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Linear code associated with symmetric matrix

Ritesh Kumar Pathak[1]*, Kanchan Singh[2] and Sheo Kumar Singh[3]

[123] *Department of Mathematics, Mahatma Gandhi Central University, Motihari-845401, Bihar, India*
[1] *email: mgcu2019math6006@mgcub.ac.in*
[2] *email: mgcu2020math6003@mgcub.ac.in*
[3] *email: sheokumarsingh@mgcub.ac.in*

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Abstract. In this paper, we introduce a new class of linear codes, called affine symmetric codes, associated with symmetric matrices. We determine their basic parameters, namely the length, dimension, and minimum distance, where the minimum distance is obtained for finite fields of odd characteristic. We also compute the weight distributions of these codes for several small parameters using `Magma`. Furthermore, we compare the basic parameters with those of affine Grassmann codes and show that affine symmetric codes exhibit a better transmission rate.

Key Words: Linear code, symmetric matrix, minimum distance, transmission rate, affine Grassmann code, affine space.
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1 Introduction

In Coding Theory, researchers are always interested in developing codes having good parameters. A code C is considered a good code if it has a large minimum distance and high transmission rate. However, achieving both of these properties simultaneously is quite difficult and, in fact, almost impossible (see pp. 173 of [12]). So, we have to trade off between the transmission rate and the minimum distance of a code. In recent decades, there has been increasing interest in studying new classes of codes obtained by evaluating polynomials from certain linear spaces at points in affine or projective spaces, leading to notable classes of codes such as Grassmann codes, determinantal codes, Schubert codes, and others (see e.g., [1, 2, 4, 8, 15, 16, 17, 18]). The study of Grassmann codes began with the work of Ryan in 1987, where these codes were studied over the binary field (cf. [17, 18]). In 1996, Nogin generalized and studied these codes over an arbitrary finite field (cf. [15]). Subsequently, in 2010, Beelen et al. further extended the study of Grassmann codes to the affine Grasssmann codes which were obtained by evaluating some suitable polynomials at points of affine space consisting of all $\ell \times \ell'$ matrices over a finite field with q elements, where ℓ, ℓ' are arbitrary fixed positive integers satisfying $\ell \leq \ell'$ (cf. [1]). Similarly, in 2015, Beelen et al. initiated the study of determinantal codes (cf. [2]), which are obtained by evaluating polynomials from a linear space over $\mathbb{F}_q$, spanned by the 1×1 minors of a generic matrix, on sets of matrices of bounded rank. Further, in recent years some researchers have extended the works and studied symmetric, skew-symmetric and Hermitian determinantal codes (cf. [4, 3, 19]). Studying such classes of codes and their parameters is also important from a geometric perspective, in addition to their significance in coding theory. Recent developments in coding theory have further highlighted its close connections

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*Corresponding Author.
All authors have contributed equally.

with algebraic geometry, finite fields, and computational algebra in the construction and analysis of linear codes. Moreover, algebraic properties of polynomial rings over finite fields and noncommutative Gröbner bases have recently been investigated in the literature (see e.g., [6, 21]). In addition, several researchers have studied linear codes associated with symmetric matrices (see e.g., [4, 7, 13]). Motivated by these developments, we study a new class of linear codes associated with symmetric matrices, closely related to affine Grassmann codes, which we call affine symmetric codes[†]. Specifically, affine symmetric codes are constructed by evaluating polynomial functions — from the linear space spanned by the minors of each order of a generic symmetric matrix — at all symmetric matrices of the corresponding order. Furthermore, affine symmetric codes can be obtained by puncturing affine Grassmann codes at those positions in the affine space of all $\ell \times \ell$ matrices where the matrices are non-symmetric. However, the parameters of affine symmetric codes do not follow directly from those of affine Grassmann codes. In this work, we determine their parameters and show that affine symmetric codes have a better transmission rate than both Grassmann codes and affine Grassmann codes.

The basic theory and terminology used in this paper are given in section 2. In section 3, we introduce the affine symmetric code and determine its length and dimension. In section 4, we compute the minimum distance of the affine symmetric code when the underlying field is of odd characteristic. Moreover, using `Magma` software [5], we determine the weight distributions of the affine symmetric code for $\ell = 2$ with $q = 2,3,4,5,7$, and for $\ell = 3$ with $q = 2,3$. Finally, we compare the parameters of the affine symmetric code with those of the affine Grassmann code.

2 Preliminaries

We denote the finite field with q elements by $\mathbb{F}_q$, where q is a power of some prime number p. From now onwards in this paper, we take ℓ as an arbitrary fixed positive integer. We have also included some basic definitions and terminologies to make the paper self-contained. For all undefined terms and notions of coding theory used in this paper, we refer to [12].

Definition 2.1. A matrix $\mathbf{X} = [X_{ij}]$ over $\mathbb{F}_q$ is said to be symmetric if $\mathbf{X}^T = \mathbf{X}$, where $\mathbf{X}^T = [X_{ji}]$ denotes the transpose of $\mathbf{X}$.

Let $\mathbf{X} = \left[X_{ij}\right]_{\ell\times\ell}$ be a symmetric matrix of order ℓ over $\mathbb{F}_q$, where X_{ij} for $1 \leq i \leq j \leq \ell$, are algebraically independent indeterminates. As $\mathbf{X}$ is symmetric, so there are exactly $\frac{\ell(\ell+1)}{2}$ linearly independent indeterminates X_{ij} in $\mathbf{X}$. Let $\mathbb{F}_q[\mathbf{X}]$ denote the polynomial ring in the indeterminates X_{ij} $(1 \leq i \leq j \leq \ell)$ with coefficients in $\mathbb{F}_q$. For a given $i = 0,1,2,\ldots,\ell$, a minor of $\mathbf{X}$ of order i is a determinant of an $i \times i$ submatrix of $\mathbf{X}$ and these minors are elements of the polynomial ring $\mathbb{F}_q[\mathbf{X}]$. A minor of $\mathbf{X}$ of order 0 is taken to be 1. For the cardinality of a finite set V, we shall use the notation $|V|$. Let $\mathcal{M}_{IJ}(\mathbf{X})$ denote the minor of a submatrix of $\mathbf{X}$ formed by the rows I and columns J, where $I, J \subseteq [\ell] = \{1,2,\ldots,\ell\}$. Clearly, $\mathcal{M}_{IJ}(\mathbf{X})$ is a principal minor when $I = J$ and a non-principal minor when $I \neq J$. Moreover, if $I = J'$ and $J = I'$, then the minors $\mathcal{M}_{I'J'}(\mathbf{X})$ and $\mathcal{M}_{IJ}(\mathbf{X})$ are equal. Next, for any $0 \leq r \leq \ell$, if we take $I = \{i_1, i_2, i_3, \ldots i_r\} \subseteq [\ell]$ and $J = \{j_1, j_2, j_3, \ldots j_r\} \subseteq [\ell]$ such that $i_1 \leq j_1, i_2 \leq j_2, \ldots, i_r \leq j_r$, then the minor $\mathcal{M}_{IJ}(\mathbf{X})$ is said to be a *doset* minor of order r.

Our main focus lies in the linear space generated by the minors of $\mathbf{X}$. Let $\mathfrak{M}_{sym}(\ell)$ denote the collection of all doset minors of $\mathbf{X}$ of orders 0 to ℓ, and let $\mathcal{V}_{sym}(\ell)$ be the linear subspace of $\mathbb{F}_q[\mathbf{X}]$ generated by $\mathfrak{M}_{sym}(\ell)$ over the field $\mathbb{F}_q$.

[†] The codes studied here are closely related to the affine symplectic Grassmann codes recently considered in [9]. In this work, we study these codes from a different viewpoint, realizing these codes as punctured affine Grassmann codes and additionally report results on transmission rates and weight distributions for certain small parameters.

