

Cohomology and deformations of compatible Hom-right-symmetric algebras

Abdellatif Saïdi

*Department of mathematics, Faculty of Sciences of Monastir, 5019, Monastir, Tunisia.
Mathematical Physics, Special Functions and Applications, ESSTHS, Tunisia.
e-mail: Abdellatif.Saïdi@fsm.rnu.tn*

Communicated by Ahmed Hamed

(Received 04 January 2026, Revised 01 May 2026, Accepted 05 May 2026)

Abstract. In this research paper, we define and investigate the concept of multiplicative compatible Hom-right-symmetric algebra and its cohomology. This algebraic structure is defined by a vector space \mathfrak{g} , a linear endomorphism α and two Hom-right-symmetric products μ, μ' with respect to α . We define up to homotopy compatible Hom-right-symmetric algebras by realization of a structure equation $[[(Q, Q'), (Q, Q')]] = 0$, where (Q, Q') is a degree one compatible coderivation of a comultiplication Δ_1 . With this structure equation, we define the cohomology of a compatible Hom-right-symmetric algebra \mathfrak{g} with value in a module M . Additionally, central extensions and linear deformations are investigated in details. Besides, we give the notion of a Nijenhuis operator on a compatible Hom-right-symmetric algebra, and we show that it can generate a trivial linear deformation of a regular compatible Hom-right-symmetric algebra.

Key Words: Up to homotopy algebras, compatible Hom-right-symmetric algebras, cohomology, deformations, Nijenhuis operator.

2020 MSC: 17A30, 17D25, 17D30, 55N20.

1 Introduction

An algebra (\mathfrak{g}, μ) over a field \mathbb{K} of characteristic zero is right-symmetric [16] provided that the associator on \mathfrak{g} is right-symmetric, i.e., is symmetric in the last two elements:

$(x, y, z) = (x, z, y)$ for all $x, y, z \in \mathfrak{g}$, where $(x, y, z) = \mu(x, \mu(y, z)) - \mu(\mu(x, y), z)$ is the associator of x, y and z . Similarly, one can define left-symmetric algebra by identity

$$(x, y, z) = (y, x, z), \quad \forall x, y, z \in \mathfrak{g}.$$

Categories of right-symmetric algebras and left-symmetric algebras are equivalent. Any right(left)-symmetric algebras will be left(right)-symmetric under new product $(x, y) \mapsto \mu(y, x)$. Right (left)-symmetric algebras, also called pre-Lie algebras, are an important generalization of associative algebras, appearing naturally in various contexts. They were first introduced by Cayley in 1857 in context of rooted trees [6]. In 1961, Koszul used them in the study of actions of affine transformations [12]. In 1963, Vinberg [22] applied these algebras to classify the convex homogeneous cones, and Gerstenhaber [10] used them to studying the deformations of algebras. Many articles related to right and left-symmetric algebras have been published since then. (see [1, 7, 13, 17, 20]). Representations and cohomology theory of right-symmetric algebras were studied in [9] by A. Dzhumadil'daev. The notion of Hom-algebra was introduced by Hartwig, Larsson, and Silvestrov in [11], as a tool for the study of deformations of Witt and Virasoro algebras. Representations and cohomologies of Hom-pre-Lie algebras were studied and developed in [14, 8].

In this paper, we are interested by an understanding of these notions in the setting of "operadic" presentation, as in [2, 3, 5, 8]. Note that compatible algebraic structures [18, 21] refer to two algebraic

structures of the same kind in a linear category such that any linear combination of multiplications corresponding to these two algebraic structures still define the same kind of algebraic structure. Compatible structures appear in various fields of mathematics and mathematical physics, it is exactly related to linear deformations.

The purpose of this paper is to study representations, cohomology and deformations of compatible Hom-right-symmetric algebras $(\mathfrak{g}, \mu, \mu', \alpha)$ in setting of "operadic" presentation. More precisely, we are considering a sequence of comultiplications Δ_r on the space $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$. Suppose Q_i is a coderivation of Δ_{r_i} ,

$(i = 1, 2)$, then the commutator $[Q_1, Q_2]$ is a coderivation of $\Delta_{r_1+r_2}$. Now, a Hom-right-symmetric product μ is exactly a bilinear coderivation Q of degree 1 of Δ_1 and $(\mathfrak{g}, Q, \alpha)$ is a Hom-right-symmetric algebra if and only if $[Q, Q] = 0$, this is the structure equation for Hom-right symmetric algebras. For compatible Hom-right-symmetric algebras, we construct a new graded Lie bracket $[[\cdot, \cdot]]$ on a new space of coderivations. For a pair of compatible coderivations (Q, Q') of Δ_1 , we obtain that $(\mathfrak{g}, Q, Q', \alpha)$ is a compatible Hom-right-symmetric algebra if and only if $[[Q, Q'], (Q, Q')] = 0$. This is the structure equation for compatible Hom-right-symmetric algebras. This structure will be needed later to construct the cohomology of compatible Hom-right-symmetric algebras.

This paper is organized as follows: in Section 2, we give a background on Hom-right-symmetric algebras and their cohomology. In Section 3, we give the notion of compatible Hom-right-symmetric algebras. In Section 4, we construct a new Lie bracket on space of $Coder_c$, and we give the structure equation of compatible-Hom-right-symmetric algebras. Using this structure equation, in Section 5, we study representations and cohomology of compatible Hom-right-symmetric algebras. As an application, in Section 6, we study trivial representation of compatible Hom-right-symmetric algebra and central extensions. In Section 7, we study linear deformations of regular compatible Hom-right-symmetric algebras, we introduce the notion of a Nijenhuis operator on a compatible Hom-right-symmetric algebras. We show that a trivial linear deformation gives rise to a Nijenhuis operator. Conversely, a Nijenhuis operator gives rise to a trivial deformation.

2 Background on Hom-right-symmetric algebras and its cohomology

In this section, we recall the cohomology theory of Hom-right-symmetric algebras. Our main references are [8, 15].

Definition 2.1. Hom-right-symmetric algebra is a triple $(\mathfrak{g}, \mu, \alpha)$ consisting of a vector space \mathfrak{g} , a bilinear map $\mu : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ and a linear map $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Hom-right-symmetric identity:

$$\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) = \mu(\alpha(x), \mu(z, y)) - \mu(\mu(x, z), \alpha(y)), \quad (1)$$

for all $x, y, z \in \mathfrak{g}$.

If α is a morphism for the product μ , the algebra $(\mathfrak{g}, \mu, \alpha)$ is a multiplicative Hom-right-symmetric algebra. If moreover α is invertible, the Hom-right-symmetric algebra is said to be regular.

Let $(x, y, z)_\alpha = \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z))$ be the Hom-associator of elements $x, y, z \in \mathfrak{g}$. In terms of Hom-associators, the Hom-right symmetric identity is $(x, y, z)_\alpha = (x, z, y)_\alpha$.

A *graded* Hom-right-symmetric algebra is a graded vector space \mathfrak{g} equipped with a degree 0, graded bilinear map μ , and a degree 0 linear map α , such that the graded Hom-symmetric relation holds, i. e. the relation (1) holds with the usual Koszul sign rule: each permutation of letters (here y and z) implies multiplication by the sign of the corresponding graded permutation. In the present setting, graded relation (1) means

$$(x, y, z)_\alpha = (-1)^{\deg(y)\deg(z)}(x, z, y)_\alpha.$$

Definition 2.2. Let $(\mathfrak{g}, \mu, \alpha)$ and $(\mathfrak{g}', \mu', \alpha')$ be two Hom-right symmetric algebras. A linear map $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ is called a morphism of Hom-right symmetric algebras if it satisfying

$$f \circ \mu = \mu' \circ (f \otimes f) \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$

In the remaining of this article, all Hom-right-symmetric algebras are multiplicative.

2.1 Up to homotopy Hom-right-symmetric algebras

Let (\mathfrak{g}, deg) be a graded vector space and $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ be a degree 0 linear map.

Consider $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1]) = \sum_{k \geq 1} \mathfrak{g}[1] \otimes S^{k-1}(\mathfrak{g}[1])$, where $S(\mathfrak{g}[1])$ is the graded symmetric algebra on the graded vector space $(\mathfrak{g}[1], | \cdot |)$, i. e. the space \mathfrak{g} with a shift of -1 on the degree: for each $x \in \mathfrak{g}$:

$$|x| = deg(x) - 1.$$

Remark that, if \mathfrak{g} is not graded, we put $deg(x) = 0$ for each x , and $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$ is simply the space $\mathfrak{g} \otimes \wedge \mathfrak{g}$ with the degree

$$|x_0 \otimes (x_1 \wedge \cdots \wedge x_{k-1})| = -k.$$

Recall the notion of shuffle permutation. A permutation σ is a $(p-1, 1, k-p-1)$ -shuffle, ($\sigma \in Sh(p-1, 1, k-p-1)$), if it is a permutation of $\{1, \dots, k-1\}$ such that $\sigma(1) < \cdots < \sigma(p-1)$, and $\sigma(p+1) < \cdots < \sigma(k-1)$.

For each natural number r , define α^r the r -times composition of α , extend α^r to $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$ by having

$$\alpha^r(x_0 \otimes x_1 \cdots x_{k-1}) = \alpha^r(x_0) \otimes \alpha^r(x_1) \cdots \alpha^r(x_{k-1}).$$

Define also the comultiplication Δ_r on $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$ by:

$$\Delta_r(x_0 \otimes x_1 \cdots x_{n-1}) = \sum_{\substack{1 \leq k \leq n-1 \\ \sigma \in Sh_{k-1, 1, n-k-1}}} \alpha^r(x_0 \otimes x_{\sigma(1)} \cdots x_{\sigma(k-1)}) \otimes \alpha^r(x_{\sigma(k)} \otimes x_{\sigma(k+1)} \cdots x_{\sigma(n-1)}).$$

(as usual, the Koszul's sign rule is understood in this formula). For instance, if \mathfrak{g} is not graded, this reads:

$$\Delta_r(x_0 \otimes x_1 \cdots x_{n-1}) = \sum_{\substack{1 \leq k \leq n-1 \\ \sigma \in Sh_{k-1, 1, n-k-1}}} \varepsilon(\sigma) \alpha^r(x_0 \otimes x_{\sigma(1)} \cdots x_{\sigma(k-1)}) \otimes \alpha^r(x_{\sigma(k)} \otimes x_{\sigma(k+1)} \cdots x_{\sigma(n-1)}).$$

where $\varepsilon(\sigma)$ is the sign of the permutation σ .

This comultiplication is Hom-permutative with respect to α^r , that is the relation:

$$(\alpha^r \otimes \Delta_r) \circ \Delta_r = (\Delta_r \otimes \alpha^r) \circ \Delta_r = (\alpha^r \otimes \tau) \circ \Delta_r \circ \Delta_r,$$

where τ is the twist: $\tau(X \otimes Y) = Y \otimes X$, holds.

A coderivation Q with degree q of Δ_r , ($Q \in Coder_q(\Delta_r)$), is a map $Q : \mathfrak{g}[1] \otimes S(\mathfrak{g}[1]) \rightarrow \mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$, such that:

$$Q \circ \alpha = \alpha \circ Q \quad \text{and} \quad (Q \otimes \alpha^r + \alpha^r \otimes Q) \circ \Delta_r = \Delta_r \circ Q.$$

Any such coderivation is characterized by its Taylor expansion. More precisely, let p be the projection on $\mathfrak{g}[1]$ parallel to $\sum_{k > 0} \mathfrak{g}[1] \otimes S^{k-1}(\mathfrak{g}[1])$, i_k the canonical injection of $\mathfrak{g}[1] \otimes S^{k-1}(\mathfrak{g}[1])$ into $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$.

