theorem 5.1. [83, Theorem 2.1] Let A be a ring and let E be a nonzero A-module. Let $R := A \propto E$ be the trivial ring extension of A by E.

1. Assume that E is a non-torsion A-module. Then R is a valuation ring if and only if A is a valuation domain and E is isomorphic to $K := qf(A)$, the field of fractions of A.

2. Assume that E is a finitely generated A-module. Then R is a valuation ring if and only if A is a field and $E \cong A$.

Lemma 5.2. *[83, Lemma 2.2] Let A be a ring, let E be a nonzero A-module, and let $R := A \propto E$ be the trivial ring extension of A by E. If R is a valuation ring, then A is a valuation domain and E is a uniserial A-module.*

Corollary 5.3. *[83, Corollary 2.3] Let A be a ring. Then $A \propto A$ is a valuation ring if and only if A is a field.*

Theorem 5.1 enriches the literature with new examples of valuation rings, as shown below.

Example 5.4. [83, Example 2.4] Let k be a field. Let $A = k[[x]]$ be the ring of formal power series with coefficients in k, and let $K = k((x))$ be its field of fractions. Then the trivial ring extension $A \propto K$ is a valuation ring.

Example 5.5. [83, Example 2.5] Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ in the p-adic topology, where p is a prime integer. The ring of p-adic integers is $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$, and $\mathbb{Q}_p$ is its field of fractions. Then the trivial ring extension $\mathbb{Z}_p \propto \mathbb{Q}_p$ is a valuation ring.

We now construct a non-valuation ring.

Corollary 5.6. *[83, Corollary 2.6] Let A be a ring and let E be a mixed module, that is, E is neither torsion nor torsion-free. Then $A \propto E$ is not a valuation ring.*

5.2 Almost valuation rings

This subsection is based on Dobbs, El Khalfi, and Mahdou [46].

Theorem 5.7. [46, Theorem 2.2] Let A be a ring, let E be a nonzero A-module, and let $R := A \propto E$. Then:

1. If R is an AV-ring, then so is A.
2. Suppose that $Z(A) = Nil(A)$ and E is a divisible A-module. Then R is an AV-ring if and only if A is an AV-ring.
3. Suppose that $\mathbb{Q} \subseteq A$, $Z(A) = Nil(A)$, and E is a torsion-free A-module. Then R is an AV-ring if and only if A is an AV-ring and E is a divisible A-module.
4. Let (A,M) be a quasi-local ring, with maximal ideal M, and let E be an A-module such that $M = \sqrt{\mathrm{Ann}(E)}$. Then R is an AV-ring if and only if A is an AV-ring.

If A is a domain and E is a divisible A-module, Theorem 5.7(2) specializes to the following result.

Corollary 5.8. *[46, Corollary 2.3] Let A be a domain and let E be a divisible A-module. Then $A \propto E$ is an* AV*-ring if and only if A is an* AV*-ring.*

If (A,M) is a quasi-local ring and E is a suitable A-module, then Theorem 5.7(4) yields the following corollary, which recovers a result of Mahdou, Mimouni, and Moutui and gives a further application.

Corollary 5.9. *[46, Corollary 2.4] Let (A,M) be a quasi-local ring, let E be an A-module, and let $R := A \propto E$. Then:*

1. *[101, Theorem 2.1(3)] Suppose that $ME = 0$. Then R is an* AV*-ring if and only if A is an* AV*-ring.*
2. *Suppose that M is the only prime ideal of A, for instance, suppose that A is an* SPIR*. Then R is an* AV*-ring if and only if A is an* AV*-ring.*

Our next two examples illustrate parts (2) and (4) of Theorem 5.7.

Example 5.10. [46, Example 2.5] Let F be a finite field and let X be an indeterminate over F. Put $H := F(X)$, the quotient field of $F[X]$, and let Y be an analytic indeterminate over H. Set $D := H + Y^3H[[Y]]$, $K := qf(D)$, $A := D \propto K$, and $E := K \propto K$. Since A is a subring of E, we can view E as an A-module. Then $A \propto E$ is an AV-ring which is neither a domain nor a valuation ring.

Example 5.11. [46, Example 2.6] Let (A,M) be a valuation domain that is not a field. Then $A \propto A/M^2$ is an AV-ring which is neither a domain nor a valuation ring.

5.3 Pseudo-valuation rings

This subsection is based on Dobbs, El Khalfi, and Mahdou [46].

The following theorem studies the possible transfer of the PV-ring property between a ring A and a trivial ring extension $A \propto E$.

Theorem 5.12. [46, Theorem 2.8] Let A be a ring, let E be a nonzero A-module, and let $R := A \propto E$. Then:

1. If R is a PV-ring, then so is A.

2. Suppose that $\text{Ann}(E) \neq 0$. Then R is a PV-ring if and only if A is a quasi-local ring with maximal ideal M such that $M^2 = 0$ and $ME = 0$.

3. Suppose that R is a PV-ring. Then A is a quasi-local ring, say with maximal ideal M; E is a divisible A-module; either A is a PVD or $M^2 = 0$; and, for each $e, f \in E$, either $Ae \subseteq Mf$ or $Mf \subseteq Ae$.

4. If E is a nonzero finitely generated A-module and R is a PV-ring, then A is quasi-local and its maximal ideal, say M, satisfies $M^2 = 0$.

Corollary 5.13. *[46, Corollary 2.9] Let A be a domain. Then:*

1. *Let E be a nonzero finitely generated A-module. Then $R := A \propto E$ is a* PV*-ring if and only if A is a field.*

2. *$A \propto A$ is a* PV*-ring if and only if A is a field.*

A ring R is finite conductor if $bR \cap cR$ is a finitely generated ideal of R for any elements b, c of R.