Proposition 2.2. *The dimension of* $\mathcal{V}_{sym}(\ell)$ *is given by*

$$dim_{\mathbb{F}_q}\mathcal{V}_{sym}(\ell) = |\mathfrak{M}_{sym}(\ell)| = \frac{1}{(\ell+2)}\binom{2\ell+2}{\ell+1}.$$

Proof. For this, we use the combinatorial identity known as Vandermonde's convolution $\sum_k \binom{p}{m+k}\binom{q}{n-k} = \binom{p+q}{m+n}$ (see [10], pp. 169). Now, there are $\frac{1}{i+1}\binom{\ell}{i}\binom{\ell+1}{i}$ doset minors of order i in an $\ell \times \ell$ symmetric matrix (cf. [11, 20]). Clearly, the set of all doset minors of order i is disjoint from the set of all doset minors of order j, for $i \neq j$, therefore, $|\mathfrak{M}_{sym}(\ell)|$ is equal to

$$\begin{aligned}\sum_{i=0}^{\ell}\frac{1}{i+1}\binom{\ell}{i}\binom{\ell+1}{i} &= \sum_{i=0}^{\ell}\frac{1}{\ell+1}\binom{\ell+1}{i+1}\binom{\ell+1}{i}\\ &= \frac{1}{\ell+1}\sum_{i=0}^{\ell}\binom{\ell+1}{\ell-i}\binom{\ell+1}{i} = \frac{1}{\ell+1}\binom{2\ell+2}{\ell} = \frac{1}{\ell+2}\binom{2\ell+2}{\ell+1}.\end{aligned}$$

Thus, $|\mathfrak{M}_{sym}(\ell)| = \frac{1}{\ell+2}\binom{2\ell+2}{\ell+1}$. Further, we know that the set of all doset minors of $\mathbf{X}$ of any order r is linearly independent (cf. [11, 20]). Hence $\mathfrak{M}_{sym}(\ell)$ is linearly independent, and therefore $\dim_{\mathbb{F}_q}\mathcal{V}_{sym}(\ell) = \frac{1}{(\ell+2)}\binom{2\ell+2}{\ell+1}$. □

Remark 2.3. In the above result, the dimension of $\mathcal{V}_{sym}(\ell)$, i.e., $\frac{1}{(\ell+2)}\binom{2\ell+2}{\ell+1}$, is in fact the $(\ell+1)^{\text{th}}$ *Catalan number*, where the ℓ^{th} Catalan number $C(\ell)$ is defined as $\frac{1}{\ell+1}\binom{2\ell}{\ell}$ (cf. [10], pp. 203). Also, it is noteworthy that among the total $\binom{2\ell}{\ell}$ minors of $\mathbf{X}$, exactly $\frac{1}{2}\binom{2\ell}{\ell} + 2^{\ell-1}$ minors are distinct.

3 Affine symmetric code

In this section, we introduce the affine symmetric code and establish its length and dimension. We also discuss some results that will be used in the subsequent section.

Let $\mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ denote the *affine space* of all symmetric matrices of order ℓ over $\mathbb{F}_q$, where $\beta = \frac{\ell(\ell+1)}{2}$. Clearly, $|\mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)| = q^{\beta}$. Let $A \in \mathbb{A}^{\beta}_{sym}$ and $f \in \mathbb{F}_q[\mathbf{X}]$, then $f(A)$ is a well-defined element of $\mathbb{F}_q$ which is obtained by evaluating the polynomial f at the matrix A. Let us fix an enumeration of the points of $\mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ as $A_1, A_2, \ldots, A_{q^{\beta}}$ and consider the evaluation map

$$\mathrm{Ev} : \mathbb{F}_q[\mathbf{X}] \to \mathbb{F}_q^{q^{\beta}} \quad \text{defined by} \quad \mathrm{Ev}(f) = \left(f(A_1), f(A_2), \ldots, f(A_{q^{\beta}})\right),$$

for every $f \in \mathbb{F}_q[\mathbf{X}]$. Evidently, the map Ev is a surjective linear map and its kernel is the ideal of $\mathbb{F}_q[\mathbf{X}]$ generated by $\left\{X_{ij}^q - X_{ij} : 1 \leq i \leq j \leq \ell\right\}$. Moreover, the restriction of Ev to $\mathcal{V}_{sym}(\ell)$ is injective.

Definition 3.1. Any $f \in \mathcal{V}_{sym}(\ell)$ can be expressed as

$$f = \sum_{\mathcal{M}_{IJ}(\mathbf{X}) \in \mathfrak{M}_{sym}(\ell)} a_{IJ}\mathcal{M}_{IJ}(\mathbf{X}), \text{ where } a_{IJ} \in \mathbb{F}_q.$$

Then the support of f, denoted by $\mathrm{supp}(f)$, is defined as:

$$\mathrm{supp}(f) = \{\mathcal{M}_{IJ}(\mathbf{X}) \in \mathfrak{M}_{sym}(\ell) : a_{IJ} \neq 0\}.$$

Definition 3.2. A code C is said to be *degenerate* if there exists a coordinate position i such that $c_i = 0$ for all $c \in C$, otherwise code C is *nondegenerate*.

With the above background, we now define the affine symmetric code studied in this paper.

Definition 3.3. The *affine symmetric code* is the image of $\mathcal{V}_{sym}(\ell)$ under the evaluation map Ev.

Let $C^{\mathbb{A}}_{sym}(\ell)$ denote the affine symmetric code. Then the following proposition determines the length and dimension of $C^{\mathbb{A}}_{sym}(\ell)$.

Proposition 3.4. *The code $C^{\mathbb{A}}_{sym}(\ell)$ is a non-degenerate linear code of length $q^{\frac{\ell(\ell+1)}{2}}$ and dimension $\frac{1}{\ell+2}\binom{2\ell+2}{\ell+1}$.*

Proof. In view of Proposition 2.2 and the injectivity of the restriction of Ev on $\mathcal{V}_{sym}(\ell)$, it is enough to check that the code $C^{\mathbb{A}}_{sym}(\ell)$ is nondegenerate. This is clear, as $\text{Ev}(1) = (1, 1, \ldots, 1)$. □

We now discuss the (*permutation*) automorphism group of $C^{\mathbb{A}}_{sym}(\ell)$. Before proceeding further, we recall the following definition from [12].

Definition 3.5. [12] Let $C \subseteq \mathbb{F}_q^n$ be a code and S_n be the symmetric group on n symbols, then the permutation automorphism group of C, denoted as $\text{PAut}(C)$, is defined as the set of all permutations $\sigma \in S_n$ such that $(c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(n)}) \in C$, whenever $(c_1, c_2, \ldots, c_n) \in C$.

Clearly, $\text{PAut}(C)$ is a subgroup of S_n. Let $\text{GL}_\ell(\mathbb{F}_q)$ denote the set of all invertible matrices of order ℓ over $\mathbb{F}_q$. For any given $S \in \mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ and $M \in \text{GL}_\ell(\mathbb{F}_q)$, we define a map $\psi_{S,M} : \mathbb{A}^{\beta}_{sym}(\mathbb{F}_q) \to \mathbb{A}^{\beta}_{sym}(\mathbb{F}_q)$ by $\psi_{S,M}(A) = MAM^T + S$, for every $A \in \mathbb{A}^{\beta}_{sym}(\mathbb{F}_q)$. Then it can be easily seen that $\psi_{S,M}$ is a bijective affine transformation on $\mathbb{A}^{\beta}_{sym}(\mathbb{F}_q)$. Let $\tau_{_{S,M}}$ denote the permutation on coordinate positions induced by $\psi_{S,M}$, i.e., $(\tau_{_{S,M}}(f(A)))_{A \in \mathbb{A}^{\beta}_{sym}(\mathbb{F}_q)} = (f(\psi_{S,M}(A)))_{A \in \mathbb{A}^{\beta}_{sym}(\mathbb{F}_q)}$.

Lemma 3.6. *If $M \in \text{GL}_\ell(\mathbb{F}_q)$ and $S \in \mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$, then $\tau_{_{S,M}} \in \text{PAut}(C^{\mathbb{A}}_{sym}(\ell))$.*

Proof. Evidently, for each $0 \leq k \leq \ell$, any minor of order k of the matrix $(M\mathbf{X}M^T + S)$ is a linear combination of minors of order i of the matrix $\mathbf{X}$, where $0 \leq i \leq k$. Hence, $f(M\mathbf{X}M^T + S) \in \mathcal{V}_{sym}(\ell)$, for each $f = f(\mathbf{X}) \in \mathcal{V}_{sym}(\ell)$. Now, define $\tau_{_{S,M}} : C^{\mathbb{A}}_{sym}(\ell) \to C^{\mathbb{A}}_{sym}(\ell)$, by $\tau_{_{S,M}}(\text{Ev}(f)) = (\tau_{_{S,M}}(f(A)))_{A \in \mathbb{A}^{\beta}_{sym}(\mathbb{F}_q)} = \text{Ev}(f(M\mathbf{X}M^T + S))$. Thus, it is easy to see that $\tau_{_{S,M}}$ is a bijective linear transformation on $C^{\mathbb{A}}_{sym}(\ell)$. Hence, we have $\tau_{_{S,M}} \in \text{PAut}(C^{\mathbb{A}}_{sym}(\ell))$. □

