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Iwasawa module of certain real quadratic number fields

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Abstract. Let k be a real quadratic number field, k_∞ its cyclotomic $\mathbb{Z}_2$-extension and $X(k_\infty)$ its Iwasawa module. Denote by $A(k_n)$ the 2-class group of the n-th layer k_n of k_∞/k, where $k_0 = k$. In this paper, we determine all fields k such that $\mathrm{rank}(X(k_\infty)) \geq 2$, k_∞/k is totally ramified and $X(k_\infty) \simeq A(k_0)$. Under these conditions, we deduce infinite families for which $X(k_\infty)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$, $m \geq 2$. In particular, Greenberg's conjecture on the vanishing of the Iwasawa λ-invariant is satisfied for new families of infinitely many real quadratic number fields. This extends several known results in the literature.

Key Words: Real quadratic fields, Hilbert class field theory, Genus theory, class group, Class field theory, Iwasawa theory.
2020 MSC: 13G05, 13A15, 13E99.

1 Introduction

For a prime integer p, an infinite Galois extension k_∞/k is called a $\mathbb{Z}_p$-extension, if its Galois group $\mathrm{Gal}(k_\infty/k)$ is topologically isomorphic to the additive group $\mathbb{Z}_p$ of p-adic integers. Note that every number field k has at least one of such $\mathbb{Z}_p$-extension, that is the cyclotomic $\mathbb{Z}_p$-extension k_∞ obtained by the compositum $k_\infty = k\mathbb{Q}_\infty$, where $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of the rational numbers $\mathbb{Q}$. For each positive integer n, let k_n denote the unique intermediate field of k_∞/k of degree p^n over k, and let $A(k_n)$ be the p-class group of k_n, that is the p-Sylow subgroup of the ideal class group of k_n. The Iwasawa module $X = X(k_\infty)$ is then defined as the inverse limit of the family $(A(k_n))_n$ with respect to the norm maps. Iwasawa proved in [14] what it is actually known as *Iwasawa's class number formula* which gives the order of $A(k_n)$. This formula states that there exist integers λ, μ and ν (known as the *Iwasawa invariants*) such that the order of $A(k_n)$, for n sufficiently large, is

$$\#A(k_n) = p^{\lambda n + \mu p^n + \nu}.$$

On the other hand, let $\mathcal{L}(k_\infty)$ (resp. $\mathcal{L}(k_n)$) be the maximal unramified pro-p-extension of k_∞ (resp. n-th layer k_n), and denote by $L(k_\infty)$ the maximal abelian subextension of $\mathcal{L}(k_\infty)/k_\infty$. The Galois group $\mathcal{G} = \mathrm{Gal}(\mathcal{L}(k_\infty)/k_\infty)$ is isomorphic to the inverse limit of the Galois groups $\mathrm{Gal}(\mathcal{L}(k_n)/k_n)$ with respect to the restriction maps, and it is well known by class field theory that the maximal abelian quotient group $\mathcal{G}^{ab}$ satisfies

$$\mathcal{G}^{ab} \simeq \mathrm{Gal}(L(k_\infty)/k_\infty) \simeq X(k_\infty)\cdot$$

Note that Iwasawa conjectured that μ vanishes for the cyclotomic $\mathbb{Z}_p$-extension k_∞ for any number field k. Ferrero and Washington in [7] proved this conjecture for the cyclotomic $\mathbb{Z}_p$-extension of an abelian number field k over $\mathbb{Q}$. Furthermore, Greenberg conjectured in [11, 12] that if k is totally real, then $\lambda = \mu = 0$, i.e., X is finite. Several works have been carried out in this direction, but this conjecture generally remains open, except for a few special cases, see for example [9, 10, 13, 15, 17, 18, 19, 23, 22, 24, 28, 29].

In what follows, let $p = 2$, $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic number field, where d is a positive square-free integer which is not equal to 2, and k_∞ be its cyclotomic $\mathbb{Z}_2$-extension. The intermediate fields k_n, $n \in \mathbb{N}$, are of the form $k_n = k\mathbb{Q}_n$, where $\mathbb{Q}_n$ are the layers of the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}$. Thus $k_n = \mathbb{Q}(\sqrt{d}, a_n)$, where $a_0 = 0$, $a_n = \sqrt{2 + a_{n-1}}$, hence, $k_0 = k$, $k_1 = \mathbb{Q}(\sqrt{2}, \sqrt{d})$, $k_2 = \mathbb{Q}(\sqrt{2+\sqrt{2}}, \sqrt{d})$, etc. The investigation of real quadratic fields has been the subject of several interesting works. In fact, Mouhib and Movahhedi listed all real quadratic fields k for which $X(k_\infty) = 0$ (cf. [22, Corollary 3.4]), as well as those for which $X(k_\infty)$ is cyclic nontrivial (cf. [22, Theorem 3.8]). Moreover, Laxmi and Saikia, and Avila investigated real quadratic number fields k such that $X(k_\infty) \simeq \mathbb{Z}/2\mathbb{Z}$ (cf. [5, 16]) and thereafter Avila constructed some families of real quadratic number fields k such that $X(k_\infty) \simeq \mathbb{Z}/2^m\mathbb{Z}$, with $m \geq 2$ (cf. [5]). Moreover, in [18, 19, 21], Mizusawa constructed several infinite families of real quadratic number fields k whose Iwasawa module is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In the light of these works, we suggest the following question:

What are real quadratic fields k such that k_∞/k is totally ramified and $X(k_\infty) \simeq A(k)$?

In this paper, we determine all real quadratic number fields k such that $\mathrm{rank}(X(k_\infty)) \geq 2, k_\infty/k$ is totally ramified and $X(k_\infty) \simeq A(k)$. Our investigations leads to the construction of certain families of real quadratic number fields whose Iwasawa module is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$, with $m \geq 2$.

Our main theorem is as follows.

Theorem 1.1 (The Main Theorem). Let $\Bbbk = \mathbb{Q}(\sqrt{d})$ be a real quadratic number field, where $d \neq 2$ is a positive square-free integer. Let $\varepsilon_d = x + y\sqrt{d}$ be the fundamental unit of $\Bbbk$, where x and y are integers or semi-integers. Then $\mathrm{rank}(X(\Bbbk_\infty)) \geq 2$, $\Bbbk_\infty/\Bbbk$ is totally ramified and $X(\Bbbk_\infty) \simeq A(\Bbbk)$ if and only if d takes one of the following forms:

1. $d = \delta p_1 p_2 q$, where $\delta \in \{1, 2\}$ and $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ are three distinct prime numbers such that $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$ and $\left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right) = -1$. **In this case** $X(\Bbbk_\infty) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

2. $d = \delta p_1 p_2 q$, where $\delta \in \{1, 2\}$ and $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ are three distinct prime numbers such that $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$ and $\left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = -(-1)^\delta$ and $\frac{2}{\delta}q(x-1)$ is not a square in $\mathbb{N}$. **In this case** $X(\Bbbk_\infty) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$, where $m \geq 2$ is an integer.