For any coderivation Q , consider the sequence of maps $Q_k = p \circ Q \circ i_k$. Each Q_k extends to a map \tilde{Q}_k defined on $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$ by:

$$\begin{aligned} \tilde{Q}_k(x_0 \otimes x_1 \dots x_{n-1}) &= \sum_{\sigma \in Sh_{k-1, n-k}} Q_k(x_0 \otimes x_{\sigma(1)} \dots x_{\sigma(k-1)}) \otimes \alpha^r(x_{\sigma(k)} \dots x_{\sigma(n-1)}) \\ &+ \sum_{\sigma \in Sh_{1, k-1, n-k-1}} (-1)^q \alpha^r(x_0) \otimes Q_k(x_{\sigma(1)} \otimes x_{\sigma(2)} \dots x_{\sigma(k)}) \cdot \alpha^r(x_{\sigma(k+1)} \dots x_{\sigma(n-1)}). \end{aligned} \quad (2)$$

Then, $Q = \sum_k \tilde{Q}_k$. Conversely, each sequence of maps $(Q_k : \mathfrak{g}[1] \otimes S^{k-1}(\mathfrak{g}[1]) \rightarrow \mathfrak{g}[1])$ ($k > 0$) defines through extension and sum a coderivation $Q = \sum_k \tilde{Q}_k$.

Let $Q \in Coder_q(\Delta_r)$ and $Q' \in Coder_{q'}(\Delta_{r'})$. Define the commutator

$$[Q, Q'] = Q \circ Q' - (-1)^{qq'} Q' \circ Q \quad (3)$$

(with Koszul's rule).

Lemma 2.3. [8] *With the above notation, the commutator $[Q, Q']$ is a coderivation of degree $q + q'$ of $(\Delta_{r+r'})$, i.e.,*

$$[Q, Q'] \in Coder_{q+q'}(\Delta_{r+r'}).$$

Define $Coder = \bigoplus_{q, r \geq 0} Coder_q(\Delta_r)$.

The following result is straightforward.

Proposition 2.4. *The space $Coder$ equipped with the commutator $[\cdot, \cdot]$ defined above is a graded Lie algebra.*

Let $\mu : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ be a bilinear map of degree zero in (\mathfrak{g}, deg) . For each $x, y \in \mathfrak{g}$, define $Q_2(x, y) = \mu(x, y)$. Obviously, Q_2 is a degree one map from $\mathfrak{g}[1] \otimes \mathfrak{g}[1] \rightarrow \mathfrak{g}[1]$. Let $Q = \tilde{Q}_2$. Then if $r = r' = 1$, we obtain the following result.

Theorem 2.5. Referring to the above notation, the triple $(\mathfrak{g}, \mu, \alpha)$ is a Hom-right-symmetric algebra, if and only if,

$$[Q, Q] = 0. \quad (4)$$

Proof. For all $x_0, x_1, x_2 \in \mathfrak{g}$, we have

$$\begin{aligned} [Q, Q]_3(x_0 \otimes x_1 \cdot x_2) &= 2(Q \circ Q)(x_0 \otimes x_1 \cdot x_2) \\ &= 2(Q(Q(x_0 \otimes x_1) \otimes \alpha(x_2)) - Q(Q(x_0 \otimes x_2) \otimes \alpha(x_1)) \\ &\quad - Q(\alpha(x_0) \otimes Q(x_1 \otimes x_2)) + Q(\alpha(x_0) \otimes Q(x_2 \otimes x_1))) \\ &= 2(\mu(\mu(x_0, x_1), \alpha(x_2)) - \mu(\mu(x_0, x_2), \alpha(x_1)) \\ &\quad - \mu(\alpha(x_0), \mu(x_1, x_2)) + \mu(\alpha(x_0), \mu(x_2, x_1))) \end{aligned}$$

Then, $[Q, Q] = 0$ if and only if for all $x_0, x_1, x_2 \in \mathfrak{g}$,

$$\mu(\alpha(x_0), \mu(x_1, x_2)) - \mu(\mu(x_0, x_1), \alpha(x_2)) = \mu(\alpha(x_0), \mu(x_2, x_1)) - \mu(\mu(x_0, x_2), \alpha(x_1)),$$

which means that $(\mathfrak{g}, \mu, \alpha)$ is a Hom-right-symmetric algebra. \square

Definition 2.6. An up to homotopy Hom-right-symmetric algebra $(\mathfrak{g}, Q, \alpha)$ is a (graded) vector space \mathfrak{g} , a degree 0 linear map $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ and a degree 1 coderivation $Q \in Coder(\Delta_1)$ of $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$ such that:

$$[Q, Q] = 0.$$

2.2 Representations and cohomology of Hom-right-symmetric algebras

Definition 2.7. A representation of a Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \alpha)$ on a vector space M with respect to $\alpha_M \in gl(M)$ consists of a pair $(L, R) : \mathfrak{g} \rightarrow gl(M)$ satisfying, for all $x, y \in \mathfrak{g}$:

$$L(\alpha(x)) \circ \alpha_M = \alpha_M \circ L(x), \quad R(\alpha(x)) \circ \alpha_M = \alpha_M \circ R(x), \quad (5)$$

$$R(\mu(x, y) - \mu(y, x)) \circ \alpha_M - R(\alpha(y)) \circ R(x) + R(\alpha(x)) \circ R(y) = 0, \quad (6)$$

$$L(\alpha(x)) \circ L(y) - L(\mu(x, y)) \circ \alpha_M - L(\alpha(x)) \circ R(y) + R(\alpha(y)) \circ L(x) = 0. \quad (7)$$

We denote (M, L, R, α_M) this representation. Simply, we say that M is a \mathfrak{g} -module [19].

Let (M, L, R, α_M) be a representation of a Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \alpha)$. Observe that, for any natural number r , if $L_r = L \circ (\alpha^r)$, $R_r = R \circ (\alpha^r)$, then (M, L_r, R_r, α_M) is also a representation of $(\mathfrak{g}, \mu, \alpha)$. If α is invertible, the same result holds for $r = -1, -2, \dots$.

The fundamental example of a representation of $(\mathfrak{g}, \mu, \alpha)$ is the adjoint-representation defined by putting:

$$M = \mathfrak{g}, \quad L(x)(y) = \mu(x, y), \quad R(x)(y) = \mu(y, x) \quad \text{and} \quad \alpha_M = \alpha.$$

Lemma 2.8. If (M, L, R, α_M) is a representation of a Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \alpha)$, then $(\mathfrak{g} \times M, \mu_{L,R}, \alpha + \alpha_M)$ is also a Hom-right-symmetric algebra, where

$$\mu_{L,R}((x, u), (y, v)) = (\mu(x, y), L(x)(v) + R(y)(u)), \quad (\alpha + \alpha_M)(x, u) = (\alpha(x), \alpha_M(u)), \quad \forall x, y \in \mathfrak{g}, u, v \in M.$$

In the situation of this lemma, The triple $(\mathfrak{g} \times M, \mu_{L,R}, \alpha + \alpha_M)$ is a Hom-right-symmetric algebra called the semi-direct product of \mathfrak{g} by the \mathfrak{g} -module M .

Definition 2.9. Let (M, L, R, α_M) be a representation of a Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \alpha)$.

- A 0-cochain is a vector $m \in M$, such that

$$\alpha_M(m) = m, \quad \text{and} \quad R(\alpha(y)) \circ R(x)m = R(\mu(x, y))m$$

for all $x, y \in \mathfrak{g}$.

- For $n \geq 1$, a n -cochain C with values in M , is a n -linear map $C : \mathfrak{g} \otimes \wedge^{n-1} \mathfrak{g} \rightarrow M$ such that $\alpha_M \circ C = C \circ \alpha$. Denote $C^n(\mathfrak{g}, M)$ the space of n -cochains.

Relying on Lemma 2.8 and Section 2.1, we can define the Hom-right symmetric algebra $(\mathfrak{g} \times M, \mu_{(R,L)}, \alpha + \alpha_M)$ and its coalgebra $((\mathfrak{g} \times M)[1] \otimes S((\mathfrak{g} \times M)[1]), \Delta_n)$. Then, for each $n \geq 1$, there is a natural bijection ψ_n from $C^n(\mathfrak{g}, M)$ onto a subspace of $Coder(\Delta_n)$, namely: let $C \in C^n(\mathfrak{g}, M)$,

1. Consider C as a n -degree map $C : (\mathfrak{g} \times M)[1] \otimes S^{n-1}((\mathfrak{g} \times M)[1]) \rightarrow (\mathfrak{g} \times M)[1]$, vanishing if one of the arguments is in M .
2. Extend this map to $\tilde{C} \in Coder(\Delta_n)$ as in (2). Put $\psi_n(C) = \tilde{C}$.

Conversely, for each coderivation $\Gamma \in \text{Coder}(\Delta_n)$ vanishing if one of its argument is in M , and with values in M (it is easy to see that $\Gamma = \psi_n(\Gamma)$), there is $C_n \in C^n(\mathfrak{g}, M)$, namely:

$$C_n(x_0 \otimes x_1 \dots x_{n-1}) = \psi_n^{-1}(\Gamma)(x_0 \otimes x_1 \dots x_{n-1}) = \Gamma_n((x_0, 0) \otimes (x_1, 0) \dots (x_{n-1}, 0)).$$

Let $Q_{L,R} \in \text{Coder}(\Delta_1)$ the coderivation defined by $Q_{L,R} = \mu_{L,R}$. For each $n \geq 1$, define the operator $d_n : C^n(\mathfrak{g}, M) \rightarrow C^{n+1}(\mathfrak{g}, M)$ by

$$d_n C = (-1)^n \psi_{n+1}^{-1}([Q_{L,R}, \psi_n(C)]).$$

Theorem 2.10. [8] The operators d_n satisfy:

$$d_{n+1} \circ d_n = 0.$$

Corollary 2.11. For each $n \geq 1$, the explicit expression of d_n is

$$(d_n C)(x_0 \otimes x_1, \dots, x_n) = \sum_{i=1}^n (-1)^{i-1} (d_n^i C)(x_0 \otimes x_1, \dots, x_n),$$

where:

$$\begin{aligned} (d_n^i C)(x_0 \otimes x_1, \dots, x_n) &= L(\alpha^n(x_0))C(x_i \otimes x_1, \dots, \hat{i}, \dots, x_n) \\ &+ R(\alpha^n(x_i))C(x_0 \otimes x_1, \dots, \hat{i}, \dots, x_n) \\ &- C(\mu(x_0, x_i) \otimes \alpha(x_1), \dots, \hat{i}, \dots, \alpha(x_n)) \\ &+ \sum_{i < j} C(\alpha(x_0) \otimes \alpha(x_1), \dots, \hat{i}, \dots, \alpha(x_{j-1}), \mu(x_i, x_j), \dots, \alpha(x_n)) \\ &- \sum_{i < j} C(\alpha(x_0) \otimes \alpha(x_1), \dots, \hat{i}, \dots, \alpha(x_{j-1}), \mu(x_j, x_i), \dots, \alpha(x_n)). \end{aligned}$$

Let m be a 0-cochain, the 0-coboundary operator is the map d_0 , where $(d_0 m)(x) = L(x)m - R(x)m$ for each $x \in \mathfrak{g}$.

Let m be a 0-cochain. For all $x_0, x_1 \in \mathfrak{g}$, we have

$$\begin{aligned} d_1 \circ d_0(m)(x_0 \otimes x_1) &= L(\alpha(x_0))(d_0 m)(x_1) + R(\alpha(x_1))(d_0 m)(x_0) - (d_0 m)(\mu(x_0, x_1)) \\ &= L(\alpha(x_0)) \circ L(x_1)m - L(\alpha(x_0)) \circ R(x_1)m + R(\alpha(x_1)) \circ L(x_0)m \\ &\quad - R(\alpha(x_1)) \circ R(x_0)m + R(\mu(x_0, x_1))m - L(\mu(x_0, x_1))m. \end{aligned}$$

Since $\alpha_M(m) = m$, and $R(\alpha(x_1)) \circ R(x_0)m = R(\mu(x_0, x_1))m$ for all $x_0, x_1 \in \mathfrak{g}$, it follows that $d_1 \circ d_0 = 0$.

