

On the integral closure of a locally finite-dimensional going-down domain in a field extension

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Abstract. Let R be an integrally closed going-down domain each of whose maximal ideals of height greater than 1 contains a height 1 prime ideal of R . Let L be an algebraic field extension of the quotient field of R , with T denoting the integral closure of R in L . Then for each maximal ideal M of R , there exists a seminormal domain \mathcal{S} that is integral over R and has quotient field L such that (the seminormal domain) $\mathcal{S}_{R \setminus M}$ is a divided domain (hence a quasi-local going-down domain) and the canonical map $\text{Spec}(\mathcal{S}_{R \setminus M}) \rightarrow \text{Spec}(R_M)$ is an order-isomorphism of posets (under inclusion). Under the above hypotheses, several sets of sufficient conditions are given for T to be a going-down domain, with special attention given to the case where R has Krull dimension 2. For that case, one upshot is (with D^+ denoting the seminormalization of a domain D) that if $R \subseteq T$ induces a finite-dimensional normal extension of quotient fields and $(R + PT)^+$ is t -closed for some nonmaximal height 1 prime ideal P of R that is contained in the Jacobson radical of R , then T is a going-down domain. Additional corollaries are given (some of which address R of arbitrary dimension), along with some multi-part remarks.

Key Words: Integral domain, prime ideal, going-down domain, integrality, quotient field, field extension, seminormalization, locally finite-conductor domain, finite fibers, simple group.

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In memory of my wife, Elaine

1 Introduction

All the rings that are considered in this paper are associative, unital and commutative, usually (integral) domains; all ring/algebra homomorphisms, modules and subrings are unital; and all inclusions of rings are subrings. As usual, if $D \subseteq E$ are domains and k is a/the quotient field of D , then we say that E is an *overring* of D (equivalently, that D is an *under-ring* of E) if $D \subseteq E \subseteq k$, that is, if E is (D -algebra isomorphic to) an overring of D .

In 1976, two questions about the possible ascent of the “going-down domain” property for certain kinds of integral extensions of domains were raised in [14]. That paper developed partial answers to each of those questions. Over the past 50 years, a number of subsequent papers have contributed additional partial answers to the first of those questions, which concerns certain integral extensions $R \subseteq T$ where R and T have the same quotient field. However, prior to the present paper, there has not been any additional work on the second question that had been raised in [14]. That second question concerns certain integral extensions $R \subseteq T$ whose corresponding quotient fields, say $K \subset L$, are distinct. (As usual, \subset and \supset denote proper containments.) **The main purpose of this paper** is to develop new partial answers to the second question that was raised in [14]. In a concrete way, the results in this paper raise our understanding of what is going on in the “ $K \subset L$ ” context of the second question (at least for a class of going-down base domains including the locally finite-dimensional

ones) to the same level that a sequence of papers managed to raise our understanding of what is going on in the “overring” context of the first question.

The main vehicle for this transference of understanding is this paper’s most important result, Theorem 2.4. Since the property of being a going-down domain is a local property of domains, we will, in order to simplify matters here, next state the most impactful part of Theorem 2.4 for the case of a quasi-local base domain. For that case, that result establishes, *i.a.*, that if $R \subset T$ is an integral extension of domains such that R is a quasi-local going-down domain whose maximal ideal properly contains a (necessarily unique) height 1 prime ideal of R , then there exists a seminormal going-down domain S such that $R \subset S \subseteq T$, the canonical contraction map $\text{Spec}(S) \rightarrow \text{Spec}(R)$ is an order isomorphism of posets under inclusion, and S and T have the same quotient field, say L (so that S and T have the same integral closure, namely, the integral closure of R in L). This result is applied in conjunction with globalization techniques and results concerning the first question in works such as [14], [16], [15].

Although Theorem 2.4 and its Corollaries illustrate and begin to accomplish what was described above as the “main purpose of this paper”, Proposition 2.13 and its corollaries accomplish more, as Theorem 2.4 does not address the possibility that the base domain R could have a maximal ideal of height 1. Remark 2.21 (a) says more about the technical device that is used in Proposition 2.13 and its corollaries to address that possibility and to thereby carry out a generalization of Theorem 2.4. The second (but shorter) half of Section 2 is more accessible, as it indicates a direct approach to the main “ascent” question that is based on classical field theory and group theory, without reference to technical ring-theoretic conditions that had been of use in earlier works such as [16] and [23]. A brief appendix provides, in Proposition 3.2 (b), a domain-theoretic consequence of a classical field-theoretic result of E. Artin and O. Schreier which, in our opinion, may provide insight into how to build an example that would give a negative answer to the main “ascent” question.

The rest of this Introduction provides some background related to going-down domains and specifies some notational conventions. As this paper has frequent occasion to mention/use the seminormalization (resp., and to a lesser extent, the t -closure) of a domain, we note that [55] (resp., [49], [50]) can be recommended as a suitable reference for the background about seminormality (resp., t -closure) that is used below.

Following Kaplansky (page 28 of [41]), we let GD, GU, LO and INC, respectively, denote the going-down, going-up, lying-over and incomparable properties of ring extensions. Our interest in a connection between Pruefer domains and GD was piqued by a result of Quentel (Corollaire 2 of [52]) that characterized one-dimensional valuation domains as the quasi-local integrally closed one-dimensional coherent domains. Moving beyond the “one-dimensional” context, we obtained Corollary 4 of [12], which generalized the result of Quentel and can now be restated as characterizing Pruefer domains as the integrally closed finite-conductor domains that are going-down domains. In the sequel paper [13], we gave the first definition of going-down domains, using Pruefer domains and one-dimensional domains as the most obvious examples of going-down domains. (The various candidates for a definition of going-down domains were all shown to be equivalent in the sequel to that paper: see Theorem 1 of [30].) Most of the subsequent examples of going-down domains have arisen as pullbacks, as in, for instance, Corollary in [30], Corollary 2.3 and pages 155-157 of [27], and Corollary 2.5 of [21].

This paragraph summarizes some information about going-down domains and their relation to treed domains that will be needed in the proofs of the initial four results in Section 2. The next three sentences are the result of editing the beginning of the third paragraph of the Introduction of [24]. Recall from [13], [30] that a *going-down domain* is a domain R such that $R \subseteq T$ satisfies GD for all domains T having R as a subring. Sufficient test domains T are provided by the set of overrings of R of the form $R[u]$ with u ranging over a fixed quotient field of R ; and also by the set of valuation overrings of R [30, Corollary]. It is well known that being a going-down domain is a local property

of domains (in the sense that a domain R is a going-down domain if and only if R_M is a going-down domain for each $M \in \text{Max}(R)$). That fact will be slightly generalized in Lemma 2.1 by using the above information. Then Lemma 2.2 will extend Lemma 2.1 in a way that is more directly applicable to the main context of interest for this paper, namely, domain extensions $R \subset T$ where T is not (R -algebra isomorphic to) an overring of R (that is, where a quotient field of T properly contains a quotient field of R). Recall also that a domain R is said to be a *treed domain* if $\text{Spec}(R)$, when viewed as a poset under inclusion, is a tree (that is, if for each $Q \in \text{Spec}(R)$, $\{P \in \text{Spec}(R) \mid P \subseteq Q\}$ is linearly ordered by inclusion). It was shown in Theorem 2.2 of [13] that each going-down domain is a treed domain, but the converse fails. The first example of a treed domain which is not a going-down domain was constructed by W. J. Lewis; the details of that construction were included in [32] with the kind permission of Dr. Lewis.

The above comments about Theorem 2.4 mentioned seminormal going-down domains. The proof of Corollary 2.6 of [15] essentially showed that these are the same as the seminormal locally divided domains. (For the definitions of a divided domain and a locally divided domain, see [15], especially its Remark 2.7 (b). For the sake of historical accuracy, we note that divided domains were introduced by Akiba, who used different terminology, in [1].) That corollary was motivated in part by an earlier result by McAdam, Corollary 11 of [45], which essentially obtained the special case of Corollary 2.6 of [15] in which “seminormal” is replaced by “integrally closed”. Recently, our understanding of such matters was significantly generalized and deepened in [26]. That paper was the second of two papers that I wrote in reaction to the surprising and clever proof by Secord [54] that every straight domain is a locally divided domain. (The first of those papers was [25] which, *i.a.*, gave both a new proof of Secord’s result, essentially by viewing Secord’s proof through a GD-tinted lens, and a “going-down to P ”-theoretic sharpening of the above-mentioned result of McAdam.) Recall from [33] that a domain R is called a *straight domain* if T/PT is a torsion-free module over R/P for each overring T of R and each $P \in \text{Spec}(R)$. In [33], G. Picavet and I proved that each locally divided domain is a straight domain; and, after defining straight rings, Picavet and I gave an example of a straight ring that is not a locally divided ring (in the sense of [4]); and Picavet and I raised the question whether every straight domain must be a locally divided domain. Secord’s affirmative answer to this question prompted my two-paper response which ultimately, in my opinion, has given new torsion-theoretic reasons to be interested in seminormal going-down domains. Moreover, Secord’s concept of a prime ideal P of a domain R being “straight with respect to a given domain extension T of R ” has led to a fruitful point of view that can be used to attack some old problems and also to open some new directions of inquiry. Interested readers are requested to read [26] from the perspective that a crucially pertinent concept is the “GD to P ” property that was famously characterized in multiple ways (one of which was torsion-theoretic) by Kaplansky in Exercise 37, pages 44–45 of [41].

More about notation: the cardinal number of a set \mathcal{S} will be denoted by $|\mathcal{S}|$; “dim” denotes Krull dimension; $\text{rad}_A(I)$ denotes the radical in A of an ideal I of a ring A (that is, the set of elements of A that are nilpotent modulo I ; equivalently, the intersection of the set of all prime ideals of A that contain I); and if D is a domain, the integral closure of D (in the quotient field k of D) will be denoted by D' , the seminormalization of D (in D' or in k) will be denoted by D^+ , the t -closure of D (in D' or in k) will be denoted by D^t , the height of a prime ideal \mathcal{P} of/in D will be denoted by $\text{ht}(\mathcal{P}) = \text{ht}_D(\mathcal{P})$, and the characteristic of D will be denoted by $\text{char}D$.

2 Results

At a couple of places in the Introduction, we mentioned that the property of being a going-down domain is a local property of domains. Although that fact has been often mentioned in papers over the years, it will be convenient to begin with a lemma that establishes it (and a bit more). Then, as the next preparatory step before Theorem 2.4, Lemma 2.2 builds on Lemma 2.1 by considering

domain extensions that are *not* overring extensions. Although the thrust of Lemma 2.2 will be familiar to readers of some papers in the references, we include its statement and its proof for the sake of completeness (and because of the lack of a convenient reference). As the next preparatory step before Theorem 2.4, Lemma 2.3 isolates the tool for proving that a certain pullback is a seminormal domain. As the final preparatory step before Theorem 2.4, we provide some information about unbranched extensions of domains (especially in the integral case) which plays an essential role in the proof of Theorem 2.4.

Kaplansky once opined that introductions are “written last and perhaps read first”, so perhaps I will be forgiven for mentioning again here that I consider Theorem 2.4 to be the main result of this paper.

For a domain D and a multiplicatively closed subset Σ of D , the proofs of Lemmas 2.1 and 2.2 will use (without further explanation) the well-known bijection between the prime ideals \mathcal{P} of D_Σ and the prime ideals P of D such that $P \cap \Sigma = \emptyset$ (cf. Proposition 3.11 (iv) of [2]), given by $\mathcal{P} \mapsto \mathcal{P} \cap D$ (with its inverse bijection being given by $P \mapsto PD_\Sigma$).

Lemma 2.1. *Let R be a domain. Then the following conditions are equivalent:*

- (1) R is a going-down domain;
- (2) R_S is a going-down domain for each multiplicatively closed subset S of R ;
- (3) R_M is a going-down domain for each $M \in \text{Max}(R)$;
- (4) R_M is a going-down domain for each $M \in \text{Max}(R)$ such that $\text{ht}_R(M) \geq 2$.

Proof. (1) \Rightarrow (2): Assume (1) and let S be a multiplicatively closed subset of R . Our task is to show that if a domain E contains R_S as a subring, then $R_S \subseteq E$ satisfies GD; that is, to show that if $\mathcal{P}_2 \subseteq \mathcal{P}_1$ are prime ideals of R_S and Q_1 is a prime ideal of E such that $Q_1 \cap R_S = \mathcal{P}_1$, then there exists a prime ideal Q_2 of E such that $Q_2 \subseteq Q_1$ and $Q_2 \cap R_S = \mathcal{P}_2$. Consider the prime ideals $P_i := \mathcal{P}_i \cap R$ of R ($i = 1, 2$). Observe that $Q_1 \cap R = (Q_1 \cap R_S) \cap R = \mathcal{P}_1 \cap R = P_1$ and that $P_2 \subseteq P_1$. Since R is a going-down domain and a subring of E , the extension $R \subseteq E$ satisfies GD. Hence, there exists a prime ideal Q_2 of E such that $Q_2 \subseteq Q_1$ and $Q_2 \cap R = P_2$. It remains only to verify that $Q_2 \cap R_S = \mathcal{P}_2$. This, in turn, follows from the fact that the formation of modules of fractions commutes with finite intersections (cf. Corollary 3.4 of [2]), since

$$Q_2 \cap R_S = Q_2 R_S \cap R R_S = (Q_2 \cap R) R_S = P_2 R_S = \mathcal{P}_2.$$

(2) \Rightarrow (3) \Rightarrow (4): Trivial.

(4) \Rightarrow (1): Assume (4). Our task is to show that if a domain E contains R as a subring, then $R \subseteq E$ satisfies GD; that is, to show that if $\mathcal{P}_2 \subseteq \mathcal{P}_1$ are prime ideals of R and Q_1 is a prime ideal of E such that $Q_1 \cap R = \mathcal{P}_1$, then there exists a prime ideal Q_2 of E such that $Q_2 \subseteq Q_1$ and $Q_2 \cap R = \mathcal{P}_2$. Without loss of generality, $0 \subset \mathcal{P}_2 \subset \mathcal{P}_1$, for if $\mathcal{P}_2 = 0$ (resp., if $\mathcal{P}_2 = \mathcal{P}_1$), it would suffice to take $Q_2 = 0$ (resp., to take $Q_2 := Q_1$). Thus, $\text{ht}_R(\mathcal{P}_1) \geq 2$. Next, choose M to be any maximal ideal of R that contains \mathcal{P}_1 (and, hence, also contains \mathcal{P}_2). Of course, this choice is possible via Zorn’s Lemma, as we are assuming the usual ZFC foundations. As $\text{ht}_R(M) \geq \text{ht}_R(\mathcal{P}_1)$, we get that $\text{ht}_R(M) \geq 2$. So, by (4), $R_M \subseteq E_{R \setminus M}$ satisfies GD. Consider the prime ideals $\mathcal{P}_i := \mathcal{P}_i R_M$ of R_M (for $i = 1, 2$) and the prime ideal $\mathcal{Q}_1 := Q_1 E_{R \setminus M}$ of $E_{R \setminus M}$. Note that $\mathcal{P}_2 \subseteq \mathcal{P}_1$. Also, since the formation of modules of fractions commutes with finite intersections,

$$\mathcal{Q}_1 \cap R_M = Q_1 E_{R \setminus M} \cap R_M = (Q_1 \cap R)_{R \setminus M} = \mathcal{P}_1 R_M = \mathcal{P}_1.$$

Thus, since $R_M \subseteq E_{R \setminus M}$ satisfies GD, there exists a prime ideal \mathcal{Q}_2 of $E_{R \setminus M}$ such that $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$ and $\mathcal{Q}_2 \cap R_M = \mathcal{P}_2$. Observe that $Q_2 := \mathcal{Q}_2 \cap E \in \text{Spec}(E)$ satisfies $Q_2 \subseteq Q_1$. It remains only to verify that $Q_2 \cap R = \mathcal{P}_2$. This, in turn, follows since

$$Q_2 \cap R = Q_2 \cap E \cap R = Q_2 \cap R = Q_2 \cap R_M \cap R = \mathcal{P}_2 \cap R = \mathcal{P}_2.$$

The proof is complete. □

Lemma 2.2. *Let $R \subseteq T$ be an integral extension of domains such that R is a going-down domain. Then the following conditions are equivalent:*

- (1) T is a going-down domain;
- (2) T_N is a going-down domain for each $N \in \text{Max}(T)$;
- (3) $T_{R \setminus M}$ is a going-down domain for each $M \in \text{Max}(R)$;
- (4) $T_{R \setminus M}$ is a going-down domain for each $M \in \text{Max}(R)$ such that $\text{ht}_R(M) \geq 2$.

Proof. (1) \Leftrightarrow (2): Apply the equivalence (1) \Leftrightarrow (3) from Lemma 2.1 to the domain T .

(2) \Rightarrow (3): Assume (2). Let $M \in \text{Max}(R)$. It will suffice to prove that if V is a valuation overring of $T_{R \setminus M}$, then $T_{R \setminus M} \subseteq V$ satisfies GD. To that end, consider prime ideals $\mathcal{Q}_2 \subseteq \mathcal{Q}_1$ of $T_{R \setminus M}$ and a prime ideal \mathcal{N} of V such that $\mathcal{N} \cap T_{R \setminus M} = \mathcal{Q}_1$. Our task is to find a prime ideal \mathcal{P} of V such that $\mathcal{P} \subseteq \mathcal{N}$ and $\mathcal{P} \cap T_{R \setminus M} = \mathcal{Q}_2$. Without loss of generality, we can assume that \mathcal{N} is the (unique) maximal ideal of V . Indeed, by *abus de langage*, we can replace V with $V_{\mathcal{N}}$, the point being that $\mathcal{N}V_{\mathcal{N}} = \mathcal{N}$ and $\mathcal{P}V_{\mathcal{N}} = \mathcal{P}$ since V is a valuation domain.

We have, for $i \in \{1, 2\}$, that $\mathcal{Q}_i = \mathcal{Q}_i T_{R \setminus M}$ where $\mathcal{Q}_i := \mathcal{Q}_i \cap T \in \text{Spec}(T)$. As $\mathcal{Q}_1 \cap R \subseteq M$ and integrality ensures that the extension $R \subseteq T$ satisfies GU (cf. Theorem 44 of [41]), there exists $N \in \text{Max}(T)$ such that $\mathcal{Q}_1 \subseteq N$ and $N \cap R = M$. Note that

$$\mathcal{N} \cap T = (\mathcal{N} \cap T_{R \setminus M}) \cap T = \mathcal{Q}_1 \cap T = \mathcal{Q}_1 \subseteq N,$$

and so $T_N \subseteq V_{\mathcal{N}} = V$. Moreover, the extension $T_N \subseteq V$ satisfies GD, since (2) ensures that T_N is a going-down domain. Next, note that $\mathcal{N} \cap T_N = \mathcal{Q}_1 T_N$. (Perhaps the easiest way to see this is to observe that $T_N \supseteq T_{R \setminus M}$ and $\mathcal{N} \cap T_{R \setminus M} = \mathcal{Q}_1 T_{R \setminus M}$.) So, since $\mathcal{Q}_2 T_N \subseteq \mathcal{Q}_1 T_N$ and $T_N \subseteq V$ satisfies GD, there exists $\mathcal{P} \in \text{Spec}(V)$ such that $\mathcal{P} \subseteq \mathcal{N}$ and $\mathcal{P} \cap T_N = \mathcal{Q}_2 T_N$. Intersecting with $T_{R \setminus M}$ then leads to

$$\mathcal{P} \cap T_{R \setminus M} = (\mathcal{P} \cap T_N) \cap T_{R \setminus M} = \mathcal{Q}_2 T_N \cap T_{R \setminus M} = \mathcal{Q}_2 T_{R \setminus M} = \mathcal{Q}_2,$$

as desired.

(3) \Rightarrow (4): Trivial.

(4) \Rightarrow (3): Assume (4). It will suffice to prove that $T_{R \setminus M}$ is a going-down domain for each $M \in \text{Max}(R)$ such that $\text{ht}_R(M) \leq 1$. Note that the extension $R_M \subseteq T_{R \setminus M}$ inherits the property of integrality from $R \subseteq T$ (cf. Proposition 5.6 (ii) of [2]) and hence, by integrality (cf. Theorem 48 of [41]), $\dim(T_{R \setminus M}) = \dim(R_M) = \text{ht}_R(M) \leq 1$. Thus, the domain $T_{R \setminus M}$ is either one-dimensional or a field. In particular, $T_{R \setminus M}$ is a going-down domain, as desired.

(3) \Rightarrow (2): Assume (3). It will suffice to prove that if $N \in \text{Max}(T)$, then T_N is a going-down domain. Since the extension $R \subseteq T$ is integral, $M := N \cap R \in \text{Max}(R)$ (cf. Corollary 5.8 of [2]). So, by (3), $T_{R \setminus M}$ is a going-down domain. Hence, by applying the implication (1) \Rightarrow (2) from Lemma 2.1 to the domain $T_{R \setminus M}$ and its multiplicatively closed subset $T \setminus N$, we get, since $R \setminus M \subseteq T \setminus N$, that

$$(T_{R \setminus M})_{T \setminus N} = T_{(R \setminus M)(T \setminus N)} = T_{T \setminus N} = T_N$$

is a going-down domain. The proof is complete. \square

Lemma 2.3. *Let $D \subseteq E$ be domains such that E is seminormal. Let I be a radical ideal of E such that the ring $D/(I \cap D)$ is seminormal (and, necessarily, reduced but not necessarily a domain). Then $D + I$ is seminormal.*

Proof. Our task is to show that if b and c are elements of $D + I$ such that $b^2 = c^3$, then there exists $a \in D + I$ such that $b = a^3$ and $c = a^2$. (Note that, thanks to $D + I$ being a reduced ring, any such element a , if it exists, is uniquely determined, by Lemma 3.1 of [55]). Since $(b + I)^2 = (c + I)^3$ in $(D + I)/I$ ($\cong D/(I \cap D)$), which is a seminormal ring, there exists $\alpha \in D + I$ such that $b + I = (\alpha + I)^3$ and $c + I = (\alpha + I)^2$ in $(D + I)/I$ and, hence, also in E/I . On the other hand, since E is seminormal, there

exists $a \in E$ such that $b = a^3$ and $c = a^2$. Consequently, by applying the canonical projection $E \rightarrow E/I$, we get that $b+I = (a+I)^3$ and $c+I = (a+I)^2$ in E/I . Hence, $(\alpha+I)^3 = (a+I)^3$ and $(\alpha+I)^2 = (a+I)^2$ in E/I . Note that E/I is a reduced ring since I is a radical ideal of E . So, by Lemma 3.1 of [55], $\alpha + I = a + I$. Thus, $a \in \alpha + I \subseteq (D + I) + I = D + I$, completing the proof. \square

Recall from [15] that an extension of domains $A \subseteq B$ is said to be *unibranched* if the canonical contraction map $c_A^B : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is a bijection. In the literature, one can find some other, inequivalent meanings assigned to the word “unibranched”. Note that our usage does not imply that a unibranched extension is integral. If $0 \leq n \leq \infty$, one example of a unibranched, but not integral, extension $A \subseteq B$ involving n -dimensional domains can be constructed as follows: take $K_1 \subset K_2$ to be fields such that K_1 is algebraically closed in K_2 , take B to be an n -dimensional valuation domain of the form $B = K_2 + N$, where N is the maximal ideal of B , and put $A := K_1 + N$.