3. $d = \delta q_1 q_2 q_3$, where $\delta \in \{1, 2\}$ and $q_1 \equiv q_2 \equiv q_3 \equiv 3 \pmod 4$ are three distinct prime numbers such that $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = -\left(\frac{2}{q_3}\right) = 1$ and $\left(\frac{q_1}{q_2}\right)\left(\frac{q_1}{q_3}\right) = -1$. **In this case** $X(\Bbbk_\infty) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

4. $d = \delta q_1 q_2 q_3$, where $\delta \in \{1, 2\}$ and $q_1 \equiv q_2 \equiv q_3 \equiv 3 \pmod 4$ are three distinct prime numbers such that $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = -\left(\frac{2}{q_3}\right) = 1$, $\left(\frac{q_1}{q_2}\right) = \left(\frac{q_1}{q_3}\right) = -(-1)^\delta$ and $\delta q_1(x-1)$ is not a square in $\mathbb{N}$. **In this case** $X(\Bbbk_\infty) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^m\mathbb{Z}$, where $m \geq 2$ is an integer.

The plan of this paper is the following. In Section 2, we collect some preliminary results that will be used later. In Section 3, we compute the unit group of some real biquadratic fields that will be useful for the sequel. Section 4 is dedicated to the investigation of the forms of the positive square-free integers d such that $\Bbbk = \mathbb{Q}(\sqrt{d})$, $\mathrm{rank}(A(\Bbbk_\infty)) \geq 2$, $\Bbbk_\infty/\Bbbk$ is totally ramified and $X(\Bbbk_\infty) \simeq A(\Bbbk)$ and in the last section (i.e. Section 5) we complete the proof of our main theorem.

Notations

Throughout this paper, we adopt the following notations:

d : A square-free positive integer such that $d \neq 2$.

$\Bbbk$: The quadratic field $\mathbb{Q}(\sqrt{d})$.

$\Bbbk'$: The quadratic field $\mathbb{Q}(\sqrt{\nu d})$, where $\nu = \frac{1}{2}$ or 2 according to whether d is even or not.

$\Bbbk_\infty$: The cyclotomic $\mathbb{Z}_2$-extension of $\Bbbk$.

$\Bbbk_1$: The biquadratic field $\mathbb{Q}(\sqrt{2}, \sqrt{d})$.

$A(\Bbbk_n)$: The 2-class group of $\Bbbk_n$, the subfield of $\Bbbk_\infty$ of degree 2^n over $\Bbbk$.

$A(\Bbbk_0)$: The 2-class group of $\Bbbk$.

h: The 2-class number of $\Bbbk_1 = \mathbb{Q}(\sqrt{2}, \sqrt{d})$.

$h(d)$: The 2-class number of $\mathbb{Q}(\sqrt{d})$.

ε_d: The fundamental unit of $\mathbb{Q}(\sqrt{d})$.

$A(k)$: The 2-class group of any number field k.

$N_{K/k}$: The relative norm map in an extension K/k of number fields.

p, p_i: Positive prime integers congruent to 1 modulo 4.

q, q_i: Positive prime integers congruent to -1 modulo 4.

Q: The unit index of $\Bbbk_1$, i.e., $Q = [E_{\Bbbk_1} : \langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle]$, where $E_{\Bbbk_1}$ denotes the unit group of $\Bbbk_1$ and ε_i, for $i = 1, 2, 3$, denote the fundamental units of its three quadratic subfields.

$\left(\frac{*}{*}\right)$: The Legendre symbol.

2 Preliminary results

We start this section by collecting some results that will be used in the sequel. The first result concerns the relative norm of a ramified extension.

Lemma 2.1 ([4]). *Let K/k be a ramified quadratic extension, then the following homomorphism is surjective*

$$\begin{array}{llll} N_{K/k}: & A(K) & \longrightarrow & A(k) \\ & [I] & \longmapsto & N_{K/k}([I]), \end{array}$$

Moreover, $A(K) \simeq A(k)$ if and only if $\#A(K) = \#A(k)$.

Theorem 2.2 ([8]). Let k_∞/k be a $\mathbb{Z}_p$-extension and n_0 be a natural number such that any prime ideal ramified in k_∞/k is totally ramified in k_∞/k_{n_0}. Let $A_p(k_n)$ denote the p-class group of the n-th layer k_n of the extension k_∞/k.

i. If $A_p(k_{n_0}) \simeq A_p(k_{n_0+1})$, then $A_p(k_{n_0}) \simeq A_p(k_n)$ for any natural number $n \geq n_0$. In this case $\lambda = \mu = 0$.

ii. If $\operatorname{rank}\left(A_p(k_{n_0})\right) = \operatorname{rank}\left(A_p(k_{n_0+1})\right)$, then $\operatorname{rank}\left(A_p(k_{n_0})\right) = \operatorname{rank}\left(A_p(k_n)\right)$ for any natural number $n \geq n_0$. In this case $\mu = 0$.

Remark 2.3. Notice that according to Lemma 2.1 and Theorem 2.2, the question that we suggested in the introduction is equivalent to the following:

> What are real quadratic fields k such that k_∞/k is totally ramified and $\#X(k_\infty) = \#A(k)$?

Lemma 2.4 ([3, Lemma 5.7]). *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be three distinct prime integers, $\delta \in \{1,2\}$ a natural number, and $\Bbbk = \mathbb{Q}\left(\sqrt{\delta p_1 p_2 q}\right)$. Then we have:*

i. The only prime ideal of $\Bbbk$ above 2 is ramified in $\Bbbk(\sqrt{2})$.

ii. $\operatorname{rank}(A(\Bbbk)) = \operatorname{rank}\left(A(\Bbbk(\sqrt{2}))\right) = 2$ if and only if $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$.

Lemma 2.5 ([26, Lemme 4.9]). *Let $q_1 \equiv q_2 \equiv q_3 \equiv 3 \pmod 4$ be three distinct prime integers, $\delta \in \{1,2\}$ a natural number, and $\Bbbk = \mathbb{Q}\left(\sqrt{\delta q_1 q_2 q_3}\right)$. Then we have:*

i. The only prime ideal of $\Bbbk$ above 2 is ramified in $\Bbbk(\sqrt{2})$.

ii. $\operatorname{rank}(A(\Bbbk)) = \operatorname{rank}\left(A(\Bbbk(\sqrt{2}))\right) = 2$ if and only if at most one element of $\left\{\left(\frac{2}{q_1}\right), \left(\frac{2}{q_2}\right), \left(\frac{2}{q_3}\right)\right\}$ is equal to 1.

Proposition 2.6 ([3, Proposition 5.4]). *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be three distinct prime integers satisfying the following conditions:*

$$\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1 \quad \text{and} \quad \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = -(-1)^\delta.$$

Let $\delta \in \{1,2\}$ be a natural number. Then the 4-rank of the class group of $k = \mathbb{Q}\left(\sqrt{\delta p_1 p_2 q}\right)$ is equal to 1.

Proposition 2.7 ([26, p. 119, Proposition 4.2]). *Let $q_1 \equiv q_2 \equiv q_3 \equiv 3 \pmod 4$ be three distinct prime integers satisfying the following conditions:*

$$\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = -\left(\frac{2}{q_3}\right) = 1.$$

Let $\delta \in \{1,2\}$ be a natural number.

i. If $\left(\frac{q_1}{q_2}\right) = \left(\frac{q_1}{q_3}\right) = -(-1)^\delta$, then the 4-rank of the class group of $k = \mathbb{Q}\left(\sqrt{\delta q_1 q_2 q_3}\right)$ is equal to 1.

ii. If $\left(\frac{q_1}{q_2}\right) = (-1)^\delta$ or $\left(\frac{q_1}{q_3}\right) = (-1)^\delta$, then the 4-rank of the class group of $k = \mathbb{Q}\left(\sqrt{\delta q_1 q_2 q_3}\right)$ is equal to 0.