Proposition 5.14. *[46, Proposition 2.10] Let (A, M) be a quasi-local ring, but not a field, such that $M^2 = 0$. Let E be an A-module such that $ME = 0$. Put $R := A \propto E$. Then:*

1. *R is a* PV*-ring.*

2. *Suppose that M is a finitely generated ideal of A and E is a finitely generated A-module. Then R is a finite-conductor ring.*

3. *Suppose that M is the only prime ideal of A and E is a nonzero finitely generated A-module. Then $R' = R$ is not a valuation ring.*

Example 5.15. [46, Example 2.11] Let $A := \mathbb{Z}/4\mathbb{Z}$ and let $E = A/M \cong \mathbb{Z}/2\mathbb{Z}$. Put $R := A \propto E$. Then R is a finite-conductor ring and a PV-ring, but $R = R'$ is not a valuation ring.

5.4 Pseudo-almost valuation rings

This subsection is based on Dobbs, El Khalfi, and Mahdou [46].

Theorem 5.16. [46, Theorem 2.13] Let A be a ring, let E be an A-module, and let $R := A \propto E$. Then:

1. If R is a PAV-ring, then so is A.

2. Suppose that $Z(A) = Nil(A)$ and E is a divisible A-module. Then R is a PAV-ring if and only if A is a PAV-ring.

3. Suppose that A is a quasi-local ring with maximal ideal M such that $M = \sqrt{\text{Ann}(E)}$. Then R is a PAV-ring if and only if A is a PAV-ring.

4. Suppose that $A \subseteq E$ is an extension of domains and $\mathbb{Q} \subseteq E$. Then R is a PAV-ring if and only if A is a PAV-ring and $qf(A) \subseteq E$.

The next two corollaries are immediate applications of parts (2) and (3) of Theorem 5.16.

Corollary 5.17. *Cf. Jahani-Nezhad and Khoshayand [77, Proposition 3.15]. Let A be a domain and let E be a divisible A-module. Then $A \propto E$ is a* PAV*-ring if and only if A is a* PAV*-ring, that is, if and only if A is a* PAVD.

Corollary 5.18. *[46, Corollary 2.15] Let (A, M) be a quasi-local ring and let E be an A-module such that $ME = 0$. Then $A \propto E$ is a* PAV*-ring if and only if A is a* PAV*-ring.*

In [17, Proposition 4.11], Badawi proved that if (R, M) is a PAVD such that M is a finitely generated ideal of R, for instance, if R is Noetherian, then R' is a valuation domain. The next example shows that [17, Proposition 4.11] does not extend from domains to the context of trivial ring extensions.

Example 5.19. [46, Example 2.16] Let (A, M) be a Noetherian valuation domain that is not a field, and let E be a nonzero finitely generated A-module such that $ME = 0$. Then $R := A \propto E$ is a Noetherian PAV-ring, but $R' = R$ is not a valuation ring.

Theorem 5.16 enriches the literature with new examples of PAV-rings, as illustrated by Examples 5.20 and 5.21.

Example 5.20. [46, Example 2.17] Let F be a finite field and let X be an indeterminate over F. Put $H := F(X)$, the quotient field of $F[X]$, and let Y be an analytic indeterminate over H. Set $R := F + HY^2 + Y^4H[[Y]]$, $K := qf(R)$, $A := R \propto K$, and $E := K \propto K$. Since A is a subring of E, we can view E as an A-module. Then $A \propto E$ is a PAV-ring which is neither a domain nor an AV-ring.

Example 5.21. [46, Example 2.18] Let (A, M) be an SPIR, for instance, $\mathbb{Z}/8\mathbb{Z}$ with $M = 2A$, and let E be an A-module. Then $A \propto E$ is a PAV-ring.

6 Elementary divisor rings, Hermite rings, Bézout rings, and almost Bézout rings in trivial ring extensions

This section reviews how elementary divisor, Hermite, Bézout, arithmetical, and almost Bézout properties behave under trivial ring extensions.

6.1 Elementary divisor rings, Hermite rings, and Bézout rings in trivial ring extensions defined by ring extensions

This subsection is based on Kabbour, Mahdou, and Mimouni [105]. Following Kaplansky [84], a ring A is said to be an *elementary divisor ring* if every matrix M over A is equivalent to a diagonal matrix, that is, there exist matrices P and Q which are invertible over A and such that PMQ is a diagonal matrix. In [90, Corollary 3.7], it is proved that a ring A is an elementary divisor ring if and only if every 2×2 matrix over A is equivalent to a diagonal matrix. Noetherian elementary divisor rings are just principal ideal rings, and the ring of entire functions is an example of a non-Noetherian, but Prüfer, elementary divisor ring; see [122]. Seeking elementary divisor rings outside the context of Noetherian rings, Kaplansky proved that an elementary divisor ring A is Hermite, that is, for every $a, b \in A$, there exist $d, a', b' \in A$ such that $a = da'$, $b = db'$, and $Aa' + Ab' = A$. Every Hermite ring is a Bézout ring, that is, every finitely generated ideal of A is principal. It is also well known that a Bézout domain is Hermite [84, Theorem 3.2].