Remark 3.7. For given $M_1, M_2 \in \text{GL}_\ell(\mathbb{F}_q)$ and $S_1, S_2 \in \mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ be two symmetric matrices of order ℓ, then we have

$$\begin{aligned}
(\psi_{S_1,M_1} \circ \psi_{S_2,M_2})(A) &= \psi_{S_1,M_1}(\psi_{S_2,M_2}(A)) \\
&= \psi_{S_1,M_1}(M_2AM_2^T + S_2) \\
&= M_1(M_2AM_2^T + S_2)M_1^T + S_1 \\
&= M_1M_2AM_2^TM_1^T + M_1S_2M_1^T + S_1 \\
&= M_1M_2A(M_1M_2)^T + M_1S_2M_1^T + S_1
\end{aligned}$$

$$\text{Hence, } (\psi_{S_1,M_1} \circ \psi_{S_2,M_2})(A) = \psi_{M_1S_2M_1^T + S_1, M_1M_2}(A), \quad \forall\, A \in \mathbb{A}^{\beta}_{sym}(\mathbb{F}_q).$$

$$\text{Thus,} \quad (\psi_{S_1,M_1} \circ \psi_{S_2,M_2}) = \psi_{M_1S_2M_1^T + S_1, M_1M_2}.$$

Now, consider $A, A_1 \in \mathbb{A}^{\beta}_{sym}(\mathbb{F}_q)$, and $\psi^{-1}_{S_1,M_1}(A_1) = A$. So,

$$\begin{aligned}
&\psi_{S_1,M_1}(A) = A_1\\
\implies\ & M_1 A M_1^T + S_1 = A_1\\
\implies\ & \qquad A = M_1^{-1}(A_1 - S_1)(M_1^T)^{-1}\\
& \qquad\ \ = M_1^{-1} A_1 (M_1^{-1})^T + (-M_1^{-1} S_1 (M_1^{-1})^T)\\
& \qquad\ \ = \psi_{-M_1^{-1}S_1(M_1^{-1})^T, M_1^{-1}}(A_1)\\
\implies\ & \psi^{-1}_{S_1,M_1}(A_1) = \psi_{-M_1^{-1}S_1(M_1^{-1})^T, M_1^{-1}}(A_1),\ \forall\ A_1 \in \mathbb{A}^{\beta}_{sym}(\mathbb{F}_q).\\
& \text{Thus,}\quad \psi^{-1}_{S_1,M_1} = \psi_{-M_1^{-1}S_1(M_1^{-1})^T, M_1^{-1}}.
\end{aligned}$$

Let $\psi_{0,I}$ denote the identity transformation on $\mathbb{A}^{\beta}_{sym}(\mathbb{F}_q)$, where 0 and I respectively denote the zero and identity matrices of order ℓ. Let Φ be the collection of all such $\psi_{S,M}$, i.e., $\Phi = \{\psi_{S,M} : M \in \mathrm{GL}_{\ell}(\mathbb{F}_q)$ and $S \in \mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)\}$. Then, the set Φ forms a group with respect to the composition of maps.

Definition 3.8. (i) For any $I, J \subseteq [\ell]$, the *spread* of minor $\mathcal{M}_{IJ}(\mathbf{X})$ is the cardinality of the set $I \cup J$.
(ii) For any $f \in \mathcal{V}_{sym}(\ell)$, a minor $\mathcal{M}_{IJ}(\mathbf{X})$ in $\mathrm{supp}(f)$ is said to be *maximal* if for each minor $\mathcal{M}_{I'J'}(\mathbf{X})$ in the $\mathrm{supp}(f)$, either $I \not\subseteq I'$ or $J \not\subseteq J'$.

Proposition 3.9. *If $f \in \mathcal{V}_{sym}(\ell)$ with $det(\mathbf{X}) \in \mathrm{supp}(f)$ and q is odd, then there exists a matrix $A \in \mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ such that $f(\mathbf{X}+A)$ has no minor of order $(\ell-1)$ in the* $\mathrm{supp}(f)$*.*

Proof. Let $\mathcal{M}_{KK}(\mathbf{X})$ denotes $det(\mathbf{X})$, i.e., $|K| = \ell$. Without loss of generality, we can assume that the coefficient of $\mathcal{M}_{KK}(\mathbf{X})$ is 1, so $f \in \mathcal{V}_{sym}(\ell)$ can be written as

$$f(\mathbf{X}) = \mathcal{M}_{KK}(\mathbf{X}) + \sum_{1 \le i \le j \le \ell} a_{K\setminus\{i\}K\setminus\{j\}} \mathcal{M}_{K\setminus\{i\}K\setminus\{j\}}(\mathbf{X})\ + h, \text{ where}$$

$\mathrm{supp}(h)$ consists of all minors of the matrix $\mathbf{X}$ up to order $(\ell-2)$ and $a_{K\setminus\{i\}K\setminus\{j\}}$ is the coefficient of the minor $\mathcal{M}_{K\setminus\{i\}K\setminus\{j\}}(\mathbf{X})$. Note that $\mathcal{M}_{K\setminus\{i\}K\setminus\{j\}}(\mathbf{X})$ denotes the minor of order $(\ell-1)$. Now, we take an arbitrary matrix $A = (\alpha_{ij}) \in \mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$, then $f(\mathbf{X}+A)$ can be written as:

$$f(\mathbf{X}+A) = \mathcal{M}_{KK}(\mathbf{X}+A) + \sum_{1 \le i \le j \le \ell} b_{K\setminus\{i\}K\setminus\{j\}} \mathcal{M}_{K\setminus\{i\}K\setminus\{j\}}(\mathbf{X})\ + h', \tag{1}$$

where, $\mathrm{supp}(h')$ contains all minors of the matrix $\mathbf{X}+A$ up to order $(\ell-2)$ and $b_{K\setminus\{i\}K\setminus\{j\}}$ is the coefficient of the minor $\mathcal{M}_{K\setminus\{i\}K\setminus\{j\}}(\mathbf{X})$. From ([1], Lemma 9), there exists $h_1 \in \mathcal{V}_{sym}(\ell)$ such that

$$\begin{aligned}
\mathcal{M}_{KK}(\mathbf{X}+A) = \mathcal{M}_{KK}(\mathbf{X}) &+ \sum_{1 \le i \le \ell} \alpha_{ii} \mathcal{M}_{K\setminus\{i\}K\setminus\{i\}}(\mathbf{X})\\
&+ 2 \sum_{1 \le i < j \le \ell} (-1)^{(i+j)} \alpha_{ij} \mathcal{M}_{K\setminus\{i\}K\setminus\{j\}}(\mathbf{X}) + h_1,
\end{aligned} \tag{2}$$

where $\mathrm{supp}(h_1)$ contains all minors of the matrix $\mathbf{X}+A$ up to order $(\ell-2)$. Since $p > 2$, we now take entries of the matrix $A = (\alpha_{ij}) \in \mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ such that

$$\alpha_{ii} = -b_{K\setminus\{i\}K\setminus\{i\}} \text{ and } \alpha_{ij} = \frac{(-1)^{(i+j+1)} b_{K\setminus\{i\}K\setminus\{j\}}}{2}, \text{ for } 1 \le i \le j \le \ell.$$

Then on putting these values of α_{ii} and α_{ij} in equation (2), we get

$$\begin{aligned}\mathcal{M}_{KK}(\mathbf{X}+A) = \mathcal{M}_{KK}(\mathbf{X}) &- \sum_{1\le i\le \ell} b_{K\setminus\{i\}K\setminus\{i\}}\mathcal{M}_{K\setminus\{i\}K\setminus\{i\}}(\mathbf{X}) \\ &- \sum_{1\le i<j\le \ell} b_{K\setminus\{i\}K\setminus\{j\}}\mathcal{M}_{K\setminus\{i\}K\setminus\{j\}}(\mathbf{X}) + h_1.\end{aligned} \tag{3}$$

Using equation (3) in equation (1), we get $f(\mathbf{X}+A) = \mathcal{M}_{KK}(\mathbf{X})+h_2$, where $\mathrm{supp}(h_2)$ contains all minors of the matrix $\mathbf{X}+A$ up to order $(\ell-2)$. □

Remark 3.10. To determine the minimum distance of our code, we need to count the number of invertible symmetric matrices of order ℓ. It is well-known that there are $q^{\frac{\ell(\ell+1)}{2}} \prod_{\substack{1\le i\le \ell \\ i\ odd}} \left(1-\frac{1}{q^i}\right)$ invertible symmetric matrices of order ℓ over $\mathbb{F}_q$ (cf. [14]).

We end this section by proving the following two propositions, which are useful in determining the minimum distance of $C^{\mathbb{A}}_{sym}(\ell)$.