3 Fundamental System of Units of some biquadratic fields

To prove our main result, we need to calculate the Fundamental System of Units (FSU) of some biquadratic number fields. We start by calculating a Fundamental System of Units of the biquadratic field $\mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2 q})$ which is a quadratic extension of $\Bbbk = \mathbb{Q}(\sqrt{p_1 p_2 q})$. Note that in [20], we fond the FSUs of some multi-quadratic number fields.

Lemma 3.1. *Let $\Bbbk_1 = \mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2 q})$ such that $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$ and $\left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = 1$. Let $\varepsilon_{p_1 p_2 q} = x + y\sqrt{p_1 p_2 q}$ be the fundamental unit of $\Bbbk = \mathbb{Q}(\sqrt{p_1 p_2 q})$. Then a FSU of $\Bbbk_1$ is*

$$\{\varepsilon_2, \varepsilon_{p_1p_2q}, \sqrt{\varepsilon_{p_1p_2q}\varepsilon_{2p_1p_2q}}\} \quad \text{or} \quad \{\varepsilon_2, \varepsilon_{p_1p_2q}, \varepsilon_{2p_1p_2q}\},$$

depending on whether $2q(x-1)$ is or not a square in $\mathbb{N}$.

Proof. Let $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$ and $\left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = 1$. Denote by $\varepsilon_{p_1p_2q} = x + y\sqrt{p_1p_2q}$ the fundamental unit of $\mathbb{Q}\left(\sqrt{p_1p_2q}\right)$ and by $\varepsilon_{2p_1p_2q} = z + t\sqrt{2p_1p_2q}$ that of $\mathbb{Q}\left(\sqrt{2p_1p_2q}\right)$, where x, y, z and t are integers. As the norm of $\varepsilon_{2p_1p_2q}$ is 1, so by the unique factorization theorem (fundamental theorem of arithmetic) applied to $z^2 - 1 = 2p_1p_2qt^2$ in $\mathbb{Z}$ and by [1, Lemma 5] there exist t_1 and t_2 in $\mathbb{Z}$ satisfying the unique possibility:

$$(1) \left\{ \begin{array}{l} z \pm 1 = 2qt_1^2, \\ z \mp 1 = p_1p_2t_2^2, \end{array} \right. \quad \text{where } t = t_1t_2. \quad \sqrt{\varepsilon_{2p_1p_2q}} = t_1\sqrt{q} + \tfrac{t_2}{2}\sqrt{2p_1p_2}.$$

Similarly, as $\varepsilon_{p_1p_2q}$ is of norm 1, so by the unique factorization theorem applied to $x^2 - 1 = p_1p_2qy^2$ in $\mathbb{Z}$ and [1, Lemma 5] there exist y_1 and y_2 in $\mathbb{Z}$ such that:

A. For $\left(\frac{p_1}{p_2}\right) = 1$, the possibilities are:

$$(2) \left\{ \begin{array}{l} x \pm 1 = 2p_1y_1^2, \\ x \mp 1 = 2p_2qy_2^2, \end{array} \right. \quad \text{where } y = 2y_1y_2. \quad \sqrt{\varepsilon_{p_1p_2q}} = y_1\sqrt{p_1} + y_2\sqrt{p_2q}.$$

or

$$(3) \left\{ \begin{array}{l} x \pm 1 = 2p_2y_1^2, \\ x \mp 1 = 2p_1qy_2^2, \end{array} \right. \quad \text{where } y = 2y_1y_2. \quad \sqrt{\varepsilon_{p_1p_2q}} = y_1\sqrt{p_2} + y_2\sqrt{p_1q}.$$

or

$$(4) \left\{ \begin{array}{l} x - 1 = 2qy_1^2, \\ x + 1 = 2p_1p_2y_2^2, \end{array} \right. \quad \text{where } y = 2y_1y_2. \quad \sqrt{\varepsilon_{p_1p_2q}} = y_1\sqrt{q} + y_2\sqrt{p_1p_2}.$$

From ((1) and (2)) or ((1) and (3)), we conclude that $\sqrt{\varepsilon_{p_1p_2q}} \notin \mathbb{k}_1$, $\sqrt{\varepsilon_{2p_1p_2q}} \notin \mathbb{k}_1$ and $\sqrt{\varepsilon_{p_1p_2q}\varepsilon_{2p_1p_2q}} \notin \mathbb{k}_1$. Therefore, an FSU of $\mathbb{k}_1$ is $\left\{\varepsilon_2, \varepsilon_{p_1p_2q}, \varepsilon_{2p_1p_2q}\right\}$.

From (1) and (4), we get that $\sqrt{\varepsilon_{p_1p_2q}} \notin \mathbb{k}_1$, $\sqrt{\varepsilon_{2p_1p_2q}} \notin \mathbb{k}_1$ and $\sqrt{\varepsilon_{p_1p_2q}\varepsilon_{2p_1p_2q}} \in \mathbb{k}_1$. Consequently, an FSU of $\mathbb{k}_1$ is $\left\{\varepsilon_2, \varepsilon_{p_1p_2q}, \sqrt{\varepsilon_{p_1p_2q}\varepsilon_{2p_1p_2q}}\right\}$.

B. For $\left(\frac{p_1}{p_2}\right) = -1$, the possibilities are:

$$(5) \left\{ \begin{array}{l} x \pm 1 = p_1y_1^2, \\ x \mp 1 = p_2qy_2^2, \end{array} \right. \quad \text{where } y = y_1y_2. \quad \sqrt{\varepsilon_{p_1p_2q}} = \tfrac{y_1}{2}\sqrt{2p_1} + \tfrac{y_2}{2}\sqrt{2p_2q}.$$

or

$$(6) \left\{ \begin{array}{l} x \pm 1 = p_2y_1^2, \\ x \mp 1 = p_1qy_2^2, \end{array} \right. \quad \text{where } y = y_1y_2. \quad \sqrt{\varepsilon_{p_1p_2q}} = \tfrac{y_1}{2}\sqrt{2p_2} + \tfrac{y_2}{2}\sqrt{2p_1q}.$$

or

$$(7) \left\{ \begin{array}{l} x - 1 = 2qy_1^2, \\ x + 1 = 2p_1p_2y_2^2, \end{array} \right. \quad \text{where } y = 2y_1y_2. \quad \sqrt{\varepsilon_{p_1p_2q}} = y_1\sqrt{q} + y_2\sqrt{p_1p_2}.$$

From ((1) and (5)) or ((1) and (6)), we conclude that $\sqrt{\varepsilon_{p_1p_2q}} \notin \mathbb{k}_1$, $\sqrt{\varepsilon_{2p_1p_2q}} \notin \mathbb{k}_1$ and $\sqrt{\varepsilon_{p_1p_2q}\varepsilon_{2p_1p_2q}} \notin \mathbb{k}_1$. Consequently, an FSU of $\mathbb{k}_1$ is $\left\{\varepsilon_2, \varepsilon_{p_1p_2q}, \varepsilon_{2p_1p_{2q}}\right\}$.