Let $\mathcal{M}_n(X)$ denote the set of all $n \times n$ matrices with entries from a set X. For any ring A, let $\mathcal{G}l_n(A)$ denote the set of invertible matrices in $\mathcal{M}_n(A)$. If A is a ring, E is an A-module, and $M = ((a_{ij}, e_{ij})) \in \mathcal{M}_n(A \propto E)$, we denote $M_a = (a_{ij}) \in \mathcal{M}_n(A)$ and $M_e = (e_{ij}) \in \mathcal{M}_n(E)$. Thus, for every $M \in \mathcal{M}_n(A \propto E)$, we write $M = M_a \propto M_e$. The product of two matrices with entries from $A \propto E$ is given by

$$(M_a \propto M_e)(N_a \propto N_e) = (M_aN_a) \propto (M_aN_e + M_eN_a).$$

We now state the main theorem of this subsection.

Theorem 6.1. [105, Theorem 2.1] Let A be an integral domain, let B be an extension of A, and let $R = A \propto B$ be the trivial ring extension of A by B. Then:

1. R is an elementary divisor ring if and only if A is an elementary divisor ring and $B = qf(A)$.
2. The following conditions are equivalent:
   (a) R is a Hermite ring.
   (b) R is a Bézout ring.
   (c) A is a Bézout domain and $B = qf(A)$.

The proof of this theorem requires the following preparatory lemmas.

Lemma 6.2. *[105, Lemma 2.2] Let A be a ring and let E be an A-module.*

1. *Let $M \in \mathcal{M}_n(A \propto E)$. Then M is invertible if and only if M_a is invertible.*
2. *If $A \propto E$ is an elementary divisor ring, then so is A.*

Lemma 6.3. *[105, Lemma 2.3] Let A be a ring and let E be an A-module. If $A \propto E$ is a Bézout ring, then A is a Bézout ring and E is a divisible A-module.*

Lemma 6.4. *[105, Lemma 2.4] Let $A \subseteq B$ be an extension of rings. If $A \propto B$ is a Bézout ring, then $B = Q(A)$.*

Remark 6.5. [105, Remark 2.5]

1. The assumption that A is an integral domain cannot be removed from Theorem 6.1. Indeed, the trivial ring extension of a ring A by its total quotient ring $Q(A)$ need not be an arithmetical ring. In fact, a characterization of when such a trivial ring extension is arithmetical is given in Theorem 6.15. It turns out that this is equivalent to A being a semihereditary ring.
2. The converse of Lemma 6.3 is not true in general. Indeed, let K be a field and let E be a K-vector space such that $\dim_K E \geq 2$. Then the ideal of $R := K \propto E$ generated by $(0,x)$ and $(0,y)$, where (x,y) is a free family of E, is not principal since $(0,x)R + (0,y)R = 0 \propto (Kx \oplus Ky)$. Thus R is not a Bézout ring. However, K is trivially a Bézout ring and E is a divisible K-module.
3. Let A be a ring, let E be an A-module, and let $R = A \propto E$ be the trivial ring extension of A by E. If R is a Hermite ring, then so is A. Indeed, let $(a,b) \in A^2$. Then there exist $(d,e),(a',x'),(b',y') \in R$ such that
$$\begin{cases} (a,0) = (a',x')(d,e), \\ (b,0) = (b',y')(d,e), \\ (a',x')R + (b',y')R = R. \end{cases}$$
Thus
$$\begin{cases} a = a'd, \\ b = b'd, \\ a'A + b'A = A. \end{cases}$$
Therefore A is a Hermite ring, as desired.

Next, we give examples illustrating Theorem 6.1.

Example 6.6. [105, Example 2.6] Let S be an elementary divisor domain and let K be its quotient field. Let $K[[x]]$ denote the ring of formal power series over K in the indeterminate x. By [73, Example 1, p. 161], $A = S + xK[[x]]$ is an elementary divisor domain. Then $A \propto K((x))$ is an elementary divisor ring.

Example 6.7. [105, Example 2.7] Let A be an elementary divisor domain which is not a valuation domain, for instance $A = \mathbb{Z}$; see [90, Proposition 2.7, p. 236] or [116, Corollary, p. 213]. Then $R := A \propto qf(A)$ satisfies the following statements:

1. R is an elementary divisor ring.
2. R is not a valuation ring.

6.2 Arithmetical rings

This subsection is based on Kabbour, Mahdou, and Mimouni [105]. A ring R is said to be arithmetical if every finitely generated ideal of R is locally principal [56, 79].

Theorem 6.8. [105, Theorem 3.1] Let A be a ring, let E be a finitely generated A-module, and let $R := A \propto E$ be the trivial ring extension of A by E. Then R is an arithmetical ring if and only if, for every maximal ideal $\mathbf{m}$ of A, one of the following two conditions holds:

1. $A_{\mathbf{m}}$ is a valuation ring and $E_{\mathbf{m}} = 0$.
2. $A_{\mathbf{m}}$ is a field and $E_{\mathbf{m}}$ is isomorphic to $A_{\mathbf{m}}$ as an $A_{\mathbf{m}}$-vector space.

The next example illustrates Theorem 6.8 and provides a chain of arithmetical rings whose direct limit is arithmetical. Recall that a ring A is said to be von Neumann regular if, for every $x \in A$, there exists $y \in A$ such that $yx^2 = x$; equivalently, $A_{\mathbf{m}}$ is a field for every maximal ideal $\mathbf{m}$ of A.

