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Compatible representations and deformation of ternary Leibniz algebras

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Abstract. In this paper, we give constructions of ternary Leibniz algebras from a given one and either Rota-Baxter bimodules, 2-cocycles or O-operators. Then, we introduce compatible ternary Leibniz algebras and we study, on the one hand, construction and characterization of bimodules and representarions on compatible ternary Leibniz algebras. On the other hand the construction and compatibility of bimodules and compatible Representations. Next, we introduce (dual) Nijenhuis pairs from the second-order deformation of ternary Leibniz algebras with a representation.

Key Words: ternary Leibniz algebra, compatibility, deformation.
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1 Introduction

A ternary algebra consists of a linear vector space V together with a trilinear map $\mu : V \times V \times V \to V$. In other words, ternary algebras are vector spaces equipped with a multiplication with three items instead of two, as in classical algebraic structures. They originate from the work of Jacobson in 1949 in the study of associative algebra $(A,\cdot)$ that are closed relative to the ternary operation $[[a,b],c]$, where $[a,b] = ab - ba$. According to some conditions satisfyied by the multiplication μ, we dispose of (partial or total) ternary associative algebras, ternary Leibniz algebras, 3-Lie algebras, ternary Leibniz-Poisson algebras, ternary Hopf algebras, ternary Heap algebras, Comstrans algebras, Akivis algebras, Lie-Yamaguti algebras, Lie triple systems [22, 23, 24], ternary Jordan algebras [25], Jordan-Lie triple systems [33], Jordan triple and so on.

It is known that certains algebraic structures admit left and right version such as right symmetric algebras (left symmetric algebras), right Leibniz algebras (left Leibniz algebras), right BiHom-Lie algebras (right BiHom-Lie algebras) and so on. In this paper, we deal with right 3-Leibniz algebras, also called right ternary Leibniz algebras and simply refer it ternary Leibniz algebra in this paper. It reads

$$[[x,y,z],t,u] = [x,y,[z,t,u]] + [x,[y,t,u],z] + [[x,t,u],y,z],$$

for any $x,y,z \in V$. Whenever the bracket $[-,-,-]$ is skew-symmetric with respect to any pair of variables, $(V,[-,-,-])$ is said to be a ternary Lie algebra or 3-Lie [2, 6, 4, 37].

It is also well-known that mathematical objects are often understood through studying operators defined on them. For instance, in Gallois theory a field is studied by its automorphisms, in analysis functions are studied through their derivations and geometry manifolds are studied through their vector fields. Fifty years ago, several operators have been found from studies in analysis, probability and combinatorics. Among these operators, one can cite, element of centroid [5, 29], averaging operator, Reynolds operator, Leroux's TD operator, Nijenhuis operator and Rota-Baxter operator [21, 38].

The Rota-Baxter operators originated from the work of G. Baxter [11, 36] on Spitzer's identity [35] in fluctuation theory. Rota- Baxter algebras (associative algebra with Rota-Baxter operator) are used in many fiels of mathematics and mathematical Physics. In mathematics, they are used in algebra, number theory, operads and combinatorics [10, 15, 34]. In mathematical physics they appear as the operator form of the classical Yang Baxter equation [3] or as the fondamental algebraic strucutre in the normalisation of quantum field theory of Connes and Kreimer [9, 17]. In non-associative algebra, the Rota-Baxter operators are used in order to produce another one of the same type or not from the previous one.

The Nijenhuis operator on an associative algebra was introduced in [14] in the study of quantum bi-Hamiltonian systems while the notion of Nijenhuis operator on a Lie algebra originated from the concept of Nijenhuis tensor that was introduced by Nijenhuis in the study of pseudo-complex manifolds and was related to the well known concept of Schouten-Nijenhuis bracket, the Frolicher-Nijenhuis bracket [19, 20, 32] and the Nijenhuis-Richardson bracket. The associative analog of the Nijenhuis relation may be regaded as the homogeneous version of Rota-Baxter relation [26].

Leibniz algebras, also known as Loday algebras, are introduced by Loday in [27] as a non-commutative version of Lie algebras. Various aspects of these algebras are intensively investigated these last dacades. For instance, H. Almutari and A. G. Ahmad studied some properties of centroid and quasicentroids [1, 30, 31] for Leibniz algebras and determine the centroids of low-dimensional Leibniz algebras and give their classification. While Elisa M. Canete and Abror Kh. Khudoyberdiyev [13] deal with some properties and classification of complexe Leibniz algebras of dimension 4. In [28] the authors characterize compatible Leibniz algebras in terms of Maurer-Cartan elements of a suitable differential graded Lie algebra, define a cohomology theory of compatible Leibniz algebras and study the abelian extension of compatible Leibniz algebras.