From (1) and (7), we conclude that $\sqrt{\varepsilon_{p_1p_2q}} \notin \mathbb{k}_1$, $\sqrt{\varepsilon_{2p_1p_2q}} \notin \mathbb{k}_1$ and $\sqrt{\varepsilon_{p_1p_2q}\varepsilon_{2p_1p_2q}} \in \mathbb{k}_1$. Consequently, an FSU of $\mathbb{k}_1$ in this case is $\left\{\varepsilon_2, \varepsilon_{p_1p_2q}, \sqrt{\varepsilon_{p_1p_2q}\varepsilon_{2p_1p_2q}}\right\}$.

□

By a process analogous to that used to prove Lemma 3.1, we justify the following lemmas.

Lemma 3.2. *Let $\mathbb{k}_1 = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2q})$ such that $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = -1$. Let $\varepsilon_{2p_1p_2q} = z + t\sqrt{2p_1p_2q}$ be the fundamental unit of $\mathbb{Q}(\sqrt{2p_1p_2q})$. Then an FSU of $\mathbb{k}_1$ is*

$$\{\varepsilon_2, \varepsilon_{p_1p_2q}, \sqrt{\varepsilon_{p_1p_2q}\varepsilon_{2p_1p_2q}}\} \quad \text{or} \quad \{\varepsilon_2, \varepsilon_{p_1p_2q}, \varepsilon_{2p_1p_2q}\}$$

depending on whether $q(z-1)$ *is or not a square in* $\mathbb{N}$.

Lemma 3.3. *Let* $\mathbb{k}_1 = \mathbb{Q}(\sqrt{2}, \sqrt{q_1q_2q_3})$ *such that* $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = -\left(\frac{2}{q_3}\right) = 1$ *and* $\left(\frac{q_1}{q_2}\right) = \left(\frac{q_1}{q_3}\right) = 1$. *Let* $\varepsilon_{q_1q_2q_3} = x + y\sqrt{q_1q_2q_3}$ *be the fundamental unit of* $\mathbb{Q}(\sqrt{q_1q_2q_3})$. *Then an FSU of* $\mathbb{k}_1$ *is*

$$\{\varepsilon_2, \varepsilon_{q_1q_2q_3}, \sqrt{\varepsilon_{q_1q_2q_3}\varepsilon_{2q_1q_2q_3}}\} \text{ or } \{\varepsilon_2, \varepsilon_{q_1q_2q_3}, \varepsilon_{2q_1q_2q_3}\},$$

depending on whether $q_1(x-1)$ *is or not a square in* $\mathbb{N}$.

Lemma 3.4. *Let* $\mathbb{k}_1 = \mathbb{Q}(\sqrt{2}, \sqrt{q_1q_2q_3})$ *such that* $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = -\left(\frac{2}{q_3}\right) = 1$ *and* $\left(\frac{q_1}{q_2}\right) = \left(\frac{q_1}{q_3}\right) = -1$. *Let* $\varepsilon_{2q_1q_2q_3} = z + t\sqrt{2q_1q_2q_3}$ *be the fundamental unit of* $\mathbb{Q}(\sqrt{2q_1q_2q_3})$. *Then an FSU of* $\mathbb{k}_1$ *is*

$$\{\varepsilon_2, \varepsilon_{q_1q_2q_3}, \sqrt{\varepsilon_{q_1q_2q_3}\varepsilon_{2q_1q_2q_3}}\} \text{ or } \{\varepsilon_2, \varepsilon_{q_1q_2q_3}, \varepsilon_{2q_1q_2q_3}\},$$

depending on whether $2q_1(z-1)$ *is or not a square in* $\mathbb{N}$.

4 Integers d such that $\mathbb{k}_1/\mathbb{k}$ is ramified, $A(\mathbb{k}) \simeq A(\mathbb{k}_1)$ and rank$(A(\mathbb{k})) \geq 2$

Recall that $\mathbb{k} = \mathbb{Q}(\sqrt{d})$, d is a square-free integer not equal to 2, and $\mathbb{k}_1$ is the biquadratic number field $\mathbb{Q}(\sqrt{2}, \sqrt{d})$. Recall also that our main task is to determine real quadratic fields $\mathbb{k}$ satisfying the conditions:

$$\text{rank}(X(\mathbb{k}_\infty)) \geq 2,\ \mathbb{k}_\infty/\mathbb{k} \text{ is totally ramified and } X(\mathbb{k}_\infty) \simeq A(\mathbb{k}).$$

By these conditions, one deduces that $\mathbb{k}_1/\mathbb{k}$ is ramified, $A(\mathbb{k}) \simeq A(\mathbb{k}_1)$ and rank$(A(\mathbb{k})) \geq 2$. In this section, we will determine integers d satisfying the three above conditions.

Note first that by the ramification theory, the extension $\mathbb{k}_1/\mathbb{k}$ is ramified if and only if one of the following two conditions is satisfied:

i. d is an odd natural number.

ii. d is an even natural number and $\frac{d}{2} \equiv 3 \pmod 4$.

On the other hand, the three quadratic subfields of $\mathbb{k}_1$ are $\mathbb{Q}(\sqrt{2})$, $\mathbb{k} = \mathbb{Q}(\sqrt{d})$, and $\mathbb{k}' = \mathbb{Q}(\sqrt{d'})$, where $d' = \nu d$ with $\nu = \frac{1}{2}$ or 2 according to whether d is even or not. So the 2-class number of $\mathbb{k}_1$ is given in [27] by:

$$h = \frac{1}{4}Qh(2)h(d)h(\mathbb{k}') = \frac{1}{4}Qh(d)h(\mathbb{k}').$$

If the extension $\mathbb{k}_1/\mathbb{k}$ is ramified, then $A(\mathbb{k}) \simeq A(\mathbb{k}_1)$ if and only if $h = h(d)$. Hence, the fact that $h = h(d)$ implies that one of the following conditions holds:

i. $Q = 1$ and $h(\mathbb{k}') = 4$.

ii. $Q = 2$ and $h(\mathbb{k}') = 2$.

iii. $Q = 4$ and $h(\mathbb{k}') = 1$.