Example 6.9. [105, Example 3.4] Let K be a field of characteristic zero. Let A denote the set of all stationary sequences of elements of K. For each $n \in \mathbb{N}$, define the element e_n of A by $e_n(k) = \delta_{kn}$, where δ_{kn} is the Kronecker symbol. Also define the sequence u by $u(k) = 1$. For each $n \in \mathbb{N}$, let E_n be the finitely generated ideal of A generated by $e_0, \ldots, e_n$, let M_n be the ideal of A given by $M_n = A(u - e_n)$, and set

$$M = K^{(\mathbb{N})} = \{x \in A \mid \exists N \in \mathbb{N},\ \forall k \geq N,\ x(k) = 0\}.$$

Then:

1. A is a von Neumann regular ring.
2. $\{M\} \cup \{M_n\}_{n \in \mathbb{N}}$ is the set of all maximal ideals of A.
3. For every $n \in \mathbb{N}$, $R_n := A \propto E_n$ is an arithmetical ring.
4. $R = \bigcup R_n = A \propto M$ is an arithmetical ring.

The next corollary characterizes when a trivial extension of a ring A by itself is arithmetical. It turns out that this is equivalent to A being von Neumann regular.

Corollary 6.10. *[105, Corollary 3.5] Let A be a ring. Then $A \propto A$ is an arithmetical ring if and only if A is a von Neumann regular ring.*

Example 6.11. [105, Example 3.6] Let A be a Boolean ring, that is, $a^2 = a$ for each $a \in A$. Then $A \propto A$ is an arithmetical ring. For instance, $(\mathbb{Z}/2\mathbb{Z})^n \propto (\mathbb{Z}/2\mathbb{Z})^n$ is an arithmetical ring, where n is a nonnegative integer.

The following corollary is an immediate consequence of the preceding corollary. It shows that, for a trivial ring extension $R := A \propto A$ of an integral domain A by itself, all the above notions collapse and are equivalent to A being a field.

Corollary 6.12. *[105, Corollary 3.7] Let A be an integral domain and let $R = A \propto A$. Then the following statements are equivalent:*

1. *R is a valuation ring.*
2. *R is an elementary divisor ring.*
3. *R is a Hermite ring.*
4. *R is a Bézout ring.*
5. *R is an arithmetical ring.*
6. *A is a field.*

We are now ready to state the second main theorem of this subsection. It characterizes when a trivial ring extension of a ring A by a non-torsion A-module E is arithmetical.

Theorem 6.13. [105, Theorem 3.8] Let A be a ring, let E be a non-torsion A-module, and let $R := A \propto E$ be the trivial ring extension of A by E. Then R is an arithmetical ring if and only if the following statements hold:

1. A is a reduced arithmetical ring.
2. For every maximal ideal $\mathbf{m}$ of A, $E_{\mathbf{m}}$ is isomorphic to $A_{\mathbf{p}}$ as an A-module, where $\mathbf{p}$ is the unique minimal prime ideal contained in $\mathbf{m}$. In this case, $\mathbf{p} = \{a \in A \mid (0 : a) \not\subseteq \mathbf{m}\}$.

The following corollary is an immediate application of Theorem 6.13. It restates when a trivial ring extension defined by an extension of integral domains is arithmetical.

Corollary 6.14. *[105, Corollary 3.9] Let $A \subseteq B$ be an extension of domains and let $R = A \propto B$ be the trivial ring extension of A by B. Then R is an arithmetical ring if and only if A is a Prüfer domain and $B = qf(A)$.*

At this point, we note that if A is a domain and $K = qf(A)$, then $A \propto K$ is an arithmetical ring if and only if A is a Prüfer domain [22, Corollary 2.4]. Next, we state the third main theorem of this subsection. It generalizes the preceding corollary to arbitrary rings.

Theorem 6.15. [105, Theorem 3.10] Let A be a ring. Then $A \propto Q(A)$ is an arithmetical ring if and only if A is a semihereditary ring.

The next examples illustrate Theorem 6.15. They show that A being a reduced arithmetical ring is not enough to ensure that $A \propto Q(A)$ is arithmetical.

Example 6.16. [105, Example 3.11] Let A be a semihereditary ring which is not a Bézout ring. Then $R = A \propto Q(A)$ is an arithmetical ring by Theorem 6.15, but R is not a Bézout ring.

Example 6.17. [105, Example 3.12] Let $(A_\alpha)_{\alpha \in I}$ be a family of semihereditary rings. Set $Q_\alpha = Q(A_\alpha)$, $A = \prod_{\alpha \in I} A_\alpha$, and $Q = \prod_{\alpha \in I} Q_\alpha$. The total ring of quotients of A is Q. Since any product of semihereditary rings is semihereditary, A is semihereditary. Therefore $A \propto Q$ is an arithmetical ring by Theorem 6.15. For instance, $\mathbb{Z}^n \propto \mathbb{Q}^n$ is an arithmetical ring for every nonnegative integer n, since $\mathbb{Z}$ is a semihereditary ring.

Example 6.18. [105, Example 3.13] Let A be the direct sum of A_n, where $A_n = \mathbb{Z}/2\mathbb{Z}$ for each $n \in \mathbb{N}$, with addition and multiplication defined componentwise in A. Let $R = \mathbb{Z} \times A$, where addition is defined componentwise and multiplication is given by

$$(m, a)(m', a') = (mm', ma' + m'a + aa').$$

Let $e_n = (\delta_{k,n})_k$ with $n \in \mathbb{N}$, and let $P_n = R(1, e_n)$. By [103, Example 3], the full set of prime ideals of R is $(m\mathbb{Z} \times A, P_n)$, where m is a prime integer or $m = 0$. Moreover, $R_{P_n} \simeq \mathbb{Z}/2\mathbb{Z}$ and $R_M \simeq \mathbb{Z}_{(m)}$, where $M = m\mathbb{Z} \times A$. Thus R is a reduced arithmetical ring. We have $Q(R) = \mathbb{Z}_{(2)} \times A$, with twisted multiplication. On the other hand, R is not a semihereditary ring since $Q(R)$ is not a von Neumann regular ring. We conclude that $R \propto Q(R)$ is not an arithmetical ring.