For our purposes, the most useful fact about any integral unibranched extension (of domains) $A \subseteq B$ is that A is a going-down domain if and only if B is a going-down domain: see Lemma 2.3 of [15]. In fact, if these equivalent conditions hold, one can conclude more, namely, that c_A^B is an order-isomorphism of posets: cf. Corollary 5.8 of [26]. One consequence of the upcoming Theorem 2.4 is that if $R \subseteq T$ is an integral extension of domains and R is a quasi-local n -dimensional integrally closed going-down domain where $2 \leq n < \infty$, then there exists a domain \mathcal{S} such that $R \subseteq \mathcal{S}$ is a unibranched extension and T is an overring of \mathcal{S} (whence \mathcal{S} is a quasi-local going-down domain having the same integral closure as T and the contraction map $c_R^{\mathcal{S}} : \text{Spec}(\mathcal{S}) \rightarrow \text{Spec}(R)$ is an order-isomorphism). That fact is one of the keys which we will use in Theorem 2.4 and beyond to obtain information about the possible transfer of the “going-down domain” property of certain going-down domains D to integral extensions of D in field extensions of the quotient field of D on the basis of known transfer results of the “going-down domain” property from such D to integral overrings of D .

Theorem 2.4. Let $R \subseteq T$ be an integral extension of domains such that R is a going-down domain. Suppose also that for each $M \in \text{Max}(R)$ such that $\text{ht}_R(M) > 1$, there exists a height 1 prime ideal P of R such that $P \subset M$ (with P possibly depending on M). Then:

(a) Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R . Put $S := R + PT$ and $\mathcal{S} := R + \text{rad}_T(PT)$. Then $S \subseteq \mathcal{S}$, with equality if and only if $\text{rad}_T(PT) = PT$; S and \mathcal{S} are each domains with the same (Krull) dimension as R ; $(S : T) \supseteq PT \supset 0$; $(\mathcal{S} : T) \supseteq \text{rad}_T(PT) \supset 0$; S and \mathcal{S} are each underrings of T ; and $S' = T' = \mathcal{S}'$.

(b) Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R . As in (a), consider $S := R + PT$ and $\mathcal{S} := R + \text{rad}_T(PT)$. Let M be any maximal ideal of R that properly contains P . Put $n := \text{ht}_R(M)$ (with $2 \leq n \leq \infty$). Then

$$S_{R \setminus M} = R_M + (PR_M)T_{R \setminus M}$$

and

$$\mathcal{S}_{R \setminus M} = R_M + (\text{rad}_T(PT))T_{R \setminus M}$$

are each quasi-local n -dimensional going-down domains with exactly $n + 1$ (pairwise distinct) prime ideals apiece (where, as usual, $\infty + 1 := \infty$). Furthermore,

$$(S_{R \setminus M} : T_{R \setminus M}) \supseteq PT_{R \setminus M} \supset 0 \text{ and}$$

$$(\mathcal{S}_{R \setminus M} : T_{R \setminus M}) \supseteq (\text{rad}_T(PT))T_{R \setminus M} \supset 0.$$

Moreover,

$$A(M) := (S_{R \setminus M})^+ = (R_M + (PR_M)T_{R \setminus M})^+ = (R + PT)^+_{R \setminus M} = S^+_{R \setminus M}$$

and

$$\mathcal{A}(M) := (\mathcal{S}_{R \setminus M})^+ = (R_M + (\text{rad}_T(PT))T_{R \setminus M})^+ =$$

$$(R + \text{rad}_T(PT))^+_{R \setminus M} = \mathcal{S}^+_{R \setminus M}$$

are each n -dimensional seminormal divided domains (hence locally divided domains and going-down domains) which are underarrings of $T_{R \setminus M}$ such that $A(M)' = T'_{R \setminus M} = \mathcal{A}(M)'$,

$$(A(M) : T_{R \setminus M}) \supseteq PT_{R \setminus M} \supseteq PT \supset 0,$$

and

$$(\mathcal{A}(M) : T_{R \setminus M}) \supseteq (\text{rad}_T(PT))T_{R \setminus M} \supseteq \text{rad}_T(PT) \supset 0.$$

(c) Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R . Assume also that P is contained in the Jacobson radical of R . As in (a), consider $S := R + PT$ and $\mathcal{S} := R + \text{rad}_T(PT)$. Then S and \mathcal{S} are each going-down domains.

(d) Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R . Assume also that R is integrally closed and T is seminormal. As above, consider $\mathcal{S} := R + \text{rad}_T(PT)$. Then \mathcal{S} is seminormal; that is, $\mathcal{S}^+ = \mathcal{S}$.

Proof. The observations in the next paragraph will be of frequent use in this proof.

Since $R \subseteq T$ is an integral extension of domains, any nonzero ideal J of T satisfies $J \cap R \neq 0$ (cf. Lemma 11.1 of [37]). Integrality also ensures that $R \subseteq T$ satisfies LO, GU and INC (cf. Theorem 44 of [41]) and that $\dim(T) = \dim(R)$ (cf. Theorem 48 of [41]); and $R \subseteq T$ satisfies GD (since R is a going-down domain). Moreover, if $Q \in \text{Spec}(T)$, then $Q \in \text{Max}(T)$ if and only if $Q \cap R \in \text{Max}(R)$ (cf. Corollary 5.8 of [2]). One easily gets the following consequences: if $\mathcal{P} \subset \mathcal{M}$ are prime ideals of R with $\text{ht}_R(\mathcal{P}) = 1$, then \mathcal{P} is the only height 1 prime ideal of R that is properly contained in \mathcal{M} (since, by Theorem 2.2 of [13], any going-down domain is a treed domain); if $\mathcal{Q} \subset \mathcal{N}$ are nonzero prime ideals of T , with $\mathcal{P} := \mathcal{Q} \cap R$ and $\mathcal{M} := \mathcal{N} \cap R$, then $\mathcal{P} \neq 0$ and $\mathcal{P} \subset \mathcal{M}$ (since $R \subseteq T$ satisfies INC); if $0 \neq \mathcal{Q} \in \text{Spec}(T)$ with $\mathcal{P} := \mathcal{Q} \cap R$, then $\text{ht}_R(\mathcal{P}) = \text{ht}_T(\mathcal{Q})$ (as $\text{ht}_R(\mathcal{P}) \leq \text{ht}_T(\mathcal{Q})$ since $R \subseteq T$ satisfies GD and the reverse inequality holds since $R \subseteq T$ satisfies INC); and if $\mathcal{P} \in \text{Spec}(R)$ and \mathcal{Q} is minimal among the prime ideals of T that contain $\mathcal{P}T$, then (and only then) $\mathcal{Q} \cap R = \mathcal{P}$ (this can be seen by a well known characterization of the ring extensions satisfying GD to P : see the equivalence (ii) \Leftrightarrow (i) in Exercise 37 on pages 44-45 of [41]). One useful consequence is that if $\mathcal{P} \in \text{Spec}(R)$ and $\{\mathcal{Q}_i \mid i \in I\}$ is the set of prime ideals of T that are minimal with respect to containing $\mathcal{P}T$, then for any element j of the index set I ,

$$\mathcal{P} \subseteq \mathcal{P}T \cap R \subseteq \text{rad}_T(\mathcal{P}T) \cap R = (\bigcap_{i \in I} \mathcal{Q}_i) \cap R \subseteq \mathcal{Q}_j \cap R = \mathcal{P},$$

whence $\mathcal{P}T \cap R = \mathcal{P} = \text{rad}_T(\mathcal{P}T) \cap R = \mathcal{Q}_j \cap R$ for all $j \in I$.

Before moving to the proofs of (a)-(h), we devote this paragraph to noting that it may not be possible to strengthen some of the observations in the preceding paragraph. Indeed, it is possible that some nonzero prime ideal \mathcal{P} of R has height 1 and is also a maximal ideal of R . It then follows from the above comments that, for any such data, if we consider some prime ideal \mathcal{Q} of T such that $\mathcal{Q} \cap R = \mathcal{P}$ (after noting that at least one such \mathcal{Q} exists, since $R \subseteq T$ satisfies LO), then $\text{ht}_T(\mathcal{Q}) = \text{ht}_R(\mathcal{P}) = 1$ and \mathcal{Q} is a maximal ideal of T . In fact, more can be stated, as one can get the following as a special case of Theorem 3.1 of [43]: for any (possibly equal) positive integers d and n , there exists a d -dimensional going-down domain (in fact, a d -dimensional Pruefer domain) D such that the set of height 1 maximal ideals of D has cardinality n . Indeed, one need only construct any finite d -dimensional tree \mathcal{T} such that \mathcal{T} has a unique minimum element and the subset of \mathcal{T} consisting of all the height 1 maximal elements of \mathcal{T} is a set of cardinality n (so that, necessarily, $|\mathcal{T}| \geq n + d + 1$ if $d \geq 2$, while $|\mathcal{T}| = n + 1$ if $d = 1$), note that the finiteness of d ensures that \mathcal{T} satisfies the properties (K1) and (K2) that are mentioned in the statement of Theorem 3.1 of [43], and then use that result to obtain a Pruefer domain whose prime spectrum is isomorphic (as a poset under inclusion) to \mathcal{T} .

We next prove the individually stated parts of this theorem.

(a) Since R is a subring of the domain T , while both PT and $\text{rad}_T(PT)$ are each ideals of T , it is clear that S and \mathcal{S} are domains. As $PT \subseteq \text{rad}_T(PT)$, it is also clear that $S \subseteq \mathcal{S}$, with equality if $PT = \text{rad}_T(PT)$. We will prove in the next paragraph that, conversely, if $S = \mathcal{S}$, then $PT = \text{rad}_T(PT)$ (that is, then PT is a radical ideal of T).

We proceed to complete the proof that if $S = \mathcal{S}$, then $PT = \text{rad}_T(PT)$. As $S \subseteq \mathcal{S}$, it suffices to show that if $\text{rad}_T(PT) \subseteq R + PT$, then $\text{rad}_T(PT) \subseteq PT$. Let $\xi \in \text{rad}_T(PT)$. By hypothesis, $\xi = r + \eta$ for some $r \in R$ and some $\eta \in PT$. So, by using the final comment in the second paragraph of the proof of this theorem,

$$r = \xi - \eta \in (\text{rad}_T(PT) + PT) \cap R = \text{rad}_T(PT) \cap R = P \subseteq PT,$$

whence $\xi = r + \eta \in PT + PT = PT$, as required.

Given the chain of rings $R \subseteq S \subseteq \mathcal{S} \subseteq T$ and the hypothesis that T is integral over R , it is clear that S is integral over R , \mathcal{S} is integral over S , and T is integral over \mathcal{S} . Thus, $\dim(T) = \dim(S) = \dim(\mathcal{S}) = \dim(R)$ (cf. Theorem 48 of [41]). As PT is a common ideal of S and T , we get $PT \subseteq (S : T)$; and $PT \supset 0$, since $0 \subset P = \{p \cdot 1 \mid p \in P\} \subseteq PT$. In particular, $(S : T) \neq 0$. Similarly, as $\text{rad}_T(PT)$ is a common ideal of \mathcal{S} and T , we get $\text{rad}_T(PT) \subseteq (\mathcal{S} : T)$, and so $(\mathcal{S} : T) \neq 0$. Hence, T is an overring of both S and \mathcal{S} ; equivalently, S and \mathcal{S} are each underings of T . Next, we will show that $S' = T' = \mathcal{S}'$. Of course, $S' \subseteq \mathcal{S}' \subseteq T'$ (because of the chain of overings $S \subseteq \mathcal{S} \subseteq T$), and so it now only remains to show that $T' \subseteq S'$. This, in turn, follows from the transitivity of integrality (more precisely, from what is actually shown in the proof of Corollary 5.4 in [2]). This completes the proof of (a).

(b) The most important part of (b) – and arguably the main technical observation in this paper – is the assertion that $\mathcal{S}_{R \setminus M}$ and $\mathcal{S}_{R \setminus M}$ are each quasi-local n -dimensional going-down domains with exactly $n + 1$ (pairwise distinct) prime ideals apiece. Let us, for the moment, take that assertion for granted. Then, after dispatching the other assertions in (b), we will devote the final part of this proof of (b) to proving what we have called its “most important part”.

First, let us note that the “Furthermore” assertion, to the effect that the conductors $(\mathcal{S}_{R \setminus M} : T_{R \setminus M})$ and $(\mathcal{S}_{R \setminus M} : T_{R \setminus M})$ each contain certain specified nonzero common ideals, is evident.

Next, it is important to note that the notations $A(M)$ and $\mathcal{A}(M)$ are not ambiguous (that is, these notations need not mention P) because P is the only height 1 prime ideal of (the treed domain) R that is properly contained in the specified maximal ideal M of R . Also, the displayed equivalent formulations of $A(M)$ and $\mathcal{A}(M)$ follow from straightforward calculations and the result that the formation of seminormalizations commutes with the formation of ring of fractions relative to any given multiplicatively closed set (cf. Proposition 2.9 of [55]).

Having just addressed the descriptions of $A(M)$ and $\mathcal{A}(M)$, we turn now to proving the other assertions about $A(M)$ and $\mathcal{A}(M)$. The details of the proofs concerning these two rings are very similar (the only differences being due to whether the symbol “ rad_T ” is relevant to the proof). Accordingly, we will prove the assertions concerning $\mathcal{A}(M)$, leaving the detailed proofs concerning $A(M)$ to the reader.

Note that $R_M \subseteq \mathcal{A}(M) = \mathcal{S}^+_{R \setminus M} \subseteq T^+_{R \setminus M}$ is a tower of integral ring extensions. As the partners in any integral ring extension have the same Krull dimension, $\dim(\mathcal{A}(M)) = \dim(R_M) = \text{ht}_R(M) = n$. Moreover, since $\mathcal{A}(M)$ is the seminormalization of $\mathcal{S}_{R \setminus M}$, it follows that $\mathcal{S}_{R \setminus M} \subseteq \mathcal{A}(M)$ is an integral unbranched extension, and so the contraction map $\text{Spec}(\mathcal{A}(M)) \rightarrow \text{Spec}(\mathcal{S}_{R \setminus M})$ is an order isomorphism of posets (under inclusion). Thus, $\mathcal{A}(M)$ inherits the property of being quasi-local from what we have assumed (for the moment and will soon prove) about $\mathcal{S}_{R \setminus M}$. Moreover, since $\mathcal{A}(M)$ is the seminormalization of a domain that we have assumed for the moment is a going-down domain (namely, $\mathcal{S}_{R \setminus M}$) and any seminormalization of a domain is a seminormal domain (cf. Corollary 3.4 of [55]), it follows from Corollary 3.6 of [4] (cf. also Corollary 2.6 of [15] and the Theorem in Remark 5.18 (f) of [26]) that $\mathcal{A}(M)$ is a locally divided domain (and hence, by Remark 2.7 (b) of [15], a going-down domain). So, as is the case for any quasi-local locally divided domain, $\mathcal{A}(M)$ must be a divided domain. Moreover, since T^+ is an overring of T and T is an overring of \mathcal{S} , we get that T^+ is

an overring of \mathcal{S} , and so $\mathcal{A}(M)$ is an underring of $T^+_{R \setminus M}$. Hence, since $T^+_{R \setminus M}$ is integral over $\mathcal{A}(M)$, the transitivity of integrality gives $\mathcal{A}(M)' = (T^+_{R \setminus M})' = (T^+)_{R \setminus M} = T'_{R \setminus M}$, with the next-to-last of these equalities following from, for instance, Proposition 10.2 of [37].

For a variant of the above proof that $\mathcal{A}(M)' = T'_{R \setminus M}$, combine the equality $\mathcal{S}' = T'$ from (a) with Proposition 10.2 of [37].

To complete the proof of (b), it remains to show that $\mathcal{S}_{R \setminus M}$ is a quasi-local n -dimensional going-down domain with exactly $n + 1$ (pairwise distinct) prime ideals. We have

$$\mathcal{S}_{R \setminus M} = R_M + (\text{rad}_T(P T))T_{R \setminus M}.$$

The last display can be usefully rewritten. Indeed,

$$\text{rad}_{T_{R \setminus M}}((P R_M)T_{R \setminus M}) = (\text{rad}_T(P T))T_{R \setminus M}.$$

(While one inclusion is not difficult to verify, it may be useful to include the following details which prove the reverse inclusion. Specifically, let $\xi \in \text{rad}_{T_{R \setminus M}}((P R_M)T_{R \setminus M})$. Then $\xi = t/z$ for some $t \in T$ and $z \in R \setminus M$ such that $\xi^m \in (P T)T_{R \setminus M}$ for some integer $m \geq 1$. Hence, there exists $\lambda \in R \setminus M$ such that $\lambda \xi^m \in P T$, that is, $\lambda t^m/z^m \in P T$, whence $\lambda t^m \in z^m P T \subseteq P T$, whence

$$(\lambda t)^m = \lambda^{m-1}(\lambda t^m) \in R(P T) = P T,$$

whence $\xi = \tau/\zeta$, where $\tau := \lambda t \in \text{rad}_T(P T)$ and $\zeta := \lambda z \in R \setminus M$, thus proving the equality in the next-to-last display). Consequently,

$$\mathcal{S}_{R \setminus M} = R_M + \text{rad}_{T_{R \setminus M}}((P R_M)T_{R \setminus M}).$$

This last display indicates that a great economy may now become available. To wit, consider replacing the 4-tuple (R, M, P, T) with the 4-tuple $(R_M, M R_M, P R_M, T_{R \setminus M})$. This replacement is available, since $R_M \subseteq T_{R \setminus M}$ is an integral extension of domains, R_M is a going-down domain (by the implication (1) \Rightarrow (3) in Lemma 2.1), $\dim(R_M) = \text{ht}_{R_M}(M R_M) = \text{ht}_R(M) = n$, $P R_M$ is a nonmaximal prime ideal of height 1 in R_M and, most importantly, the last display now reveals that the suggested replacement involving 4-tuples now reduces our task to showing (now that R is a quasi-local n -dimensional going-down domain with unique maximal ideal M and unique height 1 prime ideal P) that $\mathcal{S} (= R + \text{rad}_T(P T))$ is a quasi-local n -dimensional going-down domain with exactly $n + 1$ (pairwise distinct) prime ideals. By (a), $\dim(\mathcal{S}) = n$. Therefore, it remains only to prove that the integral extension $R \subseteq \mathcal{S}$ is unbranched (for it would then follow that \mathcal{S} is a quasi-local going-down domain by appealing to Lemma 2.3 of [15]; and since R , an n -dimensional domain whose prime ideals are linearly ordered by inclusion, necessarily has precisely $n + 1$ (pairwise distinct) prime ideals, it would then also follow that $|\text{Spec}(\mathcal{S})| = n + 1$ by observing that Corollary 5.8 of [26] ensures that the contraction map $\text{Spec}(\mathcal{S}) \rightarrow \text{Spec}(R)$ is an order isomorphism of posets under inclusion).

Observe that $\mathcal{S} = R + \text{rad}_T(P T)$ is the pullback (in the category of commutative rings and unital ring homomorphisms) of the canonical surjection $T \rightarrow T/\text{rad}_T(P T)$ and the canonical injection

$$R/P = R/(\text{rad}_T(P T) \cap R) \cong \mathcal{S}/\text{rad}_T(P T) \rightarrow T/\text{rad}_T(P T).$$

According to Theorem 1.4 of [36], $\text{Spec}(\mathcal{S})$ can be understood as a topological space (with respect to the Zariski topology) and hence as a poset under inclusion by establishing a homeomorphism (and hence an order isomorphism) between this space and a certain quotient space obtained by subjecting the disjoint union of $\text{Spec}(T)$ and $\text{Spec}(R/P)$ to a certain equivalence relation. That equivalence relation is generated by requiring that every prime ideal Q of T which contains $P T$ is identified with

$$Q/\text{rad}_T(P T) \cap \mathcal{S}/\text{rad}_T(P T) \in \text{Spec}((R + \text{rad}_T(P T))/\text{rad}_T(P T)),$$

that is, with $(Q \cap R)/P \in \text{Spec}(R/P)$. I recommend that the reader verify the last "that is" assertion, as I found it worthwhile to carry out a calculation demonstrating that for any nonzero prime ideal Q of T such that $Q \supseteq PT$, the standard isomorphism $R/P \rightarrow S/\text{rad}_T(PT)$ induces an isomorphism

$$(Q \cap R)/P \rightarrow ((Q \cap R) + \text{rad}_T(PT))/\text{rad}_T(PT) = (Q \cap S)/\text{rad}_T(PT).$$

The upshot of the resulting "gluing" is that the set of elements of $\text{Spec}(S)$ is identified with the resulting set of equivalence classes. The rest of the proof of (b) will show, *i.a.*, that there are precisely $n + 1$ such (pairwise distinct) equivalence classes.

