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On the Dynamics of Functionals under Adjoint Operators

Meysam Asadipour[1] and Hamid Rezaei[2]

[1,2] *Department of Mathematics, Yasouj University, Yasouj, Iran.*
[1] *email: Asadipour.mey@gmail.com*

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Abstract. In this article, we focus on the interplay between extended limit sets and mixing dynamics of the adjoint of a linear operators on Banach spaces. We characterize situations in which the set $J^{\mathrm{mix}}_{T^*}(x^*)$ coincides with the classical limit set $L(T^*, x^*)$, particularly under power boundedness assumptions. Moreover, we show that while mixing implies topological transitivity, the converse fails in general by presenting a specific dynamical property that is satisfied by some topologically transitive operators but fails to hold for mixing operators. This distinction reveals that the class of mixing operators is not identical to that of topologically transitive ones.

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1 Introduction

In the classical setting, dynamical systems primarily investigate the long-term behavior of nonlinear transformations or flows, typically arising from differential equations or iterative maps. However, over the past several decades, a parallel theory has evolved—known as linear dynamics—which shifts attention to the iteration of bounded linear operators acting on infinite-dimensional Banach spaces. This field began to mature following Rolewicz's seminal 1969 result, in which he showed that multiplying the backward unilateral shift on ℓ^p spaces by an appropriate scalar yields a bounded operator with a dense orbit, thereby introducing the notion of hypercyclicity [21].
A bounded operator $T\colon X \to X$ on a separable Banach space is called hypercyclic if there exists a vector $x \in X$ whose orbit

$$\{T^n x : n \in \mathbb{N} \cup \{0\},\ T^0 x = x\}$$

is dense in X. Such a vector is said to be hypercyclic, and the set of all such vectors is denoted $HC(T)$. Hypercyclicity reveals a topological manifestation of chaos: the operator drives some vector arbitrarily close to any point in the space under repeated application. This idea is not merely algebraic—it is deeply topological. In fact, Birkhoff's transitivity theorem ensures that hypercyclicity is equivalent to topological transitivity: for any pair of non-empty open subsets $U, V \subset X$, there exists an iterate $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Subsequent developments extended the scope of this theory in several directions. Kitai's Ph.D. thesis in 1982 formulated criteria for hypercyclicity based on the denseness of polynomial images of T, providing functional-analytic tools for its verification [17]. Gethner and Shapiro later expanded the setting to include spaces of holomorphic functions [11], while Bourdon and Feldman established that even partial density of an orbit implies full density under suitable conditions [7].

An essential component of this framework is its interaction with spectral theory. Specifically, the emptiness of the point spectrum of the adjoint T^* corresponds to the denseness of the ranges of all scalar shifts $T-\lambda I$. Consequently, the range of any nonzero polynomial $P(T)$ remains dense whenever T is hypercyclic. The spectral constraints are nontrivial: as noted by Kitai, the spectrum $\sigma(T)$ must intersect the unit circle, a condition later shown to extend to weaker forms of orbit-dense behavior [10]. This line of reasoning naturally brings attention to the adjoint operator T^*, whose spectral characteristics frequently mirror, yet also subtly diverge from, those of T itself. For researchers in operator theory and functional analysis, investigating T^* on the dual space X^* provides essential insight into the geometric and topological behavior of the original operator. In many cases, the failure of hypercyclicity or density conditions for T^* reflects deeper structural asymmetries that are not readily apparent from T alone.

Motivated by the desire to capture a localized version of orbit-density, Costakis and Manoussos introduced the notion of J-class operators in 2008 [8]. Instead of requiring the orbit of a single vector to be dense, this concept allows for approximating points via sequences of iterated images of nearby vectors. Formally, given $T \in B(X)$ and $x \in X$, the extended limit set (or J-set) of x is defined as:

$$J_T(x) = \left\{ y \in X : \begin{array}{l} T^{k_n}x_n \to y \text{ for some sequences } \{x_n\} \subset X,\ \{k_n\} \subset \mathbb{N}, \\ \text{with } x_n \to x \text{ and } k_n < k_{n+1} \text{ for all } n \in \mathbb{N} \end{array} \right\}.$$

We say that x is a J-class vector if $J_T(x) = X$, and refer to T as a J-class operator if it admits at least one such vector. It is evident that hypercyclic operators are necessarily J-class, but the converse fails; there exist operators for which no orbit is dense, yet the J-set of some vectors still covers the entire space.

