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Dedekind domains and property (⋆)

Elmakki Ahmed[1], Chatbouri Ridha[2] and Gam Ines[3]

[1] *Department of Mathematics of Monastir, Tunisia.*
email: elmakkiahmed@gmail.com
[2] *Department of Mathematics of Monastir, Tunisia.*
email: ridha.chatbouri@yahoo.fr
[3] *Department of Mathematics of Monastir, Tunisia.*
email: gam_ines@yahoo.fr

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Abstract. Let R be a commutative ring. We say that R satisfies (⋆) if for any ideal L of R, the ascending chain of ideals $L : I_1 \subseteq L : I_1I_2 \subseteq L : I_1I_2I_3 \subseteq \cdots$ is stationary for any sequence $< I_n >$ of ideals of R. In this paper, we prove a new characterization of Dedekind domains related to Krull's Intersection Theorem and the property (⋆). We show that a valuation domain $(R, \mathfrak{m})$ is a Dedekind domain if and only if R satisfies (⋆); if and only if for each sequence (I_n) of proper ideals of R, $\bigcap_{n=1}^{\infty} I_1I_2\cdots I_n = (0)$; if and only if for any sequence of elements $< r_n >$ of $\mathfrak{m}$, $\bigcap_{n=1}^{\infty} r_1r_2\cdots r_nR = (0)$.

Key Words: Dedekind domain, Ascending Chain Condition.
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1 Introduction

In this paper, all rings considered are assumed to be commutative rings with identity and all modules are non zero unital. Let R be a ring, M an R-module. Recall from [14, see page 172] that M *satisfies* (⋆) if the ascending chain of submodules of the form, $N :_M I_1 \subseteq N :_M I_1I_2 \subseteq N :_M I_1I_2I_3 \subseteq \cdots$ terminates for every submodule N of M and every sequence $< I_n >$ of ideals of R. Also, recall from [11] that M satisfies *accr* if the ascending chain of submodules of the form, $N :_M B \subseteq N :_M B^2 \subseteq N :_M B^3 \subseteq \cdots$ terminates for every submodule N of M and every finitely generated ideal B of R. It is clear that a ring R satisfies (⋆), then it satisfies accr.

Let R be a ring, M an R-module. We say that M is Noetherian if any submodule of M is finitely generated. Equivalently that for any ascending chain of submodules of M of the form $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ is stationary, where N_i are R-submodules of M. A ring R is said to be Noetherian if R regarded as a module over R is Noetherian. Clearly that if a module M over a ring R is Noetherian, then it satisfies (⋆).

Let R be an integral domain. We say that R is a *Dedekind domain* if it is a Noetherian ring for witch the localizations at the nonzero prime ideals are discrete valuation rings. Clearly, it is a one-dimensional integrally closed Noetherian domain. Much research has been carried out concerning Dedekind domains, and among this research is that in [10, 7]. For an extensive bibliography on Dedekind domains we refer the reader to [6].

Let R be a ring and Q be an ideal of R. We say that Q is a *primary ideal* of R if for all $a, b \in R$ with $ab \in Q$, we have either $a \in Q$ or $b \in \sqrt{Q}$. We say that a primary ideal Q of R is said to be *strongly primary* if there exists $n \in \mathbb{N}$ such that $\sqrt{Q}^n \subseteq Q$. Recall from [2, page 51] and [9] that a *primary* *decomposition* (respectively, *strong primary decomposition*) of an ideal I of R is an expression of I as a finite intersection of primary (respectively, strongly primary) ideals of R. Also we say that an ideal I of R is *decomposable* (respectively, *strongly decomposable*) if I has a primary (respectively, strong primary) decomposition. Recall from [9] that a ring R is said to be *Laskerian* (respectively, *strongly Laskerian*) if each proper ideal of R has a primary (respectively, strong primary) decomposition.

The purpose of this paper is to explore more rings and modules satisfying ($\star$). For each of the following rings, we investigate a necessary and sufficient condition in order that the ring satisfies ($\star$) : valuation rings, trivial extensions. Also, we prove new characterisations of Dedekind domains related to property ($\star$). In Section 2, we prove some classic results on modules satisfying ($\star$). We prove that if a ring R satisfies ($\star$) then every finitely generated R-module satisfies ($\star$) (Proposition 2.9). Let R be a ring and M be a finitely generated R-module. We show that M satisfies ($\star$) if and only if the ring $R/ann_R(M)$ satisfies ($\star$) (Theorem 2.11). Let R be a ring and M be an R-module. Then the trivial extension of R by M, denoted by $R(+)M$, is equal to $R\bigoplus M$ as R-modules with coordinate-wise addition and multiplication $(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. It is easy to verify that $R(+)M$ is a commutative ring. We show that $R(+)M$ satisfies ($\star$) if and only if the ring R satisfies ($\star$) and the R-module M satisfies ($\star$) (Theorem 2.17).

In Section 3, we provide a necessary and sufficient condition for a valuation ring to satisfy ($\star$). Let $(R,\mathfrak{m})$ be a valuation ring such that R is not an integral domain. If such a valuation ring R satisfies ($\star$), then for any sequence $< I_n >$ of finitely generated proper ideals of R, there exists $k \in \mathbb{N}$ such that $I_1I_2\cdots I_k = (0)$ (Proposition 3.5). Let $(R,\mathfrak{m})$ be a valuation ring. Then R satisfies ($\star$) if and only if R is strongly Laskerian if and only if $\bigcap_{n=1}^{\infty} I_1I_2\cdots I_n = (0)$ for any sequence $< I_n >$ of finitely generated proper ideals of R if and only if R is Noetherian (Theorem 3.6). It is deduced that a valuation domain $(R,\mathfrak{m})$ which is not equal to its quotient field satisfies ($\star$) if and only if R is a Dedekind domain if and only if for each sequence $< I_n >$ of proper ideals of R, $\bigcap_{n=1}^{\infty} I_1I_2\cdots I_n = (0)$ (Corollary 3.8).