Note that any prime ideal Q of T which does not contain P must be 0 . (Indeed, since $R \subseteq S$ is an integral extension involving domains, the intersection of R with any nonzero prime ideal of T must be nonzero, say \mathcal{P} . As (R, M) is quasi-local and R is treed, \mathcal{P} and P are comparable with respect to inclusion. Since $\text{ht}_R(P) = 1$ and $\mathcal{P} \neq 0$, the hypothesis that R is a domain gives $\mathcal{P} \supseteq P$.) Each nonzero prime ideal Q of T which is not minimal with respect to containing PT is thus identified with $(Q \cap R)/P$, that is, with $p/P \in \text{Spec}(R/P)$ for some $p \in \text{Spec}(R)$ such that $0 \subset P \subset p \subseteq M$. In the construction of $\text{Spec}(S)$ given by applying Theorem 1.4 of [36], let \mathcal{C}_p denote the element of $\text{Spec}(S)$ (that is, the equivalence class) represented by $p/P \in \text{Spec}(R/P)$ for any $p \in \text{Spec}(R)$ such that $p \supseteq P$. The above discussion has shown that the *only other possible kind* of element of $\text{Spec}(S)$ (that is, the only other possible kind of equivalence class in $\text{Spec}(S)$) would have a unique member, namely, the zero prime ideal of T . It will be convenient to let 0_T denote the zero prime ideal of T and to let 0 denote the element of $\text{Spec}(S)$ represented by 0_T . One consequence of the work in the next paragraph is that there does not exist a nonzero prime ideal p of R such that $\mathcal{C}_p = 0$ in $\text{Spec}(S)$ (and so it will follow that the above contemplated "only other possible kind of element of $\text{Spec}(S)$ " really does exist). At the moment, we at least have the following set-theoretic description of $\text{Spec}(S)$:

$$\text{Spec}(S) = \{\mathcal{C}_p \mid p \in \text{Spec}(R), p \supseteq P\} \cup \{0\}.$$

In particular, $|\text{Spec}(S)| \leq n + 1$. As explained above, the rest of the proof of (b) will show that the integral extension $R \subseteq S$ is unbranched. One consequence (*en passant*) of that fact will be to sharpen the just-mentioned inequality to an equation.

The folklore surrounding the topological and categorical underpinnings of Theorem 1.4 of [36] includes the following fact concerning inclusions of prime ideals occurring within a single vertex of a pullback diagram of rings (of the kind that we used in describing S as the pullback of the surjection $T \rightarrow T/\text{rad}_T(PT)$ and the injection $R/P \rightarrow T/\text{rad}_T(PT)$). Any such inclusion is preserved under any relevant spectral map induced by the ring homomorphisms in the given pullback diagram and the overall "gluing" construction. We next identify two applications of this fact. First, consider any nonzero prime ideals $p_1 \subset p_2$ of R . By the LO and GU aspects of integral extensions, there exist (nonzero) prime ideals $\mathcal{Q}_1 \subset \mathcal{Q}_2$ of T such that $\mathcal{Q}_i \cap R = p_i$ for $i \in \{1, 2\}$. When one views $\text{Spec}(T)$ as a poset under inclusion, one has $\mathcal{Q}_1 < \mathcal{Q}_2$. So, by the above folklore fact, $\mathcal{C}_{p_1} \leq \mathcal{C}_{p_2}$ when one views $\text{Spec}(S)$ as a poset under inclusion (the point being that \mathcal{C}_{p_i} is the equivalence class in $\text{Spec}(S)$ which contains \mathcal{Q}_i). (This first application may seem intuitively credible because $p_1/P < p_2/P$ when $\text{Spec}(R/P)$ is viewed as a poset under inclusion; I would be interested in learning whether readers find the following application to be intuitively credible.) Second, for each nonzero prime ideal p of R , $0 \leq \mathcal{C}_p$ when $\text{Spec}(S)$ is viewed as a poset (under inclusion). This second application can be seen as a consequence of the above folklore fact in the following way: for any such p , integrality of the extension $R \subseteq T$ ensures that some prime ideal Q of T satisfies $Q \cap R = p$; so, when one combines the above folklore fact with the datum $0_T < Q$ in the poset $\text{Spec}(T)$, we get, in the poset $\text{Spec}(S)$, that $0 \leq \mathcal{C}_p$ (the points being that 0 is the equivalence class in $\text{Spec}(S)$ which contains 0_T , and \mathcal{C}_p is the equivalence class in $\text{Spec}(S)$ which contains Q). Since R and R/P are quasi-local treed domains, the upshot is that **as a poset under inclusion, $\text{Spec}(S)$ is linearly ordered**. Therefore, S is a quasi-local treed domain. With this information in hand, it is straightforward to show that $R \subseteq S$ is a

unibranched extension, that is, that the canonical map $c_R^S : \text{Spec}(S) \rightarrow \text{Spec}(R)$ is a bijection. Indeed, c_R^S is injective since S is a quasi-local treed domain and $R \subseteq S$ satisfies INC (by virtue of integrality); and c_R^S is surjective since $R \subseteq S$ satisfies LO (by virtue of integrality). The proof of (b) is complete.

We wish to stress that the reduction to the case where R is quasi-local was justified above for the sole purpose of completing the proof of (b). The reader is cautioned that, in perusing the proof of any of the remaining parts of Theorem 2.4 or the proof of the corollaries of Theorem 2.4, one should not automatically assume that R is quasi-local.

(c) We give the proof for S , leaving the details of the similar proof for \mathcal{S} to the reader. Consider any $M \in \text{Max}(R)$ such that $\text{ht}_R(M) \geq 2$. By the implication (4) \Rightarrow (1) in Lemma 2.2, it suffices to prove that $S_{R \setminus M}$ is a going-down domain. Hence, by (b), it suffices to show that $P \subseteq M$. That, in turn, holds because P was assumed to be contained in the Jacobson radical of R . This completes the proof of (c).

(d) The assertion that \mathcal{S} is seminormal is a consequence of Lemma 2.3. The only nontrivial step in verifying this is the explanation why $R/(\text{rad}_T(PT) \cap R)$ is seminormal. In fact, more is true, as this ring is an integrally closed domain. To see this, recall from the final comment in the second paragraph of the proof of this theorem that $\text{rad}_T(PT) \cap R = P$ and, then, apply Theorem 4.1 of [5], which showed that any factor domain of an integrally closed going-down domain is integrally closed. The proof is complete. \square

Remark 2.5. (a) The proof of Lemma 2.3 was a straightforward exercise of diagram chasing that used the pullback description of $D + I$ as $D/(I \cap D) \times_{E/I} E$. While there are several results in the literature showing that seminormality is often preserved in the formation of pullbacks (such as Proposition 2.4 of [38] or Lemma 2.23 of [57]), all the results of that kind which I found in the literature had finiteness hypotheses that are not necessarily satisfied by the situation that was addressed in the proof of Theorem 2.4 (namely, where we showed that under certain conditions, a ring denoted by \mathcal{S} is seminormal). Lemma 2.3 was strong enough to be applicable to that situation and, hence, was adequate to meet our needs here. Readers with a deeper interest in seminormality *per se* are encouraged to examine possible sharpenings of the statement of Lemma 2.3 (such as the possibilities of eliminating its hypothesis on $D/(I \cap D)$ and allowing D and E to be reduced rings rather than domains).

Let me expand upon some of the preceding paragraph. Some specialists may be interested in knowing that Lemma 2.3 holds more generally than we have stated it. Indeed, in Lemma 2.23 of [57], Vitulli states a more general version of Lemma 2.3 and, in lieu of proving that result, she cites Lemma 4.2 of [38] (a result of Greco and Traverso). As all rings in [38] are assumed to be Noetherian, I considered the details of the proofs of the supporting results in [38] and I have concluded that the proof of Proposition 2.5 in [38] is most pertinent and is valid without an assumption of Noetherianity, at least insofar as its special case given in Lemma 2.3 is concerned. To read that proof nowadays, note that Greco and Traverso used “quasi-isomorphic” to mean the same as “subintegral” (in the sense of [55]); the more prevalent contemporary meaning of “quasi-isomorphic” concerns derived categories.

(b) It seems reasonable to ask why we considered similar roles for both S and \mathcal{S} in Theorem 2.4. An initial, somewhat superficial, answer would be the following. While the definition of S is intuitively easier than the definition of \mathcal{S} (in that the ideal PT of T seems intuitively more “accessible” than its radical in T), the domain \mathcal{S} is “closer” than S to the main objects of interest (namely, T and T') since $S \subseteq \mathcal{S} \subseteq T (\subseteq T')$. In our opinion, a less superficial answer is provided by the comments about seminormalization and t-closure in the next paragraph.

It is straightforward to prove a general result that implies, for the context of Theorem 2.4, that $R + PT$ is seminormal in (resp., t-closed in) T if and only if R/P is seminormal in (resp., t-closed in) T/PT . (A method for proving such pullback-theoretic assertions regarding seminormality or t-closedness is illustrated by the proof (corresponding to the “if” assertion) of the preceding statement) in Proposition 4.11 of [50].) Assertions with a flavor similar to that of this seminormality assertion can be found in Lemma 4.2 of [38] and Lemma 2.23 of [57], but those specific references do not

apply directly to the context of Theorem 2.4 because they contain certain finiteness hypotheses. For the case of seminormality, the usefulness of such a result is apparent (in view of Corollary 3.4 of [55]) if T/PT is seminormal. (A similar comment holds for the case of t-closedness.) For that reason, some parts of Theorem 2.4 could have included variants in which a statement involving PT (resp., S) was noted to be equivalent to a statement involving $\text{rad}_T(PT)$ (resp., S). Indeed, $T/\text{rad}_T(PT)$ is a reduced ring and, hence, a candidate for being a seminormal ring. (I have been unable to prove, and I frankly doubt, that PT is a radical ideal of T for the contexts that are of greatest interest here. Let me be more precise about the attendant technical obstacle in dealing with S : if $\{Q_i\}$ is the set of prime ideals of T that are minimal with respect to containing PT , then although $\bigcap Q_i$ does equal $\text{rad}_T(PT)$ (thanks to R being a going-down domain), I have been unable to find rather general hypotheses on R and T which would imply that this intersection is equal to PT .) The main upshot of such considerations for the context of Theorem 2.4 is that if R/P is t-closed in (hence, seminormal in) $T/\text{rad}_T(PT)$, then $S := R + \text{rad}_T(PT)$ is t-closed in (hence, seminormal in) T , so that if T is also assumed to be t-closed (for instance, integrally closed), then $S^+ = S$ is t-closed and, consequently,

$$\mathcal{S}_{R \setminus M}^+ = (R + \text{rad}_T(PT))_{R \setminus M} = R_M + \text{rad}_{T_{R \setminus M}}(PR_M T_{R \setminus M})$$

is t-closed. I leave it to any interested reader to (possibly by making use of the results in the preceding sentence) identify/obtain results about PT (possibly some parts of Theorem 2.4 or its corollaries) which admit variants that result upon replacing PT with $\text{rad}_T(PT)$.

(c) It also seems reasonable to ask if the proof of Theorem 2.4 yields any new information for the context where R is a going-down domain that is (for simplicity) assumed to be quasi-local (and not necessarily integrally closed) with a (necessarily unique) height 1 prime ideal P and $T := R'$. Specifically, one could ask whether the proof of Theorem 2.4, when applied to this specific kind of "overring" context involving concepts that specialize to $S := R + PR'$ or $\mathcal{S} := R + (\text{rad}_{R'}(PR'))R'$, would yield any new information (going beyond what is already in Corollary 3.6 of [4]). The answer is in the negative, because for this context, $S^+ = R^+ = \mathcal{S}^+$. We next sketch a proof of this fact. As it is clear that $R^+ \subseteq S^+ \subseteq \mathcal{S}^+$, it remains only to show that $\mathcal{S}^+ \subseteq R^+$. For that purpose, it suffices to prove that $\text{rad}_{R'}(PR') \subseteq R^+$. That, in turn, follows because the original definition of a seminormalization in [56] (cf. also Definition 2.4 of [57]) ensures that $\text{rad}_{R'}(PR')$ is a subset of the sum of R' and the Jacobson radical of R' . Indeed, in Definition 2.2 of [57], Vitulli's recapitulation of what led to the classical approach to the normalization of a domain reveals that we have called "S", in the special case of the context of Theorem 2.4 where R is quasi-local with maximal ideal M , is what is classically known as "the ring obtained from R by gluing in R' over M ."

(d) The title of this paper referred to locally finite-dimensional (in short, LFD) domains. Recall that a domain D is said to be *locally finite-dimensional* if $\text{ht}_D(P) < \infty$ for all $P \in \text{Spec}(D)$ (equivalently, if $\text{ht}_D(N) < \infty$ for all $N \in \text{Max}(D)$). It is clear that a domain D is LFD if and only if D_N is LFD for each $N \in \text{Max}(D)$.

A domain D is said to be *catenarian* if, for each pair of prime ideals $P \subseteq Q$ of D , all maximal chains of prime ideals of D going from P to Q have a common finite length. Each catenarian domain is LFD. As in [7], a (not necessarily Noetherian) domain D is said to be *universally catenarian* if the polynomial rings $D[X_1, \dots, X_n]$ are catenarian for each positive integer n . Although a universally catenarian domain need not be Noetherian, it follows from Lemma 2.3 of [7] that each universally catenarian domain is a strong S -domain (in the sense of page 29 of [41]). Moreover, it was shown in [8] that a going-down domain is universally catenarian if (and only if) it is a locally finite-dimensional strong S -domain. (This result was motivated, in part, by some results in [47], [44] and [9] that are discussed later in this paragraph). Universal catenarity was originally studied only for Noetherian rings, in part because of algebro-geometric applications of some well known behavior of Cohen-Macaulay rings. Additional interest in the Noetherian case was provided by the result (3.6) of Ratiff [53] which implies, *i.a.*, that every one-dimensional Noetherian domain is universally catenarian. This result

showed that an earlier example of Nagata [46] of a two-dimensional Noetherian domain which is not universally catenarian is, from a dimension-theoretic point of view, best possible. Some decades later, two very different proofs were given to show that a Pruefer domain is universally catenarian if (and only if) it is LFD: see page 28 of [44] and Theorem 12 of [9]. The latter paper was deepened as to detail and scope in the subsequent collaboration [7], which was motivated to introduce a theory of (not necessarily Noetherian) universally catenarian domains while respecting the overall pioneering vision of [9]. On the other hand, [44] was built, in part, on a paper of Nagata [47] (see, especially, Theorem 2 of this paper by Nagata). As those works became known, some concern was expressed as to whether the details in [47] sufficed to prove that any LFD Pruefer domain is universally catenarian. Such worries were put to rest with a privately (but widely) circulated manuscript that clarified and finalized Nagata's proof, thus erasing any doubts as to the validity of Nagata's overall approach.

We would suggest that the material in the preceding paragraph supports a contention that the continued pursuit of research on going-down domains should be of wide interest, as that material has placed a class of going-down domains in the company of two leading classes of domains that are often used for motivational purposes. To wit: the class of LFD going-down domains that are also strong S -domains exhibits a property that is also shared by one-dimensional Noetherian domains and LFD Pruefer domains (namely, the property of being universally catenarian domains).

(e) For the sake of brevity, the title of this paper referred to a base domain that was a "locally finite-dimensional going-down domain", but the base domain R in Theorem 2.4 (and in some of the later material) is more general than that. Indeed, a going-down domain R satisfying the assumptions of Theorem 2.4 can be quasi-local and infinite-dimensional. Perhaps the easiest example of such an R is a valuation domain whose prime spectrum, as a poset under inclusion, is given (up to isomorphism) by the following poset:

$$0 < P < \cdots < Q_{-n-1} < Q_{-n} < Q_{-n+1} < \cdots < Q_{-3} < Q_{-2} < Q_{-1} < M.$$

The existence of such a valuation domain can be shown by verifying the conditions in the statement of either Theorem 3.1 or Corollary 3.6 of [43].

(f) The above proof of Theorem 2.4 (b) referred to a certain piece of folklore pertinent to applications of Theorem 1.4 of [36]. I next offer a way to avoid that appeal in that proof, at least for the special (but important) case where the base domain R is assumed to be LFD. In that case, $n := \text{ht}_R(M) < \infty$. At the relevant point in the proof of Theorem 2.4 (b), R is quasi-local and (since it is a going-down domain) treed, and so $|\text{Spec}(R)| = 1 + \dim(R) = n + 1$. As noted earlier, the set-theoretic description of $\text{Spec}(\mathcal{S})$ reveals that $|\text{Spec}(\mathcal{S})| \leq n + 1$. However, by the LO aspect of integrality, the contraction map $c_R^{\mathcal{S}} : \text{Spec}(\mathcal{S}) \rightarrow \text{Spec}(R)$ is surjective. So, by the Pigeonhole Principle and its neighbors, $|\text{Spec}(\mathcal{S})| = n + 1$ and $c_R^{\mathcal{S}}$ is injective (whence $R \subseteq \mathcal{S}$ is a unibranch extension, as desired).

(g) To appreciate the importance of the role of the domain $R + PT$ (and the roles of seminormalizations inferred therefrom) in the proof of Theorem 2.4 and the overall program of this paper, it is important to observe the following fact. If D is a domain with quotient field k and F is a field extension of k , then $+_F D = +_k D$. (Consequently, each seminormal domain is seminormal in any field extension of its quotient field.) One proof of this fact uses the result (Theorem 2.8 of [55]) that $+_F D$ is the filtered union of all subrings of F which can be obtained from D by a finite number of elementary subintegral extensions. That result immediately gives that $+_k D \subseteq +_F D$. If the reverse inclusion fails, the just-cited result from [55] yields $u \in F \setminus k$ such that $u^2, u^3 \in k$, whence $u = u^3(u^2)^{-1} \in k$, the desired contradiction, thus proving the above-stated fact. (For another proof of this fact, it suffices to observe that if $A \subseteq B$ is any subintegral extension of domains (so that, necessarily, $Q := 0 \in \text{Spec}(B)$ satisfies $Q \cap A = P := 0 \in \text{Spec}(A)$), then B is an overring of A , since subintegrality ensures that the canonical injective ring homomorphism from the quotient field of A/P to the quotient field of B/Q is surjective.) This fact is also remarkable at this time because on March 29, 2026, I found that the current online literature contains a fallacious argument that alleges to prove that the field $\mathbb{Q}(i)$ of

Gaussian numbers is the quotient field of the seminormalization of the (seminormal) domain $\mathbb{Z}_5\mathbb{Z}$ in $\mathbb{Q}(i)$.

(h) In part (c) of this remark and in the proof of Theorem 2.4 (b), I had occasion to cite a corollary from the paper [4]. I wish to stress that [4] went far beyond the domain-theoretic context of papers such as [15]. In particular, the above-mentioned Corollary 3.6 of [4] resulted from work in [4] on the topic of going-down rings (possibly with nontrivial zero-divisors), a concept that I had introduced in [18] (nearly 25 years after I had defined going-down domains).

(i) The setting for Theorem 2.4 reflects a belief that I have long held, namely, that the extent to which the “going-down domain” property transfers to integral closures will eventually be understood and exemplified (at least for domains that are LFD) by the behavior of quasi-local two-dimensional base (going-down) domains. In that regard, I would point to the following result from [19]: if R is an n -dimensional quasi-local going-down domain for some positive integer n and if P is the (necessarily unique) prime ideal of R of height $n - 1$, then the integral closure of R in R_P is a going-down domain. The reader of the proof of that result in [19] will have no difficulty in discerning that I originally proved the result in case $n = 2$ and then noticed that my proof carried over, *mutatis mutandis*, for any larger positive integer n . However, although Theorem 2.4 will be generalized below to a result (Proposition 2.13) that can be applied to base going-down domains that are LFD, it will require more than a “*mutatis mutandis*” kind of comment to prove Proposition 2.13! The remark is complete.

Applications of Theorem 2.4 will be given below in several corollaries. Corollaries 2.6-2.9 are set in what is perhaps the most accessible context, namely, where the base domain is two-dimensional. The following definition will be used in Corollaries 2.6-2.9. Recall (cf. [48]) that a ring extension $A \subseteq B$ is said to *have finite fibers* if each prime ideal of A is lain over by only finitely many prime ideals of B . A classical result (cf. Proposition 3, page 40 of [6]) states that any module-finite ring extension has finite fibers. For some different kinds of sufficient conditions for a ring extension to have finite fibers, see Section 3 of [3], especially its Theorem 3.7.

Corollary 2.6 is essentially the result of combining Theorem 2.4 with Theorem 1 of [23]. Apart from changes in notation, the latter result states the following. If A is a two-dimensional going-down domain and B is a seminormal integral overring of A such that A is t -closed in B , $(A : B) \neq 0$, and the ring extension $A \subseteq B$ has finite fibers, then B is a going-down domain. (The just-mentioned result gave a strict generalization of Theorem 2.2 of [20] by replacing the latter’s hypothesis that A^+ is quadratically integrally closed in B with the hypothesis that A is t -closed in B .) Note that, in view of result(s) that have already been mentioned from [15], [4] and [26], the conclusion of (the just-mentioned) Theorem 1 of [23] can be strengthened to “ B is a locally divided domain”.

Corollary 2.6. *Let $R \subseteq T$ be an integral extension of domains such that R is a two-dimensional going-down domain. Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R . Put $S := R + PT$ and $\mathcal{S} := R + \text{rad}_T(PT)$. Then:*

(a) *Let M be any prime ideal of R that properly contains P . (Necessarily, $M \in \text{Max}(R)$). Suppose that $T_{R \setminus M}$ is seminormal. Suppose also that $S_{R \setminus M}$ (resp., $\mathcal{S}_{R \setminus M}$) is t -closed in $T_{R \setminus M}$, and that the ring extension $S_{R \setminus M} \subseteq T_{R \setminus M}$ has finite fibers (resp., and that the ring extension $\mathcal{S}_{R \setminus M} \subseteq T_{R \setminus M}$ has finite fibers). Then $T_{R \setminus M}$ is a going-down domain (in fact, a locally divided domain).*

(b) *Let M be any prime ideal of R that properly contains P . (Necessarily, $M \in \text{Max}(R)$). Suppose that $T_{R \setminus M}$ is seminormal. Suppose also that $S_{R \setminus M}^+$ (resp., $\mathcal{S}_{R \setminus M}^+$) is t -closed in $T_{R \setminus M}$, and that the ring extension $S_{R \setminus M}^+ \subseteq T_{R \setminus M}$ has finite fibers (resp., and that the ring extension $\mathcal{S}_{R \setminus M}^+ \subseteq T_{R \setminus M}$ has finite fibers). Then $T_{R \setminus M}$ is a going-down domain (in fact, a locally divided domain).*

(c) *Suppose that P is contained in the Jacobson radical of R , that T is seminormal, that S (resp., \mathcal{S}) is t -closed in T , and that the ring extension $S \subseteq T$ has finite fibers (resp., and that the ring extension $\mathcal{S} \subseteq T$ has finite fibers). Then T is a going-down domain (in fact, a locally divided domain).*

(d) Suppose that P is contained in the Jacobson radical of R , that T is seminormal, that S^+ (resp., \mathcal{S}^+) is t -closed in T , and that the ring extension $S^+ \subseteq T$ has finite fibers (resp., and that the ring extension $\mathcal{S}^+ \subseteq T$ has finite fibers). Then T is a going-down domain (in fact, a locally divided domain).