Proposition 3.11. [9] *Let* $N(\eta) = \left|\left\{(y_1,y_2)\in\mathbb{F}_q^2 : (y_1+a)(y_2+b)=\eta\right\}\right|$, *where* $a,b,\eta\in\mathbb{F}_q$. *Then*

$$N(\eta) = \begin{cases} 2q-1, & \text{if } \eta = 0, \\ q-1, & \text{if } \eta \neq 0. \end{cases}$$

Proof. For the equation $(y_1+a)(y_2+b)=\eta$, if $\eta=0$, then either $y_1=-a$ or $y_2=-b$. The first choice gives q possibilities for y_2, and the second gives q possibilities for y_1. Since $(-a,-b)$ is counted in both cases, so $N(0)=2q-1$. Now, if $\eta\neq 0$, then $y_1\neq -a$, and each $y_1\in\mathbb{F}_q\setminus\{-a\}$ determines a unique $y_2=\frac{\eta}{y_1+a}-b$. Hence $N(\eta)=q-1$. □

Proposition 3.12. [9] *If* $r_1,r_2,\ldots,r_n$, $c_{ij}\in\mathbb{F}_q$, *then the system*

$$y_i^2 = r_i, \qquad y_iy_j = c_{ij}, \quad 1\le i,j\le n,$$

has at most two solutions.

Proof. If each $r_i=0$, then clearly we have a trivial solution for the given system of equations. Now, let $r_i\neq 0$ for some i, then there are at most two values of y_i satisfying $y_i^2=r_i$. Without loss of generality, suppose that $b_i'\neq 0$ be one of the values that satisfies $y_i^2=r_i$, then it has at most one solution for $i\neq j$. Hence, we have at most 2 solutions for the given system of equations. □

4 Minimum distance of $C^{\mathbb{A}}_{sym}(\ell)$

We now determine another basic parameter of $C^{\mathbb{A}}_{sym}(\ell)$, namely its minimum distance, by computing the minimum weight of its non-zero codewords. Due to some difficulties occurred in computing the minimum distance, we assume throughout this section that q is odd, unless stated otherwise. Moreover, with the help of `Magma` software [5], we present the weight distributions of affine symmetric code $C^{\mathbb{A}}_{sym}(\ell)$ for $\ell=2$ and $q=2,3,4,5,7$ and for $\ell=3$ and $q=2,3$.

Let $d(C^{\mathbb{A}}_{sym}(\ell))$ denote the minimum distance of $C^{\mathbb{A}}_{sym}(\ell)$. For $\ell>1$, $d(C^{\mathbb{A}}_{sym}(\ell))$ will be computed by the method of mathematical induction, where the induction is performed on ℓ. To compute $d(C^{\mathbb{A}}_{sym}(\ell))$ we use specialization, as done in [1]. The notation $wt(v)$ is used for the (Hamming) weight of the codeword v.

For $\ell = 1$, a typical element of $\mathcal{V}_{sym}(1)$ is of the form $f = a_0 + a_{11}X_{11}$, where $a_0, a_{11} \in \mathbb{F}_q$. Evidently, the possible weights of codewords in $C^{\mathbb{A}}_{sym}(1)$ are 0, $q-1$, and q. Hence, $d(C^{\mathbb{A}}_{sym}(1)) = q-1$.

For $\ell = 2$, the weight bounds are obtained by considering two different cases depending on the presence or absence of $det(\mathbf{X})$ in the supp(f), where $f \in \mathcal{V}_{sym}(2)$. The least weight bounds among all these will give the minimum distance for $\ell = 2$. So, let f be a typical element of $\mathcal{V}_{sym}(2)$. Then $f = a_0 + a_{11}X_{11} + a_{12}X_{12} + a_{22}X_{22} + a_{\{1,2\}\{1,2\}}(X_{11}X_{22} - X_{12}^2)$, where a_0, a_{11}, a_{12}, a_{22}, $a_{\{1,2\}\{1,2\}} \in \mathbb{F}_q$. The following lemmas 4.1 and 4.2 will provide the weight bounds of Ev(f) when $a_{\{1,2\}\{1,2\}} = 0$ and $a_{\{1,2\}\{1,2\}} \neq 0$, respectively.

Lemma 4.1. *If $f \in \mathcal{V}_{sym}(2)$ such that $det(\mathbf{X}) \notin$ supp(f), then $wt(\mathrm{Ev}(f)) \geq q^3 - q^2$.*

Proof. Let $f = a_0 + a_{11}X_{11} + a_{12}X_{12} + a_{22}X_{22}$, where a_0, a_{11}, a_{12} and $a_{22} \in \mathbb{F}_q$. We now find the $wt(\mathrm{Ev}(f))$, for this we have to find the number of matrices in $\mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ at which f is non-zero. For $a_{11} \neq 0$, we have q^2 values for X_{12} and X_{22}, and there are at least $q-1$ values of X_{11} making $f \neq 0$. Consequently, we have at least $(q-1)q^2$ matrices A_i in the affine space $\mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ such that $f(A_i) \neq 0$. Therefore, $wt(\mathrm{Ev}(f)) \geq q^3 - q^2$. □

Lemma 4.2. *If $f \in \mathcal{V}_{sym}(2)$ and $det(\mathbf{X}) \in$ supp(f), then $wt(\mathrm{Ev}(f)) \geq q^3 - q^2 - q$.*

Proof. We consider the polynomial equation

$$f = a_0 + a_{11}X_{11} + a_{12}X_{12} + a_{22}X_{22} + (X_{11}X_{22} - X_{12}^2) = 0$$

and find its solutions. For this, we rewrite the above polynomial equation as

$$a_{11}X_{11} + a_{22}X_{22} + X_{11}X_{22} = X_{12}^2 - a_{12}X_{12} - a_0. \tag{4}$$

Now, by adding $a_{11}a_{22}$ to both sides of equation (4), we have

$$a_{11}a_{22} + a_{11}X_{11} + a_{22}X_{22} + X_{11}X_{22} = X_{12}^2 - a_{12}X_{12} - a_0 + a_{11}a_{22}. \tag{5}$$

Evidently, the right-hand side of the equation (5) is a quadratic polynomial in X_{12} and the left-hand side of the equation (5) may factor as $(X_{11} + a_{22})(X_{22} + a_{11})$. Let $g(X_{12}) = X_{12}^2 - a_{12}X_{12} - a_0 + a_{11}a_{22}$. Then, the equation (5) can be rewritten as

$$(X_{11} + a_{22})(X_{22} + a_{11}) = g(X_{12}). \tag{6}$$

Let T be the set of zeros of $g(X_{12})$. Since $g(X_{12})$ is a polynomial of degree 2, it follows that $|T| \leq 2$. To count the solutions of equation (5), we consider two different cases: (i) $\alpha \in T$ and (ii) $\theta \in \mathbb{F}_q \setminus T$.

Case (i): If $\alpha \in T$, then $g(\alpha) = 0$. Hence, by Proposition 3.11, there are $2q-1$ choices of X_{11} and X_{22} satisfying $(X_{11}+a_{22})(X_{22}+a_{11}) = g(\alpha)$. Hence, there are $(2q-1)|T|$ solutions corresponding to $g(\alpha) = 0$.

Case (ii): Now assume that $\theta \in \mathbb{F}_q \setminus T$, i.e., $g(\theta) \neq 0$. Then, by the second case of Proposition 3.11, there are $q-1$ choices of X_{11} and X_{22} satisfying $(X_{11} + a_{22})(X_{22} + a_{11}) = g(\theta)$. Hence, there are $(q-|T|)(q-1)$ solutions corresponding to $g(\theta) \neq 0$.

From the above discussion, it follows that the total number of matrices in the affine space $\mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ for which $f = 0$ is $(2q-1)|T| + (q-|T|)(q-1)$. Moreover, the set T can have cardinality at most 2. Thus, the number of such matrices is $q^2 - q$, q^2 or $q^2 + q$ depending on $|T| = 0, 1$ or 2, respectively. Furthermore, since $|\mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)| = q^3$, therefore $wt(\mathrm{Ev}(f))$ can be $q^3 - q^2 + q$, $q^3 - q^2$ or $q^3 - q^2 - q$, respectively. Thus, we obtain the result. □

Let t be an indeterminate over $\mathbb{F}_q$ and $X_{12} = t + \frac{a_{12}}{2}$. Then, $g(X_{12})$ can be expressed in terms of t as:

$$g(t+\frac{a_{12}}{2}) = t^2 - \frac{a_{12}^2}{4} + a_{11}a_{22} - a_0 = t^2 - (\frac{a_{12}^2}{4} - a_{11}a_{22} + a_0)$$

where $g(X_{12})$ & $g(t+\frac{a_{12}}{2})$ have the same number of solutions. It is easy to see that if $(\frac{a_{12}^2}{4} - a_{11}a_{22} + a_0) = 0$, then $g(t+\frac{a_{12}}{2})$ has a unique solution and if $(\frac{a_{12}^2}{4} - a_{11}a_{22} + a_0) \neq 0$, then $g(t+\frac{a_{12}}{2})$ has at most two solutions. Thus, the total number of solutions satisfying equation (6) are given as:

(i) when $(\frac{a_{12}^2}{4} - a_{11}a_{22} + a_0) = 0$, we have $(2q-1)+(q-1)(q-1) = q^2$, and (ii) when $(\frac{a_{12}^2}{4} - a_{11}a_{22} + a_0) \neq 0$, we have $2(2q-1)+(q-2)(q-1) = q^2+q$. So, above discussion may be summarised as:

$$wt(\text{Ev}(f)) \geq \begin{cases} q^3 - q^2 - q, & \text{if } (\frac{a_{12}^2}{4} - a_{11}a_{22} + a_0) \neq 0 \\ q^3 - q^2, & \text{if } (\frac{a_{12}^2}{4} - a_{11}a_{22} + a_0) = 0. \end{cases} \tag{7}$$

Remark 4.3. Suppose $f|_{\{b_{i\ell};1\leq i\leq \ell\}}$ be the polynomial of $\mathcal{V}_{sym}(\ell-1)$ obtained by putting $X_{i\ell} = b_{i\ell}$ in $f(\mathbf{X}) \in \mathcal{V}_{sym}(\ell)$, where $1 \leq i \leq \ell$. We call $f|_{\{b_{i\ell};1\leq i\leq \ell\}}$ as the partial evaluation of $f(\mathbf{X})$ along the ℓ^{th} column and ℓ^{th} row of the matrix $\mathbf{X}$.

For $\ell = 3$, 4 & 5, we now obtain the weight bounds of $\text{Ev}(f)$ when $det(\mathbf{X})$ is present in the $\text{supp}(f)$.

Lemma 4.4. *If $f \in \mathcal{V}_{sym}(3)$ and $det(\mathbf{X}) \in \text{supp}(f)$, then $wt(\text{Ev}(f)) \geq q^6 - q^5 - q^4 + q^3$.*

Proof. If f is a element of $\mathcal{V}_{sym}(3)$, then it looks like

$$\begin{aligned} f &= a_0 + a_{11}X_{11} + a_{12}X_{12} + a_{13}X_{13} + a_{22}X_{22} + a_{23}X_{23} + a_{33}X_{33} \\ &+ a_{\{1,2\}\{1,2\}}(X_{11}X_{22} - X_{12}^2) + a_{\{2,3\}\{2,3\}}(X_{22}X_{33} - X_{23}^2) \\ &+ a_{\{1,3\}\{1,3\}}(X_{11}X_{33} - X_{13}^2) + a_{\{1,2\}\{2,3\}}(X_{12}X_{23} - X_{22}X_{13}) \\ &+ a_{\{1,3\}\{2,3\}}(X_{12}X_{33} - X_{13}X_{23}) + a_{\{1,2\}\{1,3\}}(X_{11}X_{23} - X_{12}X_{13}) \\ &+ a_{\{1,2,3\}\{1,2,3\}}\mathcal{M}_{\{1,2,3\}\{1,2,3\}}(\mathbf{X}). \end{aligned} \tag{8}$$

In the equation (8), $\mathcal{M}_{\{1,2,3\}\{1,2,3\}}(\mathbf{X}) = det(\mathbf{X})$. Without loss of generality, we assume $a_{\{1,2,3\}\{1,2,3\}} = 1$. Now, consider the polynomial $f = \mathcal{M}_{\{1,2,3\}\{1,2,3\}}(\mathbf{X}) + a_0 = 0$. If we take $a_0 = 0$, then the polynomial reduces to $f = \mathcal{M}_{\{1,2,3\}\{1,2,3\}}(\mathbf{X})$. To find the weight of $\text{Ev}(f)$, we count the number of invertible symmetric matrices of order 3 which may obtained by using the formula stated in Remark 3.10. So, we have $wt(\text{Ev}(f)) = q^2(q-1)(q^3-1) = q^6 - q^5 - q^3 + q^2$. If $a_0 \neq 0$, then we have $f = \mathcal{M}_{\{1,2,3\}\{1,2,3\}}(\mathbf{X}) + a_0 = 0$, so the weight of $\text{Ev}(f)$ is equal to the number of symmetric matrices of order 3 such that $\mathcal{M}_{\{1,2,3\}\{1,2,3\}}(\mathbf{X}) \neq -a_0$. It is easy to see that there are $q^2(q^3-1)$ matrices in $\mathbb{A}^{\beta}_{sym}\left(\mathbb{F}_q\right)$ satisfying the condition, $\mathcal{M}_{\{1,2,3\}\{1,2,3\}}(\mathbf{X}) = -a_0$. Thus, $wt(\text{Ev}(f)) = q^6 - q^2(q^3-1) = q^6 - q^5 + q^2$. Now, we can use Proposition 3.9 to simplify the equation (8) as

$$\begin{aligned} f = \mathcal{M}_{\{1,2,3\}\{1,2,3\}}(\mathbf{X}) &+ a_{11}X_{11} + a_{12}X_{12} + a_{13}X_{13} \\ &+ a_{22}X_{22} + a_{23}X_{23} + a_{33}X_{33} + a_0. \end{aligned} \tag{9}$$

We now proceed with the specialization of the third column, setting $X_{13} = b_{13}$, $X_{23} = b_{23}$ and $X_{33} = b_{33}$. We find the weight bound for f by considering two different specializations depending on $b_{33} = 0$ or $b_{33} \neq 0$. Now, the partial evaluation of f is given by $f|_{\{b_{13},b_{23},b_{33}\}} = b_{33}(X_{11}X_{22} - X_{12}^2) - b_{23}(X_{11}b_{23} -$

$b_{13}X_{12}) + b_{13}(X_{12}b_{23} - b_{13}X_{22}) + a_{11}X_{11} + a_{12}X_{12} + a_{13}b_{13} + a_{22}X_{22} + a_{23}b_{23} + a_{33}b_{33} + a_0$. This can be rewritten as:

$$\begin{aligned} f|_{\{b_{13},b_{23},b_{33}\}} &= a_0 + a_{13}b_{13} + a_{23}b_{23} + a_{33}b_{33} + X_{11}(a_{11} - b_{23}^2) \\ &\quad + X_{12}(a_{12} + 2b_{13}b_{23}) + X_{22}(a_{22} - b_{13}^2) + b_{33}(X_{11}X_{22} - X_{12}^2). \end{aligned} \tag{10}$$

Case (1): Consider $b_{33} \neq 0$ in the above specialization, we now have $(q-1)$ choices for b_{33}, and it is left to find choices for b_{13} and b_{23}. Let $a_0 + a_{13}b_{13} + a_{23}b_{23} + a_{33}b_{33} = b_0$, where $b_0 \in \mathbb{F}_q$, then equation (10) reduces to the form

$$\begin{aligned} f|_{\{b_{13},b_{23},b_{33}\}} &= b_0 + X_{11}(a_{11} - b_{23}^2) + X_{12}(a_{12} + 2b_{13}b_{23}) \\ &\quad + X_{22}(a_{22} - b_{13}^2) + b_{33}(X_{11}X_{22} - X_{12}^2). \end{aligned} \tag{11}$$

We now establish the relation between the coefficients of X_{11}, X_{12}, X_{22}, i.e., $d_0 = \frac{(a_{12}+2b_{13}b_{23})^2}{4b_{33}^2} - \frac{(a_{11}-b_{23}^2)(a_{22}-b_{13}^2)}{b_{33}^2} + \frac{b_0}{b_{33}}$. Note that for any $(q-1)$ values of b_{33} and any q values for b_{13}, $\frac{(a_{12}+2b_{13}b_{23})^2}{4b_{33}^2} - \frac{(a_{11}-b_{23}^2)(a_{22}-b_{13}^2)}{b_{33}^2} + \frac{b_0}{b_{33}}$ reduces to a polynomial of degree 2 in b_{23}. If $d_0 = 0$, then there are $2q(q-1)$ possible choices for b_{13}, b_{23} and b_{33} and when $d_0 \neq 0$, we have $q(q-1)(q-2)$ choices for b_{13}, b_{23} and b_{33}. In view of equation (7), it is easy to see that $2q(q-1)$ choices contribute to the weight bound q^3-q^2 and $q(q-1)(q-2)$ choices contribute to the weight bound q^3-q^2-q. Thus, $wt(\mathrm{Ev}(f)) \geq 2q(q-1)(q^3-q^2)+q(q-1)(q-2)(q^3-q^2-q)$.