It is not known whether a trivial ring extension defined by a ring extension of rings with zero-divisors is arithmetical. However, [22, Theorem 2.1(2)] can be extended to the case of rings with zero-divisors under the assumption that either A has only finitely many minimal prime ideals or $Q(A)$ is von Neumann regular.

Corollary 6.19. *[105, Corollary 3.14] Let $A \subseteq B$ be an extension of rings. If A has only finitely many minimal prime ideals or $Q(A)$ is a von Neumann regular ring, then $A \propto B$ is an arithmetical ring if and only if A is a semihereditary ring and $B = Q(A)$.*

6.3 On almost Bézout rings

This subsection is based on Mahdou, Mimouni, and Moutui [101]. In [11], Anderson and Zafrullah introduced and studied the notion of an almost Bézout domain, or an AB-domain. An integral domain A is an AB-domain if, for $a, b \in A \setminus \{0\}$, there exists a positive integer n such that (a^n, b^n) is principal. In [12], Anderson and Zafrullah continued their study of almost Bézout domains and gave a new characterization of Cohen-Kaplansky domains. In [101], the generalization of almost Bézout domains to arbitrary commutative rings with zero-divisors is considered as follows: A is called an almost Bézout ring, or an AB-ring for short, if, for any two elements a and b in A, there exists a positive integer n such that the ideal (a^n, b^n) is principal.

Our first result studies the possible transfer of the AB-ring property between a ring A and a trivial ring extension $A \propto E$.

Theorem 6.20. [101, Theorem 2.1] Let A be a ring, let E be an A-module, and let $R := A \propto E$ be the trivial ring extension of A by E. Then the following statements hold:

1. If R is an AB-ring, then so is A.
2. Suppose that A is an integral domain, $K = qf(A)$, and E is a K-vector space. Then R is an AB-ring if and only if so is A.
3. Suppose that (A, M) is a local ring and E is an A-module such that $ME = 0$. Then R is an AB-ring if and only if so is A.
4. Let $A \subseteq B$ be an extension of integral domains and suppose that $\mathbb{Q} \subseteq B$. Then $R := A \propto B$ is an AB-ring if and only if A is an AB-ring and $qf(A) \subseteq B$.

The proof of this theorem requires the following lemma, which can be viewed as a general criterion for checking when the trivial ring extension of integral domains is not an AB-ring.

Lemma 6.21. *[101, Lemma 3.2] Let $A \subseteq B$ be an extension of integral domains and let $R := A \propto B$ be the trivial ring extension of A by B. If there exists a nonzero element $m \in A$ such that $n1_B \notin mB$ for each positive integer n, then R is not an* AB-*ring.*

Corollary 6.22. *[101, Corollary 3.3] Let A be a domain containing the field of rational numbers. Then $A \propto A$ is an* AB-*ring if and only if A is a field.*

In the next two examples, we first show how to build a new family of non-coherent AB-rings that are not Bézout rings. Second, we show that, in some specific cases where the condition $\mathbb{Q} \subseteq B$ is not satisfied, so that Theorem 6.20 does not apply, Lemma 6.21 can be useful.

Example 6.23. [101, Example 3.4] Let A be a Bézout domain, let $K = qf(A)$, let E be a K-vector space such that $\dim_K(E) \geq 2$, and let $R := A \propto E$ be the trivial ring extension of A by E. Then:

1. R is an AB-ring.
2. R is not a Bézout ring.
3. R is not coherent.

Example 6.24. [101, Example 3.5] Let $B = \mathbb{Q}((X))[[Y]] = \mathbb{Q}((X)) + Y\mathbb{Q}((X))[[Y]] = \mathbb{Q}((X)) + N$, where X and Y are indeterminates over $\mathbb{Q}$ and $N = Y\mathbb{Q}((X))[[Y]]$. Let $A = \mathbb{Q}[[X]] = \mathbb{Q} + M$, where $M = X\mathbb{Q}[[X]]$, and set $E = \mathbb{Z}_{(2)} + M$ and $F = \mathbb{Z} + M$. Then A and B are DVRs, E is a valuation domain, F is a Bézout domain, and:

1. $A \propto B$ is an AB-ring by Theorem 6.20(4).
2. $E \propto A$ is not an AB-ring. Here $\mathbb{Q} \subseteq A$, but $qf(E) \not\subseteq A$.
3. $F \propto E$ is not an AB-ring. Here $\mathbb{Q} \not\subseteq E$, and we take $m = X$ in Lemma 6.21.

7 Trivial ring extensions defined by Prüfer conditions

This section is based on Bakkari, Kabbaj, and Mahdou [22]. Let R be a commutative ring.

1. R is called *semihereditary* if every finitely generated ideal of R is projective [37].
2. R is said to have *weak global dimension* at most 1, denoted by $\text{w.dim}(R) \leq 1$, if every finitely generated ideal of R is flat [64, 66].
3. R is called a *Gaussian ring* if, for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$ [117].
4. R is called a *Prüfer ring* if every finitely generated regular ideal of R is invertible [36, 69].