Proof. (a) Of course, the parenthetical assertion that $M \in \text{Max}(R)$ holds because R is two-dimensional. In view of the final comment before the statement of this corollary, one can show that the rest of (a) results by combining parts (a) and (b) of the above Theorem 2.4 with the above statement of Theorem 1 of [23], by taking $A := S_{R \setminus M}$ (resp., by taking $A := \mathcal{S}_{R \setminus M}$) while also taking $B := T_{R \setminus M}$. In detail, note via parts (a) and (b) of Theorem 2.4 that $S_{R \setminus M}$ and $\mathcal{S}_{R \setminus M}$ are each two-dimensional going-down domains and that the conductors $(S_{R \setminus M} : T_{R \setminus M})$ and $(\mathcal{S}_{R \setminus M} : T_{R \setminus M})$ are each nonzero. This completes the proof of (a).

(b) The proof of (a) carries over if one instead takes $A := S_{R \setminus M}^+$ (resp., if one instead takes $A := \mathcal{S}_{R \setminus M}^+$).

(c) We will prove the assertion involving S , leaving to the reader the details of the (very) similar proof of the assertion that involves \mathcal{S} . By Lemma 2.2 (and the final comment before the statement of the present corollary), it suffices to show that $T_{R \setminus M}$ is a going-down domain for each $M \in \text{Max}(R)$ such that $\text{ht}_R(M) = 2$. As P is contained in the Jacobson radical of R , $P \subseteq M$. So, since $\text{ht}_R(P) < \text{ht}_R(M)$, we get that $P \subset M$. Since the “seminormal domain” property, the “ t -closed in” property and the “has finite fibers” property are each preserved under localization at $R \setminus M$, we can apply (b) to get that $T_{R \setminus M}$ is indeed a going-down domain (and hence, by virtue of also being seminormal, necessarily a locally divided domain).

(d) To prove the assertion, it is straightforward to adapt the proof of (c) by letting S^+ (resp., \mathcal{S}^+) play the role of S (resp., \mathcal{S}) in that earlier proof. This completes the proof of Corollary 2.6. \square

Some readers may have been led to wonder whether I had any significant reasons for formulating the statement of Corollary 2.6 so as to have four closely related parts. Here are two facts that may stimulate that question. First, in a couple of senses that are made precise in Remark 3.1 (a) of [3], “has finite fibers” is a local property of arbitrary ring extensions. In particular, for domains $D \subseteq E$, $D \subseteq E$ has finite fibers if and only if $D_M \subseteq E_{D \setminus M}$ has finite fibers for all $M \in \text{Max}(D)$. Second, for domains $D \subseteq E$, it is well known (and easy to prove) that D is t -closed in E if and only if D_M is t -closed in $E_{D \setminus M}$ for all $M \in \text{Max}(D)$. In response to the above question and the two facts that may seem to support it, I have the following three responses, the initial two of which may rise to the level of “significant reasons”. First, although only a single height 2 (hence maximal) prime ideal M of R was considered in parts (a) and (b) of Corollary 2.6, the proofs of parts (c) and (d) of Corollary 2.6 each needed to consider *every* maximal ideal of R . Second, carefully separating and delineating the presentation of similar items which may seem to exhibit only minor differences is often simply what mathematicians do for the sake of completeness and on behalf of posterity. Third, Aristotle would perhaps agree that too much of a good thing should be avoided. So, in order to expedite matters (and possibly also to leave some undiscussed tangential matters for any interested readers to discover), I have decided to formulate each of Corollary 5.7-5.9 so as to have only one part, namely, a part that focuses on the context of main interest in this paper (in fact, the context of Corollary 2.7 is the two-dimensional case of the context that was specified in the first and second sentences of the Abstract of this paper). Let me close by emphasizing an earlier parenthetical comment: readers are invited to augment the following presentations of Corollaries 2.7-2.9 in ways that emulate the style of the multi-part statement of Corollary 2.6.

Corollary 2.7. *Let R be an integrally closed two-dimensional going-down domain with quotient field K , let L be a field extension of K , and let T be the integral closure of R in the algebraic closure of K in L . Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R such that P is contained in the Jacobson radical of R . Suppose also that $(R+PT)^+$ is t -closed in T (resp., $R+\text{rad}_T(PT)$ is t -closed in T) and that $(R+PT)^+ \subseteq T$ (resp., $R+\text{rad}_T(PT) \subseteq T$) has finite fibers. Then T is a going-down domain (in fact, a locally divided domain).*

Proof. By Theorem 2.4 (d), $R + \text{rad}_T(PT)$ is seminormal. Moreover, any integrally closed domain (for instance, T) is seminormal. Hence, the assertion is a special case of Corollary 2.6 (d). The proof is complete. \square

Corollary 2.8. *Let R be an integrally closed two-dimensional going-down domain with quotient field K , let L be a field extension of K , and let T be the integral closure of R in the algebraic closure of K in L . Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R such that P is contained in the Jacobson radical of R . Suppose also that $(R + PT)^+$ is t -closed in T (resp., $R + \text{rad}_T(PT)$ is t -closed in T) and that $R \subseteq T$ has finite fibers. Then T is a going-down domain (in fact, a locally divided domain).*

Proof. Recall from Theorem 2.4 that $S := R + PT$ and from Theorem 2.4 (d) that $S := R + \text{rad}_T(PT)$ is seminormal (that is, $S^+ = S$). As $R \subseteq S \subseteq S^+ \subseteq S^+ (= S) \subseteq T$, it is evident that the assumption that $R \subseteq T$ has finite fibers ensures that the extensions $S^+ \subseteq T$ and $S \subseteq T$ each have finite fibers. Thus, Corollary 2.8 follows from Corollary 2.7. \square

We next present a result that includes the result that was stated in the next-to-last sentence of the Abstract of this paper.

Corollary 2.9. *Let R be an integrally closed two-dimensional going-down domain with quotient field K , let $K \subseteq L$ be a finite-dimensional (hence algebraic) field extension that is normal, and let T be the integral closure of R in L . Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R such that P is contained in the Jacobson radical of R . If either $(R + PT)^+$ or $R + \text{rad}_T(PT)$ is t -closed in T , then T is a going-down domain (in fact, a locally divided domain).*

Proof. Let G be the Galois group of L/K (that is, G is the set of K -algebra automorphisms of L). As R is integrally closed and L/K is a normal finite-dimensional field extension, it follows from Theorem 22 on page 289 of [58] that G acts transitively on the fiber (in $\text{Spec}(T)$) of any prime ideal of R . Since $[L : K] < \infty$, G is finite. Hence, any such fiber is finite; that is, $R \subseteq T$ has finite fibers. Moreover, $S^+ = S$ by Theorem 2.4 (d). Thus, Corollary 2.9 follows from Corollary 2.8. The proof is complete. \square

The above presentation of Corollaries 2.7-2.9 was especially focused. Interested readers are invited to augment the presentations of Corollaries 2.7-2.9 in ways that emulate the style of the multi-part statement of Corollary 2.6.

Corollaries 2.10 and 2.11 will present applications of Theorem 2.4 that (unlike the applications in Corollaries 2.6-2.9) do not impose an *a priori* upper bound on the (Krull) dimension of the base domain R . The following definition will be used in Corollaries 2.10 and 2.11. Recall that if D is a domain with quotient field k , then D is said to be a *finite-conductor domain* if $Da \cap Db$ is a finitely generated D -module for all $a, b \in D$ (equivalently, if $Du \cap Dv$ is a finitely generated D -module for all $u, v \in k$; equivalently, if $(D :_D Dw)$ is a finitely generated D -module for all $w \in k$). Perhaps the most familiar examples of finite-conductor domains are coherent domains (for instance, Noetherian domains, Pruefer domains, ...) and GCD-domains (also known as pseudo-Bézout domains, HCF-domains, ...). Numerous examples of finite-conductor (in fact, coherent) going-down domains are accessible: cf. Theorem 3 of [31]. Corollaries 2.10 and 2.11 will also use the following material from [16]. If D is a finite-conductor domain, then so is D_S for any multiplicatively closed subset S of D . A domain D is called a *locally finite-conductor domain* if D_N is a finite-conductor domain for each $N \in \text{Max}(D)$, equivalently (by the fact recalled in the preceding sentence), if D_Q is a finite-conductor domain for each $Q \in \text{Spec}(D)$. Another consequence of the fact that was recalled two sentences ago is that any finite-conductor domain is a locally finite-conductor domain; the converse holds for a quasi-semilocal domain.

Corollary 2.10 is essentially the result of combining Theorem 2.4 with Theorem 3.2 of [14]. Apart from changes in notation, the latter result states the following. If A is a locally finite-conductor domain and a locally divided domain, then any integral overring of A is a going-down domain.

Corollary 2.10. *Let $R \subseteq T$ be an integral extension of domains such that R is a going-down domain. Suppose also that for each $M \in \text{Max}(R)$ such that $\text{ht}_R(M) > 1$, there exists a height 1 prime ideal \mathcal{P} of R such that $\mathcal{P} \subset M$ (with \mathcal{P} possibly depending on M). Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R . Put $S := R + PT$ and $\mathcal{S} := R + \text{rad}_T(PT)$. Then:*

(a) *Let M be any maximal ideal of R that properly contains P . Suppose also that $S_{R \setminus M}$ (resp., $\mathcal{S}_{R \setminus M}$) is a locally finite-conductor domain and a locally divided domain. Then $T_{R \setminus M}$ is a going-down domain.*

(b) *Suppose that P is contained in the Jacobson radical of R . Suppose also that S (resp., \mathcal{S}) is both a finite-conductor domain and a locally divided domain. Then T is a going-down domain.*

(c) *Suppose that R is integrally closed and P is contained in the Jacobson radical of R . Suppose also that \mathcal{S} is a locally finite-conductor domain (for instance, a finite-conductor domain) and that T is seminormal. Then T is a going-down domain (in fact, a locally divided domain).*

Proof. (a) It suffices to combine parts (a) and (b) of Theorem 2.4 with the above statement of Theorem 3.2 of [14], by taking $A := S_{R \setminus M}$ (resp., by taking $A := \mathcal{S}_{R \setminus M}$) while also taking $B := T_{R \setminus M}$. (In detail, note via parts (a) and (b) of Theorem 2.4 that $S_{R \setminus M}$ and $\mathcal{S}_{R \setminus M}$ are each going-down domains, and also that $T_{R \setminus M}$ is an (integral) overring of each of these domains because the conductors $(S_{R \setminus M} : T_{R \setminus M})$ and $(\mathcal{S}_{R \setminus M} : T_{R \setminus M})$ are each nonzero.) This completes the proof of (a).

(b) By Lemma 2.2, it suffices to show that $T_{R \setminus M}$ is a going-down domain for each $M \in \text{Max}(R)$ such that $\text{ht}_R(M) \geq 2$. As P is contained in the Jacobson radical of R , $P \subseteq M$. So, since $\text{ht}_R(P) < \text{ht}_R(M)$, we get that $P \subset M$. By an earlier comment, the “finite-conductor domain” property of S (resp., of \mathcal{S}) ensures the “locally finite-conductor” domain property of $S_{R \setminus M}$ (resp., of $\mathcal{S}_{R \setminus M}$). Moreover, it was noted in Remark 2.7 (b) of [15] that any ring of fractions of a locally divided domain is a locally divided domain. (I believe that I should insert the following explanation of a historical nature here. The just-mentioned remark from [15] used the term “localization” instead of “ring of fractions” and, for details on how to prove that the “locally divided domain” property is preserved under passage to a “localization”, that remark from [15] referred the reader to the method of proof of Lemma 2.2 (a) of [15]. A reading of *that* proof makes clear that I had been using the term “localization” instead of “ring of fractions” throughout [15]. In doing so, I was emulating the usage of many arithmetic algebraic geometers of that era, but (as I learned in the late 1970s) *not* the usage of many multiplicative ideal theorists of that era. In short: when [15] was written, I had not completed my transition from being an algebraic geometer specializing in applications to algebraic number theory to being a commutative ring theorist.) An application of (a) now shows that $T_{R \setminus M}$ is a going-down domain.

(c) By parts (a) and (b) of Theorem 2.4, T is an integral overring of \mathcal{S} . So, in view of Theorem 3.2 of [14], the assertion that T is a going-down domain will follow if we show that \mathcal{S} is a locally divided domain. By parts (c) and (d), respectively, of Theorem 2.4, we get that \mathcal{S} is a going-down domain and seminormal. Hence, the conclusion that \mathcal{S} is a locally divided domain follows from the fact that any seminormal going-down domain is a locally divided domain (cf. the proof of Corollary 2.8 of [15]). That same fact also dispatches the final parenthetical assertion. The proof is complete. \square

We next give a partner for (i.e., an analogue of) Corollary 2.7.

Corollary 2.11. *Let R be an integrally closed going-down domain with quotient field K . Suppose also that for each $M \in \text{Max}(R)$ such that $\text{ht}_R(M) > 1$, there exists a height 1 prime ideal \mathcal{P} of R such that $\mathcal{P} \subset M$ (with \mathcal{P} possibly depending on M). Let L be a field extension of K , and let T be the integral closure of R in the algebraic closure of K in L . Let $P \in \text{Spec}(R)$ be a height 1 nonmaximal prime ideal of R . Suppose also that $(R + PT)^+$ (resp., $R + \text{rad}_T(PT)$) is a locally finite-conductor domain. Then T is a going-down domain (in fact, a locally divided domain).*

Proof. As in the proof of Corollary 2.7, $R + \text{rad}_T(PT)$ is seminormal. Moreover, $(R + PT)^+$ is seminormal (as is true of the seminormalization of any domain) and T is seminormal (as is true of any integrally closed domain). So, an application of Theorem 3.2 of [14] gives that T is a going-down

domain. Finally, the parenthetical assertion holds by another use of the fact that any seminormal going-down domain is locally divided. The proof is complete. \square

To a large extent, the above work was predicated on beginning with a height 1 nonmaximal prime ideal P of a base domain R and then considering some maximal ideal M of R that properly contains P . Proposition 2.13 will give a taste of an alternate approach, where one begins with a maximal ideal M of height at least 2 of a base domain R and then considers some height 1 prime ideal P of R that is contained in M . Before touching on that alternate point of view, we pause for the following multi-part remark which is especially pertinent to the above items 2.10-2.11.

Remark 2.12. (a) The preceding two corollaries presented only two analogues of (i.e., partners for) Corollaries 2.6-2.9. We next address what may have seemed like disparate treatments for the contexts surrounding those two sets of corollaries. The fuller treatment in Corollaries 2-6-2.9 was prompted by the following two reasons: these corollaries gave this paper's first opportunity to indicate the scope of possible applications of Theorem 2.4; and it seemed likely (to me) that this set of corollaries would be of greater interest to readers, in view of the rather extensive literature involving t -closure, as well as the fact that the (admittedly less extensive) literature on the "finite-conductor domain" property is well connected to several concepts of widespread interest. On the other hand, we felt that Corollaries 2.10 and 2.11 sufficed here to illustrate applications involving (possibly locally) finite-conductor domains because of the extremely sparse literature on that class of domains.

I am uncertain whether one should be optimistic about the possibility of more explicit applications of Corollaries 2.6-2.9 being developed in the future. It is significant and lamentable that at present, there are very few tractable known sufficient conditions for (to use some notation from items 2.4-2.11) pullbacks such as $S := R + PT$ or its ring of fractions relative to $R \setminus M$ to be (possibly locally) finite-conductor domains. The technical difficulty in such considerations is that the common ideal PT seems unlikely to be a radical ideal.

(b) Many known results that one might initially consider as candidates for being applied to pullbacks such as the above S impose homological assumptions that, in our opinion, are not appropriate to most work on the ascent of going-down domains. In this regard, consider the hypotheses in results such as Theorems 4.4 and 4.7 in [10]. If one could apply the first of these results to the ring extension $S \subseteq T$ (for the context assumed in Theorem 2.4), then $S = T$ (the point being that no proper overring of a domain can be both flat and integral); if one could apply the second of these results to the data and context of Theorem 2.4, then it follows at once from [12] that T is a Pruefer domain. To be fair, the authors of [10] did not express any interest in applying their results to studying ascent of going-down domains under integral extensions. The remark is complete.

It seems natural to ask if one could develop analogues of the parts of Theorem 2.4 where we assumed that P is contained in the Jacobson radical of R . One purpose of Proposition 2.13 is to answer that question affirmatively. Another purpose of Proposition 2.13 is to show that one can develop analogues of some parts of Theorem 2.4 whose conclusions had concerned $T_{R \setminus M}$ by finding new hypotheses that imply the corresponding (but stronger) conclusions about T . Both of those purposes will be achieved as a result of the following two observations. The results in Theorem 2.4 were largely predicated on behavior related to a single height 1 nonmaximal prime ideal P of R (which may not be contained in each maximal ideal of R); and many of the ring-theoretic properties that are of interest here are local properties of domains. These observations suggest taking a new point of view, namely, developing results predicated on behavior related to a single maximal ideal M of R having height at least 2. Such an approach will be effective in Proposition 2.13 and in parts of Remark 2.14 because each such M will allow us to use facts from Theorem 2.4 about the domains that were denoted by " S " and " \mathcal{S} " in Theorem 2.4 (where those domains were defined in terms of the (necessarily unique) nonmaximal height 1 prime ideal P of (the going-down domain, hence treed) domain R such that $P \subset M$).

Let R and T be as in the riding hypotheses in Theorem 2.4, and let P be a height 1 nonmaximal prime ideal of R . For the purposes of presenting the “ M before P ” theme in Proposition 2.13 and Remark 2.14, it will be convenient to enhance some notation that was introduced and used extensively earlier. The notation that is being introduced in this paragraph will be used in the statement and the proof of Proposition 2.13. The rings that had been denoted by S and \mathcal{S} will henceforth be denoted by $S(P)$ and $\mathcal{S}(P)$, respectively. In other words,

$$S(P) := R + PT \text{ and } \mathcal{S}(P) := R + \text{rad}_T(PT).$$

Furthermore, if M is a maximal ideal of R that properly contains a (necessarily uniquely determined) height 1 prime ideal P of R , it will also be convenient to write $P(M) := P$ and to enhance the earlier notations A and \mathcal{A} as follows:

$$A(M) := (S(P(M)))_{R \setminus M}^+ \text{ and } \mathcal{A}(M) := (\mathcal{S}(P(M)))_{R \setminus M}^+$$

Also, if $M \in \text{Max}(R)$ such that $\text{ht}_R(M) = 1$, it will be convenient to let $P(M) := M$ and to extend the just-displayed formal definitions of $A(M)$ and $\mathcal{A}(M)$ so as to apply to all nonzero maximal ideals M of R (that is, to all maximal ideals M of R except in the uninteresting case where R is a field).

Proposition 2.13. *Let $R \subseteq T$ be an integral extension of domains such that R is a going-down domain but not a field. Suppose also that for each $M \in \text{Max}(R)$ such that $\text{ht}_R(M) > 1$, there exists a (necessarily uniquely determined) height 1 prime ideal $P =: P(M)$ of R such that $P \subset M$; for each height 1 maximal ideal M of R , put $P(M) := M$. If $M \in \text{Max}(R)$, put $B(M) := A(M)$ and $\mathcal{B}(M) := \mathcal{A}(M)$; that is,*

$$B(M) := (S(P(M)))_{R \setminus M}^+ \text{ and } \mathcal{B}(M) := (\mathcal{S}(P(M)))_{R \setminus M}^+, \text{ where}$$

$$S(P(M)) := R + P(M)T \text{ and } \mathcal{S}(P(M)) := R + \text{rad}_T(P(M)T).$$

Then:

(a) Let $M \in \text{Max}(R)$. Then $S(P(M)) \subseteq \mathcal{S}(P(M))$, with equality if and only if $\text{rad}_T(P(M)T) = P(M)T$; $S(P(M))$ and $\mathcal{S}(P(M))$ are each domains with the same (Krull) dimension as R ; $S(P(M))$ and $\mathcal{S}(P(M))$ are each underrings of T ; and $S(P(M))' = T' = \mathcal{S}(P(M))'$.

(b) Let $M \in \text{Max}(R)$. Put $n := \text{ht}_R(M)$ (with $1 \leq n \leq \infty$). Then

$$S(P(M))_{R \setminus M} = R_M + (P(M)R_M)T_{R \setminus M}$$

and

$$\mathcal{S}(P(M))_{R \setminus M} = R_M + (\text{rad}_T(P(M)T))T_{R \setminus M}$$

are each quasi-local n -dimensional going-down domains with exactly $n + 1$ (pairwise distinct) prime ideals apiece (where, as usual, $\infty + 1 := \infty$). Furthermore,

$$(S(P(M))_{R \setminus M} : T_{R \setminus M}) \supseteq P(M)T_{R \setminus M} \supset 0 \text{ and}$$

$$(\mathcal{S}(P(M))_{R \setminus M} : T_{R \setminus M}) \supseteq (\text{rad}_T(P(M)T))T_{R \setminus M} \supset 0.$$

Moreover,

$$B(M) = (R_M + (P(M)R_M)T_{R \setminus M})^+ = (R + P(M)T)_{R \setminus M}^+ = \mathcal{S}(P(M))_{R \setminus M}^+$$

and

$$\begin{aligned} \mathcal{B}(M) &= (R_M + (\text{rad}_T(P(M)T))T_{R \setminus M})^+ = \\ &= (R + \text{rad}_T(P(M)T))_{R \setminus M}^+ = \mathcal{S}(P(M))_{R \setminus M}^+ \end{aligned}$$

are each n -dimensional seminormal divided domains (hence locally divided domains and going-down domains) which are underrings of $T_{R \setminus M}$ such that $B(M)' = T'_{R \setminus M} = \mathcal{B}(M)'$,

$$(B(M) : T_{R \setminus M}) \supseteq P(M)T_{R \setminus M} \supseteq P(M)T \supset 0,$$

and

$$(\mathcal{B}(M) : T_{R \setminus M}) \supseteq (\text{rad}_T(P(M)T))T_{R \setminus M} \supseteq \text{rad}_T(P(M)T) \supset 0.$$

(c) Let $M \in \text{Max}(R)$. Suppose also that for each $N \in \text{Max}(R)$ such that $\text{ht}_R(N) \geq 2$ and $P(M) \not\subseteq N$, one has that $T_{R \setminus N}$ is a going-down domain. Then both $S(P(M))$ and $\mathcal{S}(P(M))$ are going-down domains. Consequently, being seminormal going-down domains, both $B(M)$ and $\mathcal{B}(M)$ are locally divided domains.