From which we get that rank$(A(\mathbb{k}')) \leq 2$. Let p, p_i (resp. q, q_i) be primes congruent to 1 modulo 4 (resp. 3 modulo 4). Since rank$(A(\mathbb{k}')) \leq 2$, then, by genus theory, d' takes one of the values in the following two sets:

$$d' \in \{2p_1p_2, 2pq_1q_2, p_1p_2p_3, p_1p_2q_1q_2, q_1q_2q_3q_4, p_1p_2q, 2p_1p_2q, q_1q_2q_3, 2q_1q_2q_3\} \text{ or}$$
$$d' \in \{p, q, 2q, q_1q_2, 2p, p_1p_2, pq, 2pq, 2q_1q_2, pq_1q_2\}.$$

Hence

$$d \in \{p_1p_2, pq_1q_2, 2p_1p_2p_3, 2p_1p_2q_1q_2, 2q_1q_2q_3q_4, 2p_1p_2q, p_1p_2q, 2q_1q_2q_3, q_1q_2q_3\} \text{ or}$$
$$d \in \{2p, 2q, q, 2q_1q_2, p, 2p_1p_2, pq, 2pq, q_1q_2, 2pq_1q_2\}.$$

The forms $d = 2p$, $d = 2q$, $d = q$, $d = 2q_1q_2$. $d = p$, $d = p_1p_2$, $d = pq_1q_2$, $d = pq$, $d = 2pq$ and $d = q_1q_2$ are rejected, since $\mathrm{rank}(A(\Bbbk))$ must be ≥ 2. The forms $d = 2p_1p_2$, $d = 2p_1p_2p_3$, $d = 2p_1p_2q_1q_2$, $d = 2q_1q_2q_3q_4$ and $d = 2pq_1q_2$ are rejected too, since in these cases $\Bbbk_1/\Bbbk$ is not ramified. Thus,

$$d \in \{p_1p_2q, 2p_1p_2q, q_1q_2q_3, 2q_1q_2q_3\}\cdot \tag{1}$$

Note that in these four cases $\mathrm{rank}(A(\Bbbk))$ is exactly 2. For each of these forms of $d \in \{p_1p_2q, 2p_1p_2q, q_1q_2q_3, 2q_1q_2q_3\}$, we aim to study the cyclotomic $\mathbb{Z}_2$-extension of $\Bbbk$ and to deduce the structure of its Iwasawa module $X(\Bbbk_\infty)$. We have all the necessary background to do this.

5 The proof of the main result

According to the previous section (see the set (1)), we have four cases to distinguish.

5.1 First case: $\Bbbk = \mathbb{Q}(\sqrt{p_1p_2q})$

Theorem 5.1. Let $\Bbbk = \mathbb{Q}(\sqrt{p_1p_2q})$ where $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ are prime integers, $\varepsilon_{p_1p_2q} = x + y\sqrt{p_1p_2q}$ be the fundamental unit of $\Bbbk$ and $\Bbbk_\infty$ the cyclotomic $\mathbb{Z}_2$-extension of $\Bbbk$, denote by $A(\Bbbk_n)$ the 2-class group of the n-th layer $\Bbbk_n$. Then $A(\Bbbk) \simeq A(\Bbbk_n)$, for all $n \geq 1$, if and only if one of the following conditions holds:

i. $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$ and $\left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right) = -1$. In this case $X(\Bbbk_\infty) \simeq A(\Bbbk) \simeq (2,2)$, this result has already been proven in [19].

ii. $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1, \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = 1$ and $2q(x-1)$ is not a square in $\mathbb{N}$. In this case $X(\Bbbk_\infty) \simeq A(\Bbbk) \simeq (2, 2^m)$, where $m \geq 2$.

Proof. By Lemma 2.4, $\mathrm{rank}(A(\Bbbk)) = \mathrm{rank}(A(\Bbbk_1)) = 2$ if and only if $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$. The 2-class number h of $\Bbbk_1$ is given in [27] by the following formula:

$$h = \frac{1}{4}Qh(2)h(p_1p_2q)h(2p_1p_2q) = \frac{1}{4}Qh(p_1p_2q)h(2p_1p_2q).$$

Recall that the extension $\Bbbk_1/\Bbbk$ is ramified. It is known that $A(\Bbbk) \simeq A(\Bbbk_1)$ if and only if $h = h(p_1p_2q)$, this implies that (since $h(2p_1p_2q) \geq 4$)

$$A(\Bbbk) \simeq A(\Bbbk_1) \text{ if and only if } Q = 1 \text{ and } h(2p_1p_2q) = 4\cdot$$

On the other hand, we find in [6] that if $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$, then $h(2p_1p_2q) = 4$ if and only if we do not have

$$\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right)\cdot$$

Therefore, $A(\Bbbk) \simeq A(\Bbbk_1)$ if and only if

$$\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1, \left[\left(\frac{q}{p_1}\right) = 1 \text{ or } \left(\frac{q}{p_2}\right) = 1\right] \text{ and } Q = 1\cdot$$

We have two cases to discuss:

1. $\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=-1$ and $\left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right)=-1$, then, according to [19], we have $A(\Bbbk)\simeq A(\Bbbk_n)\simeq X(\Bbbk_\infty)\simeq(2,2)$.

2. $\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=-1, \left(\frac{q}{p_1}\right)=\left(\frac{q}{p_2}\right)=1$ and the unit index of $\Bbbk_1$ is equal to 1. According to Lemma 3.1, we have $Q=1$ if and only if $2q(x-1)$ is not a square in $\mathbb{N}$. Then with these conditions and according to Proposition 2.6, $A(\Bbbk)\simeq A(\Bbbk_1)\simeq(2,2^m)$, where $m\geq 2$.

Finally, since the extension $\Bbbk_1/\Bbbk$ is ramified, then $\Bbbk_\infty/\Bbbk$ is totally ramified. Hence using Theorem 2.2, we deduce the result. □

Example 5.2. Put $\alpha=\left(\frac{2}{p_1}\right)$, $\beta=\left(\frac{2}{p_2}\right)$, $\gamma=\left(\frac{q}{p_1}\right)$, $\delta=\left(\frac{q}{p_2}\right)$, $A(\Bbbk)$ of type n and $A(\Bbbk_1)$ of type n_1, note that a 2-group G is said to be of type (a,b) if $G\simeq\mathbb{Z}/a\mathbb{Z}\times\mathbb{Z}/b\mathbb{Z})$. By using PARI/GP ([25]), we get the following examples.

$d=p_1{\cdot}_2{\cdot}q$	α	β	γ	δ	$2q(x-1)$	n	n_1	$X_\infty(\Bbbk)$
$7511=29\cdot37\cdot7$	-1	-1	1	1	3626	$(8,2)$	$(8,2)$	$(2,8)$
$8671=13\cdot29\cdot23$	-1	-1	1	1	110582847700	$(4,2)$	$(4,2)$	$(2,4)$
$10759=29\cdot53\cdot7$	-1	-1	1	1	9922760718416	$(4,2)$	$(4,2)$	$(2,4)$
$13727=37\cdot53\cdot7$	-1	-1	1	1	373968	$(4,2)$	$(4,2)$	$(2,4)$
$15635=5\cdot53\cdot59$	-1	-1	1	1	368750	$(8,2)$	$(8,2)$	$(2,8)$
$38695=5\cdot109\cdot71$	-1	-1	1	1	34117419628009620	$(4,2)$	$(4,2)$	$(2,4)$
$38831=13\cdot29\cdot103$	-1	-1	1	1	18250697879444	$(4,2)$	$(4,2)$	$(2,4)$
$40339=13\cdot29\cdot107$	-1	-1	1	1	18232532500	$(8,2)$	$(8,2)$	$(2,8)$
$40439=53\cdot109\cdot7$	-1	-1	1	1	10649747036250	$(4,2)$	$(4,2)$	$(2,4)$
$56459=13\cdot101\cdot43$	-1	-1	1	1	242026704	$(8,2)$	$(8,2)$	$(2,8)$
$61087=13\cdot37\cdot127$	-1	-1	1	1	285427396015034	$(4,2)$	$(4,2)$	$(2,4)$

5.2 Second case: $\Bbbk=\mathbb{Q}(\sqrt{2p_1p_2q})$

We treat now the second case $\Bbbk=\mathbb{Q}(\sqrt{2p_1p_2q})$.