Case (2): We now consider the specialization when $b_{33} = 0$, i.e., specializing the third column by taking values $X_{13} = b_{13}$, $X_{23} = b_{23}$, $X_{33} = b_{33} = 0$ and on putting these values in the equation (10), we get

$$\begin{aligned} f|_{\{b_{13},b_{23},b_{33}\}} &= a_0 + (a_{11} - b_{23}^2)X_{11} + (a_{12} + 2b_{13}b_{23})X_{12} \\ &\quad + (a_{22} - b_{13}^2)X_{22} + a_{13}b_{13} + a_{23}b_{23}. \end{aligned}$$

We know from Proposition 3.12 that there are at most 2 values of b_{13} and b_{23} simultaneously such that all three coefficients are zero. Hence, there are q^2-2 choices for b_{13} and b_{23} such that $f|_{\{b_{13},b_{23},b_{33}\}}$ has a minor of order 1 in the above expression. Thus, $wt(\mathrm{Ev}(f)) \geq (q^2-2)(q^3-q^2) = q^5-q^4-2q^3+2q^2$.

Thus, on adding the weight bound obtained in case (1) and case (2), we get $wt(\mathrm{Ev}(f)) \geq 2q(q-1)(q^3-q^2) + q(q-1)(q-2)(q^3-q^2-q) + q^5-q^4-2q^3+2q^2 = q^6-q^5-q^4+q^3$.

Thus, we can conclude that $q^6-q^5-q^4+q^3$ is the minimum among all weight bounds of $\mathrm{Ev}(f)$, i.e., $wt(\mathrm{Ev}(f)) \geq q^6-q^5-q^4+q^3$ whenever $f \in \mathcal{V}_{sym}(3)$ with $det(\mathbf{X}) \in \mathrm{supp}(f)$. ☐

Lemma 4.5. *Let* $f \in \mathcal{V}_{sym}(4)$ *and* $det(\mathbf{X}) \in \mathrm{supp}(f)$, *then* $wt(\mathrm{Ev}(f)) \geq q^{10}-q^9-q^8+q^7-3q^6+2q^5+2q^4$.

Proof. Let $f \in \mathcal{V}_{sym}(4)$ and $det(\mathbf{X}) \in \mathrm{supp}(f)$. Without loss of generality, we take the coefficient of $det(\mathbf{X})$ is 1. If $f = det(\mathbf{X})$, then $wt(\mathrm{Ev}(f)) \geq q^{10}-q^9-q^7+q^6$, and if we consider the polynomial $f = a_0 + det(\mathbf{X})$, where $a_0 \in \mathbb{F}_q \setminus \{0\}$ then we have $wt(\mathrm{Ev}(f)) \geq q^{10}-q^9+q^6$. Using Proposition 3.9, f can be reduced in the form where it has no minors of order 3. Now, we specialize the last column of $\mathbf{X}$ with values $X_{14} = b_{14}, X_{24} = b_{24}, X_{34} = b_{34}$ and $X_{44} = b_{44}$. So, when $b_{44} = 0$, it is clear from Proposition 3.12 that there are total $q^3(q^3-2)$ choices for the 3^{rd} & 4^{th} columns of $\mathbf{X}$ where f reduces to the form as given in Lemma 4.2 and when $b_{44} \neq 0$, there are total $q^3(q-1)$ choices for the 4^{th} column of $\mathbf{X}$ where f reduces to the form as given in Lemma 4.4. So, using Lemma 4.2 and 4.4, we get $wt(\mathrm{Ev}(f) \geq q^3(q-1)(q^6-q^5-q^4+q^3)+q^3(q^3-2)(q^3-q^2-q) = q^{10}-q^9-q^8+q^7-3q^6+2q^5+2q^4$. Thus, among all these weight bounds the minimum is $q^{10}-q^9-q^8+q^7-3q^6+2q^5+2q^4$, i.e., $wt(\mathrm{Ev}(f) \geq q^{10}-q^9-q^8+q^7-3q^6+2q^5+2q^4$ whenever $f \in \mathcal{V}_{sym}(4)$ with $det(\mathbf{X}) \in \mathrm{supp}(f)$. ☐

Lemma 4.6. *Let* $f \in \mathcal{V}_{sym}(5)$ *and* $det(\mathbf{X}) \in \mathrm{supp}(f)$, *then* $wt(\mathrm{Ev}(f)) \geq q^{15}-q^{14}-q^{13}+q^{10}-q+q^{11}(q-3)+2q^{10}+2q^9-2q^7+q$.

Proof. Proceeding in the similar manner as done previously, we can easily obtain the above weight bound. ☐

We now deduce some results related to bounds on the weight of codewords which will be useful in finding $d(C^{\mathbb{A}}_{sym}(\ell))$, for $\ell \geq 2$. For the given ℓ, m and s with $0 < m \leq s \leq \ell$, let $\mathcal{A}^{\ell,m,s}$ be the collection of polynomials $f \in \mathcal{V}_{sym}(\ell)$ such that $\mathrm{supp}(f)$ must contain a maximal minor of order m but not more than m and s is the smallest spread size in all these minors of order m present in $\mathrm{supp}(f)$ and $W^{\ell,m,s}_{sym} := \min\{wt(\mathrm{Ev}(f) : f \in \mathcal{A}^{\ell,m,s}\}$.

Lemma 4.7. *If $f \in \mathcal{A}^{\ell,m,s}$ with $s > m$, then there exists a polynomial $f^* \in \mathcal{A}^{\ell,m,s'} \in \mathcal{V}_{sym}(\ell)$, where $s' \leq s-1$ such that $wt(\mathrm{Ev}(f)) = wt(\mathrm{Ev}(f^*))$.*

Proof. We establish the result by considering the spread size $s = m+1$ and $s \geq m+2$. Let $E_{ij}(\gamma)$ denote the elementary matrix obtained by adding γ times row j to row i in the identity matrix of order ℓ. Suppose $\mathcal{M}_{IJ}(\mathbf{X}) \in \mathrm{supp}(f)$ is a maximal minor of order m with spread size $s > m$. Without loss of generality, we take row $I = \{1, 2, \ldots, m\}$ and column $J = \{s-(m-1), \ldots, s\}$. Consider the case when $s = m+1$. Now, for any $\gamma \in \mathbb{F}_q^*$, we take $M = E_{s1}(\gamma) \in \mathrm{GL}_\ell(\mathbb{F}_q)$. As the automorphism $\mathbf{X} \mapsto M\mathbf{X}M^T$ is a weight preserving map, so $wt(\mathrm{Ev}(f(\mathbf{X})) = wt(\mathrm{Ev}(f(M\mathbf{X}M^T)))$. Notice that the matrix $Z = E_{s1}(\gamma)\mathbf{X}E_{1s}(\gamma)$ is a symmetric matrix, so by taking a suitable choice of $E_{ij}(\gamma)$, we can obtain a polynomial $f^* = f(Z) = f(E_{s1}(\gamma)\mathbf{X}E_{1s}(\gamma))$ which has a maximal minor of order m and spread size $\leq (s-1)$. Using the multilinearity property of the determinant, we have $\mathcal{M}_{IJ}(Z) = -\gamma\mathcal{M}_{IJ\cup\{1\}\setminus\{s\}}(\mathbf{X}) + \mathcal{M}_{IJ}(\mathbf{X})$. It is easy to see that the minor $\mathcal{M}_{IJ\cup\{1\}\setminus\{s\}}(\mathbf{X})$ occur in the minors: $R = \mathcal{M}_{I\cup\{s\}\setminus\{1\}J\cup\{1\}\setminus\{s\}}(Z)$ or $T = \mathcal{M}_{I\cup\{s\}\setminus\{1\}J}(Z)$. Evidently, the spread size of the minor T is $(s-1)$, which contradicts our supposition that $\mathcal{M}_{IJ}(\mathbf{X})$ is of minimal spread size s. Thus, the coefficient of T is zero, i.e., $a_{I\cup\{s\}\setminus\{1\}J} = 0$. Here, note that Z is a symmetric matrix and we have considered the doset minors only. Thus, we see that R doesn't the part of the linear combination. So, $\mathcal{M}_{IJ\cup\{1\}\setminus\{s\}}(\mathbf{X}) \in \mathrm{supp}(f^*)$ is of order m and spread size $(s-1)$. Now, we consider the case when $s \geq m+2$ and proceed in the similar fashion by considering $M = E_{(s-1)1}(\gamma) \in \mathrm{GL}_\ell(\mathbb{F}_q)$ and correspondingly $Z = E_{(s-1)1}(\gamma)\mathbf{X}E_{1(s-1)}(\gamma)$. Now, again using the multilinearity property of determinant $\mathcal{M}_{IJ}(Z) = -\gamma\mathcal{M}_{IJ\cup\{1\}\setminus\{(s-1)\}}(\mathbf{X}) + \mathcal{M}_{IJ}(\mathbf{X})$. The minor $\mathcal{M}_{IJ\cup\{1\}\setminus\{(s-1)\}}(\mathbf{X})$ may occur in the minors: $R_1 = \mathcal{M}_{I\cup\{(s-1)\}\setminus\{1\}J\cup\{1\}\setminus\{(s-1)\}}(Z)$ or $T_1 = \mathcal{M}_{I\cup\{(s-1)\}\setminus\{1\}J}(Z)$. Clearly, the spread size of the minor T_1 is $(s-1)$, which contradicts our supposition that $\mathcal{M}_{IJ}(\mathbf{X})$ is of minimal spread size s. Thus, the coefficient of T_1 is zero, i.e., $a_{I\cup\{(s-1)\}\setminus\{1\}J} = 0$. Again, Z is a symmetric matrix and we have considered the doset minors only. Therefore, it is easy to see that R_1 is not be the part of the linear combination. Hence, $\mathcal{M}_{IJ\cup\{1\}\setminus\{(s-1)\}}(\mathbf{X}) \in \mathrm{supp}(f^*)$ is of order m and spread size $(s-1)$. Thus, the result holds for both the cases. ☐