3 Compatible Hom-right-symmetric algebras

Definition 3.1. A compatible Hom-right-symmetric algebra is a 4-tuple $(\mathfrak{g}, \mu, \mu', \alpha)$, where $(\mathfrak{g}, \mu, \alpha)$, $(\mathfrak{g}, \mu', \alpha)$ are both Hom-right-symmetric algebras satisfying the following compatibility equation

$$\begin{aligned} &\mu(\alpha(x), \mu'(y, z)) + \mu'(\alpha(x), \mu(y, z)) - \mu(\mu'(x, y), \alpha(z)) - \mu'(\mu(x, y), \alpha(z)) \\ &= \mu(\alpha(x), \mu'(z, y)) + \mu'(\alpha(x), \mu(z, y)) - \mu(\mu'(x, z), \alpha(y)) - \mu'(\mu(x, z), \alpha(y)), \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.

Remark 3.2. If $(\mathfrak{g}, \mu, \mu', \alpha)$ is a compatible Hom-right-symmetric algebra, then $(\mathfrak{g}, \mu + \mu', \alpha)$ is a Hom-right-symmetric algebra.

The following result is straightforward.

Proposition 3.3. *The 4-tuple $(\mathfrak{g}, \mu, \mu', \alpha)$ is a compatible Hom-right-symmetric algebra if and only if the triple $(\mathfrak{g}, k\mu + l\mu', \alpha)$ is a Hom-right-symmetric algebra for all $k, l \in \mathbb{K}$.*

Definition 3.4. Let $(A, \mu_1, \mu'_1, \alpha)$ and $(B, \mu_2, \mu'_2, \beta)$ be two compatible Hom-right-symmetric algebras. A morphism of compatible Hom-right-symmetric algebras between A and B is a linear map $f : A \rightarrow B$ satisfying $f(\mu_1(x, y)) = \mu_2(f(x), f(y))$, $f(\mu'_1(x, y)) = \mu'_2(f(x), f(y))$ for all $x, y \in A$, as well as $f \circ \alpha = \beta \circ f$.

Definition 3.5. [4, 14] Let $(\mathfrak{g}, \mu, \alpha)$ be a Hom-right-symmetric algebra. A Nijenhuis operator on $(\mathfrak{g}, \mu, \alpha)$ is a linear map $N : \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$N \circ \alpha = \alpha \circ N,$$

$$\mu(N(x), N(y)) = N(x \cdot_N y),$$

where $x \cdot_N y = \mu(N(x), y) + \mu(x, N(y)) - N(\mu(x, y))$ for all $x, y \in \mathfrak{g}$.

Proposition 3.6. *Let N be a Nijenhuis operator on the Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \alpha)$. Then, $(\mathfrak{g}, \mu, \cdot_N, \alpha)$ is a compatible Hom-right-symmetric algebra.*

Proof. Since $(\mathfrak{g}, \mu, \alpha)$ and $(\mathfrak{g}, \cdot_N, \alpha)$ are Hom-right-symmetric algebras, then to show that $(\mathfrak{g}, \mu, \cdot_N, \alpha)$ is a compatible Hom-right-symmetric algebra, we need to show the compatibility equation between μ and \cdot_N .

For all $x, y, z \in \mathfrak{g}$ we have

$$\begin{aligned} & \mu(\alpha(x), y \cdot_N z) + \alpha(x) \cdot_N \mu(y, z) - \mu(x \cdot_N y, \alpha(z)) - \mu(x, y) \cdot_N \alpha(z) \\ & - \mu(\alpha(x), z \cdot_N y) - \alpha(x) \cdot_N \mu(z, y) + \mu(x \cdot_N z, \alpha(y)) + \mu(x, z) \cdot_N \alpha(y) \\ & = \mu(\alpha(x), \mu(N(y), z)) + \mu(\alpha(x), \mu(y, N(z))) - \mu(\alpha(x), N(\mu(y, z))) \\ & + \mu(N(\alpha(x)), \mu(y, z)) + \mu(\alpha(x), N(\mu(y, z))) - N(\mu(\alpha(x), \mu(y, z))) \\ & - \mu(\mu(N(x), y), \alpha(z)) - \mu(\mu(x, N(y)), \alpha(z)) + \mu(N(\mu(x, y)), \alpha(z)) \\ & - \mu(N(\mu(x, y)), \alpha(z)) - \mu(\mu(x, y), N(\alpha(z))) + N(\mu(\mu(x, y), \alpha(z))) \\ & - \mu(\alpha(x), \mu(N(z), y)) - \mu(\alpha(x), \mu(z, N(y))) + \mu(\alpha(x), N(\mu(z, y))) \\ & - \mu(N(\alpha(x)), \mu(z, y)) - \mu(\alpha(x), N(\mu(z, y))) + N(\mu(\alpha(x), \mu(z, y))) \\ & + \mu(\mu(N(x), z), \alpha(y)) + \mu(\mu(x, N(z)), \alpha(y)) - \mu(N(\mu(x, z)), \alpha(y)) \\ & + \mu(N(\mu(x, z)), \alpha(y)) + \mu(\mu(x, z), N(\alpha(y))) - N(\mu(\mu(x, z), \alpha(y))). \end{aligned}$$

Since N commutes with α , and $(\mathfrak{g}, \mu, \alpha)$ is a Hom-right-symmetric algebra, then

$$\mu(\alpha(x), \mu(N(y), z)) - \mu(\mu(x, N(y)), \alpha(z)) - \mu(\alpha(x), \mu(z, N(y))) + \mu(\mu(x, z), N(\alpha(y))) = 0,$$

$$\mu(\alpha(x), \mu(y, N(z))) - \mu(\mu(x, y), N(\alpha(z))) - \mu(\alpha(x), \mu(N(z), y)) + \mu(\mu(x, N(z)), \alpha(y)) = 0,$$

$$\mu(N(\alpha(x)), \mu(y, z)) - \mu(\mu(N(x), y), \alpha(z)) - \mu(N(\alpha(x)), \mu(y, z)) + \mu(\mu(N(x), z), \alpha(y)) = 0,$$

$$\begin{aligned}
& -N\left(\mu(\alpha(x), \mu(y, z))\right) + N\left(\mu(\mu(x, y), \alpha(z))\right) + N\left(\mu(\alpha(x), \mu(z, y))\right) - N\left(\mu(\mu(x, z), \alpha(y))\right) \\
& = N\left(-\mu(\alpha(x), \mu(y, z)) + \mu(\mu(x, y), \alpha(z)) + \mu(\alpha(x), \mu(z, y)) - \mu(\mu(x, z), \alpha(y))\right) = 0.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& \mu(\alpha(x), y \cdot_N z) + \alpha(x) \cdot_N \mu(y, z) - \mu(x \cdot_N y, \alpha(z)) - \mu(x, y) \cdot_N \alpha(z) \\
& - \mu(\alpha(x), z \cdot_N y) - \alpha(x) \cdot_N \mu(z, y) + \mu(x \cdot_N z, \alpha(y)) + \mu(x, z) \cdot_N \alpha(y) \\
& = 0.
\end{aligned}$$

□

4 Up to homotopy compatible Hom-right-symmetric algebras

Let \mathfrak{g} be a (graded) vector space. Let $\mu, \mu' : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ be two degree zero bilinear maps and let $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ be a degree zero linear map such that for all $x, y \in \mathfrak{g}$, $\mu(\alpha(x), \alpha(y)) = \alpha(\mu(x, y))$ and $\mu'(\alpha(x), \alpha(y)) = \alpha(\mu'(x, y))$. We adopt the same notation as in subsection 2.1, put $Q_2(x, y) = \mu(x, y)$, $Q'_2(x, y) = \mu'(x, y)$ for all $x, y \in \mathfrak{g}$, then Q_2, Q'_2 are degree one maps from $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$ to itself. Let $Q = Q_2$ and $Q' = Q'_2$ be two coderivations ($\in \text{Coder}_1(\Delta_1)$). We obtain, $(\mathfrak{g}, \mu, \mu', \alpha)$ is a compatible Hom-right-symmetric algebra if and only if for all $k, l \in \mathbb{K}$, $(\mathfrak{g}, k\mu + l\mu', \alpha)$ is a Hom-right-symmetric algebra, which means by Theorem (2.5) that $[kQ + lQ', kQ + lQ'] = 0$. Then, we get $(\mathfrak{g}, \mu, \mu', \alpha)$ is a compatible Hom-right-symmetric algebra if and only if

$$[Q, Q] = 0, \quad [Q, Q'] = 0 \quad \text{and} \quad [Q', Q'] = 0.$$

Definition 4.1. An up to homotopy compatible Hom-right-symmetric algebra $(\mathfrak{g}, Q, Q', \alpha)$ is a (graded) vector space \mathfrak{g} , a degree zero linear map $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ and a two coderivations $Q, Q' \in \text{Coder}_1(\Delta_1)$ such that

$$[Q, Q] = 0, \quad [Q, Q'] = 0 \quad \text{and} \quad [Q', Q'] = 0.$$

In the remaining of this section, we construct a new space of coderivations and define a new graded Lie algebra needed to construct a structure equation for up to homotopy compatible Hom-right-symmetric algebra.

For any $r, q \geq 0$, define $\text{Coder}_{c,q}(\Delta_r) = \bigoplus_{n \geq 0} \text{Coder}_{c,q}^n(\Delta_r)$, where

$$\text{Coder}_{c,q}^n(\Delta_r) = \underbrace{\text{Coder}_q(\Delta_r) \oplus \cdots \oplus \text{Coder}_q(\Delta_r)}_{(n+1)\text{copies}}.$$

For any $(Q_1, \dots, Q_{m+1}) \in \text{Coder}_{c,q}^m(\Delta_r)$ and $(Q'_1, \dots, Q'_{n+1}) \in \text{Coder}_{c,q'}^n(\Delta_{r'})$, define the bracket $[[\cdot, \cdot]]$ by

$$\begin{aligned}
& [[(Q_1, \dots, Q_{m+1}), (Q'_1, \dots, Q'_{n+1})]] := \\
& \left([Q_1, Q'_1], [Q_1, Q'_2] + [Q_2, Q'_1], \dots, \underbrace{\sum_{k+j=i+1} [Q_k, Q'_j], \dots, [Q_{m+1}, Q'_{n+1}]}_{i\text{-th place}} \right).
\end{aligned}$$

Lemma 4.2. Resting on the above indicated notation, if $(Q_1, \dots, Q_{m+1}) \in \text{Coder}_{c,q}^m(\Delta_r)$ and $(Q'_1, \dots, Q'_{n+1}) \in \text{Coder}_{c,q'}^n(\Delta_{r'})$, then $[[(Q_1, \dots, Q_{m+1}), (Q'_1, \dots, Q'_{n+1})]] \in \text{Coder}_{c,q+q'}^{n+m}(\Delta_{r+r'})$.

Proof. Obviously, grounded on Lemma 2.3, for all $i \in \{1, \dots, m+1\}$ and $j \in \{1, \dots, n+1\}$, the bracket $[Q_i, Q'_j]$ is a coderivation in $\text{Coder}_{q+q'}(\Delta_{r+r'})$, which implies

$$[[(Q_1, \dots, Q_{m+1}), (Q'_1, \dots, Q'_{n+1})]] \in \text{Coder}_{c,q+q'}^{n+m}(\Delta_{r+r'}).$$

□

Define $Coder_c(\Delta_r) = \bigoplus_{q \geq 0} Coder_{c,q}(\Delta_r)$ and $Coder_c = \bigoplus_{r \geq 0} Coder_c(\Delta_r)$.