Though these two classes—hypercyclic and J-class—arise from different formulations, they exhibit striking similarities: the sets $HC(T)$ and J_T are both either empty or dense, they are invariant under the action of T, and in both cases the associated spectra must intersect the unit circle. These shared features reveal an underlying unity between global and local dynamical behavior.

Nonetheless, critical distinctions emerge upon closer inspection. Hypercyclicity fundamentally depends on the separability of the underlying space, whereas J-class operators may exist even on non-separable Banach spaces, such as $\ell^\infty(\mathbb{N})$. Moreover, while the orbit of the zero vector is always the singleton $\{0\}$, its J-set can be unexpectedly large. Furthermore, although hypercyclic operators must have spectral components intersecting the unit circle, one can construct J-class operators whose isolated spectral parts lie strictly outside or inside the unit disk. Lastly, unlike hypercyclic operators, whose invertibility ensures that the inverse is also hypercyclic, J-class operators may lose this property under inversion.For more detailed explorations and generalizations, we refer the reader to [1, 2, 3, 4, 5, 6], [9], [13], [18, 19], [26], [27] and [28].

In the course of studying J-class operators, the the *limit set* of x under $T \in B(X)$ is defined as:

$$L_T(x) = \left\{ y \in X : \begin{array}{l} T^{k_n}x \to y \text{ for some strictly increasing} \\ \text{sequence of positive integers } \{k_n\} \end{array} \right\}$$

was introduced in [8]. And also, for an operator $T \in B(X)$ and $x \in X$, the *mixing extended limit* set (or J^{mix}-set) of x is defined as:

$$J_T^{\text{mix}}(x) = \left\{ y \in X : \ T^n x_n \to y \text{ for some sequences } \{x_n\} \subset X \text{ with } x_n \to x \ \right\}.$$

We say that x is a J^{mix}-class vector if $J_T^{\text{mix}}(x) = X$, and refer to T as a J^{mix}-class operator if it admits at least one such vector.

In this paper, we examine the mixing extended limit set and the classical limit set associated with an arbitrary functional $x^* \in X^*$, under the action of the adjoint operator T^*, where $T \in B(X)$. To highlight the distinction between mixing and topologically transitive operators, we present a particular dynamical behavior that appears in some topologically transitive operators but is absent in the mixing class. This observation proves that the class of mixing operators is a *strictly* contained subset of the broader class of topologically transitive ones.

2 Main Results

2.1 Inclusion and Equality of Mixing Extended Limit Set and Limit Set

In this section, we first introduce the limit set and the mixing extended limit set of $x^* \in X^*$ under $T^* \in B(X^*)$, and then we present several properties of these sets.

Definition 2.1. Let $T: X \to X$ be a linear operator acting on a Banach space X, and let x^* be a given functional in X^*. The *limit set* of x^* under T^*, denoted by $L(T^*, x^*)$, is defined as the set of all vectors $y^* \in X^*$ for which there exists a strictly increasing sequence of natural numbers $\{k_n\}$ such that

$$(T^*)^{k_n} x^* \to y^* \quad \text{as } n \to \infty. \tag{1}$$

In addition to topologically transitive operators, whose definition was provided in the introduction, there is another important class known as *mixing operators*. An operator $T: X \to X$ is said to be mixing if, for every pair of non-empty open subsets $U, V \subset X$, there exists $N \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset \quad \text{for all } n \geqslant N.$$

Therefore, in light of [12, Example 1.43], one can observe that

$$\text{Mixing} \Rightarrow \text{Topological Transitivity} \Leftrightarrow \text{Hypercyclicity}.$$

Definition 2.2. Let X be a Banach space and let $x^* \in X^*$ be a nonzero continuous linear functional. Consider an operator $T \in B(X)$. The set $J^{\mathrm{mix}}_{T^*}(x^*)$ is defined as the collection of all functionals $y^* \in X^*$ for which the following holds:
for every neighborhood U of x^* and every neighborhood V of y^*, there exists $N \in \mathbb{N}$ such that

$$(T^*)^n(U) \cap V \neq \emptyset \quad \text{for all } n \geq N. \tag{2}$$

Proposition 2.3. *Let $x^*, y^* \in X^*$. Then $y^* \in J^{mix}_{T^*}(x^*)$ if and only if there exist a sequence $\{x^*_n\} \subset X$ such that $x^*_n \to x^*$ and $(T^*)^n x^*_n \to y^*$.*