(d) Let $M \in \text{Max}(R)$. Suppose also that R is integrally closed and T is seminormal. As above, consider $\mathcal{S}(P(M)) := R + \text{rad}_T(P(M)T)$. Then $\mathcal{S}(P(M))$ is seminormal, that is, $\mathcal{S}(P(M))^+ = \mathcal{S}(P(M))$; and so, passing to rings of fractions with respect to the multiplicatively closed set $R \setminus M$, we get that $\mathcal{B}(M) = \mathcal{S}(P(M))_{R \setminus M}$.

Proof. It may be helpful to begin by augmenting the reasoning from the proof of Theorem 2.4 (a) by explaining why $\text{rad}_T(MT) \cap R = M$ for each $M \in \text{Max}(R)$. For that purpose, as one inclusion is clear and M is a maximal ideal of R , we need only explain why $1 \notin \text{rad}_T(MT)$, that is, why $1 \notin MT$. That, in turn, holds because the integrality of the extension $R \subseteq T$ ensures that at least one prime ideal of T lies over M . With this detail now proven, the reader can check that if $P = P(M) \subset M$, the arguments given for the relevant assertions in the proof of parts (a) and (b) of Theorem 2.4 carry over essentially *verbatim* to prove parts (a) and (b) of the present result in case $\text{ht}_R(M) > 1$. For the sake of completeness, we will provide some additional details about this in the next paragraph.

Suppose first that $\text{ht}_R(M) > 1$. Then $B(M)$ and $\mathcal{B}(M)$ are, respectively, the domains that were denoted by $A(M)$ and $\mathcal{A}(M)$ in Theorem 2.4 (where the implicit nonmaximal height 1 prime ideal P of R that was used in the constructions of $A(M)$ and $\mathcal{A}(M)$ in Theorem 2.4 is taken here to be $P := P(M)$). Accordingly, all the stated equivalent descriptions of $B(M)$ and $\mathcal{B}(M)$ and all the assertions in (a) and (b) follow at once from parts (a) and (b) of Theorem 2.4. (In checking the preceding assertion, one should bear in mind that for any domain D , the extension $D \subseteq D^+$ is integral and unbranched, so that $B(M)$ and $\mathcal{B}(M)$, respectively, inherit from

$$R_M + (P(M)R_M)T_{R \setminus M} \text{ and } R_M + (\text{rad}_T(P(M)T))T_{R \setminus M}$$

the property of being a quasi-local n -dimensional going-down domain with exactly $n + 1$ (pairwise distinct) prime ideals.) Furthermore, being seminormal quasi-local going-down domains, $B(M)$ and $\mathcal{B}(M)$ are each divided domains, by Corollary 2.6 of [15]. Consequently, in the proofs that are given below for (a) and (b), we can assume, without loss of generality, that $\text{ht}_R(M) = 1$.

(a), (b): As explained in the preceding paragraph, to prove the assertions in (the rest of the proof of) (a) and (b), we need only consider the case where $n := \text{ht}_R(M) = 1$ (that is, the case where $P(M) = M$). For this case, the assertions about conductors in the statement of (b) follow via the same method of proof that was given for the similar assertions in Theorem 2.4 (a). Hence, $S(P(M))$ and $\mathcal{S}(P(M))$ are each underrings of T , and it follows that $S(P(M))' = T' = \mathcal{S}(P(M))'$. This completes the proof of (a).

We turn next to the rest of the proof of (b). As explained above, it remains only to prove (b) in case the height of M in R is 1.

As $n = \text{ht}_R(M) = 1$, we have $\dim(R_M) = 1$. Since $R_M \subseteq T_{R \setminus M}$ inherits integrality from $R \subseteq T$, we get that $\dim(T_{R \setminus M}) = \dim(R_M) = 1$ (cf. Theorem 48 of [41]). Thus, since $T_{R \setminus M}$ is integral over both $S(P(M))_{R \setminus M}$ and $\mathcal{S}(P(M))_{R \setminus M}$, all of these domains are one-dimensional and, hence, going-down domains.

We now embark on the most difficult part of this proof, namely, the proof that $S(P(M))_{R \setminus M}$ and $\mathcal{S}(P(M))_{R \setminus M}$ are each quasi-local. As we proceed to adapt the reasoning in the most difficult part of the proof of Theorem 2.4 (b), the reader is advised to bear in mind that the current $P(M)R_M$ is *not*

a nonmaximal prime ideal of R_M ; indeed, the current $P(M)R_M$ is the only nonzero prime ideal of R_M . Addressing this situation will require an approach that differs from the reasoning in the proof of Theorem 2.4 (b) in some technically significant ways. Nevertheless, that reasoning *does* adapt well enough to allow us to replace the 4-tuple $(R, M, P(M), T)$ with the 4-tuple $(R_M, MR_M, MR_M, T_{R \setminus M})$ while seeking a proof that $S(P(M))_{R \setminus M}$ (now replaced by $S(M)$) is quasi-local (necessarily with exactly two pairwise distinct prime ideals). As in the proof of Theorem 2.4 (b) (cf. Corollary 5.8 of [26]), it will be enough to show that the integral extension $R \subseteq S(P(M))$ (that is, the extension $R \subseteq S(M)$) is unbranched. That proof will occupy the next three paragraphs. By the way, note that $S(P(M))_{R \setminus M}$ being quasi-local will imply that $S(P(M))_{R \setminus M}$ is also quasi-local, as a consequence of a well known property of integral extensions (cf. Corollary 5.8 of [2]).

As in the proof of Theorem 2.4 (b), view $\mathcal{S} := S(P(M)) = S(M)$ as the pullback of the canonical surjection $T \rightarrow T/\text{rad}_T(MT)$ and the canonical injection

$$R/M = R/(\text{rad}_T(MT) \cap R) \cong \mathcal{S}/\text{rad}_T(MT) \rightarrow T/\text{rad}_T(MT).$$

By Theorem 1.4 of [36], when $\text{Spec}(\mathcal{S})$ is endowed with the Zariski topology, it can be viewed, up to homeomorphism (and hence as a poset under inclusion), as a certain quotient space of the disjoint union of $\text{Spec}(T)$ and $\text{Spec}(R/M)$. That quotient space is determined by subjecting that disjoint union to a certain equivalence relation. That equivalence relation is generated by requiring every prime ideal Q of T which contains $\text{rad}_T(MT)$ (that is, every prime ideal of T that lies over M) to be identified with the unique point of $\text{Spec}(R/M)$. It remains to show that the resulting “gluing” produces precisely two (pairwise distinct) equivalence classes.

The key to proving the above *desideratum* is to show that 0 is the only prime ideal Q of T which does not contain M . This, in turn, holds by virtue of the integrality of $R \subseteq T$ and the fact that R is quasi-local and one-dimensional, because any such Q satisfies $0 \subseteq Q \cap R = Q \cap M \subset M$. It follows that $|\text{Spec}(\mathcal{S})| \leq 2$ ($= n + 1$). We next sharpen this inequality to an equation.

We claim that $|\text{Spec}(\mathcal{S})| = 2$. Suppose that the claim fails. Then $|\text{Spec}(\mathcal{S})| = 1$, whence the domain \mathcal{S} is a field, whence the integrality of the extension $R \subseteq \mathcal{S}$ forces R to be a field (cf. Proposition 5.7 of [2]), contrary to hypothesis. This proves the claim that $|\text{Spec}(\mathcal{S})| = 2$. Next, note that, by the LO aspect of integrality, the contraction map $c_R^{\mathcal{S}} : \text{Spec}(\mathcal{S}) \rightarrow \text{Spec}(R)$ is surjective. Since $|\text{Spec}(\mathcal{S})| = 2 = |\text{Spec}(R)| < \infty$, it follows from the Pigeonhole Principle that $c_R^{\mathcal{S}}$ is a bijection. Thus, $R \subseteq \mathcal{S}$ is an integral unbranched extension, and hence an order isomorphism of posets under inclusion. In particular, since we have reduced to the case where R is quasi-local, \mathcal{S} must be quasi-local. This completes the proof that in the more general case, where $n = 1$ but R need not be quasi-local, $S(P(M))_{R \setminus M}$ and $S(P(M))_{R \setminus M}$ are each quasi-local.

Finally, we note that the proofs of the assertions in the statement of (b) concerning conductors, $B(M)$ and $\mathcal{B}(M)$ now carry over essentially *verbatim* to the case $n = 1$. This completes the proof of (b).

(c) We will first prove that $S(P(M))$ is a going-down domain, leaving the details of the (very) similar proof of the corresponding assertion about $S(P(M))$ to the reader. Since (a) ensures that the domain $S(P(\mathcal{N}))$ is an integral extension of R for each $\mathcal{N} \in \text{Max}(R)$, it follows, in view of the implication (4) \Rightarrow (1) in Lemma 2.2, that it will suffice to show that $S(P(M))_{R \setminus N}$ is a going-down domain for each $N \in \text{Max}(R)$ such that $\text{ht}_R(N) \geq 2$. By (b), $S(P(M))_{R \setminus M}$ is a going-down domain. Thus, we can assume, without loss of generality, that $N \neq M$. Consider $P := P(M)$. If $P \not\subseteq N$, then $PR_N = R_N$, and so

$$S(P(M))_{R \setminus N} = (R + PT)_{R \setminus N} = R_N + PR_N T_{R \setminus N} = R_N T_{R \setminus N} = T_{R \setminus N},$$

which is, by hypothesis, a going-down domain. Thus, without loss of generality, $P \subseteq N$. Hence, $P = P(N)$ (since R is treed); that is, $P(M) = P(N)$. Therefore $S(P(M)) = S(P(N))$, whence

$$S(P(M))_{R \setminus N} = S(P(N))_{R \setminus N},$$

which, by (b), is a going-down domain. This completes the proof that $S(P(M))$ is a going-down domain.

Finally, we address the ‘‘Consequently’’ assertions. Recall (from Lemma 2.1 of [13]) that any domain that is a ring of fractions of a going-down domain must be a going-down domain. So, in view of the above, we get that $S(P(M))_{R \setminus M}$ and $\mathcal{S}(P(M))_{R \setminus M}$ are going-down domains. Moreover, $B(M)$ and $\mathcal{B}(M)$ (which are, respectively, the seminormalizations of these two domains) inherit the property of being going-down domains, by Lemma 2.3 of [15], the point being that $D \subseteq D^+$ is an integral unbranched extension for any domain D . Finally, since D^+ is seminormal for any domain D , it now suffices to recall that any seminormal going-down domain is locally divided (by any of the following: the proof of Corollary 2.8 of [15], Corollary 3.6 of [4], Theorem 5.10 of [26], the Theorem in Remark 5.18 (f) of [26]).

(d) If $\text{ht}_R(M) > 1$, the proof of Theorem 2.4 (d) applies here without change. In the remaining case, $P := P(M) = M$ is a height 1 maximal ideal of R . The just-mentioned proof applies to this case as well, where we find, since integral extensions satisfy LO (or, alternatively, use the final comment in the second paragraph of the proof of Theorem 2.4), that

$$\text{rad}_T(P(M)T) \cap R = \text{rad}_T(MT) \cap R = M,$$

whence $R/(\text{rad}_T(P(M)T) \cap R) = R/M$, which is a field and, hence, an integrally closed domain. The proof is complete. \square

Before offering some corollaries of Proposition 2.13, we pause to collect some comments.

Remark 2.14. (a) Readers will surely have noticed that the statements and proofs of parts (a), (b) and (d) of Proposition 2.13 were very much in the spirit of the correspondingly numbered parts of Theorem 2.4, while the statements and proofs of Proposition 2.13 (c) differed significantly from the statements and proofs of Theorem 2.4 (c). We next address the latter situation further. One could express the opinion that Proposition 2.13 (c) generalizes Theorem 2.4 (c), because it is straightforward to deduce the statement of Theorem 2.4 (c) from the statement of Proposition 2.13 (c). On the other hand, there is a tradition whereby some mathematicians would dispute the ‘‘generalization’’ conclusion because some material that is ‘‘nearby’’ Theorem 2.4 (c) was explicitly used in the proof of Proposition 2.13 (c). While not joining either side of that dispute here, I wish to point out that one could express the opinion that Proposition 2.13 (c) is a *strict generalization* of Theorem 2.4 (c), because it is possible to produce an integral extension of domains $R \subseteq T$ which satisfies the hypotheses of Proposition 2.13 (c) while *not* satisfying the hypotheses of Theorem 2.4 (c).

Perhaps the easiest example of such an extension $R \subseteq T$ can be constructed as follows. Take R to be a two-dimensional Pruefer domain with exactly two distinct maximal ideals, say M and N , $\text{ht}_R(M) = 2$ and $\text{ht}_R(N) = 1$. Note that in any such example, the unique nonmaximal height 1 prime ideal of R (let us denote it by P) is contained in M but is not contained in N , so $P(M) = P$. Moreover, if one takes T to be the integral closure of R in some algebraic field extension of a quotient field of R , then Theorem 2.4 (c) is not applicable to the extension $R \subseteq T$, because neither P nor N is contained in the Jacobson radical of R . (The fact that the hypotheses of part (c) of Proposition 2.13 did not need to address the Jacobson radical of R has led me to conclude that Proposition 2.13 (c) is the most innovative part of Proposition 2.13.) One way to construct an example of such a domain R , which is in fact a Pruefer domain, is to apply Theorem 3.1 of [43] (by arranging that $\text{Spec}(R)$ is order isomorphic to the poset $\{*, p, m, n\}$ such that $* < p < m$ and $* < n$).

Some readers may comment that the domain T in the above construction/example is a Pruefer domain (by Pruefer’s classic ascent result [51]). To that comment, I have the following two replies: you are correct; and your comment does not affect the assertions in the preceding paragraph to the effect that I have produced an example showing that Proposition 2.13 (c) is stronger than (even if you may not view it as a strict generalization of) Theorem 2.4 (c).

(b) In order to keep an eye on the main context of interest here, while being mindful of the size of this project, I have decided not to tilt at every windmill that may seem to relate to entities such as the above-mentioned $\mathcal{B}(M)$ and/or $A(M)$. Nevertheless, I would encourage interested readers to search for roles for such potential protagonists, possibly while universally quantifying over (some or all) maximal ideals M of the base domain R . I am confident that such searches through the statements of the various parts of Corollary 2.15 will identify at least one such windmill that combines well with the implication (4) \Rightarrow (1) in Lemma 2.2. Having motivated Corollary 2.15 by suggesting an activity for the reader that is sure to identify at least one result that is *not* explicitly stated in this article, I now declare that this remark is complete.

Grosso modo, one can obtain proofs of Corollaries 2.15-2.20 by using Proposition 2.13 (instead of Theorem 2.4), in conjunction with Theorem 1 of [23] and Theorem 3.2 of [14], to tweak the proofs of Corollaries 2.6-2.11. For the sake of completeness, we provide some detail for the proofs of the following six corollaries.

Recall that Theorem 1 of [23] states, apart from changes in notation, the following. If A is a two-dimensional going-down domain and B is a seminormal integral overring of A such that A is t -closed in B , $(A : B) \neq 0$, and the ring extension $A \subseteq B$ has finite fibers, then B is a going-down domain.

Corollary 2.15. *Let $R \subseteq T$ be an integral extension of domains such that R is a two-dimensional going-down domain. For each $\mathcal{M} \in \text{Max}(R)$, let $P(\mathcal{M})$ denote the (necessarily unique) height 1 prime ideal of R that is contained in \mathcal{M} , with $S(P(\mathcal{M})) := R + P(\mathcal{M})T$ and $\mathcal{S}(P(\mathcal{M})) := R + \text{rad}_T(P(\mathcal{M})T)$. Then:*

(a) *Let $M \in \text{Max}(R)$ and suppose that $T_{R \setminus M}$ is seminormal. Suppose also that $S(P(M))_{R \setminus M}$ (resp., $\mathcal{S}(P(M))_{R \setminus M}$) is t -closed in $T_{R \setminus M}$, and that the ring extension $S(P(M))_{R \setminus M} \subseteq T_{R \setminus M}$ has finite fibers (resp., and that the ring extension $\mathcal{S}(P(M))_{R \setminus M} \subseteq T_{R \setminus M}$ has finite fibers). Then $T_{R \setminus M}$ is a going-down domain (in fact, a locally divided domain).*

(b) *Let $M \in \text{Max}(R)$ and suppose that $T_{R \setminus M}$ is seminormal. Suppose also that $S(P(M))_{R \setminus M}^+$ (resp., $\mathcal{S}(P(M))_{R \setminus M}^+$) is t -closed in $T_{R \setminus M}$, and that the ring extension $S(P(M))_{R \setminus M}^+ \subseteq T_{R \setminus M}$ has finite fibers (resp., and that the ring extension $\mathcal{S}(P(M))_{R \setminus M}^+ \subseteq T_{R \setminus M}$ has finite fibers). Then $T_{R \setminus M}$ is a going-down domain (in fact, a locally divided domain).*

(c) *Let $M \in \text{Max}(R)$. Suppose also that for each $N \in \text{Max}(R)$ such that $\text{ht}_R(N) = 2$ and $P(M) \not\subseteq N$, one has that $T_{R \setminus N}$ is a going-down domain. Also, suppose that T is seminormal, $S(P(M))$ (resp., $\mathcal{S}(P(M))$) is t -closed in T , and the ring extension $S(P(M)) \subseteq T$ has finite fibers (resp., and the ring extension $\mathcal{S}(P(M)) \subseteq T$ has finite fibers). Then T is a going-down domain (in fact, a locally divided domain).*

Proof. For each $\mathcal{M} \in \text{Max}(R)$, $P(\mathcal{M})$ is uniquely determined because the domain R is treed and not zero-dimensional.

(a), (b): It is straightforward to check that the rest of the proof of (a) and (b) results by adapting the proofs of parts (a) and (b) of Corollary 2.6 (a) by replacing the earlier references to parts (a) and (b) of Theorem 2.4 with references to parts (a) and (b) of Proposition 2.13. Note, in particular, that the third paragraph of the proof of Proposition 2.13 observed that the relevant conductors could be shown to be nonzero by adapting the appropriate part of the proof of Theorem 2.4.

(c) The final parenthetical assertion follows from the oft-mentioned fact that all seminormal going-down domains are locally divided.

By Lemma 2.2, it suffices to prove that if $N \in \text{Max}(R)$ is such that $\text{ht}_R(N) = 2$, then $T_{R \setminus N}$ is a going-down domain. By hypothesis, we can restrict attention to those N which also satisfy $P := P(M) \subseteq N$. The case $N = M$ is handled by appealing to (a), in view of the fact that all the relevant properties are preserved when one passes to the relevant rings of fractions determined by the multiplicatively closed set $R \setminus M$. Thus, without loss of generality, $N \neq M$. Then, as in the proof of Proposition 2.13 (c), we get that $P = P(N)$, $P(M) = P(N)$, and $S(P(M)) = S(P(N))$ is a going-down domain (and, similarly, that $\mathcal{S}(P(M)) = \mathcal{S}(P(N))$ is a going-down domain). Hence, we have the following consequence of

Theorem 1 of [23] (supported by additional facts inferred from Proposition 2.13). Since $S(P(M))_{R \setminus N}$ is a two-dimensional going-down domain (by Proposition 2.13 (c)) and $T_{R \setminus N}$ is a seminormal integral overring of $S(P(M))_{R \setminus N}$ such that $S(P(M))_{R \setminus N}$ is t -closed in $T_{R \setminus N}$, $(S(P(M))_{R \setminus N} : T_{R \setminus N}) \neq 0$, and the ring extension $S(P(M))_{R \setminus N} \subseteq T_{R \setminus N}$ has finite fibers, then $T_{R \setminus N}$ is a going-down domain. The proof is complete. \square

We next give what is perhaps the most important application of Corollary 2.15.

Corollary 2.16. *Let R be an integrally closed two-dimensional going-down domain with quotient field K , let L be a field extension of K , and let T be the integral closure of R in the algebraic closure of K in L . Let $M \in \text{Max}(R)$. Let $P(M)$ denote the (necessarily unique) height 1 prime ideal of R that is contained in M , with $\mathcal{S}(P(M)) := R + \text{rad}_T(P((M))T)$. Suppose also that for each $N \in \text{Max}(R)$ such that $\text{ht}_R(N) = 2$ and $P(M) \not\subseteq N$, one has that $T_{R \setminus N}$ is a going-down domain. Also, suppose that $\mathcal{S}(P(M))$ is t -closed in T , and the ring extension $\mathcal{S}(P(M)) \subseteq T$ has finite fibers. Then T is a going-down domain (in fact, a locally divided domain).*

Proof. The assertion is a special case of Corollary 2.15 (c), since any integrally closed domain (for instance, T) is seminormal. \square

I believe that the following variant of Corollary 2.16 warrants mention.