Proposition 4.3. • Based on the above notation, the space $Coder_c$ equipped with $[[\cdot, \cdot]]$ is a graded Lie algebra.

- The map $\Phi : Coder_c \rightarrow Coder$, defined by

$$\Phi(Q_1, \dots, Q_{m+1}) = Q_1 + \dots + Q_{m+1},$$

is a morphism of graded Lie algebras.

- The pair (Q, Q') of $Coder_{c,1}^1(\Delta_1)$, is compatible up to homotopy in \mathfrak{g} if and only if

$$[[(Q, Q'), (Q, Q')]] = 0.$$

Proof. • For $(Q_1, \dots, Q_{m+1}) \in Coder_{c,q}^m(\Delta_r)$, $(Q'_1, \dots, Q'_{n+1}) \in Coder_{c,q'}^n(\Delta_{r'})$ and $(Q''_1, \dots, Q''_{p+1}) \in Coder_{c,q''}^p(\Delta_{r''})$. We have

$$\begin{aligned} & [[(Q_1, \dots, Q_{m+1}), [[(Q'_1, \dots, Q'_{n+1}), (Q''_1, \dots, Q''_{p+1})]]]] \\ &= [[(Q_1, \dots, Q_{m+1}), ([Q'_1, Q'_1], \dots, \underbrace{\sum_{j+k=i+1} [Q'_j, Q'_k], \dots, [Q'_{n+1}, Q'_{p+1}]}_{i\text{-th place}})]] \\ &= ([Q_1, [Q'_1, Q'_1]], \dots, \underbrace{\sum_{j+k+l=i+2} [Q_l, [Q'_j, Q'_k], \dots, [Q_{m+1}, [Q'_{n+1}, Q'_{p+1}]]}_{i\text{-th place}}) \\ &= ([[Q_1, Q'_1], Q''_1] + (-1)^{qq'} [Q'_1, [Q_1, Q'_1]], \dots, \underbrace{\sum_{j+k+l=i+2} [[Q_l, Q'_j], Q''_k] + (-1)^{qq'} [Q'_j, [Q_l, Q''_k]]}_{i\text{-th place}}, \\ & \quad \dots, [[Q_{m+1}, Q'_{n+1}], Q''_{p+1}] + (-1)^{qq'} [Q'_{n+1}, [Q_{m+1}, Q''_{p+1}]]) \\ &= [[[(Q_1, \dots, Q_{m+1}), (Q'_1, \dots, Q'_{n+1})], (Q''_1, \dots, Q''_{p+1})]] \\ & \quad + (-1)^{qq'} [[(Q'_1, \dots, Q'_{n+1}), [[(Q_1, \dots, Q_{m+1}), (Q''_1, \dots, Q''_{p+1})]]]]. \end{aligned}$$

- Let $(Q_1, \dots, Q_{m+1}) \in Coder_{c,q}^m(\Delta_r)$ and $(Q'_1, \dots, Q'_{n+1}) \in Coder_{c,q'}^n(\Delta_{r'})$. We have

$$\begin{aligned} & \Phi([[(Q_1, \dots, Q_{m+1}), (Q'_1, \dots, Q'_{n+1})]]) \\ &= \Phi([[Q_1, Q'_1], \dots, \underbrace{\sum_{j+k=i+1} [Q_j, Q'_k], \dots, [Q_{m+1}, Q'_{n+1}]]_{i\text{-th place}}]) \\ &= \sum_{i=1}^{m+n+1} \sum_{k+j=i+1} [Q_j, Q'_k] \\ &= [Q_1 + \dots + Q_{m+1}, Q'_1 + \dots + Q'_{n+1}] \\ &= [\Phi(Q_1, \dots, Q_{m+1}), \Phi(Q'_1, \dots, Q'_{n+1})]. \end{aligned}$$

- Let $(Q, Q') \in Coder_{c,1}^1(\Delta_1)$. We therefore have

$$\begin{aligned} [[(Q, Q'), (Q, Q')]] &= ([Q, Q], [Q, Q'] + [Q', Q], [Q', Q']) \\ &= ([Q, Q], [Q, Q'] - (-1)^{1 \times 1} [Q, Q'], [Q', Q']) \\ &= ([Q, Q], 2[Q, Q'], [Q', Q']). \end{aligned}$$

Then, (Q, Q') is up to homotopy compatible if and only if $[[(Q, Q'), (Q, Q')]] = 0$.

□

We obtain the following theorem:

Theorem 4.4. An up to homotopy compatible Hom-right-symmetric algebra $(\mathfrak{g}, Q, Q', \alpha)$ is a (graded) vector space \mathfrak{g} , a degree zero map $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$ and a coderivation $(Q, Q') \in \text{Coder}_{c,1}^1(\Delta_1)$ of $\mathfrak{g}[1] \otimes S(\mathfrak{g}[1])$ such that

$$[[(Q, Q'), (Q, Q')]] = 0.$$

5 Representations and cohomologies of compatible Hom-right-symmetric algebras

5.1 Representations

Definition 5.1. A representation of a compatible Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \mu', \alpha)$ on a vector space M with respect the linear map $\alpha_M : M \rightarrow M$, consist of (L, R, L', R') , where $L, R, L', R' : \mathfrak{g} \rightarrow gl(M)$ are four linear maps such that for any $k, l \in \mathbb{K}$, the 4-tuple $(M, kL + lL', kR + lR', \alpha_M)$ is a representation of the Hom right-symmetric algebra $(\mathfrak{g}, k\mu + l\mu', \alpha)$.

In this case, $(M, L, R, L', R', \alpha_M)$ is called a compatible representation of $(\mathfrak{g}, \mu, \mu', \alpha)$, or simply we say that M is a compatible \mathfrak{g} -module.

Proposition 5.2. Let $(\mathfrak{g}, \mu, \mu', \alpha)$ be a compatible Hom-right-symmetric algebra. Then, $(M, L, R, L', R', \alpha_M)$ is a compatible representation of $(\mathfrak{g}, \mu, \mu', \alpha)$ if and only if (M, L, R, α_M) is a representation of the Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \alpha)$, (M, L', R', α_M) is a representation of the Hom-right-symmetric algebra $(\mathfrak{g}, \mu', \alpha)$ and the two following (compatible) identities holds

$$\begin{aligned} &R(\mu'(x, y) - \mu'(y, x)) \circ \alpha_M - R(\alpha(y)) \circ R'(x) + R(\alpha(x)) \circ R'(y) \\ &+ R'(\mu(x, y) - \mu(y, x)) \circ \alpha_M - R'(\alpha(y)) \circ R(x) + R'(\alpha(x)) \circ R(y) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} &L(\alpha(x)) \circ L'(y) - L(\mu'(x, y)) \circ \alpha_M - L(\alpha(x)) \circ R'(y) + R(\alpha(y)) \circ L'(x) \\ &+ L'(\alpha(x)) \circ L(y) - L'(\mu(x, y)) \circ \alpha_M - L'(\alpha(x)) \circ R(y) + R'(\alpha(y)) \circ L(x) \\ &= 0. \end{aligned}$$

Remark 5.3. If $(M, L, R, L', R', \alpha_M)$ is a compatible representation of $(\mathfrak{g}, \mu, \mu', \alpha)$, then for each natural number r , $(M, L_r, R_r, L'_r, R'_r, \alpha_M)$ is also a representation, where $L_r = L \circ (\alpha^r)$, $R_r = R \circ (\alpha^r)$, $L'_r = L' \circ (\alpha^r)$ and $R'_r = R' \circ (\alpha^r)$.

Therefore, if α invertible, the same result hold for each integer r .

Example 5.4. The simplest example of compatible respresentation is the trivial representation, i.e., $M = \mathbb{K}$, $L = R = L' = R' = 0$ and $\alpha_M = id_{\mathbb{K}}$.

The fundamental example of compatible representation is the adjoint-representation. Let : $M = \mathfrak{g}$, $L(x)(y) = \mu(x, y)$, $R(x)(y) = \mu(y, x)$, $L'(x)(y) = \mu'(x, y)$, $R'(x)(y) = \mu'(y, x)$ and $\alpha_M = \alpha$.

Proposition 5.5. Let $(M, L, R, L', R', \alpha_M)$ be a compatible representation of the compatible Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \mu', \alpha)$. For all $x, y \in \mathfrak{g}$, $u, v \in M$, define

$$\mu_{L,R}((x, u), (y, v)) = (\mu(x, y), L(x)(v) + R(y)(u)), \tag{8}$$

$$\mu'_{L',R'}((x, u), (y, v)) = (\mu'(x, y), L'(x)(v) + R'(y)(u)), \tag{9}$$

$$(\alpha + \alpha_M)(x, u) = (\alpha(x), \alpha_M(u)). \tag{10}$$

Then, $(\mathfrak{g} \times M, \mu_{L,R}, \mu'_{L',R'}, \alpha + \alpha_M)$ is a compatible Hom-right-symmetric algebra.

Proof. Let $(M, L, R, L', R', \alpha_M)$ be a compatible representation of the compatible Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \mu', \alpha)$.

Let $Q_{L,R} = \mu_{L,R}$ and $Q'_{L',R'} = \mu'_{L',R'}$. Then, $(\mathfrak{g} \times M, \mu_{L,R}, \mu'_{L',R'}, \alpha + \alpha_M)$ is a compatible Hom-right-symmetric algebra if and only if $[[(Q_{L,R}, Q'_{L',R'}), (Q_{L,R}, Q'_{L',R'})]] = 0$.

Since (M, L, R, α_M) is a representation of $(\mathfrak{g}, \mu, \alpha)$, then by Proposition 2.8, $(\mathfrak{g} \times M, \mu_{L,R}, \alpha + \alpha_M)$ is a Hom-right-symmetric algebra then

$$[Q_{L,R}, Q_{L,R}] = 0 \quad (11)$$

Since (M, L', R', α_M) is a representation of $(\mathfrak{g}, \mu', \alpha)$, according to Lemma 2.8, the triple $(\mathfrak{g} \times M, \mu'_{L',R'}, \alpha + \alpha_M)$ is a Hom-right-symmetric algebra, then

$$[Q'_{L',R'}, Q'_{L',R'}] = 0. \quad (12)$$

We have

$$[Q_{L,R}, Q'_{L',R'}] = Q_{L,R} \circ Q'_{L',R'} + Q'_{L',R'} \circ Q_{L,R}.$$

For all $x, y, z \in \mathfrak{g}$, $u, v, w \in M$, we have

$$\begin{aligned} & Q_{L,R} \circ Q'_{L',R'}((x, u) \otimes (y, v) \cdot (z, w)) = \\ & -Q_{L,R}(Q'_{L',R'}((x, u), (z, w)), (\alpha(y), \alpha_M(v))) \\ & -Q_{L,R}((\alpha(x), \alpha_M(u)), Q'_{L',R'}((y, v), (z, w))) \\ & +Q_{L,R}((\alpha(x), \alpha_M(u)), Q'_{L',R'}((z, w), (y, v))) \\ = & \left(\mu(\mu'(x, y), \alpha(z)), L(\mu'(x, y))(\alpha_M(w)) + R(\alpha(z)) \circ L'(x)(v) + R(\alpha(z)) \circ R'(y)(u) \right) \\ & - \left(\mu(\mu'(x, z), \alpha(y)), L(\mu'(x, z))(\alpha_M(v)) + R(\alpha(y)) \circ L'(x)(w) + R(\alpha(y)) \circ R'(z)(u) \right) \\ & - \left(\mu(\alpha(x), \mu'(y, z)), L(\alpha(x)) \circ L'(y)(w) + L(\alpha(x)) \circ R'(z)(v) + R(\mu'(y, z))(\alpha_M(u)) \right) \\ & + \left(\mu(\alpha(x), \mu'(z, y)), L(\alpha(x)) \circ L'(z)(v) + L(\alpha(x)) \circ R'(y)(w) + R(\mu'(z, y))(\alpha_M(u)) \right). \end{aligned}$$