Let R be an integral domain. We prove that R is a Dedekind domain if and only if R is strongly Laskerian and $R_\mathfrak{p}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R if and only if R satisfies ($\star$) and $R_\mathfrak{p}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R (Theorem 3.10). It is deduced that an integral domain R which is not equal to its quotient field is a Dedekind domain if and only if R is strongly Laskerian and $R_\mathfrak{m}$ is a discrete valuation domain for every maximal ideal $\mathfrak{m}$ of R if and only if R satisfies ($\star$) and $R_\mathfrak{m}$ is a discrete valuation domain for every maximal ideal $\mathfrak{m}$ of R (Corollary 3.11). Also it is deduced that an integral domain R satisfying ($\star$) which is not equal to its quotient field is Dedekind domain if and only if $R_\mathfrak{m}$ is a discrete valuation domain for every maximal ideal $\mathfrak{m}$ of R (Corollary 3.12).

2 Some classic results on modules over rings satisfying ($\star$)

Definition 2.1. [14, see page 172] Let R be a commutative ring and M an R-module. We say that M satisfies ($\star$) if for any R-submodule N of M, the ascending sequence of submodules $N : I_1 \subseteq N : I_1I_2 \subseteq N : I_1I_2I_3 \subseteq \cdots$ is stationary for any sequence $< I_n >$ of ideals of R. That is, there exists $k \in \mathbb{N}$ such that $N :_M I_1I_2\cdots I_n = N :_M I_1I_2\cdots I_k$ for all $n \geq k$. We say that a ring R satisfies ($\star$) if R regarded as a module over R satisfies ($\star$).

Example 2.2. Let R be a commutative ring and M an R-module. If M is Noetherian, then M satisfies ($\star$).

We provide Example 2.4 to illustrate that the converse of Example 2.2 can fail to hold. The proof of Example 2.4 need the following lemma.

Lemma 2.3. *Let V be a vector space over a field K. Then V satisfies ($\star$).*

Proof. Let $\{W : I_1 I_2 \cdots I_n\}_{n\in\mathbb{N}}$ be an ascending sequence of subspace of V, where W is any subspace of V and $< I_n >$ be any sequence of ideals of K. Let I be an ideal of K, note that $(W :_V I) = V$ if $I = (0)$ and if $I \neq (0)$, then $I = K$; so $(W :_V I) = W$. We consider two cases.
Case (1) : If $I_k = (0)$ for some $k \in \mathbb{N}$, then $(W :_V I_1 I_2 \cdots I_n) = (W :_V I_1 I_2 \cdots I_k) = (W :_V (0)) = V$ for all $n \geq k$.
Case (2) : If $I_n \neq (0)$ for all $n \in \mathbb{N}$, then $I_n = K$ for each $n \in \mathbb{N}$. Hence $(W :_V I_1 I_2 \cdots I_n) = (W :_V I_1 I_2 \cdots I_k) = (W :_V K) = W$ for all $n, k \in \mathbb{N}$.
This shows that V satisfies (★) in all cases. □

Example 2.4. Let V be an infinite dimensional vector space over a field K. Then V satisfies (★) but V is not Noetherian.

Proof. We know from Lemma 2.3 that V satisfies (★). But V is not Noetherian by [16, Example 2.3] □

Proposition 2.5. *Let R be a ring and M an R-module satisfying (★). We consider the following statements:*

1. *For any sequence of ideals $< I_n >$ of R the ascending sequence of submodules of M, $ann_M(I_1) \subseteq ann_M(I_1 I_2) \subseteq ann_M(I_1 I_2 I_3) \subseteq \cdots$ is stationary.*
2. *For every submodule N of M and any sequence of ideals $< I_n >$ of R, the ascending sequence of ideals of R of the form : $ann_R(I_1 N) \subseteq ann_R(I_1 I_2 N) \subseteq ann_R(I_1 I_2 I_3 N) \subseteq \cdots$ is stationary.*

Then $(1) \Rightarrow (2)$.

Proof. Assume that (1) holds, then there exists a positive integer k such that $ann_M(I_1 I_2 \cdots I_n) = ann_M(I_1 I_2 \cdots I_k)$ for all $n \geq k$. Now let $x \in ann_R(I_1 I_2 \cdots I_n N)$; so $xN \subseteq ann_M(I_1 I_2 \cdots I_n) = ann_M(I_1 I_2 \cdots I_k)$. Then $x \in ann_R(I_1 I_2 \cdots I_k N)$. Therefore $ann_R(I_1 I_2 \cdots I_n N) = ann_R(I_1 I_2 \cdots I_k N)$ for all $n \geq k$. This proves that the ascending sequence of ideals of R of the form : $ann_R(I_1 N) \subseteq ann_R(I_1 I_2 N) \subseteq ann_R(I_1 I_2 I_3 N) \subseteq \cdots$ is stationary. □

Theorem 2.6. Let R be a ring. Let M be a module over R and let N be a submodule of M. Then the following statements are equivalent :