In the domain context, all these notions coincide with the notion of a Prüfer domain. Glaz [65] provides examples showing that all these notions are distinct in the context of arbitrary rings. The following diagram of implications summarizes the relations between them:

$$\text{semihereditary} \Rightarrow \text{w.dim}(R) \leq 1 \Rightarrow \text{arithmetical} \Rightarrow \text{Gaussian} \Rightarrow \text{Prüfer}.$$

7.1 Extensions of domains

This subsection explores trivial ring extensions of the form $R := A \propto B$, where $A \subseteq B$ is an extension of integral domains. Notice that, in this context, $(a, b) \in R$ is regular if and only if $a \neq 0$. The main result, Theorem 7.1, examines the transfer of Prüfer conditions to R and hence generates new examples of non-arithmetical Gaussian rings and of arithmetical rings with weak global dimension greater than 1.

In 1969, Osofsky proved that the weak global dimension $\text{w.dim}(R)$ of an arithmetical ring is either at most 1 or infinite [112]. In 2005, Glaz proved Osofsky's result in the class of coherent Gaussian rings [64, Theorem 3.3]. Recently, Bazzoni and Glaz conjectured that the weak global dimension of a Gaussian ring is 0, 1, or ∞ [30]. Theorem 7.1 validates this conjecture for the class of all Gaussian rings arising from these constructions. Moreover, Example 7.7 widens its scope of validity beyond coherent Gaussian rings.

Theorem 7.1. [22, Theorem 2.1] Let $A \subseteq B$ be an extension of domains and let $K := qf(A)$. Let $R := A \propto B$ be the trivial ring extension of A by B. Then:

1. R is Gaussian if and only if R is Prüfer if and only if A is Prüfer and $K \subseteq B$.
2. R is arithmetical if and only if A is Prüfer and $K = B$.
3. $\text{w.dim}(R) = \infty$.

The proof of the theorem involves the following lemmas of independent interest.

Lemma 7.2. *[22, Lemma 2.2] Let A be a ring, let E be a nonzero A-module, and let $R := A \propto E$. If R is Gaussian, respectively arithmetical, then so is A.*

Notice that Lemma 7.2 does not hold for the Prüfer property, as shown by Example 7.8.

Lemma 7.3. *[22, Lemma 2.3] Let K be a field, let E be a nonzero K-vector space, and let $R := K \propto E$. Then* $\text{w.dim}(R) = \infty$.

The following corollary is an immediate consequence of Theorem 7.1.

Corollary 7.4. *[22, Corollary 2.4] Let D be a domain, let $K := qf(D)$, and let $R := D \propto K$. Then the following statements are equivalent:*

1. *D is a Prüfer domain;*
2. *R is an arithmetical ring;*
3. *R is a Gaussian ring;*
4. *R is a Prüfer ring.*

Recall Jensen's 1966 result: for a ring R, $\text{w.dim}(R) \leq 1$ if and only if R is arithmetical and reduced [79]. Classical examples of arithmetical rings with weak global dimension greater than 1 arise from Jensen's result as non-reduced principal rings, for example $\mathbb{Z}/n^2\mathbb{Z}$ for any integer $n \geq 2$. In this vein, Theorem 7.1 generates a new family of examples that are far from being principal, as shown below.

Example 7.5. [22, Example 2.5] Let D be any Prüfer domain that is not a field and let $K := qf(D)$. Then $R := D \propto K$ is an arithmetical ring with $\text{w.dim}(R) = \infty$. Moreover, R is not a finite conductor ring and hence is not coherent.

Theorem 7.1 also enriches the literature with new examples of non-arithmetical Gaussian rings, as shown below.

Example 7.6. [22, Example 2.6] Let $K \subsetneq L$ be a field extension. Then $R := K \propto L$ is a Gaussian ring that is not arithmetical.

The next example shows that Theorem 7.1 widens the scope of validity of the Bazzoni–Glaz conjecture beyond the class of coherent Gaussian rings.

Example 7.7. [22, Example 2.7] Let $\mathbb{Z}$ and $\mathbb{R}$ denote the ring of integers and the field of real numbers, respectively. Then $R := \mathbb{Z}_{(2)} \propto \mathbb{R}$ satisfies the following statements:

1. R is a Gaussian ring;
2. R is not an arithmetical ring;

3. R is not a coherent ring;
4. $\text{w.dim}(R) = \infty$.

The next example illustrates the failure of Theorem 7.1, in general, beyond the context of domain extensions.

Example 7.8. [22, Example 2.8] Let (A, M) be a non-valuation local domain, let E be a nonzero A-module with $ME = 0$, and let $B := A \propto E$. Then $R := A \propto B$ is a Prüfer ring that is not Gaussian.

7.2 A class of total rings of quotients

This subsection investigates Prüfer conditions in a particular class of total rings of quotients, namely those arising as trivial ring extensions of local rings by vector spaces over their residue fields. The main result, Theorem 7.9, enriches the literature with new examples of non-arithmetical Gaussian total rings of quotients, as well as non-Gaussian total rings of quotients, which are necessarily Prüfer.

Theorem 7.9. [22, Theorem 3.1] Let (A, M) be a local ring and let E be a nonzero A/M-vector space. Let $R := A \propto E$ be the trivial ring extension of A by E. Then:

1. R is a total ring of quotients and hence a Prüfer ring.
2. R is Gaussian if and only if A is Gaussian.
3. R is arithmetical if and only if $A = K$ is a field and $\dim_K E = 1$.
4. $\text{w.dim}(R) > 1$. If M admits a minimal generating set, then $\text{w.dim}(R) = \infty$.

Theorem 7.9 generates new examples of rings with zero-divisors satisfying Prüfer conditions, as shown below.

Example 7.10. [22, Example 3.2] Let (V, M) be a nontrivial valuation domain. Then $R := V \propto V/M$ is a non-arithmetical Gaussian total ring of quotients.