Similarly

$$\begin{aligned} & Q'_{L',R'} \circ Q_{L,R}((x, u) \otimes (y, v) \cdot (z, w)) = \\ = & \left(\mu'(\mu(x, y), \alpha(z)), L'(\mu(x, y))(\alpha_M(w)) + R'(\alpha(z)) \circ L(x)(v) + R'(\alpha(z)) \circ R(y)(u) \right) \\ & - \left(\mu'(\mu(x, z), \alpha(y)), L'(\mu(x, z))(\alpha_M(v)) + R'(\alpha(y)) \circ L(x)(w) + R'(\alpha(y)) \circ R(z)(u) \right) \\ & - \left(\mu'(\alpha(x), \mu(y, z)), L'(\alpha(x)) \circ L(y)(w) + L'(\alpha(x)) \circ R(z)(v) + R'(\mu(y, z))(\alpha_M(u)) \right) \\ & + \left(\mu'(\alpha(x), \mu(z, y)), L'(\alpha(x)) \circ L(z)(v) + L'(\alpha(x)) \circ R(y)(w) + R'(\mu(z, y))(\alpha_M(u)) \right). \end{aligned}$$

Then,

$$\begin{aligned} & [Q_{L,R}, Q'_{L',R'}]((x, u) \otimes (y, v) \cdot (z, w)) = \\ = & \left(\mu(\mu'(x, y), \alpha(z)), L(\mu'(x, y))(\alpha_M(w)) + R(\alpha(z)) \circ L'(x)(v) + R(\alpha(z)) \circ R'(y)(u) \right) \\ & - \left(\mu(\mu'(x, z), \alpha(y)), L(\mu'(x, z))(\alpha_M(v)) + R(\alpha(y)) \circ L'(x)(w) + R(\alpha(y)) \circ R'(z)(u) \right) \\ & - \left(\mu(\alpha(x), \mu'(y, z)), L(\alpha(x)) \circ L'(y)(w) + L(\alpha(x)) \circ R'(z)(v) + R(\mu'(y, z))(\alpha_M(u)) \right) \\ & + \left(\mu(\alpha(x), \mu'(z, y)), L(\alpha(x)) \circ L'(z)(v) + L(\alpha(x)) \circ R'(y)(w) + R(\mu'(z, y))(\alpha_M(u)) \right) \\ & + \left(\mu'(\mu(x, y), \alpha(z)), L'(\mu(x, y))(\alpha_M(w)) + R'(\alpha(z)) \circ L(x)(v) + R'(\alpha(z)) \circ R(y)(u) \right) \\ & - \left(\mu'(\mu(x, z), \alpha(y)), L'(\mu(x, z))(\alpha_M(v)) + R'(\alpha(y)) \circ L(x)(w) + R'(\alpha(y)) \circ R(z)(u) \right) \\ & - \left(\mu'(\alpha(x), \mu(y, z)), L'(\alpha(x)) \circ L(y)(w) + L'(\alpha(x)) \circ R(z)(v) + R'(\mu(y, z))(\alpha_M(u)) \right) \\ & + \left(\mu'(\alpha(x), \mu(z, y)), L'(\alpha(x)) \circ L(z)(v) + L'(\alpha(x)) \circ R(y)(w) + R'(\mu(z, y))(\alpha_M(u)) \right). \end{aligned}$$

Since $(\mathfrak{g}, \mu, \mu', \alpha)$ is a compatible Hom-right-symmetric algebra, then

$$\begin{aligned} & \mu(\mu'(x, y), \alpha(z)) - \mu(\mu'(x, z), \alpha(y)) - \mu(\alpha(x), \mu'(y, z)) + \mu(\alpha(x), \mu'(z, y)) \\ & + \mu'(\mu(x, y), \alpha(z)) - \mu'(\mu(x, z), \alpha(y)) - \mu'(\alpha(x), \mu(y, z)) + \mu'(\alpha(x), \mu(z, y)) \\ & = 0. \end{aligned}$$

Since $(M, L, R, L', R', \alpha_M)$ is a compatible representation, then by using the compatibility identities (Proposition 5.2) we obtain

$$\begin{aligned} & R(\alpha(z)) \circ R'(y)(u) - R(\alpha(y)) \circ R'(z)(u) - R(\mu'(y, z))(\alpha_M(u)) - R(\mu'(z, y))(\alpha_M(u)) \\ & + R'(\alpha(z)) \circ R(y)(u) - R'(\alpha(y)) \circ R(z)(u) - R'(\mu(y, z))(\alpha_M(u)) + R'(\mu(z, y))(\alpha_M(u)) \\ & = 0, \end{aligned}$$

$$\begin{aligned} & R(\alpha(z)) \circ L'(x)(v) - L(\mu'(x, z))(\alpha_M(v)) - L(\alpha(x)) \circ R'(z)(v) + L(\alpha(x)) \circ L'(z)(v) \\ & R'(\alpha(z)) \circ L(x)(v) - L'(\mu(x, z))(\alpha_M(v)) - L'(\alpha(x)) \circ R(z)(v) + L'(\alpha(x)) \circ L(z)(v) \\ & = 0, \end{aligned}$$

$$\begin{aligned} & L(\mu'(x, y))(\alpha_M(w)) - R(\alpha(y)) \circ L'(x)(w) - L(\alpha(x)) \circ L'(y)(w) + L(\alpha(x)) \circ R'(y)(w) \\ & + L'(\mu(x, y))(\alpha_M(w)) - R'(\alpha(y)) \circ L(x)(w) - L'(\alpha(x)) \circ L(y)(w) + L'(\alpha(x)) \circ R(y)(w) \\ & = 0. \end{aligned}$$

Thus, $[Q_{L,R}, Q'_{L',R'}] = 0$, and with (11) and (12), we conclude that

$[[(Q_{L,R}, Q'_{L',R'}), Q_{L,R}, Q'_{L',R'}]]$ = 0, which implies that $(\mathfrak{g} \times M, \mu_{L,R}, \mu'_{L',R'}, \alpha + \alpha_M)$ is a compatible Hom-right-symmetric algebra. \square

5.2 Cohomology

Let $(M, L, R, L', R', \alpha_M)$ be a compatible representation of a compatible Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \mu', \alpha)$. For $n \geq 1$, define

$$C_c^n(\mathfrak{g}, M) = \underbrace{C^n(\mathfrak{g}, M) \oplus \cdots \oplus C^n(\mathfrak{g}, M)}_{n\text{-copies}}$$

Let now define the operator $d_n^c : C_c^n(\mathfrak{g}, M) \rightarrow C_c^{n+1}(\mathfrak{g}, M)$ by:

$$d_n^c(C_1, \dots, C_n) = (-1)^n \psi_{n+1}^{-1} \left(\left[(Q_{L,R}, Q'_{L',R'}), (\psi_n(C_1), \dots, \psi_n(C_n)) \right] \right), \quad \forall n \geq 1. \quad (13)$$

Theorem 5.6. With the above notation, for each $n \geq 1$, we have

$$d_{n+1}^c \circ d_n^c = 0. \quad (14)$$

Proof. Let $(C_1, \dots, C_n) \in C_c^n(\mathfrak{g}, M)$. we have

$$d_n^c(C_1, \dots, C_n) = (-1)^n \psi_{n+1}^{-1} \left(\left[(Q_{L,R}, Q'_{L',R'}), (\psi_n(C_1), \dots, \psi_n(C_n)) \right] \right)$$

Then $d_{n+1}^c \circ d_n^c(C_1, \dots, C_n) = -\psi_{n+2}^{-1} \left(\left[(Q_{L,R}, Q'_{L',R'}), \left[(Q_{L,R}, Q'_{L',R'}), (\psi_n(C_1), \dots, \psi_n(C_n)) \right] \right] \right)$. Since $[[(Q_{L,R}, Q'_{L',R'}), (Q_{L,R}, Q'_{L',R'})]]$ = 0, then with regard to Jacobi identity of the graded Lie bracket $[[\cdot, \cdot]]$, we obtain

$$d_{n+1}^c \circ d_n^c = 0.$$

\square

Let $(M, L, R, L', R', \alpha_M)$ be a compatible representation of a compatible Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \mu', \alpha)$.

For each $n \geq 0$, denote by $d_n : C^n(\mathfrak{g}, M) \rightarrow C^{n+1}(\mathfrak{g}, M)$ the coboundary operator associated to the representation (M, L, R, α_M) of the Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \alpha)$ and denote by d'_n the coboundary operator associated to the representation (M, L', R', α_M) of the Hom-right-symmetric algebra $(\mathfrak{g}, \mu', \alpha)$. Recall that for each $n \geq 1$ and $C \in C^n(\mathfrak{g}, M)$, we have

$$d_n C = (-1)^n \psi_{n+1}^{-1}([Q_{L,R}, \psi_n(C)]) \quad \text{and} \quad d'_n C = (-1)^n \psi_{n+1}^{-1}([Q'_{L',R'}, \psi_n(C)]).$$

Proposition 5.7. For each $n \geq 1$ and $(C_1, \dots, C_n) \in C_c^n(\mathfrak{g}, M)$, we have

$$d_n^c(C_1, \dots, C_n) = (d_n C_1, d_n C_2 + d'_n C_1, \dots, \underbrace{d_n C_i + d'_n C_{i-1}}_{i\text{-th place}}, \dots, d'_n C_n).$$

Proof. Let $(C_1, \dots, C_n) \in C_c^n(\mathfrak{g}, M)$. We have

$$\begin{aligned} d_n^c(C_1, \dots, C_n) &= (-1)^n \psi_{n+1}^{-1}([[(Q_{L,R}, Q'_{L',R'}), (\psi_n(C_1), \dots, \psi_n(C_n))]]) \\ &= (-1)^n \psi_{n+1}^{-1}([Q_{L,R}, \psi_n(C_1)], \dots, \underbrace{[Q_{L,R}, \psi_n(C_i)] + [Q'_{L',R'}, \psi_n(C_{i-1})]}_{i\text{ th-place}}, \\ &\quad \dots, [Q'_{L',R'}, \psi_n(C_n)]) \\ &= (-1)^n (\psi_{n+1}^{-1}[Q_{L,R}, \psi_n(C_1)], \dots, \psi_{n+1}^{-1}[Q_{L,R}, \psi_n(C_i)] + \psi_{n+1}^{-1}[Q'_{L',R'}, \psi_n(C_{i-1})], \\ &\quad \dots, \psi_{n+1}^{-1}[Q'_{L',R'}, \psi_n(C_n)]) \\ &= (d_n C_1, d_n C_2 + d'_n C_1, \dots, d_n C_i + d'_n C_{i-1}, \dots, d'_n C_n). \end{aligned}$$

□

Definition 5.8. A zero-cochain is a vector $m \in M$ such that $\alpha_M(m) = m$ and for all $x, y \in \mathfrak{g}$:

$$\begin{aligned} R(\alpha(y)) \circ R(x)(m) &= R(\mu(x, y))(m), \\ R'(\alpha(y)) \circ R'(x)(m) &= R'(\mu'(x, y))(m), \\ L(x)(m) - R(x)(m) &= L'(x)(m) - R'(x)(m). \end{aligned}$$

For a zero-cochain m , define $d_0^c(m)(x) = L(x)(m) - R(x)(m)$. Then,

$$d_1^c(d_0^c(m))(x) = (d_1(d_0^c(m)))(x), d'_1(d'_0(m))(x) = (0, 0).$$

Corollary 5.9. For each $n \geq 0$, we have

$$d_{n+1}^c \circ d_n^c = 0.$$

Notation: Let $(M, L, R, L', R', \alpha_M)$ be a compatible representation of a compatible Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \mu', \alpha)$. Denote $H_c^0(\mathfrak{g}, M) = Z_c^0 = \ker(d_0^c)$, and for each $n \geq 1$, denote $Z_c^n = \ker(d_n^c)$, $B_c^n = \text{Im}(d_{n-1}^c)$. Denote by $H_c^n(\mathfrak{g}, M) = Z_c^n/B_c^n$ the space of n -th-cohomology group.