1. M satisfies (★).
2. N and M/N satisfy (★).

Proof. $(1) \Rightarrow (2)$ It's obvious that N satisfies (★). On the other hand, let $\pi : M \longrightarrow M/N$ be the natural homomorphism. Let $< I_n >$ be any sequence of ideals of R and K be an R-submodule of M. Then, $(\pi(K) :_{M/N} \prod_{i=1}^{n} I_i) = ((K + N) :_M \prod_{i=1}^{n} I_i)/N$. Hence, M/N satisfies (★) if M does it.
$(2) \Rightarrow (1)$ Let L be a submodule of M and let $< I_n >$ be a sequence of ideals of R. We verify that the ascending sequence $L :_M I_1 \subseteq L :_M I_1 I_2 \subseteq L :_M I_1 I_2 I_3 \subseteq \cdots$ of M is stationary. Since M/N satisfies (★), there exists $k_1 \in \mathbb{N}$ such that for all $n \geq k_1$, $(L+N)/N :_{M/N} I_1 I_2 \cdots I_n = (L+N)/N :_{M/N} I_1 I_2 \cdots I_{k_1}$. This implies that $L + N :_M I_1 I_2 \cdots I_n = L + N :_M I_1 I_2 \cdots I_{k_1}$ for all $n \geq k_1$. As N satisfies (★), it follows that the ascending sequence of submodules $N \cap L :_M I_{k_1+1} \subseteq N \cap L :_M I_{k_1+1} I_{k_1+2} \subseteq N \cap L :_M I_{k_1+1} I_{k_1+2} I_{k_1+3} \subseteq \cdots$ of N is stationary. Hence, there exists $k_2 \in \mathbb{N}$ such that $N \cap L :_M I_{k_1+1} I_{k_1+2} \cdots I_{k_1+k_2+j} = N \cap L :_M I_{k_1+1} I_{k_1+2} \cdots I_{k_1+k_2}$ for all $j \in \mathbb{N}^*$. We show that $L :_M I_1 I_2 \cdots I_n = L :_M I_1 I_2 \cdots I_{k_1+k_2}$ for all $n \geq k_1 + k_2$. Let $n \geq k_1 + k_2$. Then $n = k_1 + k_2 + r$ for some $r \geq 0$. Let $m \in L :_M I_1 I_2 \cdots I_n$, this implies that $m I_1 I_2 \cdots I_n \subseteq L \subseteq L + N$. Hence, $m I_1 I_2 \cdots I_{k_1} \subseteq L + N$. Now for each $q \in I_1 I_2 \cdots I_{k_1}$, there exist $y_q \in L$ and $z_q \in N$, such that $mq = y_q + z_q$. Let $r \in I_{k_1+1} I_{k_1+2} \cdots I_n$, we have $mqr = y_q r + z_q r$. Therefore $z_q r = mqr - y_q r \in N \cap L$. So $z_q I_{k_1+1} I_{k_1+2} \cdots I_n \subseteq N \cap L$. Hence $z_q \in N \cap L :_N I_{k_1+1} I_{k_1+2} \cdots I_n = N \cap L :_N I_{k_1+1} I_{k_1+2} \cdots I_{k_1+k_2}$. Let $f \in I_{k_1+1} I_{k_1+2} \cdots I_{k_1+k_2}$, we have $z_q f \in N \cap L \subseteq L$. Therefore $mqf = y_q f + z_q f \in L$. Hence $m I_1 I_2 \cdots I_{k_1} I_{k_1+1} I_{k_1+2} \cdots I_{k_1+k_2} \subseteq L$. So $m \in L :_M I_1 I_2 \cdots I_{k_1+k_2}$. Hence $L :_M I_1 I_2 \cdots I_n = L :_M I_1 I_2 \cdots I_{k_1+k_2}$. Therefore, $L :_M I_1 I_2 \cdots I_n = L :_M I_1 I_2 \cdots I_{k_1+k_2}$ for all $n \geq k_1 + k_2$. □

Corollary 2.7. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules. Then M satisfies $(\star)$ if and only if M' and M'' satisfy $(\star)$.*

Proof. Let $f : M' \longrightarrow M$ and $g : M \longrightarrow M''$.
Since $M' \cong Im(f) = ker(g)$ and $M'' \cong M/ker(g)$, by Theorem 2.6 M satisfies $(\star)$ if and only if M' and M'' satisfy $(\star)$. □

Lemma 2.8. *Let R be a ring and let $n \in \mathbb{N}$ with $n \geq 2$. If the R-modules M_1, M_2,...,M_n satisfy $(\star)$, then their direct product $M_1 \times M_2 \times \cdots \times M_n$ satisfies $(\star)$.*

Proof. We prove by induction in $n \geq 2$. If $n = 2$, then the lemma follows from Corollary 2.7 applied to the exact sequence of R-modules $0 \longrightarrow M_1 \longrightarrow M_1 \times M_2 \longrightarrow M_2 \longrightarrow 0$. Suppose that $n > 2$ and that the R-module $M_1 \times M_2 \times \cdots \times M_{n-1}$ satisfies $(\star)$. Applying Corollary 2.7 one more, this time to the exact sequence of R-modules $0 \longrightarrow M_n \longrightarrow M_1 \times M_2 \times \cdots \times M_n \longrightarrow M_1 \times M_2 \times \cdots \times M_{n-1} \longrightarrow 0$, we complete the proof. □

Proposition 2.9. *If a ring R satisfies $(\star)$, then every finitely generated R-module satisfies $(\star)$.*

Proof. Let $M =< x_1, x_2, ..., x_n >$ be any finitely generated R-module. Let $g : R^n \longrightarrow M$, $(a_1, a_2, ..., a_n) \longmapsto \sum_{i=1}^{n} a_i x_i$. Then g is a surjective homomorphism of R-modules. If $n = 1$, then as M is a homomorphic image of the R-module R, we obtain that M satisfies $(\star)$. Hence, we assume that $n \geq 2$. Let $M_i = R$ for each $i \in \{1, 2, ..., n\}$. Note that $R^n = M_1 \times M_2 \times \cdots \times M_n$. As M_i satisfies $(\star)$ for each $i \in \{1, 2, ..., n\}$, we obtain that the R-module R^n satisfies $(\star)$ by Lemma 2.8. Since M is a homomorphic image of R^n, we get that M satisfies $(\star)$. □

Proposition 2.10. *Let $N_1, N_2, ..., N_k$ be submodules of an R-module M. Suppose that, for each $i \in \mathbb{N}$, M/N_i satisfies $(\star)$. Then $M/(\bigcap_{i=1}^{k} N_i)$ satisfies $(\star)$.*

Proof. There is nothing to prove if $k = 1$. Hence, we can assume that $k \geq 2$. Let $f : M \rightarrow M/N_1 \times M/N_2 \times \cdots \times M/N_k$ be given by $f(m) = (m + N_1, m + N_2, ..., m + N_k)$. It is not hard to verify that f is a homomorphism of R-modules and it has kernel $\bigcap_{i=1}^{k} N_i$. Therefore, $M/\bigcap_{i=1}^{k} N_i \cong Im(f)$ as R-modules. As M/N_i satisfies $(\star)$ for each $i \in \{1, 2, ..., n\}$, $M/N_1 \times M/N_2 \times \cdots \times M/N_k$ satisfies $(\star)$ by Lemma 2.8. Hence, its R-submodule $Im(f)$ satisfies $(\star)$. Therefore, $M/(\bigcap_{i=1}^{k} N_i)$ satisfies $(\star)$. □

Our next result gives a necessary and sufficient condition for a finitely generated module over a ring to satisfy $(\star)$.