Corollary 2.17. *Let R be an integrally closed two-dimensional going-down domain with quotient field K , let L be a field extension of K , and let T be the integral closure of R in the algebraic closure of K in L . Let $M \in \text{Max}(R)$. Let $P(M)$ denote the (necessarily unique) height 1 prime ideal of R that is contained in M , with $\mathcal{S}(P(M)) := R + P(M)T$ and $\mathcal{S}(P(M)) := R + \text{rad}_T(P((M))T)$. Suppose also that for each $N \in \text{Max}(R)$ such that $\text{ht}_R(N) = 2$ and $P(M) \not\subseteq N$, one has that $T_{R \setminus N}$ is a going-down domain. Also, suppose that $\mathcal{S}(P(M))^+$ is t -closed in T (resp., $\mathcal{S}(P(M))^+$ is t -closed in T) and that $R \subseteq T$ has finite fibers. Then T is a going-down domain (in fact, a locally divided domain).*

Proof. Since $\mathcal{S}(P(M))^+ = \mathcal{S}(P(M))$ by Proposition 2.13 (d), one can assert that Corollary 2.17 now admits the same kind of proof that was given for Corollary 2.16. Indeed, as $R \subseteq \mathcal{S}(P(M)) \subseteq \mathcal{S}(P(M))^+ \subseteq \mathcal{S}(P(M))^+ (= \mathcal{S}(P(M))) \subseteq T$ is a chain of integral extensions (each of which necessarily satisfies LO), it is evident that the assumption that $R \subseteq T$ has finite fibers ensures that the extensions $\mathcal{S}(P(M))^+ \subseteq T$ and $\mathcal{S}(P(M)) \subseteq T$ each have finite fibers. The proof is complete. \square

Corollary 2.18. *Let R be an integrally closed two-dimensional going-down domain with quotient field K , let $K \subseteq L$ be a finite-dimensional (hence algebraic) field extension that is normal, and let T be the integral closure of R in L . Let $M \in \text{Max}(R)$. Let $P(M)$ denote the (necessarily unique) height 1 prime ideal of R that is contained in M , with $\mathcal{S}(P(M)) := R + P(M)T$ and $\mathcal{S}(P(M)) := R + \text{rad}_T(P((M))T)$. Suppose also that for each $N \in \text{Max}(R)$ such that $\text{ht}_R(N) = 2$ and $P(M) \not\subseteq N$, one has that $T_{R \setminus N}$ is a going-down domain. Also, suppose that $\mathcal{S}(P(M))^+$ is t -closed in T (resp., $\mathcal{S}(P(M))^+$ is t -closed in T). Then T is a going-down domain (in fact, a locally divided domain).*

Proof. To show that Corollary 2.18 is a special case of Corollary 2.17, it will suffice to show that $R \subseteq T$ has finite fibers. This, in turn, follows, precisely as in the proof of Corollary 2.9, from the assumption that L/K is a finite-dimensional normal field extension. The proof is complete. \square

In the interest of not belaboring some points, the above presentation of Corollaries 2.16-2.18 was especially focused on the most important context. That fact leads me to emphasize an analogue of an earlier comment. Interested readers are invited to augment the presentations of Corollaries 2.16-2.18 in ways that emulate the style of the multi-part statement of Corollary 2.15.

We next present two applications of Proposition 2.13 that do not restrict the Krull dimension of the base domain R . Corollary 2.19 is essentially the result of combining Proposition 2.13 with Theorem

3.2 of [14]. Recall that, apart from changes in notation, the latter result states the following. If A is a locally finite-conductor domain and a locally divided domain, then any integral overring of A is a going-down domain.

Corollary 2.19. *Let $R \subseteq T$ be an integral extension of domains such that R is a going-down domain but not a field. Suppose also that for each $\mathcal{M} \in \text{Max}(R)$, there exists a (necessarily unique) height 1 prime ideal $\mathcal{P}(\mathcal{M})$ of R such that $\mathcal{P}(\mathcal{M}) \subseteq \mathcal{M}$ (with $\mathcal{P}(\mathcal{M})$ possibly depending on \mathcal{M}). For each $\mathcal{M} \in \text{Max}(R)$, put $S(\mathcal{P}(\mathcal{M})) := R + \mathcal{P}(\mathcal{M})T$ and $\mathcal{S}(\mathcal{P}(\mathcal{M})) := R + \text{rad}_T(\mathcal{P}(\mathcal{M})T)$. Let M be any maximal ideal of R . Put $P := \mathcal{P}(M)$. Then:*

(a) *Suppose also that $S(P)_{R \setminus M}^+$ (resp., $\mathcal{S}(P)_{R \setminus M}^+$) is a locally finite-conductor domain. Then $T_{R \setminus M}^+$ is a going-down domain.*

(b) *Suppose also that for each $N \in \text{Max}(R)$ such that $\text{ht}_R(N) > 1$ and $P(M) \not\subseteq N$, one has that $T_{R \setminus N}$ is a going-down domain. Suppose also that $S(P)^+$ (resp., $\mathcal{S}(P)^+$) is a locally finite-conductor domain (for instance, a finite-conductor domain). Then T^+ is a going-down domain. So, if one also assumes that T is seminormal, then T is a locally divided domain.*

(c) *Suppose that R is integrally closed and that T is seminormal. Suppose also that for each $N \in \text{Max}(R)$ such that $\text{ht}_R(N) > 1$ and $P(M) \not\subseteq N$, one has that $T_{R \setminus N}$ is a going-down domain. Suppose also that $S(P)$ is a locally finite-conductor domain (for instance, a finite-conductor domain). Then T is a going-down domain (and, in fact, a locally divided domain).*

Proof. (a) By Proposition 2.13 (b), both $S(P)_{R \setminus M}$ and $\mathcal{S}(P)_{R \setminus M}$ are quasi-local n -dimensional going-down domains, where $n := \text{ht}_R(M)$. It follows that their respective seminormalizations, $S(P)_{R \setminus M}^+$ and $\mathcal{S}(P)_{R \setminus M}^+$, are each quasi-local locally divided domains, that is, divided domains (cf. Lemma 2.3 of [15] and the proof of Corollary 2.6 of [15], in conjunction with Proposition 2.9 of [55]). As Proposition 2.13 (a) ensures that $S(P)_{R \setminus M}^+ \subseteq \mathcal{S}(P)_{R \setminus M}^+ \subseteq T_{R \setminus M}^+$ is a chain of integral overring extensions (and $T_{R \setminus M}^+ = (T_{R \setminus M})^+$ is seminormal), an application of Theorem 3.2 of [14] completes the proof of (a).

(b) By Proposition 2.13 (c), both $S(P)$ and $\mathcal{S}(P)$ are going-down domains. Hence, their respective seminormalizations, $S(P)^+$ and $\mathcal{S}(P)^+$, are locally divided domains. By the proof of Proposition 2.13, $S(P)^+ \subseteq \mathcal{S}(P)^+ \subseteq T^+$ is a chain of integral overring extensions. An application of Theorem 3.2 of [14] shows that T^+ is a going-down domain. Then the final assertion also holds, for if T is seminormal, then (and only then) $T = T^+$ (of course, the proof of the final assertion also uses, once again, the fact that any seminormal going-down domain is locally divided). This completes the proof of (b).

(c) By Proposition 2.13 (d), $\mathcal{S}(P)$ is seminormal; that is, $\mathcal{S}(P) = \mathcal{S}(P)^+$. Hence, (c) is a special case of (b). The proof is complete. \square

We next give what is perhaps the most important application of Corollary 2.19.

Corollary 2.20. *Let R be an integrally closed going-down domain with quotient field $K \neq R$, let L be a field extension of K , and let T be the integral closure of R in the algebraic closure of K in L . Suppose also that for each $\mathcal{M} \in \text{Max}(R)$, there exists a (necessarily unique) height 1 prime ideal $\mathcal{P}(\mathcal{M})$ of R such that $\mathcal{P}(\mathcal{M}) \subseteq \mathcal{M}$ (with $\mathcal{P}(\mathcal{M})$ possibly depending on \mathcal{M}). For each $\mathcal{M} \in \text{Max}(R)$, put $\mathcal{S}(\mathcal{P}(\mathcal{M})) := R + \text{rad}_T(\mathcal{P}(\mathcal{M})T)$. Let M be any maximal ideal of R . Put $P := \mathcal{P}(M)$. Suppose also that for each $N \in \text{Max}(R)$ such that $\text{ht}_R(N) > 1$ and $P \not\subseteq N$, one has that $T_{R \setminus N}$ is a going-down domain. Suppose also that $\mathcal{S}(P)$ is a locally finite-conductor domain (for instance, a finite-conductor domain). Then T is a going-down domain (and, in fact, a locally divided domain).*

Proof. The assertion is a special case of Corollary 2.19 (c), since any integrally closed domain (for instance, T) is seminormal. \square

Remark 2.21. (a) The context in which Proposition 2.13 and its corollaries were developed is more general than the context in which Theorem 2.4 and its corollaries were developed. The differences

between those contexts are largely due to the fact that Theorem 2.4 did not address the possibility that the base domain R could have what are sometimes referred to as “low maximals”, that is, maximal ideals M of height 1. Proposition 2.13 and its corollaries were able to address the possibility that the base going-down domain (hence, treed domain) R has low maximals largely because of the use of the following technical condition: if M and N are maximal ideals of R and P denotes the (unique) height 1 prime ideal of R that is contained in M , with $\text{ht}_R(N) > 1$ and $P \not\subseteq N$, then (the assumed integral extension T is such that) $T_{R \setminus N}$ is a going-down domain. Notice that we avoided having to address such possibilities in parts of Theorem 2.4 and in some of its corollaries by assuming that P was contained in the Jacobson radical of R . Indeed, if *that* were to hold, it would not be possible to have that $P \not\subseteq N$, and so in any such situation, it would not be necessary to assume that certain domains of the form $T_{R \setminus N}$ are going-down domains. I was willing to pay the price of adding such an assumption in order to develop a result such as Proposition 2.13 which would admit corollaries that could address actual examples with low maximals. Because of the above consequences of that assumption, I have concluded that Proposition 2.13 is the second most important result in this paper.

(b) Some of the above comments have already indicated my view that, in some ways, Proposition 2.13 is more general or more powerful than Theorem 2.4. It likely seems natural to many readers to ask why I did not develop Proposition 2.13 first or why I did not develop the two results together in the form of one (somewhat large) result. The next two paragraphs share my two reasons for the separate developments of Theorem 2.4 and Proposition 2.13. Before giving those reasons, I would like to state that, in furtherance of the above “more general or more powerful” comment, I do not view Proposition 2.13 as a competitor of Theorem 2.4. Indeed, I view Proposition 2.13 as a natural completion of the ideas which motivated Theorem 2.4. From that point of view, it is perhaps not surprising that our successful approach to the more general context of Proposition 2.13 required the introduction of the new technical condition that was discussed in (a).

Let me first address what I referred to above as “the ideas which motivated Theorem 2.4”. I wish to ask/advise readers to (re)read the following material: the third and fourth sentences of the Abstract of this paper; and the final two sentences of the Introduction of this paper. In short, my most important aim here was to construct a vehicle that would refocus our attention from a given integral overring extension $R \subseteq T$ of a going-down domain R to a tractable integral overring extension $\mathcal{R} \subseteq \mathcal{T}$ for some going-down domain \mathcal{R} and some (integral overring) \mathcal{T} that have as their quotient field a *preassigned* algebraic field extension of a quotient field of R . It was expedient to use domains of the form $R + PT$ or $R + \text{rad}_T(PT)$ (or their seminormalizations) to play the role of \mathcal{R} . It was also expedient to avoid some unnecessary generality in formulating Theorem 2.4 *in order to more quickly reach* the substantive applications in Corollaries 2.6-2.11. I do not believe that I need to elaborate further on this point to any reader who has followed the advice in the second sentence of this paragraph.

My second reason for the separate developments of Theorem 2.4 and Proposition 2.13 is pedagogic. Thanks to Proposition 2.13, we did eventually reach the applications in Corollary 2.20-2.21. Moreover, a “linear” reader who had already worked through the proof of Theorem 2.4 had, no doubt, less difficulty in navigating the proof of Proposition 2.13. This gradual revisiting of an idea that is set in increasingly complicated situations is part of a long-honored pedagogic tradition that I called the “cyclic method” in a Precalculus textbook that I coauthored nearly 40 years ago. It is how we as students came to understand one-variable calculus before studying calculus of functions of several variables. It is why many of us were made to study metric spaces before we were allowed to study general topology (formerly called “point set topology”). It is how students used to (and my generation of undergraduate mathematics honours students at the University of Manitoba 60+ years ago still did) learn about polynomial rings in one indeterminate over a field in a course on the “theory of equations” before being allowed to take a course on “abstract/modern algebra” which introduced more general concepts such as principal ideal domains and unique factorization domains. In short, a store of examples that has been rigorously studied can provide workers at *any* level with three desir-

able consequences: an increased level of comfort with the general area of study, an increased level of skill in reasoning in that general area, and a toolchest of examples to call upon when subsequently testing hypotheses that are couched in terms of more general concepts.

(c) Since the work to this point has, to some extent, been focused on transferring attention from “integral overring” contexts to more general integral extensions involving domains, I have not considered the possibility of using “transferring vehicles” other than those of the form $R + PT$ or $R + \text{rad}_T(PT)$ (or their seminormalizations). (After all, this is meant to be a paper, not a book or an encyclopedia.) Interested readers are encouraged to investigate whether other tractable vehicles are available for such purposes. I have not addressed other possible ring-theoretic properties of domains of the form $R + PT$ or $R + \text{rad}_T(PT)$, and I would also encourage interested readers to look into such matters as well.

(d) This part of Remark 2.21 will complete the first (and longer) half of Section 2. At this point, I consider it appropriate to provide some historical background that seems pertinent to future studies of the possible “ascent” (or “descent”) of the “going-down domain” property under integral extensions. The rest of this paragraph is the result of editing Example 2.1 of [14]. It was shown by Heinzer and Ohm in [39] that there exists a chain of integral extensions $V \subset D \subset V^*$ such that D is a quasi-local G-domain (in the sense of [41]) with infinitely many prime ideals of height 1, V is a two-dimensional valuation domain (hence, a Pruefer domain), and V^* is the integral closure of V in an algebraic closure of a quotient field of V . (Of course, it follows from Pruefer’s ascent result that V^* is a Pruefer domain.) The reasons that were explicitly given by Heinzer and Ohm for developing such data had no explicit connection with the topics of treed domains and the GD property. Note, however, that D is not a treed domain and, hence, D is not a going-down domain. Moreover, both V^* and V are going-down domains because any Pruefer domain is a going-down domain. In view of the integral extension $V \subset D$, it follows that the going-down domain property need not ascend under an arbitrary integral extension of domains; and in view of the integral extension $D \subset V^*$, it follows that the going-down domain property need not descend under an arbitrary integral extension of domains. Furthermore, the going-down domain property need not ascend under an arbitrary integral overring extension. Indeed, by suitably applying the classical $D + M$ construction (and facts about that construction, as in, for instance, pages 202–203 of [37]), to the extension $V \subset D$ and then using the Corollary in [30], we see that for each d such that $3 \leq d \leq \infty$, there exists a d -dimensional quasi-local going-down domain A and an integral overring B of A such that B is not a going-down domain. The rest of Example 2.1 of [14] goes on to identify certain other facts about the above chain of rings $V \subset D \subset V^*$ and to indicate how those additional facts serve to motivate certain results appearing later in [14].

The fifth paragraph of the Introduction mentioned one of my results from [12] that generalized a result of Quentel by providing a new characterization of Pruefer domains. At the time that [12] was written, I was still considered (by others and myself) as someone would nowadays be called an arithmetic algebraic geometer because my doctoral thesis had involved, *i.a.*, some applications to algebraic number theory of a certain cohomological dimension theory that I had invented in order to answer some then-current questions in modern algebraic geometry. As one would expect, my doctoral studies had acquainted me with much of what was then known about Dedekind domains. But that study had also led me (to the extent that time permitted) to read more widely about the more general class of Pruefer domains. Even though much of what I read about Pruefer domains as a graduate student essentially cast that topic as being merely a formal generalization of Dedekind domains, I recall that those few, hurried readings as a student led me to wonder about the place that Pruefer domains should occupy in the hierarchy (such as it may be) of integral domains. The paper [12] was submitted three years after I had received my doctoral degree. By that time, I had taught graduate courses in both homological algebra and commutative algebra. In particular, reading an early version of [37] had shown me that the topics of Pruefer domains and Krull domains each played a central

and motivational role in multiplicative ideal theory (an area of research that I had not even heard about until nearly the end of my postdoctoral year). However, I found some noteworthy differences when comparing some classical results about these two kinds of domains. Some experiences during the intervening years have led me to conclude (with appropriate acknowledgment of the current multi-faceted interest in factorization theory) that from some points of view, Pruefer domains are more important than (their special case of) Dedekind domains or Krull domains. As evidence for this (perhaps minority) opinion, the next paragraph collects some information.

The integral closure of a Dedekind domain in a finite-dimensional (hence algebraic) field extension of the quotient field is a Dedekind domain (cf. Theorem 98 in [41], Corollary 40.10 in [37]), but the analogous assertion fails for (some) infinite-dimensional algebraic field extensions (the most famous example of this failure being provided by the base ring \mathbb{Z} and the extension ring being the ring of all algebraic integers inside some algebraic closure of \mathbb{Q}). Similarly, the integral closure of a Krull domain in a finite-dimensional field extension of the quotient field is a Krull domain (cf. Theorem 43.13 (1) in [37]), but the analogous assertion fails for (some) infinite-dimensional algebraic field extensions (cf. Exercise 16, page 539 in [37]). On the other hand, the integral closure of a Pruefer domain in a (possibly infinite-dimensional) algebraic field extension of the quotient field is a Pruefer domain, by Pruefer's ascent result (cf. [51], Theorem 22.3 in [37]). In order for the just-mentioned evidence to be persuasive, one would need to show that it is often the case that the integral closure of an integrally closed going-down domain in an infinite-dimensional algebraic field extension of the quotient field is a going-down domain. This, I would like to suggest, would, in turn, be achieved if one were able to prove that the integral closure of an integrally closed going-down domain in a *finite*-dimensional algebraic field extension of the quotient field is a going-down domain, since it was shown in Corollary 2.7 of [29] that the class of going-down domains is stable under the formation of direct limits.

For the proof of Pruefer's ascent result that appears most often in current textbooks and appropriate graduate courses, see the proof of Theorem 22.3 of [37]. (In [41], Kaplansky explicitly admits that his textbook is presenting essentially that same proof of Pruefer's ascent result.) A reading of that proof (including a reading of what *it* cites) reveals that the "modern" proof of Pruefer's ascent result does not specifically use the GD property. What should that fact lead us to conclude, when considered in conjunction with the above-mentioned fact that in Corollary 4 of [12], I characterized Pruefer domains (without using the as-yet uninvented "going-down domain" terminology) as the integrally closed finite-conductor going-down domains? My answer to this question is threefold. First, Pruefer's ascent result is about Pruefer domains – it is not, *per se*, about going-down domains. Second, it is possible to prove Pruefer's ascent result by using the property INC while making no use of the property GD: see [17]. Third, in view of the preceding two parts of my overall answer, I suggest that an attempt to find the place that going-down domains should occupy in the hierarchy (such as it may be) of integral domains *need not* succeed in proving that *every* integral closure of *every* integrally closed going-down domain D in *every* algebraic field extension L of a /the quotient field of D is a going-down domain. Instead, I suggest that we should try to develop enough additional results that could be used to help the community to reach a consensus as to *how often* one should expect such data D, L to produce an affirmative answer to the ascent question.

Before closing this first half of Section 2, I wish to add some thoughts in connection with the suggestion that I made in the third of the answers that were given in the preceding paragraph. As commutative ring theorists, we value highly a result that can be stated simply, especially when the result's statement does not break into multiple cases or ignore certain contexts which would appear, *prima facie*, to be relevant. That preference for elegant statements has been fostered by some excellent exposition, especially in a number of prominent textbooks on modern/abstract algebra. To a considerable extent, the material in those textbooks concerning integral domains consists of generalizations of results in the theory of (polynomial) equations which took hundreds of years to develop.

However, despite the impression that some readers infer from slick, excellently written textbooks, the storehouse of information collected about the classical theory of equations includes many special cases, many counterexamples to overly broad assertions, etc. This fact of life can be glossed over in textbooks, for example, when treating polynomials that are solvable by radicals, by imposing various restrictions on the characteristic of the relevant domains/fields or by assuming the presence of appropriate primitive roots of unity. Different textbooks can choose to define the relevant general concepts in slightly different ways: in this regard, compare/contrast the treatments in [40] and [42], for instance. Moreover, without some of the simplifying assumptions, an “answer” can break unavoidably into a list of results for several cases (which admit different actual answers): in this regard, consider the complexity of the statement of Theorem 16 on page 221 in [42] concerning polynomials of the form $X^n - a$. When I encounter a result featuring such a complicated statement, I often examine the literature to see whether some authors of other presentations of similar results decided to exclude the consideration of some apparently relevant context (and to wonder, if so, why). To paraphrase someone (history records many credible candidates for this honor), “things should be made as simple as possible, but not simpler”. To paraphrase some well-known advice: progress sometimes consists of taking one step back and (hopefully) two steps forward. In an attempt to appeal to what I consider to be a simplistic aesthetic criterion, a highly regraded analytic number theorist recently complained (in an article published by the American Mathematical Society) of the “tyranny of small numbers”. His remark was made in the context of seeking/preferring results on the distribution of prime numbers that hold for all sufficiently large parameters (such as left-hand endpoints of open intervals of real numbers). While numerical analysts and actuaries often need/love to consider operators such as $\lim_{a \rightarrow \infty}$, I would suggest that as long as mathematics remains a human activity and as long as humans remain mortal, it would be appropriate to maintain a respect for (and an interest in) accessible results involving accessible data, even when such research involves proofs featuring case analyses and statements that break into cases admitting different answers. The remark is complete.

The remainder of Section 2 will involve classical field theory and group theory more explicitly as we now focus entirely on the possible ascent of the “going down domain” property for ring extensions $R \subset T$ where R is an integrally closed going-down domain and T is the integral closure of R in an algebraic field extension of the quotient field of R . (In particular, I expect the rest of Section 2 to be more accessible, as it will not consider concepts such as locally finite-conductor domains, t -closure, seminormalizations, ring extensions with the finite fiber property, etc.) In recalling Corollary 2.7 of [29], the third paragraph of Remark 2.21 (d) has sharpened that focus by allowing us to consider only finite-dimensional (rather than arbitrary algebraic) field extensions. Proposition 2.22 presents some facts that will help to further sharpen that focus. Although much of the statement and proof of Proposition 2.22 holds for an arbitrary algebraic field extension L/K , we will simplify the exposition of Proposition 2.22 and sharpen its focus with the blanket assumption that $[L : K] < \infty$. The presentation of Proposition 2.22 and its corollaries assumes familiarity with results on field theory and Galois theory that are typically covered in a first-year graduate course on abstract algebra.