Theorem 5.3. Let $\Bbbk=\mathbb{Q}(\sqrt{2p_1p_2q})$ where $p_1\equiv p_2\equiv -q\equiv 1 \pmod 4$ are prime integers, $\varepsilon_{2p_1p_2q}=z+t\sqrt{2p_1p_2q}$ be the fundamental unit of $\Bbbk$ and $\Bbbk_\infty$ the cyclotomic $\mathbb{Z}_2$-extension of $\Bbbk$, denote by $A(\Bbbk_n)$ the 2-class group of the n-th layer $\Bbbk_n$. Then $A(\Bbbk)\simeq A(\Bbbk_n)$, for all $n\geq 1$, if and only if one of the following conditions holds:

i. $\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=-1$ and $\left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right)=-1$. In this case $A(\Bbbk)\simeq X(\Bbbk_\infty)\simeq(2,2)$, this result has already been proven in [19].

ii. $\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=-1$, $\left(\frac{q}{p_1}\right)=\left(\frac{q}{p_2}\right)=-1$ and $q(z-1)$ is not a square in $\mathbb{N}$. In this case $A(\Bbbk)\simeq X(\Bbbk_\infty)\simeq(2,2^m)$, where $m\geq 2$.

Proof. We proceed as in the proof of Theorem 5.1 using the following points:

- $\mathrm{rank}(A(\Bbbk))=\mathrm{rank}(A(\Bbbk_1))=2$ if and only if $\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=-1$ (Lemma 2.4).

- By the 2-class number formula ([27]) $h=\frac{1}{4}Qh(p_1p_2q)h(2p_1p_2q)$, we get $A(\Bbbk)\simeq A(\Bbbk_1)$ if and only if $h=h(2p_1p_2q)$, this implies, since $h(p_1p_2q)\geq 4$, that

$$A(\Bbbk)\simeq A(\Bbbk_1)\ \text{ if and only if }\ Q=1\ \text{ and }\ h(p_1p_2q)=4\cdot$$

1. By [6], if $\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=-1$, then $h(p_1p_2q)=4$ if and only if

$$\left[\left(\frac{q}{p_1}\right)=-1 \text{ or } \left(\frac{q}{p_2}\right)=-1\right].$$

- Hence $A(\mathbb{k})\simeq A(\mathbb{k}_1)$ if and only if

$$\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=-1,\ \left[\left(\frac{q}{p_1}\right)=-1 \text{ or } \left(\frac{q}{p_2}\right)=-1\right] \text{ and } Q=1\cdot$$

- If $\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=-1$ and $\left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right)=-1$, then according to [19], we have $A(\mathbb{k})\simeq A(\mathbb{k}_n)\simeq X(\mathbb{k}_\infty)\simeq (2,2)$.

- Assuming $\left(\frac{2}{p_1}\right)=\left(\frac{2}{p_2}\right)=-1$, $\left(\frac{q}{p_1}\right)=\left(\frac{q}{p_2}\right)=-1$, the unit index Q of $\mathbb{k}_1$ is equal to 1 if and only if $q(z-1)$ is not a square in $\mathbb{N}$ (see Lemma 3.2). Then with these conditions and according to Proposition 2.6, we have $A(\mathbb{k})\simeq A(\mathbb{k}_1)\simeq(2,2^m)$, where $m\geq 2$.

Since the extension $\mathbb{k}_1/\mathbb{k}$ is ramified, then $\mathbb{k}_\infty/\mathbb{k}$ is totally ramified. Using Theorem 2.2, we deduce the result. ☐

Example 5.4. Put $\alpha=\left(\frac{2}{p_1}\right)$, $\beta=\left(\frac{2}{p_2}\right)$, $\gamma=\left(\frac{q}{p_1}\right)$, $\delta=\left(\frac{q}{p_2}\right)$, $A(\mathbb{k})$ of type n and $A(\mathbb{k}_1)$ of type n_1. By using PARI/GP ([25]), we get the following examples.

$d=2\cdot p_1\cdot p_2\cdot q$	α	β	γ	δ	$q(z-1)$	n	n_1	$X_\infty(\mathbb{k})$
$910=2\cdot 5\cdot 13\cdot 7$	-1	-1	-1	-1	1260	$(4,2)$	$(4,2)$	$(2,4)$
$8294=2\cdot 13\cdot 29\cdot 11$	-1	-1	-1	-1	14014	$(4,2)$	$(4,2)$	$(2,4)$
$12470=2\cdot 5\cdot 29\cdot 43$	-1	-1	-1	-1	1930270	$(4,2)$	$(4,2)$	$(2,4)$
$13630=2\cdot 5\cdot 29\cdot 47$	-1	-1	-1	-1	29277086727830	$(4,2)$	$(4,2)$	$(2,4)$
$14030=2\cdot 5\cdot 61\cdot 23$	-1	-1	-1	-1	108552870	$(4,2)$	$(4,2)$	$(2,4)$
$18278=2\cdot 13\cdot 37\cdot 19$	-1	-1	-1	-1	667850	$(4,2)$	$(4,2)$	$(2,4)$
$23374=2\cdot 13\cdot 29\cdot 31$	-1	-1	-1	-1	78470297334	$(8,2)$	$(8,2)$	$(2,8)$
$26182=2\cdot 13\cdot 53\cdot 19$	-1	-1	-1	-1	169059150	$(8,2)$	$(8,2)$	$(2,8)$
$29822=2\cdot 13\cdot 37\cdot 31$	-1	-1	-1	-1	224812	$(8,2)$	$(8,2)$	$(2,8)$
$35510=2\cdot 5\cdot 53\cdot 67$	-1	-1	-1	-1	3636090	$(4,2)$	$(4,2)$	$(2,4)$
$38918=2\cdot 29\cdot 61\cdot 11$	-1	-1	-1	-1	15502762	$(4,2)$	$(4,2)$	$(2,4)$
$40774=2\cdot 29\cdot 37\cdot 19$	-1	-1	-1	-1	19474506	$(4,2)$	$(4,2)$	$(2,4)$
$49166=2\cdot 13\cdot 61\cdot 31$	-1	-1	-1	-1	48878662550	$(4,2)$	$(4,2)$	$(2,4)$
$53846=2\cdot 109\cdot 13\cdot 19$	-1	-1	-1	-1	1304751720283466	$(4,2)$	$(4,2)$	$(2,4)$
$58406=2\cdot 29\cdot 53\cdot 19$	-1	-1	-1	-1	688750	$(8,2)$	$(8,2)$	$(2,8)$

5.3 Third case: $\mathbb{k}=\mathbb{Q}(\sqrt{q_1q_2q_3})$

We consider, in this subsection, the quadratic field $\mathbb{k}=\mathbb{Q}(\sqrt{q_1q_2q_3})$.