Theorem 2.11. Let R be a ring, M an R-module and let $I = ann_R(M)$. Suppose that M is a finitely generated R-module. Then the following statements are equivalent:

1. M satisfies $(\star)$ as an R-module
2. R/I satisfies $(\star)$ as a ring.

Proof. (1) $\Rightarrow$ (2) Let $M = Rx_1 + Rx_2 + \cdots + Rx_k$ and put $I_i = ann_R(Rx_i)$. Then $I = \bigcap_{i=1}^{k} I_i$ (because $\{x_1, x_2, ..., x_k\}$ is linearly independent). Next the surjective homomorphism $R \longrightarrow Rx_i$ in which $r \longmapsto rx_i$ has kernel I_i. Therefore R/I_i and Rx_i are isomorphic as R-modules. Now suppose that the R-module M satisfies $(\star)$. Since $R/I_i \cong Rx_i$ an R-submodules of M, the R-module R/I_i satisfies $(\star)$ for each $i \in \{1, 2, ..., k\}$. Accordingly, by Proposition 2.10, the R-module $R/(\bigcap_{i=1}^{k} I_i) = R/I$ satisfies $(\star)$. However the R-submodules of R/I are just the ideals of the ring R/I. Thus the ring R/I satisfies $(\star)$.
(2) $\Rightarrow$ (1) Follows from [13, Proposition 24, p. 65], that we can regard M as an R/I-module and non-empty subset N of M is an R-submodule if and only if it is an R/I-submodule. Now by assumption, the ring R/I satisfies $(\star)$ and M is a finitely generated R-module, and so it is a finitely generated R/I-module. Hence we obtain from Proposition 2.9 that M as an R/I-module satisfies $(\star)$. Note that the R/I-submodules of M are the same as its R-submodules. Therefore the R-module M satisfies $(\star)$. □

In the following example, we provide a module M over $\mathbb{Z}$ such that M does not satisfy $(\star)$, thereby illustrating that Theorem 2.11 may fail to hold if the assumption that the module is finitely generated is omitted.

Example 2.12. Let us denote the set of all prime numbers by $\mathbb{P}$. Let $\mathbb{P} = \{p_1 = 2 < p_2 = 3 < p_3 < p_4 < \cdots\}$. Consider the $\mathbb{Z}$-module M given by $M = \bigoplus_{p\in\mathbb{P}} \frac{\mathbb{Z}}{p\mathbb{Z}}$. Then $\mathbb{Z}/ann_{\mathbb{Z}}(M)$ satisfies $(\star)$. But the $\mathbb{Z}$-module M does not satisfy $(\star)$.

Proof. It is clear that $ann_{\mathbb{Z}}(M) = 0$. Therefore $\mathbb{Z}/ann_{\mathbb{Z}}(M) \cong \mathbb{Z}$ satisfies $(\star)$. Now, it is clear that M is not a finitely generated $\mathbb{Z}$-module and it is not hard to verify that $(0) :_M (p_1)(p_2)\cdots(p_n) = \frac{\mathbb{Z}}{(p_1)} \bigoplus \frac{\mathbb{Z}}{(p_2)} \bigoplus \cdots \bigoplus \frac{\mathbb{Z}}{(p_n)}$, where $(0) = (\bar{0}, \bar{0}, \bar{0}, \ldots)$ is the zero element of M. We show that the ascending sequence of submodules of M, $(0) :_M (p_1) \subset (0) :_M (p_1)(p_2) \subset (0) :_M (p_1)(p_2)(p_3) \subset \cdots$ is not stationary. Suppose that the above sequence of submodules of M is stationary. Then there exists $k \in \mathbb{N}$ such that $(0) :_M (p_1)(p_2)\cdots(p_n) = (0) :_M (p_1)(p_2)\cdots(p_k)$ for all $n \geq k$. Hence, $(0) :_M (p_1)(p_2)\cdots(p_{k+1}) = (0) :_M (p_1)(p_2)\cdots(p_k) = \frac{\mathbb{Z}}{(p_1)} \bigoplus \frac{\mathbb{Z}}{(p_2)} \bigoplus \cdots \bigoplus \frac{\mathbb{Z}}{(p_k)}$. This implies that $\frac{\mathbb{Z}}{(p_{k+1})} \subseteq \frac{\mathbb{Z}}{(p_1)} \bigoplus \frac{\mathbb{Z}}{(p_2)} \bigoplus \cdots \bigoplus \frac{\mathbb{Z}}{(p_k)}$, which is impossible. Therefore, the ascending sequence of submodules $(0) :_M (p_1) \subset (0) :_M (p_1)(p_2) \subset (0) :_M (p_1)(p_2)(p_3) \subset \cdots$ is not stationary. This shows that M does not satisfy $(\star)$. □

Let R be a ring and M be an R-module. Then the trivial extension of R by M, denoted by $R(+)M$, is equal to $R \bigoplus M$ as R-modules with coordinate-wise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. It is easy to verify that $R(+)M$ is a commutative ring.