Proof. Assume that $y^* \in J^{\mathrm{mix}}_{T^*}(x^*)$. For each $n \in \mathbb{N}$, consider the neighborhoods

$$U_n = B(x^*, \frac{1}{n}) \subset X^* \text{ and } V_n = B(y^*, \frac{1}{n}) \subset X^*.$$

By definition (2), there exists $k_n \in \mathbb{N}$ such that for all $m \geq k_n$, we have

$$T^m(U_n) \cap V_n \neq \emptyset.$$

In particular, for each n, one can choose $x^*_m \in U_n$ such that $(T^*)^m x^*_m \in V_n$. In the next step, one can choose $k_{n+1} > m$ and subsequently there exists a sequence $\{x^*_n\} \subset X^*$ with $x^*_m \to x^*$ and $(T^*)^m x^*_m \to y^*$, as desired. Note that, the converse direction is immediate from the convergence assumptions. ☐

In a manner similar to the proof shown in the proposition above, the proof of the following proposition can be observed.

Proposition 2.4. *Let $x^*, y^* \in X^*$. Then $y^* \in L(T^*, x^*)$ if and only if for every $N \in \mathbb{N}$ and every neighborhoods U, V of x^*, y^*, respectively, there exists an integer $n > N$ such that*

$$(T^*)^n(U) \cap V \neq \emptyset.$$

Based on the explanations provided so far, it can be observed that for $x^* \in X^*$ and $T \in B(X)$, it is not difficult to observe that

$$L(T^*, x^*) \subseteq J^{\text{mix}}_{T^*}(x^*). \tag{3}$$

The next theorem brings with it a beautiful result in this context.

Theorem 2.5. Let T be a bounded linear operator on a Banach space X and $\{x_n^*\}, \{y_n^*\}$ be two sequences in X such that $x_n^* \longrightarrow x^*$ and $y_n^* \longrightarrow y^*$ for some $x^*, y^* \in X^*$. If

$$y_n^* \in J^{\text{mix}}_{T^*}(x_n^*), \quad \text{for all } n \in \mathbb{N},$$

then $y^* \in J^{\text{mix}}_{T^*}(x^*)$.

Proof. We can find a positive integer k_1 such that

$$\|x^*_{k_1} - x^*\| < \frac{1}{2} \quad \text{and} \quad \|y^*_{k_1} - y^*\| < \frac{1}{2}.$$

Since $y^*_{k_1}$ lies in $J^{\text{mix}}_{T^*}(x^*_{k_1})$, it follows that there exists an integer l_1 and a sequence $\{z_n^*\} \subset X^*$ such that

$$\|z_n^* - x^*_{k_1}\| < \frac{1}{2} \quad \text{and} \quad \|(T^*)^n z_n^* - y^*_{k_1}\| < \frac{1}{2} \quad \text{for all}\, n \geq l_1.$$

Consequently, we obtain the following estimates:

$$\|z_n^* - x^*\| < 1 \quad \text{and} \quad \|(T^*)^n z_n^* - y^*\| < 1 \quad \text{for all}\, n \geq l_1.$$

Repeating this process, we can find a new integer $l_2 > l_1$ and a sequence $\{w_n^*\} \subset X^*$ such that

$$\|w_n^* - x^*\| < \frac{1}{2} \quad \text{and} \quad \|(T^*)^n w_n^* - y^*\| < \frac{1}{2} \quad \text{for all}\, n \geq l_2.$$

Now define $v_n^* = z_n^*$ for all $l_1 \leq n < l_2$, so that

$$\|v_n^* - x^*\| < 1 \quad \text{and} \quad \|(T^*)^n v_n^* - y^*\| < 1.$$

By continuing this procedure inductively, we can construct a strictly increasing sequence of integers $\{n_k\}$ and a sequence $\{v_n^*\} \subset X^*$ such that for $n \geq n_k$, we have

$$\|v_n^* - x^*\| < \frac{1}{k} \quad \text{and} \quad \|(T^*)^n v_n^* - y^*\| < \frac{1}{k}.$$