6 Trivial representation and central extensions

Let $(\mathfrak{g}, \mu, \mu', \alpha)$ be a compatible Hom-right-symmetric algebra. The compatible trivial representation is $(M, L, R, L', R', \alpha_M)$, where $M = \mathbb{K}$, $L = R = L' = R' = 0$ and $\alpha_M = id$.

By a simple computation, we show that

$$H_c^0(\mathfrak{g}, \mathbb{K}) = \mathbb{K} \text{ and } H_c^1(\mathfrak{g}, \mathbb{K}) = \{C \in C_c^1(\mathfrak{g}, \mathbb{K}) : C(\mu(x, y)) = C(\mu'(x, y)) = 0, \forall x, y \in \mathfrak{g}\}.$$

Let $(C, C') \in C_c^2(\mathfrak{g}, \mathbb{K})$, for all $x, y \in \mathfrak{g}$ and $s, t \in \mathbb{K}$ define on $\mathfrak{g} \oplus \mathbb{K}$,

$$\mu^+((x, s), (y, t)) = (\mu(x, y), C(x, y)), \quad \mu'^+((x, s), (y, t)) = (\mu'(x, y), C'(x, y)),$$

$$\alpha^+(x, s) = (\alpha(x), s).$$

Theorem 6.1. With the above stated notation, the 4-tuple $(\mathfrak{g} \oplus \mathbb{K}, \mu^+, \mu'^+, \alpha^+)$ is a compatible Hom-right-symmetric algebra if and only if $d_2^c(C, C') = 0$.

Proof. On the one hand, we have $\alpha^+(\mu^+((x, s), (y, t))) = (\alpha(\mu(x, y)), C(x, y))$.

On the other hand, we have $\mu^+(\alpha^+(x, s), \alpha^+(y, t)) = (\mu(\alpha(x), \alpha(y)), C(\alpha(x), \alpha(y)))$.

Since $\alpha(\mu(x, y)) = \mu(\alpha(x), \alpha(y))$ and by definition of the cochain C , we have

$C(\alpha(x), \alpha(y)) = \alpha_M \circ C(x, y) = C(x, y)$, then $\alpha^+(\mu^+((x, s), (y, t))) = \mu^+(\alpha^+(x, s), \alpha^+(y, t))$. We deduce that α^+ is a morphism for the product μ^+ .

Similarly, by the same method, it is easy to show that α^+ is multiplicative for the product μ'^+ .

Now, to show that $(\mathfrak{g} \oplus \mathbb{K}, \mu^+, \mu'^+, \alpha^+)$ is a compatible Hom-right-symmetric algebra, we need to prove $[[Q^+, Q^+], (Q^+, Q'^+)] = 0$, where $Q^+ = \mu^+$ and $Q'^+ = \mu'^+$. Or $[[Q^+, Q^+], (Q^+, Q'^+)] = 0$ equivalent to prove $[Q^+, Q^+] = 0$, $[Q'^+, Q'^+] = 0$ and $[Q^+, Q'^+] = 0$. Let $x, y, z \in \mathfrak{g}$ and $s, r, t \in \mathbb{K}$. We have

$$\begin{aligned} \frac{1}{2}[Q^+, Q^+](x, s) \otimes (y, r) \cdot (z, t) &= Q^+(Q^+((x, s), (y, r)), \alpha^+(z, t)) \\ &\quad - Q^+(\alpha^+(x, s), Q^+((y, r), (z, t))) \\ &\quad - Q^+(Q^+((x, s), (z, t)), \alpha^+(y, r)) \\ &\quad + Q^+(\alpha^+(x, s), Q^+((z, t), (y, r))) \\ &= (\mu(\mu(x, y), \alpha(z)), C(\mu(x, y), \alpha(z))) \\ &\quad - (\mu(\alpha(x), \mu(y, z)), C(\alpha(x), \mu(y, z))) \\ &\quad - (\mu(\mu(x, z), \alpha(y)), C(\mu(x, z), \alpha(y))) \\ &\quad + (\mu(\alpha(x), \mu(z, y)), C(\alpha(x), \mu(z, y))) \\ &= (0, -d_2 C(x \otimes y, z)), \end{aligned}$$

We obtain $[Q^+, Q^+] = 0$ if and only if $d_2 C = 0$.

Similarly, by the same computation, it is easy to see that $[Q'^+, Q'^+] = 0$ if and only if $d_2' C' = 0$.

Let's now compute $[Q^+, Q'^+](x, r) \otimes (y, s) \cdot (z, t)$.

We have $[Q^+, Q'^+] = Q^+ \circ Q'^+ + Q'^+ \circ Q^+$, then

$$\begin{aligned}
[Q^+, Q'^+]((x, s) \otimes (y, r), (z, t)) &= (Q^+ \circ Q'^+ + Q'^+ \circ Q^+)((x, s) \otimes (y, r), (z, t)) \\
&= Q^+(Q'^+((x, s), (y, r)), \alpha^+(z, t)) \\
&\quad - Q^+(\alpha^+(x, s), Q'^+((y, r), (z, t))) \\
&\quad - Q^+(Q'^+((x, s), (z, t)), \alpha^+(y, r)) \\
&\quad + Q^+(\alpha^+(x, s), Q'^+((z, t), (y, r))) \\
&\quad + Q'^+(Q^+((x, s), (y, r)), \alpha^+(z, t)) \\
&\quad - Q'^+(\alpha^+(x, s), Q^+((y, r), (z, t))) \\
&\quad - Q'^+(Q^+((x, s), (z, t)), \alpha^+(y, r)) \\
&\quad + Q'^+(\alpha^+(x, s), Q^+((z, t), (y, r))) \\
&= (\mu(\mu'(x, y), \alpha(z)), C(\mu'(x, y), \alpha(z))) \\
&\quad - (\mu(\alpha(x), \mu'(y, z)), C(\alpha(x), \mu'(y, z))) \\
&\quad - (\mu(\mu'(x, z), \alpha(y)), C(\mu'(x, z), \alpha(y))) \\
&\quad + (\mu(\alpha(x), \mu'(z, y)), C(\alpha(x), \mu'(z, y))) \\
&\quad + (\mu'(\mu(x, y), \alpha(z)), C'(\mu(x, y), \alpha(z))) \\
&\quad - (\mu'(\alpha(x), \mu(y, z)), C'(\alpha(x), \mu(y, z))) \\
&\quad - (\mu'(\mu(x, z), \alpha(y)), C'(\mu(x, z), \alpha(y))) \\
&\quad + (\mu'(\alpha(x), \mu(z, y)), C'(\alpha(x), \mu(z, y)))
\end{aligned}$$

The first component in the last sum is zero (because μ and μ' are compatible). The second component is exactly $-(d'_2 C + d_2 C')(x \otimes y, z)$. Then $[Q^+, Q'^+] = 0$ if and only if

$$d'_2 C + d_2 C' = 0.$$

Thus, we conclude that $[[Q^+, Q'^+], [Q^+, Q'^+]] = 0$ if and only if $d_2^c(C, C') = 0$. This finishing the proof. \square

Definition 6.2. With the above notation, the compatible Hom-right-symmetric algebra $(\mathfrak{g}^+, \mu^+, \mu'^+, \alpha^+)$ associated to the cochain (C, C') is called a central extension of $(\mathfrak{g}, \mu, \mu', \alpha)$.

Two central extensions $(\mathfrak{g}_1^+, \mu_1^+, \mu_1'^+, \alpha^+)$ and $(\mathfrak{g}_2^+, \mu_2^+, \mu_2'^+, \alpha^+)$ associated respectively to cochains (C_1, C'_1) and (C_2, C'_2) are said to be equivalent, if there exists a morphism $f^+ : \mathfrak{g}_1^+ \rightarrow \mathfrak{g}_2^+$ of compatible Hom-right-symmetric algebras of the form $f^+(x, s) = (x, s + f(x))$, with $f : \mathfrak{g} \rightarrow \mathbb{K}$ such that $f \circ \alpha = f$.

Proposition 6.3. Two central extensions $(\mathfrak{g}_1^+, \mu_1^+, \mu_1'^+, \alpha^+)$ and $(\mathfrak{g}_2^+, \mu_2^+, \mu_2'^+, \alpha^+)$ associated respectively to cochains (C_1, C'_1) and (C_2, C'_2) are equivalent if and only if $(C_1, C'_1) - (C_2, C'_2)$ is exact.

Proof. Since f^+ is a morphism of compatible Hom-right-symmetric algebra, then from $f^+(\mu_1^+((x, s), (y, r))) = \mu_2^+(f^+(x, s), f^+(y, r))$ for all $x, y \in \mathfrak{g}, s, r \in \mathbb{K}$, we obtain

$$C_1(x, y) - C_2(x, y) = -f(\mu(x, y)) = d_1 f(x, y). \quad (15)$$

Similarly, from $f^+(\mu_1'^+((x, s), (y, r))) = \mu_2'^+(f^+(x, s), f^+(y, r))$ for all $x, y \in \mathfrak{g}, s, r \in \mathbb{K}$, we obtain

$$C'_1(x, y) - C'_2(x, y) = -f(\mu'(x, y)) = d'_1 f(x, y). \quad (16)$$

The equations (15) and (16) give that

$$(C_1, C'_1) - (C_2, C'_2) = d_1^c f, \quad (17)$$

which means that $(C_1, C'_1) - (C_2, C'_2)$ is exact. \square

7 Linear deformations of regular compatible Hom-right-symmetric algebras

Let $(\mathfrak{g}, \mu, \mu', \alpha)$ be a regular compatible Hom-right-symmetric algebra and $(\mathfrak{g}, L, R, L', R', \alpha)$ be the adjoint representation of it. i.e., for all $x, y \in \mathfrak{g}$:

$$L(x)(y) = \mu(x, y), \quad R(x)(y) = \mu(y, x), \quad L'(x)(y) = \mu'(x, y), \quad R'(x)(y) = \mu'(y, x).$$

For the remaining of this section, we study linear deformations of $(\mathfrak{g}, \mu, \mu', \alpha)$ by using the cohomology with the compatible representation $(\mathfrak{g}, L_{-1,-1}, R_{-1,-1}, L'_{-1,-1}, R'_{-1,-1}, \alpha)$.

Let $(C, C') \in C_c^2(\mathfrak{g}, \mathfrak{g})$. For all $x, y \in \mathfrak{g}$, $t \in \mathbb{K}$, define

$$\mu_t(x, y) = \mu(x, y) + tC(x, y),$$

$$\mu'_t(x, y) = \mu'(x, y) + tC'(x, y),$$

Definition 7.1. With the above notation, if $(\mathfrak{g}, \mu_t, \mu'_t, \alpha)$ is a compatible Hom-right-symmetric algebra, we say that (C, C') generates a (one parameter) linear deformation of $(\mathfrak{g}, \mu, \mu', \alpha)$.