Proposition 2.13. *[4, Theorem 3.1., Corollary 3.4.]. Let R be a ring, I an ideal of R, M an R-module and N a submodule of M.*

1. *$I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$.*
2. *The ideals of $R(+)M$ containing $0(+)M$ are of the form $J(+)M$ for some ideal J of R.*
3. *The ideals of $R(+)M$ contained in $0(+)M$ are of the form $0(+)K$ for some submodule K of M.*

Proposition 2.14. *Let R be a ring and M an R-module. Suppose that for any sequence $< I_n >$ of ideals of R and for any sequence $< N_n >$ of R-submodules of M, $I_i M \subseteq N_i$ for all $i \in \mathbb{N}$. Then:*

$$\prod_{i=1}^{n}(I_i(+)N_i) = \prod_{i=1}^{n} I_i(+)\sum_{i=1}^{n}\Big(\prod_{j=1, j\neq i}^{n} I_j\Big)N_i.$$

Proof. By induction on $n \geq 2$. For $n = 2$, it follows from [3, Theorem 3.3 (2)]

$$\begin{aligned}\prod_{i=1}^{2}(I_i(+)N_i) &= (I_1(+)N_1)(I_2(+)N_2)\\ &= I_1 I_2(+)(I_1 N_2 + I_2 N_1)\\ &= \prod_{i=1}^{2} I_i(+)\sum_{i=1}^{2}\Big(\prod_{j=1, j\neq i}^{2} I_j\Big)N_i.\end{aligned}$$

Let $n \geq 2$ and suppose that :

$$\prod_{i=1}^{n}(I_i(+)N_i) = \prod_{i=1}^{n} I_i(+)\sum_{i=1}^{n}\Big(\prod_{j=1, j\neq i}^{n} I_j\Big)N_i.$$

We show that :

$$\prod_{i=1}^{n+1}(I_i(+)N_i) = \prod_{i=1}^{n+1} I_i(+)\sum_{i=1}^{n+1}\Big(\prod_{j=1, j\neq i}^{n+1} I_j\Big)N_i.$$

We have

$$\begin{aligned}\prod_{i=1}^{n+1}(I_i(+)N_i) &= \prod_{i=1}^{n}(I_i(+)N_i)(I_{n+1}(+)N_{n+1})\\ &= \Big(\prod_{i=1}^{n} I_i(+)\sum_{i=1}^{n}\Big(\prod_{j=1, j\neq i}^{n} I_j\Big)N_i\Big)(I_{n+1}(+)N_{n+1})\\ &= \prod_{i=1}^{n} I_i I_{n+1}(+)\Big(I_{n+1}\sum_{i=1}^{n}\Big(\prod_{j=1, j\neq i}^{n} I_j\Big)N_i + \prod_{i=1}^{n} I_i N_{n+1}\Big)\\ &= \prod_{i=1}^{n+1} I_i(+)\Big(\sum_{i=1}^{n}\Big(\prod_{j=1, j\neq i}^{n+1} I_j\Big)N_i + \prod_{j=1, j\neq n+1}^{n+1} I_j N_{n+1}\Big)\\ &= \prod_{i=1}^{n+1} I_i(+)\sum_{i=1}^{n+1}\Big(\prod_{j=1, j\neq i}^{n+1} I_j\Big)N_i.\end{aligned}$$

□

Let R, M be as in Proposition 2.14. Let W any R-module. Then the R-module W has the structure of an $R(+)M$-module with the help of the following map given by
$\bullet : (R(+)M) \times W \to W$
$\bullet((r,m),w) = \pi(r,m)w = rw$.

In particular, M (respectively, R) has the structure of an $R(+)M$-module. Using the same idea as in the proof of [5, Lemma 4.2], and with help of Proposition 2.14 we can prove the following lemma

Lemma 2.15. *Let R be a ring and let M be any R-module. For any R-module W, the following statements are equivalent:*

1. *W satisfies $(\star)$ as an R-module.*
2. *W satisfies $(\star)$ as an $R(+)M$-module.*

Let R, M be as in the statement of Lemma 2.15. Note that a ring T satisfies $(\star)$ if and only if it satisfies $(\star)$ as a module over T. As R and M are R-modules, they can be regarded as modules over $R(+)M$ using $\bullet$. Hence, by applying Lemma 2.15 with $W = R$ (respectively, $W = M$), we get the following corollary.

Corollary 2.16. *Let R be a ring and let M be a module over R. Then the following statements are equivalent:*

1. *R (respectively, M) satisfies $(\star)$ as an R-module.*
2. *R (respectively, M) satisfies $(\star)$ as an $R(+)M$-module.*

Our next result gives a necessary and sufficient condition for $R(+)M$ to satisfy strong accr*. It is useful to mention here that the mapping $i : M \to R(+)M$ defined by $i(m) = (0,m)$ is an injective homomorphism of $R(+)M$-modules.

Theorem 2.17. Let R be a ring and M an R-module. Then the following conditions are equivalent:

1. The ring $R(+)M$ satisfies $(\star)$.
2. R and M satisfy $(\star)$ regarded as $R(+)M$-modules.
3. R and M satisfy $(\star)$ regarded as R-modules.

Proof. (1) $\Leftrightarrow$ (2) It follows from Theorem 2.6 and the fact that

$$0 \longrightarrow M \xrightarrow{\ i\ } R(+)M \xrightarrow{\ \pi\ } R \longrightarrow 0$$

is an exact sequence of $R(+)M$-modules.
(2) $\Leftrightarrow$ (3) Follows from Lemma 2.15 and Corollary 2.16. □

3 Dedekind domain and property $(\star)$

Let R be a ring and Q be an ideal of R. We say that Q is a *primary ideal* of R if for all $a, b \in R$ with $ab \in Q$, and $a \notin Q$ then $b \in \sqrt{Q}$. We say that a primary ideal Q of R is *strongly primary* if there exists $n \in \mathbb{N}$ such that $\sqrt{Q}^n \subseteq Q$. Recall from [2, page 51] and [9] that a *primary decomposition* (respectively, *strong primary decomposition*) of an ideal I of R is an expression of I as a finite intersection of primary (respectively, strongly primary) ideals of R. Also we say that an ideal I of R is *decomposable* (respectively, *strongly decomposable*) if I has a primary (respectively, strong primary) decomposition. Recall from [9] that a ring R is said to be *Laskerian* (respectively, *strongly Laskerian*) if each proper ideal of R has a primary (respectively, strong primary) decomposition.

Proposition 3.1. *Let R be a ring. If R is strongly Laskerian, then R satisfies* (⋆).