Now, consider any $\epsilon > 0$. Since $\frac{1}{k_0} < \epsilon$ for some integer k_0, it follows that for $n \geq n_{k_0}$, we obtain

$$\|v_n^* - x^*\| < \frac{1}{k_0} < \epsilon \quad \text{and} \quad \|(T^*)^n v_n^* - y^*\| < \frac{1}{k_0} < \epsilon.$$

Thus, the proof is concluded, as required by Proposition 2.3. ☐

An immediate consequence of the previous lemma is the following corollary:

Corollary 2.6. *For all $x^* \in X^*$, the set $J^{mix}_{T^*}(x^*)$ is closed and T^*-invariant.*

Subsequently, when T^* is invertible, the following important results can be verified:

Proposition 2.7. *Let $T^*\colon X^* \to X^*$ be an invertible operator. Then*

$$(T^*)^{-1}(J_{T^*}^{mix}(x^*)) = J_{T^*}^{mix}(x^*)$$

for every $x^* \in X^*$.

Proof. According to Corollary 2.6, we know that

$$J_{T^*}^{\mathrm{mix}}(x^*) \subseteq (T^*)^{-1}(J_{T^*}^{\mathrm{mix}}(x^*))$$

for every $x^* \in X^*$. To prove the equality, let $y^* \in (T^*)^{-1}(J_{T^*}^{\mathrm{mix}}(x^*))$. Then by definition, $T^*y^* \in J_{T^*}^{\mathrm{mix}}(x^*)$. Thus, there exist a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$ and a sequence $\{x_n^*\} \subset X^*$ such that

$$x_n^* \to x^* \quad \text{and} \quad (T^*)^{k_n} x_n^* \to T^*y^*.$$

Applying the continuity of T^*, we consider the sequence $(T^*)^{k_n-1}x_n$. Since

$$(T^*)^{k_n} x_n^* = (T^*)\Big((T^*)^{k_n-1}x_n\Big) \to T^*y^*,$$

and T^* is continuous, it follows that

$$(T^*)^{k_n-1}x_n^* \to y^*.$$

Therefore, $y^* \in J_{T^*}^{\mathrm{mix}}(x^*)$, completing the proof that $(T^*)^{-1}J_{T^*}^{\mathrm{mix}}(x^*) \subset J_{T^*}^{\mathrm{mix}}(x^*)$. ☐

Proposition 2.8. *Let T^* be an invertible operator on a Banach space X^*, and let $x^*, y^* \in X^*$. Then $y^* \in J_{T^*}^{mix}(x^*)$ if and only if $x^* \in J_{(T^*)^{-1}}^{mix}(y^*)$.*

Proof. Assume that $y^* \in J_{T^*}^{\mathrm{mix}}(x^*)$. Then there exist a strictly increasing sequence of natural numbers $\{k_n\}$ and a sequence $\{x_n^*\} \subset X^*$ satisfying

$$x_n^* \to x^* \quad \text{and} \quad (T^*)^{k_n} x_n^* \to y^*.$$

Define $y_n^* := (T^*)^{k_n} x_n^*$, so we have $y_n^* \to y^*$. Since T^* is an invertible operator, applying the inverse powers yields

$$(T^*)^{-k_n} y_n^* = x_n^* \to x^*.$$

This shows that $x^* \in J_{(T^*)^{-1}}^{\mathrm{mix}}(y^*)$. The converse implication can be shown in a similar manner, completing the proof. ☐

In the next theorem, we present a condition under which equality holds in expression (3).

Proposition 2.9. *Let $T^*\colon X^* \to X^*$ be a bounded linear operator. If T^* is power bounded, then for every $x^* \in X^*$, we have $J_{T^*}^{mix}(x^*) = L(T^*, x^*)$.*