The aim of this paper is to study the compatibility of ternary Leibniz algebras, their modules and the deformations of order 2 of ternary Leibniz algebras with representations. First, we introduce compatible ternary Leibniz algebras, gives some constructions using averaging operators and establish some properties connecting compatible Leibniz algebra to the associated ternary Leibniz algebra. The notion of O-operators for ternary Leibniz algebras are introduced and a new construction of ternary Leibniz algebras from O-operators is done. Next, we study modules and representations of (compatible) ternary Leibniz algebras as well the compatibility concept of modules and representations of compatible ternary Leibniz algebras. Finally,we introduce Nijenhuis operators and (dual)-Nijenhuis pair for ternary Leibniz algebras.

The paper is organized as follows. Section 2 is devoted to a systematic review of the fundamental concepts pertaining to Leibniz algebras and ternary Leibniz algebras. We present their formal definitions, provide representative examples, and discuss the principal construction techniques, with particular emphasis on the associated theories of modules and representations. In Section 3, we develop a construction of compatible ternary Leibniz algebras induced by averaging operators. Furthermore, we investigate the extent to which these operators are preserved under the transition from Leibniz algebras to their associated ternary structures. Section 4 focuses on formal deformations of order two in the presence of representations. We further elaborate the theory of bimodules and representations for compatible ternary Leibniz algebras, and introduce the notion of Nijenhuis pairs in this setting, together with a detailed analysis of their structural properties.

2 Basics on Leibniz and ternary Leibniz algebras

This section is devoted to the recalls of some definitions and results on (compatible) Leibniz algebras and their modules. We also recall bimodules on ternary Leibniz algebras.

Definition 2.1. Let L be a vector over a field $\mathbb{K}$ and $x,y,z\in L$.
a) A Leibniz algebra structure on L is a bilinear map $[-,-]:L\otimes L\to L$ satisfying

$$[[x,y],z]=[x,[y,z]]+[[x,z],y] \tag{1}$$

b) Two Leibniz algebras $(L,[-,-]_1)$ and $(L,[-,-]_2)$ are called compatible if for any $\lambda_1,\lambda_2\in\mathbb{K}$, the following bracket

$$[x,y]=\lambda_1[x,y]_1+\lambda_2[x,y]_2,\quad \forall x,y\in L, \tag{2}$$

defines a Leibniz algebra structure on L.

See [28] for examples.

Remark 2.2. The braket (2) defines a Leibniz algebra structure on L if and only if

$$[[x,y]_1,z]_2+[[x,y]_2,z]_1=[x,[y,z]_1]_2+[x,[y,z]_2]_1+[[x,z]_1,y]_2+[[x,z]_2,y]_1. \tag{3}$$

Definition 2.3. The triple $(L,[-,-]_1,[-,-]_2)$ is said to be a compatible Leibniz algebra if $(L,[-,-]_1)$ and $(L,[-,-]_2)$ are both Leibniz algebras and (3) holds.

Definition 2.4. A linear map $R:L\to L$ on a Leibniz algebra is said to be a **Rota-Baxter operator** of weight $\lambda\in\mathbb{K}$ if

$$[R(x),R(y)])=R\Big([R(x),y]+[x,R(y)]+\lambda[x,y]\Big),$$

for any $x,y\in L$.

In this case, $(L,[-,-],R)$ is said to be a Rota-Baxter Leibniz algebra. And recall that it is well known that if $(L,[-,-],R)$ is a Rota-Baxter Leibniz algebra, then $(L,[-,-]_R)$ is a Leibniz algebra and $R:[-,-]_R\to[-,-]$ is a morphism of Leibniz algebra.
Let us recall bimodules over Leibniz algebras ([28]).

Definition 2.5. Let $(L,[-,-])$ be a Leibniz algebra, M be a linear vector space, $l:L\times M\to M$ and $r:M\times L\to M$ be two bilinear maps. The triple (M,l,r) is said to be a bimodule over L if

$$\begin{aligned} l([x,y],m) &= l(x,l(y,m))+r(l(x,m),y),\\ r(l(x,m),y) &= l(x,r(m,y))+l([x,y],m),\\ r(r(m,x),y) &= r(m,[x,y])+r(r(m,y),x), \end{aligned}$$

for all $x,y\in L, m\in M$.

This definition is borrowed from [7].

Definition 2.6. Let $(L,[-,-])$ be a Leibniz algebra, $R:L\to L$ a Rota-Baxter operator on L, (M,l,r) a bimodule on M and $R_M:M\to M$ a linear map on M. We say that (M,l,r,R_M) is a Rota-Baxter bimodule on L if, for any $x,y\in L$, $m\in M$, we have

$$\begin{aligned} l(R(x),R_M(m)) &= R_M\Big(l(R(x),m)+l(x,R_M(m))\Big),\\ r(R_M(m),R(x)) &= R_M\Big(r(R_M(m),x)+r(m,R(x))\Big). \end{aligned}$$

The proof of the following proposition is straightforward.