Theorem 5.5. Let $\mathbb{k}=\mathbb{Q}(\sqrt{q_1q_2q_3})$ where $q_1\equiv q_2\equiv q_3\equiv -1 \pmod 4$ are prime integers, we denote by $\varepsilon_{q_1q_2q_3}=x+y\sqrt{q_1q_2q_3}$ the fundamental unit of $\mathbb{k}$, by $\mathbb{k}_\infty$ the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{k}$ and by $A(\mathbb{k}_n)$ the 2-class group of the n-th layer $\mathbb{k}_n$. Then $A(\mathbb{k})\simeq A(\mathbb{k}_n)$, for all $n\geq 1$, if and only if one of the following conditions holds:

i. $\left(\frac{2}{q_1}\right)=-\left(\frac{2}{q_2}\right)=-\left(\frac{2}{q_3}\right)=1$ and $\left(\frac{q_1}{q_2}\right)\left(\frac{q_1}{q_3}\right)=-1$. In this case $A(\mathbb{k})\simeq X(\mathbb{k}_\infty)\simeq(2,2)$, this result has already been proven in [19].

ii. $\left(\frac{2}{q_1}\right)=-\left(\frac{2}{q_2}\right)=-\left(\frac{2}{q_3}\right)=1$, $\left(\frac{q_1}{q_2}\right)=\left(\frac{q_1}{q_3}\right)=1$ and $q_1(x-1)$ is not a square in $\mathbb{N}$. In this case $A(\mathbb{k})\simeq X(\mathbb{k}_\infty)\simeq(2,2^m)$, where $m\geq 2$.

Proof. By the Lemma 2.5, $\operatorname{rank}(A(\mathbb{k}))=\operatorname{rank}(A(\mathbb{k}_1))=2$ if and only if at most one element of $\left\{\left(\frac{2}{q_1}\right),\left(\frac{2}{q_2}\right),\left(\frac{2}{q_3}\right)\right\}$ is equal to 1. The 2-class number of $\mathbb{k}_1$ is ([27]):

$$h=\frac{1}{4}Qh(2)h(q_1q_2q_3)h(2q_1q_2q_3)=\frac{1}{4}Qh(q_1q_2q_3)h(2q_1q_2q_3).$$

Hence $A(\mathbb{k})\simeq A(\mathbb{k}_1)$ if and only if $h=h(q_1q_2q_3)$, this implies that

$$A(\mathbb{k})\simeq A(\mathbb{k}_1)\ \text{ if and only if }\ Q=1\ \text{ and }\ h(2q_1q_2q_3)=4.$$

If $\left(\frac{2}{q_1}\right)=\left(\frac{2}{q_2}\right)=\left(\frac{2}{q_3}\right)=-1$, then according to [2], we can not have both $Q=1$ and $h(2q_1q_2q_3)=4$, so $A(\mathbb{k}_1)$ can never be isomorphic to $A(\mathbb{k})$.
If $\left(\frac{2}{q_1}\right)=-\left(\frac{2}{q_2}\right)=-\left(\frac{2}{q_3}\right)=1$, then according to Proposition 2.7, we have

$$h(2q_1q_2q_3)=4\ \text{ if and only if }\left(\frac{q_1}{q_2}\right)=1\text{ or }\left(\frac{q_1}{q_3}\right)=1.$$

Therefore, $A(\mathbb{k})\simeq A(\mathbb{k}_1)$ if and only if

$$\left(\frac{2}{q_1}\right)=-\left(\frac{2}{q_2}\right)=-\left(\frac{2}{q_3}\right)=1,\ \left[\left(\frac{q_1}{q_2}\right)=1\text{ or }\left(\frac{q_1}{q_3}\right)=1\right]\text{ and }Q=1.$$

There are two cases to consider:

i. $\left(\frac{2}{q_1}\right)=-\left(\frac{2}{q_2}\right)=-\left(\frac{2}{q_3}\right)=1$ and $\left(\frac{q_1}{q_2}\right)\left(\frac{q_1}{q_3}\right)=-1$, according [19], we have $A(\mathbb{k})\simeq A(\mathbb{k}_n)\simeq X(\mathbb{k}_\infty)\simeq(2,2)$.

ii. $\left(\frac{2}{q_1}\right)=-\left(\frac{2}{q_2}\right)=-\left(\frac{2}{q_3}\right)=1$ and $\left(\frac{q_1}{q_2}\right)=\left(\frac{q_1}{q_3}\right)=1$, but the unit index Q of $\mathbb{k}_1$ is equal to 1 if and only if $q_1(x-1)$ is not a square in $\mathbb{N}$ (see Lemma 3.3). So with these conditions and according to the Proposition 2.7, we have $A(\mathbb{k})\simeq A(\mathbb{k}_1)\simeq(2,2^m)$, where $m\geq 2$.

Since the extension $\mathbb{k}_1/\mathbb{k}$ is ramified, then $\mathbb{k}_\infty/\mathbb{k}$ is totally ramified. Using Theorem 2.2, we deduce the result. □

Example 5.6. Put $\alpha=\left(\frac{2}{q_1}\right)$, $\beta=\left(\frac{2}{q_2}\right)$, $\gamma=\left(\frac{2}{q_3}\right)$, $\delta=\left(\frac{q_1}{q_2}\right)$, $\theta=\left(\frac{q_1}{q_3}\right)$, $A(\mathbb{k})$ of type n and $A(\mathbb{k}_1)$ of type n_1. By using PARI/GP ([25]), we get the following examples.

$d=q_1\cdot q_2\cdot q_3$	α	β	γ	δ	θ	$q_1(x-1)$	n	n_1	$X_\infty(\mathbb{k})$
$9823=47\cdot 11\cdot 19$	1	−1	−1	1	1	41877	(8,2)	(8,2)	(2,8)
$14663=31\cdot 43\cdot 11$	1	−1	−1	1	1	41261	(4,2)	(4,2)	(2,4)
$22231=47\cdot 43\cdot 11$	1	−1	−1	1	1	48460487071638213	(4,2)	(4,2)	(2,4)
$28303=31\cdot 83\cdot 11$	1	−1	−1	1	1	2278606404555496	(4,2)	(4,2)	(2,4)
$29279=23\cdot 19\cdot 67$	1	−1	−1	1	1	35397	(8,2)	(8,2)	(2,8)
$34279=7\cdot 83\cdot 59$	1	−1	−1	1	1	894772512047559208	(4,2)	(4,2)	(2,4)
$34639=47\cdot 11\cdot 67$	1	−1	−1	1	1	1968125	(4,2)	(4,2)	(2,4)
$38399=47\cdot 43\cdot 19$	1	−1	−1	1	1	20769747710014906	(4,2)	(4,2)	(2,4)
$46079=71\cdot 59\cdot 11$	1	−1	−1	1	1	6310636200	(4,2)	(4,2)	(2,4)
$46759=23\cdot 107\cdot 19$	1	−1	−1	1	1	24961877	(4,2)	(4,2)	(2,4)
$54103=7\cdot 131\cdot 59$	1	−1	−1	1	1	35372454314442980677	(4,2)	(4,2)	(2,4)
$59831=47\cdot 67\cdot 19$	1	−1	−1	1	1	774996143138938	(4,2)	(4,2)	(2,4)

5.4 Fourth case: $\mathbb{k} = \mathbb{Q}(\sqrt{2q_1q_2q_3})$

We now conclude with the last case: $\mathbb{k} = \mathbb{Q}(\sqrt{2q_1q_2q_3})$.