Proof. Assume that T^* is power bounded. Then there exists a constant $M > 0$ such that

$$\|(T^*)^n\| \leq M \quad \text{for all } n \in \mathbb{N}.$$

Fix an arbitrary $x^* \in X^*$. If $J_{T^*}^{\mathrm{mix}}(x^*) = \emptyset$, the result holds trivially. Since it is always true that $L(T^*, x^*) \subseteq J_{T^*}^{\mathrm{mix}}(x^*)$, (see (3)), it is enough to show that $J_{T^*}^{\mathrm{mix}}(x^*) \subseteq L(T^*, x^*)$.
To prove this, suppose $y^* \in J_{T^*}^{\mathrm{mix}}(x^*)$. By the definition of $J_{T^*}^{\mathrm{mix}}(x^*)$, there exists a sequence $\{x_n^*\} \subset X^*$ such that

$$x_n^* \to x^* \quad \text{and} \quad (T^*)^n x_n^* \to y^*.$$

Using the triangle inequality and the power boundedness of T^*, we obtain the following estimate:

$$\begin{aligned} \|(T^*)^n x^* - y^*\| &\leq \|(T^*)^n x^* - (T^*)^n x_n^*\| + \|(T^*)^n x_n^* - y^*\| \\ &\leq M\|x^* - x_n^*\| + \|(T^*)^n x_n^* - y^*\|. \end{aligned} \tag{4}$$

As $n \to \infty$, both terms on the right-hand side of (4) tend to zero. Therefore, we conclude that

$$y^* \in L(T^*, x^*),$$

which shows that $J_{T^*}^{\mathrm{mix}}(x^*) \subseteq L(T^*, x^*)$. This completes the proof. □

The lemma below serves as a fundamental step in the proof of Proposition 2.11.

Lemma 2.10. *Let $T\colon X \to X$ be a linear operator. If $J_{T^*}^{mix}(x^*) = X^*$ for some non-zero vector $x^* \in X^*$, then $J_{T^*}^{mix}(\lambda x^*) = X^*$ for every scalar $\lambda \in \mathbb{C}$.*

Proof. First, observe that for any nonzero scalar $\lambda \in \mathbb{C}$, the equality $J_{T^*}^{\mathrm{mix}}(\lambda x^*) = X^*$ follows directly from Proposition 2.3, due to the continuity and linearity of scalar multiplication.
It remains to verify that $J_{T^*}^{\mathrm{mix}}(0) = X^*$. To that end, let $\{\lambda_n\} \subset \mathbb{C} \setminus \{0\}$ be a sequence converging to zero and $y^* \in X^*$. Since $J_{T^*}^{\mathrm{mix}}(\lambda_n x^*) = X^*$, we have $y^* \in J_{T^*}^{\mathrm{mix}}(\lambda_n x^*)$ for each $n \in \mathbb{N}$. Since $\lambda_n x \to 0$, an application of Lemma 2.5 implies $y^* \in J_{T^*}^{\mathrm{mix}}(0)$. As this holds for arbitrary $y^* \in X^*$, it follows that $J_{T^*}^{\mathrm{mix}}(0) = X^*$. □

Proposition 2.11. *Let $T\colon X \to X$ be a bounded linear operator on a Banach space X. Define the set*

$$J_{T^*}^{mix} := \{x^* \in X^* : J_{T^*}^{mix}(x^*) = X^*\}.$$

Then $J_{T^*}^{mix}$ is a closed, connected, and T^*-invariant subset of X^*.

Proof. The T^*-invariance of $J_{T^*}^{\mathrm{mix}}$ follows directly from the property that each $J_{T^*}^{\mathrm{mix}}(x^*)$ is invariant under the action of T^*, i.e.,

$$T^*(J_{T^*}^{\mathrm{mix}}(x^*)) \subset J_{T^*}^{\mathrm{mix}}(T^* x^*).$$

Hence, if $J_{T^*}^{\mathrm{mix}}(x^*) = X^*$, then $J_{T^*}^{\mathrm{mix}}(T^* x^*) = X^*$, implying $T^* x^* \in J_{T^*}^{\mathrm{mix}}$.
To establish that $J_{T^*}^{\mathrm{mix}}$ is closed, let $\{x_n^*\} \subset J_{T^*}^{\mathrm{mix}}$ be a sequence converging to some $x^* \in X^*$. By Lemma 2.5, we know that if $y^* \in J_{T^*}^{\mathrm{mix}}(x_n^*) = X$ for all n, then $y^* \in J_{T^*}^{\mathrm{mix}}(x^*)$. Therefore, $J_{T^*}^{\mathrm{mix}}(x^*) = X^*$, implying $x^* \in J_{T^*}^{\mathrm{mix}}$. Hence, $J_{T^*}^{\mathrm{mix}}$ is sequentially closed and thus closed.
To prove that $J_{T^*}^{\mathrm{mix}}$ is connected, take any $x^* \in J_{T^*}^{\mathrm{mix}}$. From Lemma 2.10, we know that for any scalar $\lambda \in \mathbb{C}$, $J_{T^*}^{\mathrm{mix}}(\lambda x^*) = X^*$, implying $\lambda x^* \in J_{T^*}^{\mathrm{mix}}$. Thus, every complex scalar multiple of x^* lies in $J_{T^*}^{\mathrm{mix}}$, and this scalar line is connected. As $x^* \in J_{T^*}^{\mathrm{mix}}$ was arbitrary, the union of such lines forms a connected set, so $J_{T^*}^{\mathrm{mix}}$ is connected because $0 \in J_{T^*}^{\mathrm{mix}}$. □