Proof. Let $N : I_1 \subseteq N : I_1I_2 \subseteq N : I_1I_2I_3 \subseteq \cdots$ be an ascending chain of ideals of R, where N is an ideal of R and $< I_n >$ is any sequence of ideals of R. Since R is a strongly Laskerian ring, then $N = \bigcap_{j=1}^{m} Q_j$, where Q_j is strongly primary ideal of R for each $j \in \{1,...,m\}$. Since for each $j \in \{1,...,m\}$, Q_j is strongly primary, the ascending sequence of ideals of R of the form, $Q_j : I_1 \subseteq Q_j : I_1I_2 \subseteq Q_j : I_1I_2I_3 \subseteq \cdots$ is stationary by [15, Lemma 2.10]. Thus for each $j \in \{1,...,m\}$ there exists $k_j \in \mathbb{N}$ such that $Q_j : I_1I_2\cdots I_n = Q_j : I_1I_2\cdots I_k$ for all $n \geq k_j$. Put $k = max\{k_j \mid j \in \{1,...,m\}\}$, then $Q_j : I_1I_2\cdots I_n = Q_j : I_1I_2\cdots I_k$ for all $n \geq k$. Hence

$$\begin{aligned} \textstyle\bigcap_{j=1}^{m} Q_j : I_1I_2\cdots I_n &= \textstyle\bigcap_{j=1}^{m}(Q_j : I_1I_2\cdots I_n) \\ &= \textstyle\bigcap_{j=1}^{m}(Q_j : I_1I_2\cdots I_k) \\ &= \textstyle\bigcap_{j=1}^{m} Q_j : I_1I_2\cdots I_k. \end{aligned}$$

Therefore $N : I_1I_2\cdots I_n = N : I_1I_2\cdots I_k$ for all $n \geq k$. Then R satisfies (⋆). □

Let R be a ring. Recall from [12, page 7] that R is said to be a *valuation ring* (or *generalized valuation ring*) if it satisfies one of the following three equivalent conditions:

1. For any two elements a and b, either a divides b or b divides a.
2. The ideals of R are linearly ordered by inclusion.
3. R is a quasi-local ring and every finitely generated ideal is principal.

We shall denote a valuation ring R with the maximal ideal $\mathfrak{m}$ by $(R,\mathfrak{m})$.

Let R be a ring, M an R-module. Recall from [11] that M satisfies *accr* if the ascending chain of submodules of the form, $N :_M B \subseteq N :_M B^2 \subseteq N :_M B^3 \subseteq \cdots$ terminates for every submodule N of M and every finitely generated ideal B of R. We say that a ring R satisfies accr if R regarded as a module over R satisfies accr.

Remark 3.2. Let R be a ring. If R satisfies (⋆), then R satisfies accr.

The following example illustrates that the reverse of the above remark can fail to hold.

Example 3.3. Let $\mathbb{P}$, M be as in Example 2.12. It follows from [11, Example 1], that M satisfies accr. But it is already verified in Example 2.12 that M does not satisfy (⋆).

Lemma 3.4. *Let $(R,\mathfrak{m})$ be a valuation ring such that R is not an integral domain. Let $(0) \neq I$ be a finitely generated ideal of R. If R satisfies accr, then $I^n = (0)$ for some n.*

Proof. Assume that R satisfies accr. Let $(0) \neq I$ be a finitely generated ideal of R. As R is a valuation ring by hypothesis, by [1, Theorem 3.1], exactly one of the following statements holds:

(i) $I = I^2$ is a prime ideal in R.

(ii) $I^n \supsetneq I^{n+1}$ for all $n \in \mathbb{N}$, $\bigcap_{n=1}^{\infty} I^n$ is a prime ideal in R, and $\bigcap_{n=1}^{\infty} I^n = \bigcap_{n=1}^{\infty}(i)^n$ for any $i \in I - I^2$. In particular, $\bigcap_{n=1}^{\infty} I^n = I\bigcap_{n=1}^{\infty} I^n$.

(iii) $I^n = (0)$ for some n.

Suppose that $I = I^2$. Since R is a valuation ring and I is a finitely generated ideal of R, I is a principal ideal of R. Let $I = (Rx)$ for some $x \in \mathfrak{m}$. Clearly that $x \neq 0$, since by hypothesis $I \neq (0)$. Therefore $I = (Rx) \neq (Rx)^2 = I^2$.

Since R satisfies accr by assumption and, since I is a finitely generated ideal contained in the Jacobson radical $\mathfrak{m}$ of R, $\bigcap_{n=1}^{\infty} I^n = (0)$ by [11, Theorem 3]. Thus, the statement (ii) can not hod because (0) is not prime ideal of R.

Hence, (iii) must hold and so, $I^n = (0)$ for some n. □

Note that the proof of the next proposition is inspired by [14, Lemma 1.2]

Proposition 3.5. *Let $(R,\mathfrak{m})$ be a valuation ring such that R is not an integral domain. If R satisfies $(\star)$, then for any sequence $< I_n >$ of finitely generated ideals of R, there exists $k \in \mathbb{N}$ such that $I_1 I_2 \cdots I_k = (0)$.*

Proof. Let $< I_n >$ be any sequence of finitely generated ideals of R. Suppose that R satisfies $(\star)$. We show that there exists $k \in \mathbb{N}$ such that $I_1 I_2 \cdots I_k = 0$. Since R satisfies $(\star)$, there exists a positive integer k such that $(0) : I_1 I_2 \cdots I_n = (0) : I_1 I_2 \cdots I_k$ for all $n \geq k$. We assert that $I_1 I_2 \cdots I_{k+1} = 0$. This is clear if $I_{k+1} = (0)$. So we may assume that $I_{k+1} \neq (0)$. Since R satisfies $(\star)$, R satisfies accr. As R is a valuation ring and is not an integral domain, we obtain from Lemma 3.4 that $I_{k+1}^t = (0)$ for some $t \geq 2$, since $I_{k+1} \neq 0$ and I_{k+1} is a finitely generated ideal of R. Now $I_{k+1}^{t-1} \subseteq (0) : I_1 I_2 \cdots I_{k+1} = (0) : I_1 I_2 \cdots I_k$. Therefore $I_1 I_2 \cdots I_k I_{k+1}^{t-1} = (0)$. This implies that $I_{k+1}^{t-2} \subseteq (0) : I_1 I_2 \cdots I_{k+1} = (0) : I_1 I_2 \cdots I_k$. So we obtain $I_1 I_2 \cdots I_k I_{k+1}^{t-2} = 0$. Proceeding like this, it follows that $I_1 I_2 \cdots I_{k+1} = (0)$. Since $(0) : I_1 I_2 \cdots I_k = (0) : I_1 I_2 \cdots I_{k+1} = R$. We obtain in fact $I_1 I_2 \cdots I_k = (0)$. □

Now we are ready to show the main result of this section. We provide a necessary and sufficient condition for a valuation ring to satisfy $(\star)$.