First, it will be convenient to pause to recall some terminology. If F/k is an algebraic (for instance, a finite-dimensional) field extension, we define F_s and F_i as being, respectively, the *separable part* of F/k and the *purely separable part* of F/k . In other words, F_s (resp., F_i) is the subset of F consisting of those elements of F that are separable (resp., purely inseparable) over k . If $\text{char}(k) = 0$, then $F_s = F$ and $F_i = k$. Note that if $\text{char}(k) = p > 0$, then

$$F_i = \{u \in F \mid u^{p^e} \in k \text{ for some integer } e \geq 1\}.$$

Proposition 2.22. *Let R be an integrally closed going-down domain with quotient field K . Let L be a finite-dimensional (hence algebraic) field extension of K and let N be the normal closure of L/K in some algebraic closure of K (that contains L). Let T be the integral closure of R in L and let S be the integral*

closure of R in N . Let L_s (resp., N_s) be the separable part of L (resp., of N) over K . Let L_i (resp., N_i) be the purely inseparable part of L (resp., of N) over K . Let A be the integral closure of R in N_i . Let G be the Galois group of N/K (that is, the group of K -algebra automorphisms of N). Then:

(a) The integral ring extension $R \subseteq A$ is unbranched. Consequently, the canonical map $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is an order isomorphism of posets under inclusion and A is a going-down domain.

(b) The quotient field of A (inside N) is N_i , and N/N_i is a finite-dimensional Galois field extension whose Galois group is G .

(c) S is the integral closure of A in N .

(d) If S is a going-down domain, then T is a going-down domain.

Proof. (a) Suppose first that $\text{char}(K) = 0$. Then, since $N_i = K$ and R is integrally closed, we get $A = A \cap K = R$, in which case the assertions in (a) are clear.

Without loss of generality, $p := \text{char}(K) > 0$. If the integral ring extension $R \subseteq A$ is unbranched, then the canonical map $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is an order isomorphism of posets under inclusion (cf. Corollary 5.8 of [26]) and A is a going-down domain by Lemma 2.3 of [15]. Thus, since integrality ensures that $R \subseteq A$ satisfies LO, it will suffice to show that if \mathcal{P}_1 and \mathcal{P}_2 are prime ideals of A such that $\mathcal{P}_1 \cap R = \mathcal{P}_2 \cap R$, then $\mathcal{P}_1 = \mathcal{P}_2$. It will suffice to show that $\mathcal{P}_1 \subseteq \mathcal{P}_2$. Let $u \in \mathcal{P}_1$. As $u \in A \subseteq N_i$, there exists an integer $e \geq 1$ such that $u^{p^e} \in K$. Hence, $u^{p^e} \in A \cap K = R$, whence $u^{p^e} \in \mathcal{P}_1 \cap R = \mathcal{P}_2 \cap R \subseteq \mathcal{P}_2$. As $u \in A$ and $\mathcal{P}_2 \in \text{Spec}(A)$, we get $u \in \mathcal{P}_2$, thus completing the proof of (a).

(b) Consider these facts: A is the integral closure of R in the field N_i ; N_i is algebraic over K ; and K is a quotient field of R . It then follows via the “clearing of denominators” trick (as in the proof of Theorem 7, page 264 of [58]) that each $\xi \in N_i$ can be written as $\xi = u/r$ for some $u \in A$ and some nonzero element $r \in R$. Consequently, N_i is a subfield of the quotient field of A inside N . As N_i clearly contains that quotient field, N_i must be that quotient field, thus proving the first assertion in (b). Next, the definition of N ensures that the field extension N/K inherits finite-dimensionality from the field extension L/K . As the chain of fields $K \subseteq N_i \subseteq N$ leads to $[N : N_i][N_i : K] = [N : K]$ and neither $[N_i : K]$ nor $[N : K]$ is 0, we get that $[N : N_i] < \infty$, that is, N/N_i is a finite-dimensional field extension. Another consequence of N being a normal extension of K is that the “fixed field”

$$N^G := \{v \in N \mid \sigma(v) = v \text{ for all } \sigma \in G\}$$

coincides with N_i . Hence, N is both normal and separable as an algebraic field extension of N_i ; that is, N is a Galois field extension over N_i , necessarily with Galois group G . (By the way, although many textbooks prove that $N^G = N_i$ for the case of positive characteristic, this fact also holds easily for characteristic 0, for then N is both normal and separable, hence Galois, over $K = N_i$, with Galois group G .) The proof of (b) is complete.

(c) This assertion follows from standard facts about integrality, including a subtlety concerning the transitive property of integral extensions. For the sake of completeness, we provide the details. In view of the chain of rings $R \subseteq A \subseteq S$ and the fact that S is integral over R , we get that S is integral over A . It remains only to prove that if $w \in N$ is integral over A , then $w \in S$. This, in turn, follows from the definition of S , the assumption that w is integral over A , and what was actually shown in the proof of Corollary 5.4 in [2]. The proof of (c) is complete.

(d) Since $T \subseteq S$ is an integral extension of domains and T is integrally closed, the assertion in (d) follows from the second “descent” assertion in the statement of Theorem 2.4 of [14]. The proof is complete. \square

The main purpose of Proposition 2.23 is to provide a proof that essentially shows how Proposition 2.22 can be used to reduce the search for an affirmative answer to the question about the possible ascent of the “integrally closed going-down domain” property to integral closures in arbitrary field extensions to the context where the relevant field extension is a finite-dimensional Galois field extension whose Galois group is a (finite) simple group. (This “main purpose” will be explicitly isolated

as Corollary 2.24.) In the proof of Proposition 2.23, which addresses the case where the relevant field extension is a finite-dimensional Galois field extension with Galois group G , the relevant simple groups are shown to be the factor groups induced by adjacent members of a composition series of G . Although the proof of Proposition 2.23 will be mostly taken from what was already in (or what is strongly suggested by what was in) Section 3 of [14], we have decided to call item 2.23 a “Proposition” rather than a “Lemma” because of our stylistic view that a lemma should not be immediately followed by a corollary.

Proposition 2.23. *Let R be an integrally closed going-down domain with quotient field K . Let L be a finite-dimensional (hence algebraic) Galois field extension of K and let T be the integral closure of R in L . Let G denote the (necessarily finite) Galois group of L/K (that is, the group of K -algebra automorphisms of L). Let $\{1\} = G_0 \subset G_1 \subset \dots \subset G_\lambda = G$ be a composition series of G . Necessarily, if $1 \leq j \leq \lambda$, then $H_j := G_j/G_{j-1}$ is a (finite) simple group (possibly such that $H_{j_1} \cong H_{j_2}$ for some $j_1 \neq j_2$). For each j such that $0 \leq j \leq \lambda$, consider the fixed field F_j corresponding to G_j , that is,*

$$F_j := L^{G_j} := \{v \in L \mid \sigma(v) = v \text{ for all } \sigma \in G_j\}.$$

(So, $F_0 = L$ and $F_\lambda = K$.) If $0 \leq j \leq \lambda$, let S_j denote the integral closure of R in F_j . (So, $S_0 = T$ and $S_\lambda = R$.) Then:

(a) Suppose, for each j such that $1 \leq j \leq \lambda$, that if \mathcal{R}_j is an integrally closed going-down domain with quotient field \mathcal{K}_j , \mathcal{L}_j is a finite-dimensional Galois field extension of \mathcal{K}_j and the Galois group of $\mathcal{L}_j/\mathcal{K}_j$ is isomorphic to H_j , then the integral closure of \mathcal{R}_j in \mathcal{L}_j is a going-down domain. Then T is a going-down domain.

(b) Suppose, in addition, that the finite-dimensional Galois field extension L/K is solvable (in the usual sense that its Galois group G is a solvable group). Necessarily, if $1 \leq j \leq \lambda$, then $H_j \cong \mathbb{Z}/p_j\mathbb{Z}$ for some prime number p_j dividing $|G| (= [L : K])$, possibly such that $p_{j_1} = p_{j_2}$ for some $j_1 \neq j_2$. Also suppose, for each j such that $1 \leq j \leq \lambda$, that if \mathcal{R}_j is an integrally closed going-down domain with quotient field \mathcal{K}_j , \mathcal{L}_j is a finite-dimensional Galois field extension of \mathcal{K}_j and the Galois group of $\mathcal{L}_j/\mathcal{K}_j$ is isomorphic to $\mathbb{Z}/p_j\mathbb{Z}$, then the integral closure of \mathcal{R}_j in \mathcal{L}_j is a going-down domain. Then T is a going-down domain.

Proof. (a) It follows from standard Galois theory that if $1 \leq j \leq \lambda$, then F_{j-1}/F_j is a (finite-dimensional) Galois field extension whose Galois group is isomorphic to H_j . Hence, since $S_\lambda = R$ is assumed to be an integrally closed going-down domain, the hypothesis for $j = \lambda$ ensures that the integrally closed domain $S_{\lambda-1}$ is a going-down domain, then the hypothesis for $j = \lambda - 1$ ensures that the integrally closed domain $S_{\lambda-2}$ is a going-down domain, \dots , and then (after sufficiently many applications of the hypothesis) the hypothesis for $j = 1$ ensures that the integrally closed domain $S_0 (= T)$ is a going-down domain.

(b) This is the special case of (a) where the finite group G is solvable. In this case, each of the factor groups H_j (inferred from the given composition series of G) is a simple abelian group and, hence, necessarily isomorphic to $\mathbb{Z}/p_j\mathbb{Z}$ for some prime number p_j (possibly such that $p_{j_1} = p_{j_2}$ for some $j_1 \neq j_2$). Of course, each p_j divides $|G|$ by Lagrange’s Theorem and $|G| = [L : K]$ by Galois Theory. In view of (a), the proof is complete. \square

Corollary 2.24. *The following conditions are equivalent:*

- (1) *Let R be an integrally closed going-down domain with quotient field K . Let L be a (not necessarily algebraic) field extension of K and let T be the integral closure of R in L . Then T is a going-down domain.*
- (2) *Let \mathcal{R} be an integrally closed going-down domain with quotient field \mathcal{K} , let \mathcal{L} be a finite-dimensional (hence algebraic) Galois field extension of \mathcal{K} whose (necessarily finite) Galois group is a simple group, and let T be the integral closure of \mathcal{R} in \mathcal{L} . Then T is a going-down domain.*

Proof. (1) \Rightarrow (2): Trivial.

(2) \Rightarrow (1): Assume (2). Let R, K, L and T be as in (1). Since the class of going-down domains is stable under direct limit (by Corollary 2.7 of [29]), we may assume, without loss of generality, that $[L : K] < \infty$. Let N be the normal closure of L/K in some algebraic closure of K (that contains L), let S be the integral closure of R in N , let N_i be the purely inseparable part of N/K , and let A be the integral closure of R in N_i . Let G denote the Galois group of N/K (that is, the group of K -algebra automorphisms of N). Next, consider the following three facts: the quotient field of A is N_i ; N/N_i is a (finite-dimensional) Galois field extension with Galois group G ; and S is the integral closure of A in N . (The second of these facts is a well known result in field theory that was mentioned above, while the first and third of these facts are from parts (b) and (c) of Proposition 2.22.) Therefore, by Proposition 2.22 (d), we may, without loss of generality, make the following replacements: replace R with A , replace K with N_i , and replace L with N . The upshot of these replacements is that we can now assume, without loss of generality, that L/K is a (finite-dimensional) Galois field extension with (finite) Galois group G .

Let $\{1\} = G_0 \subset G_1 \subset \dots \subset G_\lambda = G$ be a composition series of G . Necessarily, if $1 \leq j \leq \lambda$, then $H_j := G_j/G_{j-1}$ is a (finite) simple group possibly such that $H_{j_1} \cong H_{j_2}$ for some $j_1 \neq j_2$. For each j such that $0 \leq j \leq \lambda$, consider the fixed field F_j corresponding to G_j , that is,

$$F_j := L^{G_j} := \{v \in L \mid \sigma(v) = v \text{ for all } \sigma \in G_j\}.$$

(So, $F_0 = L$ and $F_\lambda = K$.) By Galois theory, if $1 \leq j \leq \lambda$, the Galois group of F_j/F_{j-1} is isomorphic to H_j and, hence, is a (finite) simple group. If $0 \leq j \leq \lambda$, let S_j denote the integral closure of R in F_j . (So, $S_0 = T$ and $S_\lambda = R$.) It follows that if $1 \leq j \leq \lambda$, then S_j is the integral closure of S_{j-1} in F_j . Successive applications of (a) show that $S_{\lambda-1}$ inherits the “going-down domain” property from $S_\lambda (= R)$, then $S_{\lambda-2}$ inherits the “going-down domain” property from $S_{\lambda-1}$, ..., and then finally $S_0 (= T)$ inherits the “going-down domain” property from S_1 , as desired. The proof is complete. \square

The purpose of Corollary 2.25 is to use Corollary 2.24 in order to reduce the search for an affirmative answer to the main “ascent” question in case the Galois group G is a solvable group to its special case where $G \cong \mathbb{Z}/p\mathbb{Z}$ for some prime number $p \geq 5$.

Corollary 2.25. *The following conditions are equivalent:*

(1) *Let R be an integrally closed going-down domain with quotient field K . Let L be a finite-dimensional (hence algebraic) field extension of K and let T be the integral closure of R in L . Let N be the normal closure of L/K in some algebraic closure of K (that contains L). Assume that the (necessarily finite) Galois group of N/K is solvable. Then T is a going-down domain.*

(2) *Let \mathcal{R} be an integrally closed going-down domain with quotient field \mathcal{K} , let \mathcal{L} be a finite-dimensional (hence algebraic) Galois field extension of \mathcal{K} whose (necessarily finite) Galois group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime number $p \geq 5$, and let \mathcal{T} be the integral closure of \mathcal{R} in \mathcal{L} . Then \mathcal{T} is a going-down domain.*

Proof. (1) \Rightarrow (2): This implication follows at once from the following equivalence(s). A nontrivial finite group is a simple solvable group if and only if (it is a simple abelian group if and only if) it is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime number p .

(2) \Rightarrow (1): Let us rework the proof that (2) \Rightarrow (1) from Corollary 2.24 (while modifying that proof as needed for the present context). By the first paragraph of that proof (ignoring its third sentence), we can assume, without loss of generality, that the finite-dimensional field extension L/K is Galois and that its (finite) Galois group, G , is a solvable group.

Next, let us rework the second paragraph of the proof that (2) \Rightarrow (1) from Corollary 2.24. By that proof (while using its notation but modifying that proof as needed for the present context), we get that $H_j := G_j/G_{j-1}$ is a (finite) simple solvable group possibly such that $H_{j_1} \cong H_{j_2}$ for some $j_1 \neq j_2$. (Note that we have also used the standard fact that nontrivial subgroups and factor groups

of finite solvable groups are solvable: cf. Theorem 7.11 (i), page 103 of [40].) So, by a comment from the proof that (1) \Rightarrow (2) in the present result, we get, for each j , that $H_j \cong \mathbb{Z}/p_j\mathbb{Z}$ for some prime number p_j . In view of the statement of (2) in the present result, there is *only one* remaining obstacle to invoking/adapting the remaining part of the proof that (2) \Rightarrow (1) from Corollary 2.24. That obstacle is our present need to give an independent proof in case $p_j = 2$ (resp., in case $p_j = 3$) that S_{j-1} inherits the “going-down domain” property from S_j . That, in turn, follows by invoking the proof of Case 1 (resp., the proof of Case 2) in the proof of Theorem 3.3 of [14]. (Incidentally, both of the just-mentioned proofs from [14] made crucial use of the fact that a quasi-local integrally closed domain is a divided domain.) The proof is complete. \square

The above proof of Corollary 2.25 illustrated our occasional need to address the role of a specific prime number p (inferred from a step in a composition series of an ambient finite Galois group). That point of view leads to a reworking of some of the above arguments, with the upshot being the proof, in Corollary 2.26, of a sufficient condition (in terms of prime numbers greater than 3) for the ascent of the “integrally closed going-down domain” property in the presence of a finite solvable Galois group.

Corollary 2.26. *Let p_1, p_2, \dots, p_k be a finite nonempty list of pairwise distinct prime numbers such that $p_j \geq 5$ for all j . Let $n := 2^a 3^b \prod_{j=1}^k (p_j)^{e_j}$ for some nonnegative integers a and b and some positive integers e_j . Consider the following two statements:*

(i) *Let R be an integrally closed going-down domain with quotient field K . Let L be a finite-dimensional (hence algebraic) field extension of K and let T be the integral closure of R in L . Let N be the normal closure of L/K in some algebraic closure of K (that contains L). Assume that the (necessarily finite) Galois group G of N/K is solvable and that $|G| = n$. Then T is a going-down domain.*

(ii) *Let \mathcal{R} be an integrally closed going-down domain with quotient field \mathcal{K} , let \mathcal{L} be a finite-dimensional (hence algebraic) Galois field extension of \mathcal{K} whose (necessarily finite) Galois group is isomorphic to $\mathbb{Z}/p_j\mathbb{Z}$ for some j such that $1 \leq j \leq k$, and let \mathcal{T} be the integral closure of \mathcal{R} in \mathcal{L} . Then \mathcal{T} is a going-down domain.*

Then (ii) \Rightarrow (i).

Proof. Assume (ii). Let R, K, L, T, N and G be as in (i). As in the first paragraph of the proof that (2) \Rightarrow (1) in Corollary 2.24, we can (by harmless changes of notation) assume, without loss of generality, that the finite-dimensional field extension L/K is Galois, with Galois group G .

Next, let us rework the second paragraph of the proof that (2) \Rightarrow (1) in Corollary 2.24. By that proof (while using its notation but modifying that proof as needed for the present context), we have that if $1 \leq j \leq \lambda$, then $H_j := G_j/G_{j-1}$ is a (finite) simple group whose order must, by Lagrange’s Theorem, divide $|G| = n$. Moreover, for each such j , H_j inherits the property of being a solvable group from G . (We have just, once again, used the fact that nontrivial subgroups and factor groups of finite solvable groups are solvable.) Thus, for each j , H_j is a nontrivial finite simple solvable group, hence isomorphic to $\mathbb{Z}/q_j\mathbb{Z}$ for some prime number q_j . As Lagrange’s Theorem gives that $q_j = |H_j|$ divides $|G| = n$, it follows from the Fundamental Theorem of Arithmetic that q_j is one of $2, 3, p_1, \dots, p_k$. It remains only to confirm that if $1 \leq j \leq \lambda$, then S_j inherits the “integrally closed domain” property from S_{j-1} . This, in turn, follows if q_j equals some p_j (resp., if $q_j = 2$; resp., if $q_j = 3$) from the hypothesis in (ii) (resp., from the proof of Case 1 in the proof of Theorem 3.3 of [14]; resp., from the proof of Case 2 in the proof of Theorem 3.3 of [14]). The proof is complete. \square

Remark 2.27. (a) By the celebrated Feit-Thompson Theorem [35], every nontrivial finite group of odd order is solvable, and so every finite nonabelian simple group is of even order. However, not every nonabelian finite group of even order is solvable. Perhaps the most famous example illustrating this fact is S_5 , the symmetric group on 5 letters. Indeed, $H := S_5$ has a composition series $\{1\} = H_0 \subset H_1 \subset H_2 = H$ with $H_1 \cong A_5$ (the alternating group on 5 letters), which has composition series factors $H_1/H_0 \cong A_5$ and $H_2/H_1 \cong \mathbb{Z}/2\mathbb{Z}$. So, if R is an integrally closed going-down domain with

quotient field K and L/K is a Galois field extension with Galois group isomorphic to S_5 , not all the composition series factors of that Galois group are cyclic of prime order. As the order of that Galois group is $5! = 120 = 2^3 3^1 5^1$, this example serves to illustrate why it was necessary to insert the adjective “solvable” when stating condition (i) in Corollary 2.26.

(b) Because the class of going-down domains is stable under direct limits, it was expedient in Corollary 2.25 to consider only field extensions that were finite-dimensional. Readers who are interested in determining how/whether Corollary 2.25 can be extended to infinitistic contexts are invited to consider that question by taking advantage of the extensive and accessible theories for infinite solvable groups and for infinite-dimensional algebraic Galois field extensions (and the associated profinite topology on their Galois groups).

(c) It seems natural to ask about possible applications of the above results involving finite solvable groups to polynomials that are “solvable by radicals”. Apparently, all standard textbooks agree on the definition of this concept if the relevant fields have characteristic 0, but those same textbooks exhibit some mild differences in their definitions of this concept if the relevant fields have positive characteristic. In teaching the standard first-year graduate algebra course, I tended to adopt the definitions in Section 9 of Chapter V of [40] for this concept. For the sake of completeness, I repeat the relevant definitions next. A *radical (field) extension* is a field extension F/E such that $F = E(u_1, \dots, u_m)$ for some finite subset $\{u_1, \dots, u_m\}$ of F such that there exist positive integers e_1, \dots, e_m satisfying $u_1^{e_1} \in E$ and, whenever $2 \leq i \leq m$, $u_i^{e_i} \in E(u_1, \dots, u_{i-1})$. If $f \in K[X]$ for a field K , we say that f is *solvable by radicals (over K)* if there exist a radical (field) extension F of K and a splitting field E of f over K such that $K \subseteq E \subseteq F$.

For illustrative purposes, I will consider only fields of characteristic 0 here. The result (cf. Corollary 3.7 of [40]) which shows that the terms “solvable by radicals” and “solvable group” were well chosen implies the following fact: if K is a field of characteristic 0 and $f \in K[X]$, then f is solvable by radicals (over K) if and only if, for some (equivalently, for every) splitting field L of f over K , the Galois group of L/K is a solvable group. (That Galois group, which is well defined up to isomorphism of groups, is commonly called the *Galois group of f over K* .) To get some applications of the material in Proposition 2.23 (b) and Corollaries 2.25-2.26, the reader is invited to consider an integrally closed going-down domain R with quotient field K of characteristic 0, a polynomial $f \in K[X]$ that is solvable by radicals, and a splitting field L of f over K (noting that the Galois group of L/K , that is the Galois group of f over K , is a finite solvable group).