Theorem 2.12. The adjoint of an operator $T\colon X \to X$ is mixing if and only if, for every $x^* \in X^*$, the set $J_{T^*}^{\mathrm{mix}}(x^*)$ coincides with the entire space X^*; that is,

$$J_{T^*}^{\mathrm{mix}}(x^*) = X^* \quad \text{for all } x^* \in X^*.$$

Proof. Suppose that T^* is mixing. Let $x^*, y^* \in X^*$ and let U and V be neighborhoods of x^* and y^*, respectively. By the mixing property, there exists $N \in \mathbb{N}$ such that

$$(T^*)^n(U) \cap V \neq \emptyset \quad \text{for all } n \geq N,$$

which implies $y^* \in J_{T^*}^{\mathrm{mix}}(x^*)$. Since $y^* \in X^*$ was arbitrary, it follows that $J_{T^*}^{\mathrm{mix}}(x^*) = X^*$.
Conversely, assume that $J_{T^*}^{\mathrm{mix}}(x^*) = X^*$ for all $x^* \in X^*$. Let $U, V \subset X^*$ be nonempty open sets, and pick $x_0^* \in U$, $y_0^* \in V$. Since $y_0^* \in J_{T^*}^{\mathrm{mix}}(x_0^*)$, there exists $N \in \mathbb{N}$ such that

$$(T^*)^n(U) \cap V \neq \emptyset \quad \text{for all } n \geq N,$$

which shows that T^* is mixing. □

2.2 Dual Topologically Transitive

To familiarize the reader with some properties of topologically transitive operators, we present the theorem and its corollary below.

Theorem 2.13. Let $T\colon X \to X$ be a linear operator. The following statements are equivalent:

1. T^* is topologically transitive.
2. X^* cannot be decomposed as $X^* = A \cup B$, where A and B are disjoint sets, $T^*(A) \subset A$ and A, B have non-empty interior.
3. For every non-empty open set $U \subset X^*$, the set $\bigcup_{n=0}^{\infty}(T^*)^n(U)$ is dense in X^*.
4. For every non-empty open set $U \subset X^*$, the set $\bigcup_{n=0}^{\infty}(T^*)^{-n}(U)$ is dense in X^*.

Proof. **(1) ⇒ (2)**: Suppose that $X^* = A \cup B$, with $A \cap B = \emptyset$, $T^*(A) \subset A$, and $\mathrm{int}(B) \neq \emptyset$. Consider the open sets $\mathrm{int}(A)$ and $\mathrm{int}(B)$. We then have

$$(T^*)^n(\mathrm{int}(A)) \cap \mathrm{int}(B) \subset A \cap B = \emptyset \quad \text{for some } n \in \mathbb{N},$$

By assumption (1). This forces either $\mathrm{int}(A) = \emptyset$ or $\mathrm{int}(B) = \emptyset$ and subsequently $\mathrm{int}(A) = \emptyset$.

(2) ⇒ (3): Let $A = \bigcup_{n=0}^{\infty}(T^*)^n(U)$ and $B = X^* \setminus A$. We know that $(T^*)(A) \subset A$ and that $\mathrm{int}(A) \supset U$, which is non-empty. From condition (2), we deduce that $\mathrm{int}(B) = \emptyset$, hence A is dense in X.

(3) ⇒ (1): This follows immediately from the definitions of topological transitivity and density.