Example 7.11. [22, Example 3.3] Let K be a field and let E be a K-vector space with $\dim_K E \geq 2$. Then $R := K \propto E$ is a non-arithmetical Gaussian total ring of quotients.

Example 7.12. [22, Example 3.4] Let (A, M) be a non-valuation local domain. Then $R := A \propto A/M$ is a non-Gaussian total ring of quotients.

Recently, Bazzoni and Glaz proved that a Gaussian ring with a maximal ideal M such that the nilradical of R_M is nonzero and nilpotent has infinite weak global dimension [30, Theorem 6.4]. The next example widens the scope of validity of the Bazzoni–Glaz conjecture and illustrates this setting beyond coherent Gaussian rings.

Example 7.13. [22, Example 3.5] Let $\mathbb{R}$ denote the field of real numbers and let x be an indeterminate over $\mathbb{R}$. Then $R := \mathbb{R} \propto \mathbb{R}[x]$ satisfies the following statements:

1. R is a Gaussian ring;
2. R is not an arithmetical ring;
3. R is not a coherent ring;
4. R is local with nonzero nilpotent maximal ideal;
5. $\text{w.dim}(R) = \infty$.

7.3 Kaplansky–Tsang–Glaz–Vasconcelos conjecture

Let R be a ring. A polynomial f over R is said to be Gaussian if $c(fg) = c(f)c(g)$ holds for every polynomial g over R.

A problem initially associated with Kaplansky and his student Tsang [8, 31, 68, 94], and also called the Tsang–Glaz–Vasconcelos conjecture in [75], asserts that every nonzero Gaussian polynomial over a domain has an invertible, equivalently locally principal, content ideal. It is well known that a polynomial over any ring is Gaussian if its content ideal is locally principal. The converse is precisely the object of the Kaplansky–Tsang–Glaz–Vasconcelos conjecture, extended to those rings for which every Gaussian polynomial has locally principal content ideal.

For convenience, we note that the conjecture has a local character, since the Gaussian condition is a local property; that is, a polynomial is Gaussian over a ring R if and only if its image is Gaussian over R_M for each maximal ideal M of R. This fact enables a natural extension of the conjecture from domains to arbitrary rings.

Significant progress has been made on this conjecture. Glaz and Vasconcelos proved it for normal Noetherian domains [68]. Then Heinzer and Huneke established its validity over locally approximately Gorenstein rings and over locally Noetherian domains [75, Theorem 1.5 and Corollary 3.4]. Recently, Loper and Roitman settled the conjecture for locally domains [93, Theorem 4], and Lucas extended their result to arbitrary rings by restricting to polynomials with regular content [94, Theorem 6]. Obviously, the conjecture is true in arithmetical rings. Moreover, trivial ring extensions make it possible to widen the scope of its validity to a large family of rings distinct from the above contexts. This gives rise to a new class of rings that properly contains the three classes of arithmetical rings, locally domains, and locally approximately Gorenstein rings; see Figure 1. We use the following terminology.

Definition 7.14. [22, Definition 4.1] A ring R is pseudo-arithmetical if every Gaussian polynomial over R has locally principal content ideal.

We first prove a transfer result, Theorem 7.15, for trivial ring extensions. Then Conjecture 7.18 will relate the pseudo-arithmetical property to the local irreducibility of the zero ideal. If true, this conjecture would provide an optimal solution to the Kaplansky–Tsang–Glaz–Vasconcelos conjecture and recover all previous results.

Theorem 7.15. [22, Theorem 4.2]

1. Let $R := A \propto K$ be the trivial ring extension of a domain A by its quotient field K. Then R is a pseudo-arithmetical ring.
2. Let $A \subseteq B$ be an extension of rings and let $R := A \propto B$. If R is a pseudo-arithmetical ring, then so is A.

Obviously, a ring is arithmetical if and only if it is Gaussian and pseudo-arithmetical. In this context, note that Examples 7.6 and 7.11 illustrate the failure of Theorem 7.15(1) for trivial ring extensions $R := A \propto E$ with $E \neq qf(A)$.

Example 7.16. [22, Example 4.3] Let (A, M) be a local ring which is not a field and let E be a nonzero vector space over A/M. Then $R := A \propto E$ is a Prüfer ring which is not pseudo-arithmetical. Indeed, Theorem 7.9 ensures that R is a non-arithmetical total ring of quotients, and hence Prüfer. We claim that the polynomial $f := (a,0) + (0,e)x$, where $0 \neq a \in M$ and $0 \neq e \in E$, is Gaussian but $c(f)$ is not principal in R. To see this, let $g \in R[x]$. If $g \notin (M \times E)[x]$, then Gauss's lemma ensures that $c(fg) = c(f)c(g)$ since R is local with maximal ideal $M \propto E$. Assume that $g \in (M \times E)[x]$. Then $ME = 0$ yields $c(f)c(g) = (a,0)c(g) = c((a,0)g) = c(fg)$. Now adapt the proof of Theorem 7.9(3) to obtain that $c(f)$ is not principal. Therefore R is not pseudo-arithmetical.

Remark 7.17. [22, Remark 4.4]

1. Pick any non-Prüfer domain A with $K := qf(A)$ and consider the trivial extension $R := A \propto K$. Then by Corollary 7.4, R is not a Prüfer ring; a fortiori, R is not arithmetical. Moreover, there are plenty of non-regular Gaussian polynomials over R, for example $f := \sum(0,k_i)x^i$. However, Theorem 7.15 ensures that every Gaussian polynomial over R has locally principal content ideal; that is, R is pseudo-arithmetical.