Proposition 7.2. With the above notation, (C, C') generates a linear deformation of $(\mathfrak{g}, \mu, \mu', \alpha)$ if and only if $d_2^c(C, C') = 0$ and $(\mathfrak{g}, C, C', \alpha)$ is a compatible Hom-right-symmetric algebra.

Proof. It is easy to see that $\alpha \circ \mu_t = \mu_t(\alpha, \alpha)$, $\alpha \circ \mu'_t = \mu'_t(\alpha, \alpha)$.

Then, $(\mathfrak{g}, \mu_t, \mu'_t, \alpha, \beta)$ is a compatible Hom-right-symmetric algebra if and only if for all $t \in \mathbb{K}$,

$$[[Q_t, Q'_t], (Q_t, Q'_t)] = 0, \tag{18}$$

where $Q_t = \mu_t$ and $Q'_t = \mu'_t$.

To shown (18), we need to show that $[Q_t, Q_t] = 0$, $[Q_t, Q'_t] = 0$ and $[Q'_t, Q'_t] = 0$.

It is easy to see that $Q_t(x, y) = Q(x, y) + tC(x, y)$, $Q'_t(x, y) = Q'(x, y) + tC'(x, y)$

where $Q(x, y) = \mu(x, y)$ and $Q'(x, y) = \mu'(x, y)$.

Let's first compute $[Q_t, Q_t]$. For all $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned} \frac{1}{2}[Q_t, Q_t](x \otimes y, z) &= Q_t(Q_t(x \otimes y), \alpha(z)) - Q_t(Q_t(x, z), \alpha(y)) \\ &\quad - Q_t(\alpha(x), Q_t(y, z)) + Q_t(\alpha(x), Q_t(z, y)). \end{aligned}$$

By a simple computation, we prove

$$\begin{aligned} \frac{1}{2}[Q_t, Q_t](x \otimes y, z) &= \mu(\mu(x, y), \alpha(z)) - \mu(\mu(x, z), \alpha(y)) \\ &\quad - \mu(\alpha(x), \mu(y, z)) + \mu(\alpha(x), \mu(z, y)) \\ &\quad + t(C(\mu(x, y), \alpha(z)) + \mu(C(x, y), \alpha(z)) - C(\mu(x, z), \alpha(y)) \\ &\quad - \mu(C(x, z), \alpha(y)) - C(\alpha(x), \mu(y, z)) - \mu(\alpha(x), C(y, z)) \\ &\quad + C(\alpha(x), \mu(z, y)) + \mu(\alpha(x), C(z, y))) \\ &\quad + t^2(C(C(x, y), \alpha(z)) - C(C(x, z), \alpha(y)) \\ &\quad - C(\alpha(x), C(y, z)) + C(\alpha(x), C(z, y))) \end{aligned}$$

Since $(\mathfrak{g}, \mu, \alpha)$ is a Hom-right-symmetric algebra, then

$$\mu(\mu(x, y), \alpha(z)) - \mu(\mu(x, z), \alpha(y)) - \mu(\alpha(x), \mu(y, z)) + \mu(\alpha(x), \mu(z, y)) = 0.$$

Therefore, it is easy to observe that, for each $t \in \mathbb{K}$, $[Q_t, Q_t] = 0$ if and only if

$$\begin{aligned} & C(\mu(x, y), \alpha(z)) + \mu(C(x, y), \alpha(z)) - C(\mu(x, z), \alpha(y)) \\ & - \mu(C(x, z), \alpha(y)) - C(\alpha(x), \mu(y, z)) - \mu(\alpha(x), C(y, z)) \\ & + C(\alpha(x), \mu(z, y)) + \mu(\alpha(x), C(z, y)) = 0, \end{aligned}$$

and $C(C(x, y), \alpha(z)) - C(C(x, z), \alpha(y)) - C(\alpha(x), C(y, z)) + C(\alpha(x), C(z, y)) = 0$, which means that $d_2 C = 0$ and $(\mathfrak{g}, C, \alpha)$ is a Hom-right-symmetric algebra.

Secondly, using the same computation, it is easily to show that, $[Q'_t, Q'_t] = 0$ for each $t \in \mathbb{K}$ if and only if $d'_2 C' = 0$ and $(\mathfrak{g}, C', \alpha)$ is a Hom-right-symmetric algebra.

Now, let's compute $[Q_t, Q'_t](x \otimes y, z)$ for all $x, y, z \in \mathfrak{g}$. Then, we have

$$\begin{aligned} [Q_t, Q'_t](x \otimes y, z) &= Q_t(Q'_t(x \otimes y), \alpha(z)) - Q_t(Q'_t(x, z), \alpha(y)) \\ &\quad - Q_t(\alpha(x), Q'_t(y, z)) + Q_t(\alpha(x), Q'_t(z, y)) \\ &\quad + Q'_t(Q_t(x \otimes y), \alpha(z)) - Q'_t(Q_t(x, z), \alpha(y)) \\ &\quad - Q'_t(\alpha(x), Q_t(y, z)) + Q'_t(\alpha(x), Q_t(z, y)) \\ &= \mu(\mu'(x, y), \alpha(z)) + t\mu(C'(x, y), \alpha(z)) \\ &\quad + tC(\mu'(x, y), \alpha(z)) + t^2C(C'(x, y), \alpha(z)) \\ &\quad - \mu(\mu'(x, z), \alpha(y)) - t\mu(C'(x, z), \alpha(y)) \\ &\quad - tC(\mu'(x, z), \alpha(y)) - t^2C(C'(x, z), \alpha(y)) \\ &\quad - \mu(\alpha(x), \mu'(y, z)) - t\mu(\alpha(x), C'(y, z)) \\ &\quad - tC(\alpha(x), \mu'(y, z)) - t^2C(\alpha(x), C'(y, z)) \\ &\quad + \mu(\alpha(x), \mu'(z, y)) + t\mu(\alpha(x), C'(z, y)) \\ &\quad + tC(\alpha(x), \mu'(z, y)) + t^2C(\alpha(x), C'(z, y)) \\ &\quad + \mu'(\mu(x, y), \alpha(z)) + t\mu'(C(x, y), \alpha(z)) \\ &\quad + tC'(\mu(x, y), \alpha(z)) + t^2C'(C(x, y), \alpha(z)) \\ &\quad - \mu'(\mu(x, z), \alpha(y)) - t\mu'(C(x, z), \alpha(y)) \\ &\quad - tC'(\mu(x, z), \alpha(y)) - t^2C'(C(x, z), \alpha(y)) \\ &\quad - \mu'(\alpha(x), \mu(y, z)) - t\mu'(\alpha(x), C(y, z)) \\ &\quad - tC'(\alpha(x), \mu(y, z)) - t^2C'(\alpha(x), C(y, z)) \\ &\quad + \mu'(\alpha(x), \mu(z, y)) + t\mu'(\alpha(x), C(z, y)) \\ &\quad + tC'(\alpha(x), \mu(z, y)) + t^2C'(\alpha(x), C(z, y)). \end{aligned}$$

Since μ, μ' are compatible, then

$$\begin{aligned} & \mu(\mu'(x, y), \alpha(z)) - \mu(\mu'(x, z), \alpha(y)) - \mu(\alpha(x), \mu'(y, z)) + \mu(\alpha(x), \mu'(z, y)) \\ & \mu'(\mu(x, y), \alpha(z)) - \mu'(\mu(x, z), \alpha(y)) - \mu'(\alpha(x), \mu(y, z)) + \mu'(\alpha(x), \mu(z, y)) \\ & = 0, \end{aligned}$$

then, we obtain

$$\begin{aligned} [Q_t, Q'_t](x \otimes y, z) &= t(\mu(C'(x, y), \alpha(z)) + C(\mu'(x, y), \alpha(z)) - \mu(C'(x, z), \alpha(y)) \\ &\quad - C(\mu'(x, z), \alpha(y)) - \mu(\alpha(x), C'(y, z)) - C(\alpha(x), \mu'(y, z)) \\ &\quad + \mu(\alpha(x), C'(z, y)) + C(\alpha(x), \mu'(z, y)) + \mu'(C(x, y), \alpha(z)) \\ &\quad + C'(\mu(x, y), \alpha(z)) - \mu'(C(x, z), \alpha(y)) - C'(\mu(x, z), \alpha(y)) \\ &\quad - \mu'(\alpha(x), C(y, z)) - C'(\alpha(x), \mu(y, z)) + \mu'(\alpha(x), C(z, y)) + C'(\alpha(x), \mu(z, y))) \\ &\quad + t^2(C(C'(x, y), \alpha(z)) - C(C'(x, z), \alpha(y)) - C(\alpha(x), C'(y, z)) \\ &\quad + C(\alpha(x), C'(z, y)) + C'(C(x, y), \alpha(z)) - C'(C(x, z), \alpha(y)) \\ &\quad - C'(\alpha(x), C(y, z)) + C'(\alpha(x), C(z, y))) \end{aligned}$$

Observe that, in the last equation, the terms in t sum up to $(-d'_2C - d_2C')(x \otimes y.z)$, then we obtain $[Q_t, Q'_t] = 0$ if and only if the terms in t and t^2 vanish, which means that $(d'_2C + d_2C') = 0$ and C, C' are compatible. Thus, we conclude that (C, C') generate a linear deformation of $(\mathfrak{g}, \mu, \mu', \alpha)$ if and only if $d_2^c(C, C') = 0$ and $(\mathfrak{g}, C, C', \alpha)$ is a compatible Hom-right-symmetric algebra. \square

Definition 7.3. Two linear deformations $(\mathfrak{g}, \mu_{t,1}, \mu'_{t,1}, \alpha)$ and $(\mathfrak{g}, \mu_{t,2}, \mu'_{t,2}, \alpha)$ of $(\mathfrak{g}, \mu, \mu', \alpha)$ generated respectively by (C_1, C'_1) and (C_2, C'_2) are said to be equivalent, if there exists a linear operator $\Phi \in gl(\mathfrak{g})$ commute with α , such that $\Phi_t = Id + t\Phi$ is a compatible Hom-right-symmetric algebras morphism from $(\mathfrak{g}, \mu_{t,1}, \mu'_{t,1}, \alpha)$ to $(\mathfrak{g}, \mu_{t,2}, \mu'_{t,2}, \alpha)$.

In particular, a linear deformation $(\mathfrak{g}, \mu_t, \mu'_t, \alpha)$ generated by (C, C') is said to be trivial if there exists a linear operator $\Phi \in gl(\mathfrak{g})$ commuting with α , such that $\Phi_t = Id + t\Phi$ is a compatible morphism from $(\mathfrak{g}, \mu_t, \mu'_t, \alpha)$ to $(\mathfrak{g}, \mu, \mu', \alpha)$.

Theorem 7.4. If two linear deformations $(\mu_{t,1}, \mu'_{t,1})$ and $(\mu_{t,2}, \mu'_{t,2})$ generated respectively by (C_1, C'_1) and (C_2, C'_2) are equivalent, then (C_1, C'_1) and (C_2, C'_2) are in the same cohomology class of $H_c^2(\mathfrak{g}, \mathfrak{g})$.