(1) ⇔ (4): The equivalence follows from the fact that $(T^*)^n(U) \cap V \neq \emptyset$ if and only if $U \cap (T^*)^{-n}(V) \neq \emptyset$. □

Corollary 2.14. *Let $T\colon X \to X$ be an operator. The following statements are equivalent:*

1. *T^* is topologically transitive.*
2. *Any open set $U \subset X$ such that $(T^*)^{-n}(U) \subset U$ for all $n \geq 0$ is either empty or dense in X^*.*

Definition 2.15. We say that an operator $T\colon X \to X$ is *dual topologically transitive* when both the operator itself and its adjoint T^* are topologically transitive.

The first examples of dual topologically transitive operators were presented in paper [22], and further results in this area were published in papers [20], [23] and [24]. Recall from [12, Example 1.43] that not every topologically transitive operator is mixing. This observation motivates the following question:

problem Is the concept of a dual mixing operator feasible, where both the operator T and its adjoint T^* exhibit mixing?

The following discussions aim to provide the necessary background for answering the above question.

Proposition 2.16. *Let X be a Banach space and let $T\colon X \to X$ be a linear operator. In this case, the closure of the orbit of an arbitrary nonzero $x^* \in X^*$ under T^* is the entire space X^* if and only if the limit set of x^* under T^* is equal to X^*.*

Proof. Assume that the orbit $\mathrm{Orb}(T^*, x^*)$ is dense in X^*. For any $\varepsilon > 0$ and integer $m \geq 0$, the intersection

$$B((T^*)^m x^*, \varepsilon) \cap X^* \setminus \{(T^*)^m x\} = B((T^*)^m x, \varepsilon) \cap \overline{\mathrm{Orb}((T^*), x^*)} \setminus \{(T^*)^m x\}$$

is nonempty and since $\mathrm{Orb}(T^*, x^*)$ is dense in X^*, there exists some vector

$$(T^*)^{m_\varepsilon} x \in B((T^*)^m x, \varepsilon),$$

with $m_\varepsilon > m$. Therefore, we conclude that

$$X^* = \overline{\mathrm{Orb}(T^*, x^*)} = L(T^*, x^*).$$

The converse follows immediately from the definitions. $\square$

Remark 2.17. Following an analogous argument, one can derive a corresponding result for $L(T,x)$ for some $x \in X$.

Based on the above remark, the following valuable result can be observed.

Corollary 2.18. *Let $T\colon X \to X$ be a dual topologically transitive operator. Then for some nonzero vector $x^* \in X^*$ the set*

$$L(T^*, x^*)$$

is nonempty.

Theorem 2.19. Suppose $T\colon X \to X$ is a mixing operator acting on a Banach space X. Then, for every nonzero functional $x^* \in X^*$, the limit set of x^* under the adjoint operator T^* is empty; that is,

$$L(T^*, x^*) = \emptyset, \text{ for all } x^* \in X^* \setminus \{0\}.$$

Proof. Assume, for the sake of contradiction, that there exists a nonzero functional $x^* \in X^*$ such that $L(T^*, x^*)$ is nonempty. Then, there exists an increasing sequence of positive integers $\{k_n\}$ such that $(T^*)^{k_n} x^*$ converges. Consequently, this sequence is bounded. Thus, there exists a constant M such that

$$\|(T^*)^{k_n} x^*\| \leq M,$$

for every $n \in \mathbb{N}$. Now, consider the open sets

$$U = \{x \in X \mid \|x\| < 1\}, \quad V = (x^*)^{-1}\left(\overline{B(0,M)}^c\right),$$

where $B(0,M)$ denotes the ball of radius M centered at the origin. By the mixing property of T, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we can find $x \in U$ satisfying $T^{k_n} x \in V$. Therefore,

$$M < |\langle T^{k_n} x, x^* \rangle| = |\langle x, (T^*)^{k_n} x^* \rangle| \leq \|(T^*)^{k_n} x^*\| \leq M,$$

which leads to a contradiction. $\square$

The following important result is obtained by combining Corollary 2.18 and Theorem 2.19. Indeed, it serves as a negative response to the previously stated question.

Corollary 2.20. *No mixing operator has a mixing adjoint. Equivalently, dual mixing operators do not exist.*

This corollary further reinforces the fact that the class of topologically transitive operators is not equivalent to the class of mixing operators, thereby emphasizing their structural distinction.

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