2. We next examine the Noetherian case. From [75], a local ring (R,M) is said to be approximately Gorenstein if R is Noetherian and, for every integer $n > 0$, there is an ideal $I \subseteq M^n$ such that R/I is Gorenstein; for example, any local Noetherian ring (R,M) with reduced M-adic completion $\widehat{R}$. Heinzer and Huneke proved that every locally approximately Gorenstein ring is pseudo-arithmetical [75, Theorem 1.5]. This result, combined with [75, Remark 1.6], asserts that Noetherianity has no direct effect on the pseudo-arithmetical property, even in low dimension, in the sense that non-Gorenstein Artinian local rings are not pseudo-arithmetical. Finally, note that the above example $R := A \propto K$ is not Noetherian since it is not coherent.

3. From [1], a ring R is called a PF ring if all principal ideals of R are flat, or equivalently, if R is locally a domain [66, Theorem 4.2.2(3)]. A ring R is called a PP ring or a weak Baer ring if all principal ideals of R are projective. In the class of Gaussian rings, the PP and PF properties coincide, respectively, with the notions of semihereditary ring and ring with weak global dimension at most 1. Clearly, the above example $R := A \propto K$ is not locally a domain.

In view of Example 7.16 and Remark 7.17, Figure 1 summarizes the relations between all these classes of rings, where the implications are irreversible in general.

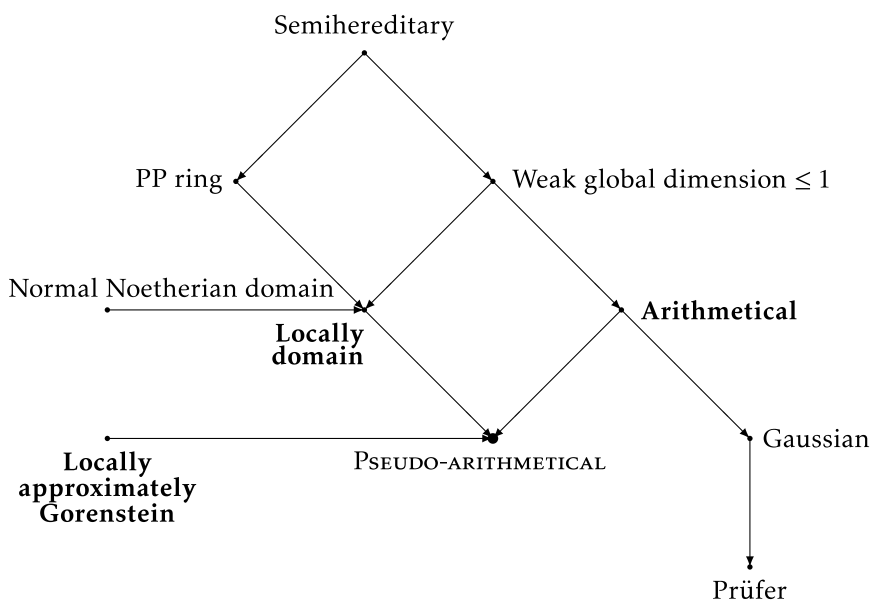

Figure 1: Pseudo-arithmetical rings in perspective

From the above discussion, it is natural to ask whether the pseudo-arithmetical property can be characterized by a local irreducibility condition on the zero ideal. This question was motivated by the fact that the known positive cases include arithmetical rings, locally domains, and locally approximately Gorenstein rings; see Figure 1. In this direction, Bakkari, Kabbaj, and Mahdou proposed the following conjecture.

Conjecture 7.18. [22, Conjecture 4.5] A ring R is pseudo-arithmetical if and only if the zero ideal is locally irreducible.

The conjecture would have provided a unified explanation for several known positive cases of the Kaplansky–Tsang–Glaz–Vasconcelos conjecture. The following example shows that the equivalence in Conjecture 7.18 is false in general. Thus the local irreducibility of the zero ideal cannot serve as a complete characterization of pseudo-arithmetical rings.

Example 7.19. [85] Let k be a field and set $R = k[[X,Y]]/(XY)$. Let x, y denote the images of X, Y in R. Since $(XY) = (X) \cap (Y)$ in the unique factorization domain $k[[X,Y]]$, the ring R is reduced. It is local and is not a domain, because $xy = 0$ while $x \neq 0$ and $y \neq 0$. Therefore R is pseudo-arithmetical by [85, Lemma 2]. On the other hand, $xR \cap yR = 0$, so the zero ideal of this local ring is reducible. Thus R is a one-dimensional complete Noetherian local counterexample to [22, Conjecture 4.5].

Remark 7.20.

1. Fuchs, Heinzer, and Olberding studied irreducibility in commutative rings [57, 58] and observed that a ring R is arithmetical if and only if, for each proper ideal I of R, I_M is an irreducible ideal of R_M for every maximal ideal M of R containing I [57].
2. The known positive cases remain valid independently of Conjecture 7.18. If R is locally a domain, then the zero ideal is locally irreducible, and Loper and Roitman proved that a polynomial over R is Gaussian if and only if its content is locally principal [93, Theorem 4]. If R is locally approximately Gorenstein, then Heinzer and Huneke proved the corresponding assertion [75, Theorem 1.5]. Their proof uses the fact that a Gaussian polynomial $f := \sum a_i x^i$ over R induces a Gaussian polynomial $\overline{f} := \sum \overline{a_i} x^i$ over R/I for every ideal I of R, together with the locality of the Gaussian condition. In the approximately Gorenstein case, the argument reduces to the zero-dimensional local Gorenstein case, where the zero ideal is irreducible.

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