Corollary 4.8. *If $f \in \mathcal{A}^{\ell,m,s}$ with $s > m$, then there is a polynomial $f^* \in \mathcal{A}^{\ell,m,m}$ such that $wt(\mathrm{Ev}(f)) = wt(\mathrm{Ev}(f^*))$.*

Proof. Let $f \in \mathcal{V}_{sym}(\ell)$ such that $\mathrm{supp}(f)$ has a maximal minor of order m and spread size s. Then, from Lemma 4.7, we can easily see that with the help of a suitable choice of elementary matrices $E_{ij}(\gamma) = M$ and the automorphism $\mathbf{X} \mapsto M\mathbf{X}M^T$, we obtain $f_1 \in \mathcal{A}^{\ell,m,s'}$, where $s' \leq s-1$ such that $wt(\mathrm{Ev}(f)) = wt(\mathrm{Ev}(f_1))$. Thus, we obtain $f^* \in \mathcal{A}^{\ell,m,m}$ with the desired property, i.e., $wt(\mathrm{Ev}(f)) = wt(\mathrm{Ev}(f^*))$ by using the above Lemma 4.7 repeatedly. ☐

We now determine a lower bound for $W^{m,m,m}_{sym}$ based on the above discussion.

Lemma 4.9. *Let m be a positive integer such that $2 \leq m \leq \ell$. Then* $W^{m,m,m}_{sym} \geq q^{\frac{m(m+1)}{2}} - q^{\frac{m(m+1)}{2}-1} - q^{\frac{m(m+1)}{2}-2}$.

Proof. In view of Lemmas 4.2, 4.4 and 4.5, it is enough to show that the above inequality holds when $5 \leq m \leq \ell$. For this, we will show that the following inequality 12 holds for $5 \leq m \leq \ell$ using the method of mathematical induction.

$$W_{sym}^{m,m,m} \geq q^{\frac{m(m+1)}{2}} - q^{\frac{m(m+1)}{2}-1} - q^{\frac{m(m+1)}{2}-2} + q^{\frac{m(m-1)}{2}} - q. \tag{12}$$

From Lemma 4.6, it is clear that the inequality (12) holds for $m = 5$. Now, assume that the inequality (12) holds for $m = t \geq 5$, i.e., $W_{sym}^{t,t,t} \geq q^{\frac{t(t+1)}{2}} - q^{\frac{t(t+1)}{2}-1} - q^{\frac{t(t+1)}{2}-2} + q^{\frac{t(t-1)}{2}} - q$. Then we have to show that the result will also hold for $m = t+1$. We calculate the minimum weight of codewords corresponding to the polynomial $f \in \mathcal{A}^{t+1,t+1,t+1}$ by specializing the last column of the underlying matrix $\mathbf{X}$, viz. $X_{i(t+1)} = b_{i(t+1)}$, $\forall\ i = 1, \dots, t+1$. In this specialization, it is easy to see that when $b_{(t+1)(t+1)} \neq 0$, then the supp(f) must contains a maximal minors of order t and when $b_{(t+1)(t+1)} = 0$, then supp(f) contains only minors of order up to $(t-1)$. So, let $f \in \mathcal{A}^{t+1,t+1,t+1}$ such that supp(f) contain the $det(\mathbf{X})$. Note that, using Proposition 3.9 we may reduce f such that there is no minor of order t in the supp(f). We now specialize the $(t+1)^{th}$ column and correspondingly it fixes the row. If $b_{(t+1)(t+1)} \neq 0$, then there are exactly $q^t(q-1)$ choices for the $(t+1)^{th}$ column such that supp(f) has a maximal minor of order t. Also, if $b_{(t+1)(t+1)} = 0$, then there is no minor of order t in the supp(f), so, from Proposition 3.12 it is clear that there are $q^t - 2$ choices for the $(t+1)^{th}$ column and q^t choices for t^{th} column such that supp(f) has a maximal minor of order $t-1$. Therefore,

$$W_{sym}^{t+1,t+1,t+1} \geq q^t(q-1)W_{sym}^{t,t,t} + q^t(q^t-2)W_{sym}^{t-1,t-1,t-1}.$$

Putting the values of $W_{sym}^{t,t,t}$ and $W_{sym}^{t-1,t-1,t-1}$ in the above expression, we get

$$\begin{aligned} W_{sym}^{t+1,t+1,t+1} \geq\ & (q^{t+1} - q^t)(q^{\frac{t(t+1)}{2}} - q^{\frac{t(t+1)}{2}-1} - q^{\frac{t(t+1)}{2}-2} + q^{\frac{t(t-1)}{2}} - q) \\ & + (q^{2t} - 2q^t)(q^{\frac{t(t-1)}{2}} - q^{\frac{t(t-1)}{2}-1} - q^{\frac{t(t-1)}{2}-2} + q^{\frac{(t-1)(t-2)}{2}} - q). \end{aligned}$$

On solving the above expression, we have the following

$$\begin{aligned} W_{sym}^{t+1,t+1,t+1} \geq\ & (q^{\frac{(t+2)(t+1)}{2}} - q^{\frac{(t+2)(t+1)}{2}-1} - q^{\frac{(t+2)(t+1)}{2}-2}) \\ & + (2q^{\frac{t(t+1)}{2}+1} - 3q^{\frac{t(t+1)}{2}} - q^{t+2} + 3q^{t+1} - q^{2t+1}) \\ & + (-2q^{\frac{t(t-1)}{2}+1}) + 2q^{\frac{t(t+1)}{2}-2}(q+1), \end{aligned}$$

which reduces to the following inequality after some algebraic simplification,

$$W_{sym}^{t+1,t+1,t+1} \geq q^{\frac{(t+2)(t+1)}{2}} - q^{\frac{(t+2)(t+1)}{2}-1} - q^{\frac{(t+2)(t+1)}{2}-2} + q^{\frac{(t)(t+1)}{2}} - q.$$

So, the inequality (12) is also true for $m = t+1$. Thus, for all m with $5 \leq m \leq \ell$ the inequality (12) and hence the inequality stated in the Lemma holds. Evidently, for $2 \leq m \leq \ell$, $W_{sym}^{m,m,m} \geq q^{\frac{m(m+1)}{2}} - q^{\frac{m(m+1)}{2}-1} - q^{\frac{m(m+1)}{2}-2}$. ☐

Theorem 4.10. For $\ell > 1$, $d(C_{sym}^{\mathbb{A}}(\ell)) = q^{\frac{\ell(\ell+1)}{2}} - q^{\frac{\ell(\ell+1)}{2}-1} - q^{\frac{\ell(\ell+1)}{2}-2}$.

Proof. To prove the result, we first obtain the weight bound for given $1 < m \leq s \leq \ell$ and $f \in \mathcal{A}^{\ell,m,s}$. So, let $f \in \mathcal{A}^{\ell,m,s}$, i.e., $f \in \mathcal{V}_{sym}(\ell)$ such that supp(f) has a maximal minor of order m and minimal spread size s. With this assumption, it is easy to see that we are left with free $\ell - s$ rows and columns, and for these $\ell - s$ rows and columns we have $\frac{\ell(\ell+1)}{2} - \frac{s(s+1)}{2}$ linearly independent indeterminates. Clearly, there are $q^{\frac{\ell(\ell+1)}{2} - \frac{s(s+1)}{2}}$ choices for these rows and columns, and with these specializations f appears as a linear combination of minors of order s having the same maximal minors. Note that the weight bound $W_{sym}^{s,m,s}$ contributes with each of such specialization therefore,

$$wt(\mathrm{Ev}(f)) \geq q^{\frac{\ell(\ell+1)}{2}-\frac{s(s+1)}{2}}(W_{sym}^{s,m,s}). \tag{13}$$