(d) As many textbooks on abstract algebra point out, certain Galois field extensions L/K of prime degree admit elegant generator-relation descriptions if their base fields have appropriate primitive roots of unity. This happens, in particular, if K is large enough that it contains a primitive n^{th} root of unity for each integer $n \geq 2$. For instance, that happens if K contains an algebraic closure of \mathbb{Q} . We will say more about such matters (and possible implications for some naturally occurring integrally closed domains R and S within K and L , respectively) in the Appendix.

Apropos of the above Corollaries and rather “large” fields K : the Galois group of a finite-dimensional Galois field extension L/K can be a rather small kind of solvable group, namely a metacyclic group, even if K is algebraic over (but not necessarily finite-dimensional over) a global field F (for instance, when F is an algebraic number field). Indeed, the main result of [34] shows that the phenomenon of a metacyclic Galois group occurs if the following additional hypotheses hold (where A denotes the ring of integers in the global field F , while B and C denote the integral closure of A in K and L , respectively): there exists a nonzero prime ideal P of B such that only one prime ideal of C lies over P and $\text{char}(B/P)$ does not divide $[L : K]$. I hope that this result will stimulate readers’ interest in developing additional applications of Corollaries 2.25 and 2.26.

(e) My work on “ascent” of the “going-down domain” property under integral extensions has often had occasion to consider unbranched extensions. However, the cardinality of the fiber above a maximal ideal of the base ring was typically of no importance in such considerations. In fact, that

cardinality can be at least 2 for *each* maximal ideal of the base ring. Indeed, the above-mentioned collaboration [34] produced the following result. Let $2 \leq m \leq \infty$ and let p be any odd prime number. Then there exists a going-down domain R such that R is 2-root closed, $\dim(R) = m$, $\text{char}(R) = p$, $|\text{Max}(R)|$ is infinite, and there exists an integral overring T of R such that each non-maximal prime ideal of R has only one prime ideal of T lying over it, each maximal ideal of R has more than one prime ideal of T lying over it, and the conductor $(R : T)$ is nonzero. Hence, by Lemma 2.2 of [22], T is a going-down domain; in fact, by Remark 2.6 (c) of [34], any domain S such that $R \subseteq S \subseteq T$ is a going-down domain.

(f) Readers of [34] will/did come across domains D that contain a primitive n^{th} root of unity for each integer $n \geq 2$. I would expect such domains D to be of interest to folks who deplore the “tyranny of small numbers”, as they probably also probably hate the “tyranny of domains that are ‘close’ to their prime subring”. Of course, any domain D as described above is not “small” (in the sense of being “close” to its prime subring) because a well known result from classical algebraic number theory states that no algebraic number field can contain infinitely many roots of unity (inside any preassigned algebraic closure of \mathbb{Q}).

(g) The corollaries of Proposition 2.22 furthered a program that had been initiated in [14], namely, the study of the possible ascent of the “going-down domain” property under integral extension by using the composition series factors arising from the Galois group of a suitable field extension related to the given extension of domains $R \subseteq T$. This program may be more feasible now than it seemed to some ring-theory experts whom I queried about it at the ICM in Vancouver in 1974.

I expect that the following anecdote may have a salutary effect on some readers. Some time prior to the 1974 ICM, I had submitted (what eventually became) [14] for publication to a prestigious group-theorist who was on the editorial board of a prestigious journal. After the manuscript had languished on someone’s desk for 17 months, it was rejected by a referee whose report indicated to me that the referee had not understood the paper at all. In view of how quickly research on going-down domains had progressed during those 17 months (especially by one of my then-current doctoral research students), I submitted the (rejected) manuscript, with no changes to it, to another prestigious group theorist, Bernard Neumann, whose instructions for authors submitting papers to the Bulletin of the Australian Mathematical Society requested a terse and clear argument for quick acceptance and quick publication. I made my case as best I could and Prof. Neumann responded quickly and positively.

I would like to pause here to apologize to any reader who may have found that some of the material toward the end of Section 2 may have seemed repetitive. In writing the statements and proofs of those corollaries with so much explicit detail, my intent was to not emulate the somewhat dense (but logically complete) exposition in Section 3 of [14]. None of the expected readers of *this* paper should require 17 months to read it!

In view of Corollary 2.24 (whose statement could easily have been discerned from the methods in some proofs in [14] 50 years ago), I hope that future workers in this area will be encouraged to further our above-mentioned approach that utilized composition series factors. Specifically, although the program had success in [14] only when dealing with the composition series factors $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ (fortified by applications of Burnside’s celebrated $p^\alpha q^\beta$ result [11]), I hope that future workers will strive to go beyond what I did with *those* (simple) abelian composition series factors, by taking advantage of the (mammoth, but now complete) classification of all simple finite groups.

(h) The “2-root closed” property appeared naturally in part (e) of this remark. That property will appear again in this final part of the remark. No discussion of the possible ascent of the “going-down domain” property under integral extensions would be complete without mentioning the existence of applications that involve [28]. By using an assumption about domains that implies (but is not implied by) “seminormal” (namely, the “2-root closed” property) and by placing a restriction on the characteristic of relevant domains (namely, that the given base two-dimensional going-down domain

R has no factor domain of characteristic 2), [28] proved a result (its Theorem 2.2) stating that (under the just-mentioned assumption and restriction) *any* integral overring of the given two-dimensional going-down domain R is a going-down domain (and so the integral closure of such a domain R must be a locally divided domain). Combining that result with the present paper's Theorem 2.4 and Proposition 2.13, Theorem 1 of [23], and Theorem 3.2 of [14] can produce a number of interesting analogues of Corollaries 2.6-2.11 and 2.15-2.20. We invite interested readers to flesh out the detailed statements and proofs of those applications. The remark is complete.

Although this paper will be my final publication and the second half of Section 2 is essentially complete, I wish to make one more contribution. It is given in the Appendix, where I pursue the method of proof of the reduction of the main ascent problem for the "solvable" context that was given in Corollaries 2.25 and 2.26. It was shown in Corollary 2.26 that if the main "ascent" question has an affirmative answer for the context when the relevant Galois group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for a specific prime number p , the upshot is an affirmative answer when the relevant solvable Galois group has order $2^i 3^j p^k$ for nonnegative integers i, j and k . On the other hand, if the main "ascent" problem has a negative answer, the main result of the Appendix will show that if the quotient field of the base domain R in such a counterexample has enough primitive roots of unity and the pertinent prime number $p \geq 5$ is both the ambient characteristic and the degree of the field extension formed by quotient fields, then any such counterexample R has a few additional properties which are potentially useful and which may not have been evident at first glance. We would hope that access to those additional properties would be of help to workers who are eager to construct an example giving a negative answer to the main "ascent" problem. The material in the appendix is being excluded from the present Section 2 only because the Appendix has no need of the main concepts (such as seminormality, t -closure, locally finite-conductor domains, pullbacks, and "gluing") that surrounded the main results of Section 2 (namely, Theorem 2.4 and Proposition 2.13). In particular, the Appendix can be read by anyone who is familiar with the content in the above results 2.22-2.26, as well as the work on going-down domains that was published in or before 1976.

3 Appendix

Many of us share a fondness for the historical developments leading up to the current state of commutative algebra, including work on the concept of polynomials that are solvable by radicals. For readers sharing an interest in such matters, Corollary 2.26 gave a concrete reason for wanting to know the answer to the following question about a given prime number $p \geq 5$. If \mathcal{R} is an integrally closed going-down domain with quotient field \mathcal{K} and \mathcal{L}/\mathcal{K} is a finite-dimensional Galois field extension whose Galois group has order (that is, cardinality) p , must it then be the case that \mathcal{T} , the integral closure of \mathcal{R} in \mathcal{L} , is a going-down domain?

The restriction $p \geq 5$ in the just-mentioned question arises because the underlying question was already answered in the affirmative for the prime numbers 2 and 3 in [14]. The question itself may become more tractable if \mathcal{K} contains enough primitive roots of unity, because it is classically known that, under those conditions, \mathcal{L} has an attractive generator-relation description as a \mathcal{K} -algebra. This classical fact breaks naturally into a pair of facts whose special cases that are relevant here are summarized in Proposition 3.1.

In keeping with the style in most of this paper and most of the relevant literature, I will now change fonts (so that $\mathcal{K}, \mathcal{L}, \mathcal{R}, \mathcal{T}, \dots$ will be typed, respectively, as K, L, R, T, \dots).

Proposition 3.1. *Let p be a prime number and let $K \subset L$ be an extension of (distinct) fields contained in an algebraic closure \bar{K} of K . Then:*

(a) *(cf. the proof of Proposition 7.8 on page 293 of [40]) Suppose that $\text{char}(K) = p$. Then L/K is a Galois field extension such that $[L : K] = p$ (and, necessarily, the Galois group of L/K is isomorphic to $\mathbb{Z}/p\mathbb{Z}$)*

if and only if L is a splitting field over K of a (necessarily irreducible) polynomial $f \in K[X]$ of the form $f(X) = X^p - X - a$. Moreover, if these equivalent conditions hold, then $L = K(u)$ for any root u of f which is contained in \overline{K} .

(b) (cf. the proof of Theorem 7.11 on page 295 of [40]) Suppose that K contains a primitive p^{th} root of unity. Then L/K is a Galois field extension such that $[L : K] = p$ (and, necessarily, the Galois group of L/K is isomorphic to $\mathbb{Z}/p\mathbb{Z}$) if and only if L is a splitting field over K of a (necessarily irreducible) polynomial $g \in K[X]$ of the form $g(X) = X^p - b$. Moreover, if these equivalent conditions hold, then $L = K(u)$ for any root u of g which is contained in \overline{K} .

If p , R , K , L and T are (apart from the choice of type font) as in the question that was raised in the first paragraph of this Appendix, Proposition 3.1 has led me to wonder whether an answer to the following question may be of use. Does T have an attractive generator-relation description as an R -algebra? A study of this question breaks naturally into an analysis of two rather different contexts, just as was the case in Proposition 3.1. Proposition 3.2 gives what we have noticed in beginning such an analysis. The first paragraph of Remark 3.3 addresses whether the information in Proposition 3.2 can be expected to play a role in answering the underlying question. To aid readers who may not have read the entirety of the paper to this point, I am making the proof of Proposition 3.2 (b) markedly self-contained by intentionally including in it some explanatory details that have appeared earlier in this paper.

Proposition 3.2. *Let p be a prime number, let R be an integrally closed going-down domain with quotient field K , let L/K be a finite-dimensional Galois field extension with Galois group $G \cong \mathbb{Z}/p\mathbb{Z}$, and let T be the integral closure of R in L . Then:*

(a) *Suppose that each nonzero nonmaximal prime ideal of R is lain over by only one prime ideal of T (for instance, suppose that $R \subseteq T$ is unbranched). Then T is a going-down domain.*

(b) *Suppose that T is not a going-down domain. Then there exists at least one nonzero nonmaximal prime ideal of R which is lain over by more than one prime ideal of T . Let P be a nonzero nonmaximal prime ideal of R with this property. (Note that if R is quasi-local and two-dimensional, then P is uniquely determined.) Then*

$$|\{Q \in \text{Spec}(T) \mid Q \cap R = P\}| = p.$$

Write $\mathcal{F} := \{Q \in \text{Spec}(T) \mid Q \cap R = P\}$ as $\{Q_1, \dots, Q_p\}$. Pick any nonzero element r of P . Let σ be any nonidentity element of G (that is, any generator of the cyclic group G). If we also suppose that $\text{char}(K) = p$, then there exists a nonzero element $v \in PT$ ($\subseteq \bigcap_{k=1}^p Q_k = \text{rad}_T(PT) \subseteq T$) such that $L = K(v)$, the integral closure of $R[v]$ is T , the minimum polynomial of v over K is of the form $f(X) = X^p + aX + b \in R[X]$ with $a, b \in Rr$ ($\subseteq P$), the elements of \mathcal{F} can be relabeled so that, for each i such that $0 \leq i \leq p-1$,

$$v_i := \sigma^i(v) \in \bigcap_{j=1}^p Q_j$$

and $v_\lambda - v_\mu \in P$ for all distinct elements λ, μ of $\{0, \dots, p-1\}$.

(c) *Suppose that T is not a going-down domain. Then there exists at least one nonzero nonmaximal prime ideal of R which is lain over by more than one prime ideal of R . Let P be a nonzero nonmaximal prime ideal of R with this property. (Note that if R is quasi-local and two-dimensional, then P is uniquely determined.) Then*

$$|\{Q \in \text{Spec}(T) \mid Q \cap R = P\}| = p.$$

Write $\mathcal{F} := \{Q \in \text{Spec}(T) \mid Q \cap R = P\}$ as $\{Q_1, \dots, Q_p\}$. Pick any nonzero element r of P . Let σ be any nonidentity element of G (that is, any generator of the cyclic group G). If we also suppose that K contains a primitive p^{th} root of unity ζ , then $\text{char}(K) \neq p$, there exists a nonzero element $v \in PT$ ($\subseteq \bigcap_{k=1}^p Q_k = \text{rad}_T(PT) \subseteq T$) such that $L = K(v)$, the integral closure of $R[v]$ is T , the minimum polynomial of v over K

is of the form $f(X) = X^p - b \in R[X]$ for some nonzero element $b \in Rr(\subseteq P)$, and the elements of \mathcal{F} can be relabeled so that, for each i such that $1 \leq i \leq p$,

$$v_i := \sigma^{i-1} v = \zeta^{i-1} v \in \cap_{j=1}^p Q_j.$$

Proof. (a) This assertion follows at once from Lemma 2.2 of [22].

(b) The existence of a nonzero nonmaximal prime ideal of R that is lain over by more than one prime ideal of T is immediate from (a). The parenthetical assertion about the case of a two-dimensional quasi-local domain is immediate since R , being a going-down domain, is necessarily treed.

We next prove the assertion that $|\mathcal{F}| = p$. Pick any $Q_1 \in \mathcal{F}$. (Such Q_1 exists since integrality ensures that $R \subseteq T$ satisfies LO.) The hypotheses ensure that G acts transitively on \mathcal{F} (by Theorem 22, page 289 of [58]). Thus, $\mathcal{F} = \{\sigma^i(Q_1) \mid 0 \leq i \leq p-1\}$. Now, by the fundamental theorem of permutation groups (cf. Theorem 4.3, page 89 of [40]), $|\mathcal{F}|$ is the index in G of the stabilizer of Q_1 (under the action of G). So, since $|G| = p$ is a prime number, it follows from Lagrange's Theorem that $|\mathcal{F}|$ is either 1 or p . By the first sentence of this proof of (b), P and, consequently, \mathcal{F} were chosen (with the aid of (a)) so that $|\mathcal{F}| \neq 1$. Therefore, $|\mathcal{F}| = p$, as asserted.

Next, note that $\cap_{k=1}^p Q_k = \text{rad}_T(PT)$, by the reasoning that used the "GD to P " concept in the fourth sentence of the second paragraph of the proof of Theorem 2.4. Moreover, the integral closure of $R[v]$ is T since T is an integrally closed overring of $R[v]$ which is integral over $(R$ and hence also integral over) $R[v]$.

By a celebrated result of E. Artin and O. Schreier on cyclic extensions (we will need to use only its special case that was stated in Proposition 3.1 (a)), there exists $u \in L \setminus K$ such that $L = K(u)$ and the minimum polynomial of u over K is of the form $X^p - X - a \in K[X]$. Necessarily, $a \neq 0$ (for otherwise, the nonzero element u would be a root of the polynomial $X^{p-1} - 1$ in $K[X]$, whence $p = [L : K] = [K(u) : K] \leq p-1$, a contradiction). Since the above element a is algebraic over the quotient field of R and T is the integral closure of R in L , it follows by "clearing denominators" (as in the proof of Theorem 7, page 264 of [58]) that $u = t/r$ for some elements $t \in T$ and $r \in R \setminus \{0\}$. Note that $t \notin K$ (for otherwise, $u = t/r \in K$ and $L = K(u) = K$, a contradiction). Also, by algebraically rewriting the equation $r^p(u^p - u - a) = 0$, we get $t^p = r^{p-1}t + r^p a$, whence

$$t^p - r^{p-1}t = r^p a \in T \cap K = R.$$

Hence, there exist elements r_1 and r_0 in R such that $t^p = r_1 t + r_0$.

Next, pick an element $\xi \in P \setminus \{0\}$. Note that $v := \xi t \notin K$ (for otherwise, $t = v/\xi$ would be in K , a contradiction). Since $K \subset K(v) \subseteq L$ and $[L : K]$ is a prime number, it follows that $K(v) = L$ (cf. Theorem 1.2, page 231 of [40]). Next, observe that

$$v^p = \xi^p t^p = \xi^p (r_1 t + r_0) = (\xi^{p-1} r_1)(v) + \xi^p r_0,$$

with $\xi^{p-1} r_1 \in Pr_1 \subseteq P$ and $\xi^p r_0 \in P$. By more *abus de langage*, we now replace the ordered triple (u, r_1, r_0) with the ordered triple $(v, \xi^{p-1} r_1, \xi^p r_0)$. After this change of notation), the upshot is that $L = K(u)$ where $u \in T$ satisfies $u^p = r_1 u + r_0$ for some elements r_1 and r_0 of P .

Since R is integrally closed (with quotient field K) and u is a root of the polynomial $f := f(X) := X^p - r_1 X - r_0$, it follows that f is the minimum polynomial of u over K (cf. page 261 of [58]). As f is relatively prime to its formal derivative, $f' = pX^{p-1} - r_1 = -r_1 (\neq 0)$, in $K[X]$, it follows that f is separable over K . Let $u = u_1, u_2, \dots, u_p$ denote the p (pairwise distinct) conjugates of u (that is, the p roots of f) in $L (= N)$. If $1 \leq i < j \leq p$, we have

$$u_i^p = r_1 u_i + r_0 \quad \text{and} \quad u_j^p = r_1 u_j + r_0,$$

whence

$$(r_1(u_i - u_j))^p = r_1^p (u_i - u_j)^p = r_1^p (u_i^p - u_j^p) = r_1^p [(r_1 u_i + r_0) - (r_1 u_j + r_0)] =$$

$r_1^{p+1}(u_i - u_j)$. Consequently, since r_1 and $u_i - u_j$ are each nonzero, cancellation gives $(u_i - u_j)^{p-1} = r_1$. This equation shows that $F := K(u_i - u_j)$ is a field contained between K and L such that $[F : K] \leq p - 1$. As $[F : K]$ is an integral divisor of $[L : K] = p$ and p is a prime number, it follows that $[F : K] = 1$, whence $F = K$, whence $u_i - u_j \in K$. As u_i and u_j are each integral over R and R is integrally closed, we next get that $u_i - u_j \in R$. Moreover, $u_i - u_j \in P$, as a consequence of the facts that $(u_i - u_j)^{p-1} = r_1 \in P$ and P is a prime ideal of R .

It remains only to prove that $u_i := \sigma^i(u) \in Q_j$ for all $0 \leq i \leq p - 1$ and $1 \leq j \leq p$. As $u \in PT$ and σ^i fixes P elementwise, we get

$$u_i \in \sigma^i(PT) \subseteq \sigma^i(P)\sigma^i(T) = P\sigma^i(T) \subseteq PT \subseteq Q_j,$$

thus completing the proof of (b).

(c) One can prove (c) by adjusting the above proof of (b) in light of the following four observations. First, note that the existence of ζ , a primitive p^{th} root of unity in K , ensures that $\text{char}K \neq p$: cf. Remarks, page 295 of [40]. Second, by using part (b) (rather than part (a)) of Lemma 3.1, one finds the appropriate minimum polynomial to be $g := g(X) := X^p - b \in R[X]$, with $b \in P \setminus \{0\}$. Third, note that g is separable over K because the roots of g in L (inside some algebraic closure of K) are the pairwise distinct elements $u_1 = u$, $u_2 = \zeta u$, $u_3 = \zeta^2 u$, \dots , $u_p = \zeta^{p-1} u$. Fourth, note that one can rearrange the elements of S so that if $1 \leq j \leq p$, then $Q_j = \sigma^{j-1} Q_1$. This completes the proof. \square

Remark 3.3. I would not presume to predict how useful Proposition 3.2 will prove to be in the future. Perhaps the greater detail in Proposition 3.2 (b) will be of some use for that very special context (where $p = [L : K] = \text{char}(K)$). I do consider the final assertion in the statement of Proposition 3.2 (b) to be the most significant result in the Appendix, in part because it is the only part of that statement for which I could not find an analogue while stating and proving part (c) of Proposition 3.2. Indeed, history's verdict may turn out to be that the conclusions in Proposition 3.2 (c) (for the more prevalent context where $p = [L : K] \neq \text{char}(K)$) amount to no more than standard applications of fundamental facts.

Despite the preceding sentence, I try to be optimistic. And I am nostalgic by nature. So, I am led to ask whether the method of proof that worked for $p = 2$ and $p = 3$ in Section 3 of [14] could, with suitable ingenuity (perhaps fortified with abundant primitive roots of unity), be generalized to work for (at least *some* values of) $p \geq 5$. For some specifics in this regard, I offer the following two comments. First, the calculations in the just-mentioned analyses of Cases 1 and 2 in Section 3 of [14] were made possible by an early passage to the case of a quasi-local base (integrally closed going-down) domain and then taking advantage of the fact that this base domain is a divided domain; note that, by Lemma 2.2, that reduction to the case where the base (integrally closed) domain is a divided domain is also available to us for the problem at hand here. Second, I would ask interested readers to consider the suggestions in the Remark on page 261 of [14]; in view of the advances in computational algorithms during the past 60 years, it may be appropriate to ask if some of those advances could be used to carry out those suggestions. For the more general question arising in the “not necessarily solvable” context that is suggested by Corollary 2.24, I would ask interested readers to (re)read the final sentence of Remark 2.27 (g). This remark is complete.

I close with a heartfelt salute to my readers, without whom my task would have been harder and less meaningful.

Ave atque valete.

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