Theorem 3.6. Let $(R,\mathfrak{m})$ be a valuation ring. Then the following statements are equivalent :

1. R is strongly Laskerian.
2. R satisfies $(\star)$.
3. $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n = (0)$ for any sequence $< I_n >$ of finitely generated proper ideals of R.
4. R is Noetherian.

Proof. (1) ⇒ (2) This follows from Proposition 3.1.
(2) ⇒ (3) Let $< I_n >$ be any sequence of finitely generated proper ideals of R. We consider the following two cases.
Case 1: R is not an integral domain.
In this case, we obtain by (1) ⇒ (2) of Proposition 3.5, that there exists $k \in \mathbb{N}$ such that $I_1 I_2 \cdots I_k = (0)$. Thus, $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n = (0)$, since $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n \subseteq I_1 I_2 \cdots I_k = (0)$.
Case 2: R is an integral domain.
Since R is a valuation domain satisfying $(\star)$, for any ideal I of R the ascending chain of ideals of R of the forme $I : r_1 \subseteq I : r_1 r_2 \subseteq I : r_1 r_2 r_3 \subseteq \cdots$ is stationary for any sequence $< r_n >$ of elements of R. Therefore, we obtain from [14, Proposition 2.2], that R satisfies acc on principal ideals. It follows that a valuation ring satisfies acc on principal ideals if and only if it is Noetherian. Therefore, R is a local Noetherian domain. Hence, we obtain from Krull's intersection Theorem that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. Observe that $I_1 I_2 \cdots I_n \subseteq \mathfrak{m}^n$ for each $n \geq 1$ and so, $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n \subseteq \bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. Hence, $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n = (0)$.
(3) ⇒ (4) Assume that (3) holds. Since R is a valuation ring with unique maximal ideal $\mathfrak{m}$ such that $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n = (0)$ for any sequence $< I_n >$ of finitely generated proper ideals of R, $\bigcap_{n=1}^{\infty} r_1 r_2 \cdots r_n R = (0)$ for any sequence $< r_n >$ of elements of $\mathfrak{m}$. Therefore, we obtain from (2) ⇒ (3) of [5, Theorem 3.10], that R is Noetherian.
(4) ⇒ (1) It is obvious, since any Noetherian ring is strongly Laskerian. □

Definition 3.7. Let K be a field. A *discrete valuation* on K is a function $\nu : K \longrightarrow \mathbb{Z}$ satisfying:

1. $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in K^*$;
2. ν is surjective;
3. $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in K^*$ with $x + y \neq 0$.

Let ν be a discrete valuation on an field K. Then the set $R = \{x \in K^* \mid \nu(x) \geq 0\} \cup \{0\}$ is a subring of K which is called the *valuation ring of* ν. Consider the set $\mathfrak{m} = \{x \in K^* \mid \nu(x) > 0\}$. It is easy to verify that $\mathfrak{m}$ is a maximal ideal in R. An integral domain R is called a *discrete valuation domain* if there is a valuation ν on its quotient field such that R is the valuation ring of ν. Let R be a ring and $\mathfrak{p}$ a prime ideal of R. We denote by $R_{\mathfrak{p}}$ the localization of R at $\mathfrak{p}$. An *Dedekind domain* is an integral domain R such that it is a one-dimensional integrally closed Noetherian domain. Equivalently, that R is Noetherian and $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R. Equivalently, that R is Noetherian Prüfer domain.

The following corollary provides a new characterization of a valuation domain to be a Dedekind domain.

Corollary 3.8. *Let $(R, \mathfrak{m})$ be a valuation domain that is not equal to its quotient field. Then the following statements are equivalent*

1. *R is a Dedekind domain.*
2. *R is strongly Laskerian.*
3. *R satisfies (⋆).*
4. *$\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n = (0)$ for any sequence $< I_n >$ of finitely generated proper ideals of R.*
5. *R is Noetherian.*
6. *$\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n = (0)$ for any sequence $< I_n >$ of proper ideals of R.*
7. *$\bigcap_{n=1}^{\infty} r_1 r_2 \cdots r_n R = (0)$ for any sequence $< r_n >$ of elements of $\mathfrak{m}$.*

Proof. (1) ⇒ (5) It is clear.
(5) ⇒ (1) Assume that R is Noetherian. We show that R is Dedekind domain. Since R is Noetherian and R is a valuation domain, R is Noetherian and $R_{\mathfrak{m}} = R$ is a valuation dimain. Therefore R is Noetherian Prüfer domain and, so R is a Dedekind domain.
(2) ⇔ (3) ⇔ (4) ⇔ (5) Follows from Theorem 3.6.
(5) ⇒ (6) Let $< I_n >$ be any sequence of proper ideals of R. Suppose that (1) holds. We show that $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n = (0)$. As R is a local Noetherian domain with unique maximal ideal $\mathfrak{m}$, we obtain from Krull's intersection Theorem that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. Thus $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n = (0)$, since $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n \subseteq \bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. Therefore, $\bigcap_{n=1}^{\infty} I_1 I_2 \cdots I_n = (0)$ for any sequence $< I_n >$ of proper ideals of R and $R_{\mathfrak{m}}$ is a discrete valuation domain.
(6) ⇒ (7) This is obvious.
(7) ⇒ (5) Assume that (5) holds. As R is a valuation ring and $\bigcap_{n=1}^{\infty} r_1 r_2 \cdots r_n R = (0)$ for any sequence $< r_n >$ of elements of $\mathfrak{m}$, we obtain from (2) ⇒ (3) of [5, Theorem 3.10], that R is Noetherian. □