Theorem 5.7. Let $\mathbb{k} = \mathbb{Q}(\sqrt{2q_1q_2q_3})$ where $q_1 \equiv q_2 \equiv q_3 \equiv -1 \pmod 4$ are prime integers, $\varepsilon_{2q_1q_2q_3} = x + y\sqrt{2q_1q_2q_3}$ be the fundamental unit of $\mathbb{k}$ and $\mathbb{k}_\infty$ be the cyclotomic $\mathbb{Z}_2$-extension of $\mathbb{k}$. We denote by $A(\mathbb{k}_n)$ the 2-class group of the n-th layer $\mathbb{k}_n$. Then $A(\mathbb{k}) \simeq A(\mathbb{k}_n)$, for all $n \geq 1$, if and only if one of the following conditions holds:

i. $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = -\left(\frac{2}{q_3}\right) = 1$ and $\left(\frac{q_1}{q_2}\right)\left(\frac{q_1}{q_3}\right) = -1$. In this case $A(\mathbb{k}) \simeq X(\mathbb{k}_\infty) \simeq (2,2)$.

ii. $\left(\frac{2}{q_1}\right) = -\left(\frac{2}{q_2}\right) = -\left(\frac{2}{q_3}\right) = 1$, $\left(\frac{q_1}{q_2}\right) = \left(\frac{q_1}{q_3}\right) = -1$ and $2q_1(x-1)$ is not a square in $\mathbb{N}$. In this case $A(\mathbb{k}) \simeq X(\mathbb{k}_\infty) \simeq (2, 2^m)$, where $m \geq 2$.

Proof. We proceed as in the proof of Theorem 5.5. □

Example 5.8. Put $\alpha = \left(\frac{2}{q_1}\right)$, $\beta = \left(\frac{2}{q_2}\right)$, $\gamma = \left(\frac{2}{q_3}\right)$, $\delta = \left(\frac{q_1}{q_2}\right)$, $\theta = \left(\frac{q_1}{q_3}\right)$, $A(\mathbb{k})$ of type n and $A(\mathbb{k}_1)$ of type n_1. By using PARI/GP ([25]), we get the following examples.

$d = 2 \cdot q_1 \cdot q_2 \cdot q_3$	α	β	γ	δ	θ	$2q_1(x-1)$	n	n_1	$X_\infty(\mathbb{k})$
$16638 = 2 \cdot 47 \cdot 3 \cdot 59$	1	−1	−1	−1	−1	1042648	(4,2)	(4,2)	(2,4)
$23406 = 2 \cdot 47 \cdot 3 \cdot 83$	1	−1	−1	−1	−1	1466776	(8,2)	(8,2)	(2,8)
$40334 = 2 \cdot 7 \cdot 43 \cdot 67$	1	−1	−1	−1	−1	16856	(4,2)	(4,2)	(2,4)
$45582 = 2 \cdot 71 \cdot 3 \cdot 107$	1	−1	−1	−1	−1	60492	(4,2)	(4,2)	(2,4)
$154318 = 2 \cdot 31 \cdot 19 \cdot 131$	1	−1	−1	−1	−1	146072	(8,2)	(8,2)	(2,8)
$186798 = 2 \cdot 191 \cdot 3 \cdot 163$	1	−1	−1	−1	−1	528064372012	(8,2)	(8,2)	(2,8)
$197918 = 2 \cdot 7 \cdot 67 \cdot 211$	1	−1	−1	−1	−1	25847500	(8,2)	(8,2)	(2,8)
$251502 = 2 \cdot 167 \cdot 3 \cdot 251$	1	−1	−1	−1	−1	334668	(4,2)	(4,2)	(2,4)
$267406 = 2 \cdot 31 \cdot 19 \cdot 227$	1	−1	−1	−1	−1	283733247276	(4,2)	(4,2)	(2,4)
$268142 = 2 \cdot 7 \cdot 107 \cdot 179$	1	−1	−1	−1	−1	14912173528	(8,2)	(8,2)	(2,8)
$480238 = 2 \cdot 263 \cdot 11 \cdot 83$	1	−1	−1	−1	−1	45928216	(8,2)	(8,2)	(2,8)
$482942 = 2 \cdot 71 \cdot 19 \cdot 179$	1	−1	−1	−1	−1	110720524	(4,2)	(4,2)	(2,4)
$491854 = 2 \cdot 79 \cdot 11 \cdot 283$	1	−1	−1	−1	−1	608830295084	(8,2)	(8,2)	(2,8)
$494846 = 2 \cdot 271 \cdot 11 \cdot 83$	1	−1	−1	−1	−1	160694489516	(16,2)	(16,2)	(2,16)

Remark 5.9. Let $G = \mathrm{Gal}(\mathcal{L}(\mathbb{k}_\infty)/\mathbb{k}_\infty)$ be the Galois group of the maximal unramified pro-2-extension $\mathcal{L}(\mathbb{k}_\infty)$ of $\mathbb{k}_\infty$, where $\mathbb{k}_\infty$ is the cyclotomic $\mathbb{Z}_2$-extension of a real quadratic field $\mathbb{k}$. Let $\delta = 1$ or 2.

1. Let $\mathbb{k} = \mathbb{Q}(\sqrt{\delta p_1p_2q})$, where $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ are prime integers, we denote by $\varepsilon_{\delta p_1p_2q} = x + y\sqrt{\delta p_1p_2q}$ the fundamental unit of $\mathbb{k}$.

   i. When $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$ and $\left(\frac{q}{p_1}\right)\left(\frac{q}{p_2}\right) = -1$, Y. Mizusawa proved, in [19], that G is isomorphic to a dihedral group or a generalized quaternion group with finite 2-power order. Furthermore, if the absolute norm of the fundamental unit of $\mathbb{Q}\left(\sqrt{p_1p_2}\right)$ is positive, then G is isomorphic to a dihedral group of order $4h(p_1p_2)$ the 2-class number of $\mathbb{Q}\left(\sqrt{p_1p_2}\right)$ (see [21, Theorem 3.3]).

   ii. When $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = -1$, $\left(\frac{q}{p_1}\right) = \left(\frac{q}{p_2}\right) = -(-1)^\delta$ and $\left(\frac{2}{\delta}\right)q(x-1)$ is not a square in $\mathbb{N}$, then, by [3], G is a meta-procyclic pro-2-group.

2. Let $\mathbb{k} = \mathbb{Q}(\sqrt{\delta q_1q_2q_3})$, where $q_1 \equiv q_2 \equiv q_3 \equiv -1 \pmod 4$ are prime integers, we denote by $\varepsilon_{\delta q_1q_2q_3} = x + y\sqrt{\delta q_1q_2q_3}$ the fundamental unit of $\mathbb{k}$.

i. When $\left(\frac{2}{q_1}\right)=-\left(\frac{2}{q_2}\right)=-\left(\frac{2}{q_3}\right)=1$ and $\left(\frac{q_1}{q_2}\right)\left(\frac{q_1}{q_3}\right)=-1$, then by [19] G is isomorphic to the Klein four group $(2,2)$ (see also [28]).

ii. When $\left(\frac{2}{q_1}\right)=-\left(\frac{2}{q_2}\right)=-\left(\frac{2}{q_3}\right)=1$, $\left(\frac{q_1}{q_2}\right)=\left(\frac{q_1}{q_3}\right)=-(-1)^{\delta}$ and $\delta q_1(x-1)$ is not a square in $\mathbb{N}$, then [26, Theorem 4.8] G is an abelian pro -2-group of rank 2.

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