Proof. Suppose that, the two linear deformations $(\mu_{t,1}, \mu'_{t,1})$ and $(\mu_{t,2}, \mu'_{t,2})$ generated respectively by (C_1, C'_1) and (C_2, C'_2) are equivalent. Let $\Phi_t(x) = x + t\Phi(x)$ be a morphism of compatible Hom-right-symmetric algebras from $(\mathfrak{g}, \mu_{t,1}, \mu'_{t,1}, \alpha)$ to $(\mathfrak{g}, \mu_{t,2}, \mu'_{t,2}, \alpha)$. Then, for all $x, y \in \mathfrak{g}$, we have

$$\Phi_t(\mu_{t,1}(x, y)) = \mu_{t,2}(\Phi_t(x), \Phi_t(y)), \quad (19)$$

$$\Phi_t(\mu'_{t,1}(x, y)) = \mu'_{t,2}(\Phi_t(x), \Phi_t(y)). \quad (20)$$

Departing from equation (19), we obtain the three following equations

$$C_1(x, y) - C_2(x, y) = \mu(x, \Phi(y)) + \mu(\Phi(x), y) - \Phi(\mu(x, y)), \quad (21)$$

$$\Phi(C_1(x, y)) = \mu(\Phi(x), \Phi(y)) + C_2(x, \Phi(y)) + C_2(\Phi(x), y), \quad (22)$$

$$C_2(\Phi(x), \Phi(y)) = 0. \quad (23)$$

Similarly, from equation (20), we get

$$C'_1(x, y) - C'_2(x, y) = \mu'(x, \Phi(y)) + \mu'(\Phi(x), y) - \Phi(\mu'(x, y)), \quad (24)$$

$$\Phi(C'_1(x, y)) = \mu'(\Phi(x), \Phi(y)) + C'_2(x, \Phi(y)) + C'_2(\Phi(x), y), \quad (25)$$

$$C'_2(\Phi(x), \Phi(y)) = 0. \quad (26)$$

Equation (21) means that $C_1 - C_2 = d_1\Phi$, and Equation (24) means that $C'_1 - C'_2 = d'_1\Phi$.

We deduce thus that, $(C_1, C'_1) - (C_2, C'_2) = d_1^c\Phi$, which implies (C_1, C'_1) and (C_2, C'_2) are in the same cohomology class of $H_c^2(\mathfrak{g}, \mathfrak{g})$. \square

Remark 7.5. With the notation of theorem 7.4, suppose $C_2 = C'_2 = 0$, we obtain the following result: the linear deformation $(\mu_{t,1}, \mu'_{t,1})$ generated by (C_1, C'_1) is trivial if and only if

$$C_1(x, y) = \mu(x, \Phi(y)) + \mu(\Phi(x), y) - \Phi(\mu(x, y)), \quad (27)$$

$$\Phi(C_1(x, y)) = \mu(\Phi(x), \Phi(y)), \quad (28)$$

$$C'_1(x, y) = \mu'(x, \Phi(y)) + \mu'(\Phi(x), y) - \Phi(\mu'(x, y)), \quad (29)$$

$$\Phi(C'_1(x, y)) = \mu'(\Phi(x), \Phi(y)). \quad (30)$$

The equations (27), (28), (29) and (30) motivate us to introduce the notion of Nijenhuis operator on a regular compatible Hom-right-symmetric algebra.

Definition 7.6. Let $(\mathfrak{g}, \mu, \mu', \alpha)$ be a regular compatible Hom-right-symmetric algebra. A linear map $N : \mathfrak{g} \rightarrow \mathfrak{g}$ is called a Nijenhuis operator on $(\mathfrak{g}, \mu, \mu', \alpha)$ if N is both a Nijenhuis operator on the Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \alpha)$ and a Nijenhuis operator on the Hom-right-symmetric algebra $(\mathfrak{g}, \mu', \alpha)$, i.e.,

$$N \circ \alpha = \alpha \circ N, \quad (31)$$

$$\mu(N(x), N(y)) = N(x \cdot_N y), \quad (32)$$

$$\mu'(N(x), N(y)) = N(x \cdot'_N y), \quad (33)$$

for all $x, y \in \mathfrak{g}$, where $x \cdot_N y = \mu(N(x), y) + \mu(x, N(y)) - N(\mu(x, y))$ and $x \cdot'_N y = \mu'(N(x), y) + \mu'(x, N(y)) - N(\mu'(x, y))$.

Proposition 7.7. Let $(\mathfrak{g}, \mu, \mu', \alpha)$ be a regular compatible Hom-right-symmetric algebra and $N \in gl(\mathfrak{g})$. Then, N is a Nijenhuis operator on $(\mathfrak{g}, \mu, \mu', \alpha)$ if and only if, for all $k, l \in \mathbb{K}$, N is a Nijenhuis operator on the Hom-right-symmetric algebra $(\mathfrak{g}, k\mu + l\mu', \alpha)$.

Proof. Let N be a Nijenhuis operator on the regular compatible Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \mu', \alpha)$. Consider $\mu'' = k\mu + l\mu'$, then for all $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} \mu''(N(x), N(y)) &= k\mu(N(x), N(y)) + l\mu'(N(x), N(y)) \\ &= kN(\mu(N(x), y) + \mu(x, N(y)) - N(\mu(x, y))) \\ &\quad + lN(\mu'(N(x), y) + \mu'(x, N(y)) - N(\mu'(x, y))) \\ &= N((k\mu + l\mu')(N(x), y) + (k\mu + l\mu')(x, N(y)) - N((k\mu + l\mu')(x, y))) \\ &= N(\mu''(N(x), y) + \mu''(x, N(y)) - N(\mu''(x, y))) \\ &= N(x \cdot''_N y), \end{aligned}$$

where $x \cdot''_N y = \mu''(N(x), y) + \mu''(x, N(y)) - N(\mu''(x, y))$. Thus implies that N is a Nijenhuis operator on the Hom-right-symmetric algebra $(\mathfrak{g}, k\mu + l\mu', \alpha)$.

It is easy to see that $\cdot''_N = k \cdot_N + l \cdot'_N$.

Conversely, it is easy to prove the opposite. □

Theorem 7.8. Let N be a Nijenhuis operator on the regular compatible Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \mu', \alpha)$. Then a linear deformation $(\mathfrak{g}, \mu_{t,1}, \mu'_{t,1}, \alpha, \beta)$ of $(\mathfrak{g}, \mu, \mu', \alpha)$ can be obtained by putting

$$C_1(x, y) = x \cdot_N y \quad \text{and} \quad C'_1(x, y) = x \cdot'_N y,$$

for all $x, y \in \mathfrak{g}$.

Furthermore, this linear deformation $(\mathfrak{g}, \mu_{t,1}, \mu'_{t,1}, \alpha)$ is trivial.

Proof. Let N be a Nijenhuis operator on a regular compatible Hom-right-symmetric algebra $(\mathfrak{g}, \mu, \mu', \alpha)$. Let $C_1(x, y) = x \cdot_N y$ and $C'_1(x, y) = x \cdot'_N y$ for all $x, y \in \mathfrak{g}$. By definition of the product \cdot_N and \cdot'_N , we have

$$C_1(x, y) = \mu(N(x), y) + \mu(x, N(y)) - N(\mu(x, y)) = (d_1 N)(x, y),$$

and

$$C'_1(x, y) = \mu'(N(x), y) + \mu'(x, N(y)) - N(\mu'(x, y)) = (d'_1 N)(x, y).$$

Then, gives $(C_1, C'_1) = d_1^c N$ which implies that $d_2^c(C_1, C'_1) = 0$.

Since N is a Nijenhuis operator on the Hom-right-symmetric algebra $(\mathfrak{g}, k\mu + l\mu', \alpha)$, then with notation

of Proposition 7.7, $(\mathfrak{g}, \cdot, \alpha)$ is a Hom-right-symmetric algebra, which means that $(\mathfrak{g}, k \cdot_N + l \cdot'_N, \alpha)$ is a Hom-right-symmetric algebra for all $k, l \in \mathbb{K}$. Thus, $(\mathfrak{g}, \cdot_N, \cdot'_N, \alpha)$ is a compatible Hom-right-symmetric algebra. We conclude that (C_1, C'_1) generate a linear deformation.

Let $\Phi = N$. By definition of the Nijenhuis operator on the compatible Hom-right-symmetric algebra, it is easy to observe that, C_1 verify the equations (27), (28) and C'_1 verify the equations (28) and (30). Observe that, $N_t = id + tN$ is a morphism of compatible Hom-right-symmetric algebras from $(\mathfrak{g}, \mu_{t,1}, \mu'_{t,2}, \alpha)$ to $(\mathfrak{g}, \mu, \mu', \alpha)$. This implies that, the linear deformation of $(\mathfrak{g}, \mu, \mu', \alpha)$ generated by (C_1, C'_1) is trivial. \square

References

- [1] A. A. Agrachev and R. V. Gamkrelidze, Chronological algebras and nonstationary vector fields, *J. Sov. Math.* 17(1) (1981), 1650-1675.
- [2] W. Aloulou, D. Arnal and R. Chatbouri, Algèbres et cogèbres de Gerstenhaber et cohomologies de Chevalley-Harrison, *Bull. Sci. Math* 133 (1) (2009), 1-50.
- [3] W. Aloulou and R. Chatbouri, Algèbres Hom-Gerstenhaber à homotopie près, *Bull. Sci. Math.* 140 (2016), 36-63.
- [4] S. Attan, Representations and \mathcal{O} -operators of Hom-(pre)-Jacobi-Jordan algebras, *Moroccan J. Algebra Geom. Appl.* (2025), to appear.
- [5] M. Bordemann, O. Elchinger and A. Makhlouf, Twisting Poisson algebras coPoisson algebras and quantization, *Trav. Math. Fac. Sci. Luxembourg* 20 (2012), 83-119.
- [6] A. Cayley, On the theory of analytical forms called trees, in the collected Mathematical papers of Arthur Cayley, Cambridge Univ. Press, Vol. 3 (1890), 242-246.
- [7] F. Chapoton and M. Livernet, Pre-Lie algebras and the rooted trees operad, *Int. Math. Res. Not.* 2001(8) (2001), 395-408.
- [8] R. Chatbouri, Hom-right symmetric algebras representations, *J. Lie Theory* 27(4) (2017), 1089-1105.
- [9] A. Dzhumadil'daev, Cohomologies and deformations of right-symmetric algebras, *J. Math. Sci.* 93 (6) (1999), 836-876.
- [10] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. Math.* 78 (1963), 267-288.
- [11] J. Hartwig, D. Larsson and S. Silvestrov, Deformations of Lie algebras using σ -derivation, *J. Algebra* 295 (2006), 314-361.
- [12] J. L. Koszul, Domaines bornés homogènes et orbites de groupes de transformations affines, *Bull. Soc. Math. France* 89 (1961), 515-533.
- [13] M. Livernet, A rigidity theorem for pre-Lie algebras, *J. Pure Appl. Algebra* 207 (2006), 1-18.
- [14] S. Liu, L. Song and R. Tang, Representations and cohomologies of regular hom-pre-Lie algebras, *J. Algebra Appl.* 19(08) (2020), 2050149.
- [15] A. Makhlouf, and S. D. Silvestrov, Hom-algebras structures, *J. Gen. Lie Theory Appl.* 2(2) (2008), 51-64.

- [16] D. Manchon, A short survey on pre-Lie algebras, In: Non commutative Geometry and physics: Renormalisation, Motives, Index Theory, (2011), 89-102.
- [17] D. Manchon and A. Saïdi, Lois pre-Lie en interaction, Commun. Algebra 39(10) (2011), 3662-3680.
- [18] A. Odesskii and V. Sokolov, Algebraic structures connected with pairs of compatible associative algebras, Int. Math. Res. Not. 2006 (2006), 43734.
- [19] P. G. Romeo and K. K. Sneha, Generalized module groupoids, Moroccan J. Algebra Geom. Appl. 1(1) (2022), 76-85.
- [20] A. Saïdi, On a pre-Lie algebra defined by insertion of rooted trees, Lett. Math. Phys. 92 (2010), 181-196.
- [21] H. Strohmer, Operads of compatible structures and weighted partitions, J. Pure. Appl. Algebra 212 (2008), 2522-2534.
- [22] E. B. Vinberg, Convex homogeneous cones, Transl. Moscow Math. Soc. 12 (1963), 340-403.