Lemma 3.9. *Let M be a module over a ring R, and let S be a multiplicative set of R. If M satisfies (⋆), then the $S^{-1}R$-module $S^{-1}M$ satisfies (⋆).*

Proof. Let W be any $S^{-1}R$-submodule of $S^{-1}M$. Let $< Q_n >$ be a sequence of ideals of $S^{-1}R$. Note that $W = S^{-1}N$ and $< Q_n = S^{-1}I_n >$, where N is an R-submodule of M and $< I_n >$ be a sequence of ideals of R. We prove that the ascending sequence of submodule of $S^{-1}M$, $W :_{S^{-1}M} Q_1 \subseteq W :_{S^{-1}M} Q_1 Q_2 \subseteq W :_{S^{-1}M} Q_1 Q_2 Q_3 \subseteq \cdots$ is stationary. Since M satisfies (⋆) by hypothesis, we obtain that the ascending sequence of submodules of M, $N :_M I_1 \subseteq N :_M I_1 I_2 \subseteq N :_M I_1 I_2 I_3 \subseteq \cdots$ is stationary. Hence, there exists $k \in \mathbb{N}$ such that $N :_M I_1 I_2 \cdots I_n = N :_M I_1 I_2 \cdots I_k$ for all $n \geq k$. This implies that $S^{-1}(N :_M I_1 I_2 \cdots I_n) = S^{-1}(N :_M I_1 I_2 \cdots I_k)$ for all $n \geq k$. Hence $S^{-1}N :_{S^{-1}M} S^{-1}I_1 S^{-1}I_2 \cdots S^{-1}I_n = S^{-1}N :_{S^{-1}M} S^{-1}I_1 S^{-1}I_2 \cdots S^{-1}I_k$ for all $n \geq k$. Therefore $W :_{S^{-1}M} Q_1 Q_2 \cdots Q_n = W :_{S^{-1}M} Q_1 Q_2 \cdots Q_k$ for all $n \geq k$. This shows that the $S^{-1}R$-module $S^{-1}M$ satisfies (⋆). □

Now we are ready to show the second main result of this section. We provide a new characterisation of Dedekind domain.

Theorem 3.10. Let R be an integral domain. Then the following statements are equivalent :

1. R is a Dedekind domain.
2. R is strongly Laskerian and $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R.
3. R satisfies $(\star)$ and $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R.

Proof. $(1) \Rightarrow (2)$ Let R be a Dedekind domain, and let $\mathfrak{p}$ be a nonzero prime ideal of R. The properties of being Noetherian, being integrally closed, and having dimension at most one are preserved under localization. Then $R_{\mathfrak{p}}$ is a local domain having mentioned properties. Hence $R_{\mathfrak{p}}$ is a discrete valuation domain. Therefore R is Noetherian and $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R. Then R is strongly Laskerian and $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R.
$(2) \Rightarrow (3)$ Assume that (3) holds. As R is strongly Laskerian, we obtain from Proposition 3.1, that R satisfies $(\star)$. Since by assumption $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R, we obtain in fact that R satisfies $(\star)$ and $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R.
$(3) \Rightarrow (1)$ Assume that R satisfies $(\star)$ and $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R. We show that R is Noetherian and $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R, and so, R is a Dedekind domain.

Firstly, we claim that $R_{\mathbf{p}}$ is Noetherian for every prime ideal $\mathbf{p}$ of R. Let $\mathbf{p}$ be any prime ideal of R. We consider two cases.
Case 1: $\mathbf{p} = (0)$. Then $R_{\mathbf{p}}$ is a field, and so is Noetherian.
Case 2: $\mathbf{p} \neq (0)$. As R satisfies $(\star)$, we obtain from Lemma 3.9 that $R_{\mathbf{p}}$ satisfies $(\star)$, and by hypothesis $R_{\mathbf{p}}$ is a discrete valuation domain. Hence $R_{\mathbf{p}}$ is a valuation domain satisfying $(\star)$. Therefore, we obtain from $(2) \Rightarrow (4)$ of Theorem 3.6, that $R_{\mathbf{p}}$ is Noetherian.

Now, we prove that R is Noetherian. From the above $R_{\mathbf{p}}$ is Noetherian for every prime ideal $\mathbf{p}$ of R. So $dimR_{\mathbf{p}} \leq 1$ for every prime ideal $\mathbf{p}$ of R. Therefore we obtain that $dimR \leq 1$. We consider two cases.
Case 1: $dimR = 0$. Then R is a field and so is Noetherian.
Case 2: $dimR = 1$. As R satisfies $(\star)$, for any ideal I of R the ascending chain of ideals of R of the form, $I :_R r_1 \subseteq I :_R r_1 r_2 \subseteq I :_R r_1 r_2 r_3 \subseteq \cdots$ is stationary for any sequence of elements $< r_n >$ of R. Moreover, R is an integral domain and $dimR = 1$. Therefore we obtain from [14, Proposition 2.1], that R is Laskerian. Further R is locally Noetherian. Hence, we obtain from [8, Proposition, p 74] that R is Noetherian. Since $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R by hypothesis, we obtain in fact that R is Noetherian and $R_{\mathfrak{p}}$ is a discrete valuation domain for every nonzero prime ideal $\mathfrak{p}$ of R, and so, R is a Dedekind domain. □

Corollary 3.11. *Let R be an integral domain that is not equal to its quotient field. Then the following statements are equivalent :*

1. *R is a Dedekind domain.*
2. *R is strongly Laskerian and $R_{\mathfrak{m}}$ is a discrete valuation domain for every maximal ideal $\mathfrak{m}$ of R.*
3. *R satisfies $(\star)$ and $R_{\mathfrak{m}}$ is a discrete valuation domain for every maximal ideal $\mathfrak{m}$ of R.*

Corollary 3.12. *Let R be an integral domain satisfying $(\star)$ which is not equal to its quotient field. Then R is a Dedekind domain if and only if $R_{\mathfrak{m}}$ is a discrete valuation domain for every maximal ideal $\mathfrak{m}$ of R.*

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