By Corollary 4.8, we get a polynomial f^* in $\mathcal{A}^{\ell,m,m}$ such that $wt(\mathrm{Ev}(f)) = wt(\mathrm{Ev}(f^*))$. Now, applying the equation (13) for the polynomial f^*, we get $wt(\mathrm{Ev}(f^*)) \geq q^{\frac{\ell(\ell+1)}{2}-\frac{m(m+1)}{2}}(W_{sym}^{m,m,m})$. Thus, $wt(\mathrm{Ev}(f)) \geq q^{\frac{\ell(\ell+1)}{2}-\frac{m(m+1)}{2}}(W_{sym}^{m,m,m})$. Evidently, using Lemma 4.9, we get

$$\begin{aligned} wt(\mathrm{Ev}(f)) &\geq q^{\frac{\ell(\ell+1)}{2}-\frac{m(m+1)}{2}}\left(q^{\frac{m(m+1)}{2}} - q^{\frac{m(m+1)}{2}-1} - q^{\frac{m(m+1)}{2}-2}\right) \\ &= q^{\frac{\ell(\ell+1)}{2}} - q^{\frac{\ell(\ell+1)}{2}-1} - q^{\frac{\ell(\ell+1)}{2}-2}. \end{aligned}$$

Hence, $d(C_{sym}^{\mathbb{A}}(\ell)) \geq q^{\frac{\ell(\ell+1)}{2}} - q^{\frac{\ell(\ell+1)}{2}-1} - q^{\frac{\ell(\ell+1)}{2}-2}$, where equality can be attained for the polynomial $f = 1 + X_{11} + X_{12} + X_{22} + \mathcal{M}_{\{1,2\}\{1,2\}}(\mathbf{X})$. Thus, $d(C_{sym}^{\mathbb{A}}(\ell)) = q^{\frac{\ell(\ell+1)}{2}} - q^{\frac{\ell(\ell+1)}{2}-1} - q^{\frac{\ell(\ell+1)}{2}-2}$. □

Using `Magma` software [5], we now present the weight distributions of affine symmetric code $C_{sym}^{\mathbb{A}}(\ell)$ for $\ell = 2$ with $q = 2,3,4,5,7$ and for $\ell = 3$ with $q = 2,3$, as given in Tables 1 and 2, respectively.

Table 1: Weight distributions of the code $C_{sym}^{\mathbb{A}}(\ell)$ for $\ell = 2$ and $q = 2,3,4,5,7$.

$q = 2$	
Weight	Multiplicity
0	1
2	4
4	22
6	4
8	1

$q = 3$	
Weight	Multiplicity
0	1
15	54
18	132
21	54
27	2

$q = 4$	
Weight	Multiplicity
0	1
44	288
48	444
52	288
64	3

$q = 5$	
Weight	Multiplicity
0	1
95	1000
100	1120
105	1000
125	4

$q = 7$	
Weight	Multiplicity
0	1
287	6174
294	4452
301	6174
343	6

4.1 Comparison of parameters

In this section, we compare the length, dimension, and minimum distance of $C_{sym}^{\mathbb{A}}(\ell)$ with affine Grassmann codes (of [1]). We also compute and compare the transmission rate of affine symmetric code and affine Grassmann code. Before proceeding, let us recall the following definition from [12].

Definition 4.11. [12] If C is an $[n,k]_q$-linear code, then the ratio $\frac{k}{n}$ is said to be the transmission rate of the code C.

Table 2: Weight distribution of the code $C^{\mathbb{A}}_{sym}(\ell)$ for $\ell = 3$ and $q = 2, 3$.

For $q = 2$	
Weight	Multiplicity
0	1
16	28
20	224
24	448
28	3872
32	7238
36	3872
40	448
44	224
48	28
64	1

For $q = 3$	
Weight	Multiplicity
0	1
405	702
414	18954
432	12636
459	12636
468	721710
477	682344
486	1536054
495	1367604
504	341172
513	25272
522	56862
540	6318
567	702
729	2

For convenience of our discussion, we denote the affine Grassmann code (of [1]) by $C^{\mathbb{A}}(\ell, 2\ell)$. It can be seen that the length and dimension of $C^{\mathbb{A}}(\ell, 2\ell)$ are q^{ℓ^2} and $\binom{2\ell}{\ell}$ respectively (cf. [1]), while the length and dimension of $C^{\mathbb{A}}_{sym}(\ell)$ are respectively $q^{\frac{\ell(\ell+1)}{2}}$ and $\frac{1}{\ell+2}\binom{2\ell+2}{\ell+1}$. Thus, the transmission rates of $C^{\mathbb{A}}_{sym}(\ell)$ and $C^{\mathbb{A}}(\ell, 2\ell)$ are respectively $\frac{\frac{1}{\ell+2}\binom{2\ell+2}{\ell+1}}{q^{\frac{\ell(\ell+1)}{2}}}$ and $\frac{\binom{2\ell}{\ell}}{q^{\ell^2}}$. Let us denote $\frac{\frac{1}{\ell+2}\binom{2\ell+2}{\ell+1}}{q^{\frac{\ell(\ell+1)}{2}}}$ by R_1 and $\frac{\binom{2\ell}{\ell}}{q^{\ell^2}}$ by R_2, then we have the following cases:

(i) when $\ell = 1$, we have $R_1 = R_2 = \frac{2}{q}$, and

(ii) when $\ell > 1$, we obtain $R_1 > R_2$ from the following inequality:

$$1 < q^{\frac{\ell(\ell-1)}{2}}\left(\frac{2(2\ell+1)}{(\ell+1)(\ell+2)}\right) \Rightarrow \binom{2\ell}{\ell} < q^{\frac{\ell(\ell-1)}{2}}\left(\frac{2(2\ell+1)}{(\ell+1)(\ell+2)}\right)\binom{2\ell}{\ell}$$

$$\Rightarrow \binom{2\ell}{\ell} < q^{\frac{\ell(\ell-1)}{2}}\left(\frac{2(\ell+1)(2\ell+1)}{(\ell+1)^2(\ell+2)}\right)\binom{2\ell}{\ell} \Rightarrow \frac{\binom{2\ell}{\ell}}{q^{\frac{\ell(\ell-1)}{2}}} < \frac{1}{(\ell+2)}\binom{2\ell+2}{\ell+1}$$

$$\Rightarrow \frac{\binom{2\ell}{\ell}}{q^{\ell^2}} < \frac{\frac{1}{(\ell+2)}\binom{2\ell+2}{\ell+1}}{q^{\frac{\ell(\ell+1)}{2}}}.$$ Hence, $R_1 = R_2$ for $\ell = 1$ and $R_1 > R_2$ for $\ell > 1$.

The above comparisons can be summarized in the following table:

Remark 4.12. For every q and $\ell > 1$, the transmission rate of affine symmetric code is *strictly greater* than that of affine Grassmann code.

5 Conclusion

In this paper, we introduced the affine symmetric code over a finite field $\mathbb{F}_q$ and showed that it is a non-degenerate linear code. The length and dimension of the code are $q^{\frac{\ell(\ell+1)}{2}}$ and $\frac{1}{\ell+2}\binom{2\ell+2}{\ell+1}$,

Parameters	$C^{\mathbb{A}}(\ell,2\ell)$	$C^{\mathbb{A}}_{sym}(\ell)$
Length	q^{ℓ^2}	$q^{\frac{\ell(\ell+1)}{2}}$
Dimension	$\binom{2\ell}{\ell}$	$\frac{1}{\ell+2}\binom{2\ell+2}{\ell+1}$
Minimum distance	$\prod_{i=0}^{\ell-1}(q^{\ell}-q^{i})$	$(q-1)$, for $\ell=1$; $q^{\frac{\ell(\ell+1)}{2}}-q^{\frac{\ell(\ell+1)}{2}-1}-q^{\frac{\ell(\ell+1)}{2}-2}$, for $\ell>1$ and q odd
Transmission rate	$\frac{\binom{2\ell}{\ell}}{q^{\ell^2}}$	$\frac{\frac{1}{\ell+2}\binom{2\ell+2}{\ell+1}}{q^{\frac{\ell(\ell+1)}{2}}}$

Table 3: Comparison of affine symmetric code with affine Grassmann code.

respectively. We determined the minimum distance when q is odd, while the case of even q remains open. We also computed the weight distributions of the affine symmetric code $C^{\mathbb{A}}_{sym}(\ell)$ for $\ell=2$ with $q=2,3,4,5,7$, and for $\ell=3$ with $q=2,3$. Furthermore, we compared the our findings of the affine symmetric code with affine Grassmann codes (of [1]) and found that it has a better transmission rate. An interesting direction for future work is to determine the minimum weight codewords and the full weight spectrum of the affine symmetric code.

ORCID

Ritesh Kumar Pathak - `https://orcid.org/0000-0001-9344-2876`
Kanchan Singh - `https://orcid.org/0000-0002-3568-6079`
Sheo Kumar Singh - `https://orcid.org/0000-0003-2239-2642`

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