Proposition 2.7. *Let (M, l, r, R_M) be a Rota-Baxter bimodule over the Rota-Baxter Leibniz algebra $(L, [-,-], R)$. Let us define the two bilinear maps $l' : L \times M \rightarrow, r' : M \times L \rightarrow M$ as follows*

$$\begin{aligned} l'(x, m) &= l(R(x), m) + l(x, R_M(m)), \\ r'(m, x) &= r(R_M(m), x) + r(m, R(x)), \end{aligned}$$

for any $x \in L, m \in M$. Then (M, l', r') is a bimodule over the Leibniz algebra $L_R = (L, [-,-]_R)$.

Now let us recall bimodules over compatible Leibniz algebras ([28]).
The following theorem comes from a straightforward computation.

Theorem 2.8. Let (M, l_1, r_1, l_2, r_2) be a compatible bimodule over the compatible Leibniz algebra $(L, [-,-]_1, [-,-]_2)$. Then, $L \oplus M$ has a compatible Leibniz structure with the following operations

$$[x + m_1, y + m_2]_i := [x, y]_i + l_i(x, m_2) + r_i(m_1, y), \tag{4}$$

for all $x, y \in L, m_i \in M, i = 1, 2$.

Definition 2.9. A ternary Leibniz algebra L is said to be a ternary Leibniz algebra if the bracket satisfies the following identity:

$$[[x, y, z], t, u] = [x, y, [z, t, u]] + [x, [y, t, u], z] + [[x, t, u], y, z] \tag{5}$$

for any $x, y, z, t, u \in L$.

The following result asserts that one may associate a ternary Leibniz algebra to a Leibniz algebra. It will be useful later.

Theorem 2.10. [16, 8] Let $(L, [-,-])$ be a Leibniz algebra. Then

$$\bar{L} = (L, \{x, y, z\} := [x, [y, z]]),$$

is a ternary Leibniz algebra, for any $x, y, z \in L$.

Proof. Applying twice relation (1), for any $x, y, z \in L$, we have

$$\begin{aligned} \{\{x, y, z\}, t, u\} &= \{[x, [y, z]], t, u\} = [[x, [y, z]], [t, u]] = \\ &= [x, [[y, z], [t, u]]] + [[x, [t, u]]], [y, z]] \\ &= [x, [y, [z, [t, u]]]] + [x, [[y, [t, u]], z]] + [[x, [t, u]]], [y, z]] \\ &= \{x, y, [z, [t, u]]\} + [x, [\{y, t, u\}, z]] + [\{x, t, u\}, [y, z]] \\ &= \{x, y, \{z, t, u\}\} + \{x, \{y, t, u\}, z\} + \{\{x, t, u\}, y, z\}. \end{aligned}$$

This completes the proof. □

3 Representations of compatible ternary Leibniz algebras

We give constructions of ternary Leibniz algebras from a given and either Rota-Baxter bimodules, 2-cocycles or O-operators. Then we introduce compatible representations and give a characterization of ternary Leibniz algebras.

The following definition is inspired from the work on bimodules and Rota-Baxter relation ([7]).

Definition 3.1. Let $(L,[-,-,-])$ be a ternary Leibniz algebra. A bimodule M over L (or, equivalently, a representation of L on M) is given by three trilinear maps

$$l_1 : M\times L\times L \to M,\quad l_2 : L\times M\times L \to M,\quad l_3 : L\times L\times M \to M,$$

satisfying the following system of axioms:

$$\begin{aligned}
l_1(l_1(m,x,y),z,t) &= l_1(m,x,[y,z,t])+l_1(m,[x,z,t],y)+l_1(l_1(m,z,t),x,y),\\
l_1(l_2(x,m,y),z,t) &= l_2(x,m,[y,z,t])+l_2(x,l_1(m,z,t),y)+l_2([x,z,t],m,y),\\
l_1(l_3(x,y,m),z,t) &= l_3(x,y,l_1(m,z,t))+l_3(x,[y,z,t],m)+l_3([x,z,t],y,m),\\
l_2([x,y,z],m,t) &= l_3(x,y,l_2(z,m,t))+l_2(x,l_2(y,m,t),z)+l_1(l_2(x,m,t),y,z),\\
l_3([x,y,z],t,m) &= l_3(x,y,l_3(z,t,m))+l_2(x,l_3(y,t,m),z)+l_1(l_3(x,t,m),y,z),
\end{aligned}$$

for all $x,y,z,t\in L$ and $m\in M$.

Equivalently, a representation of L on M is given by three bilinear maps $\lambda,\mu,\rho : L\times L\to \mathrm{End}(M)$ such that

$$\lambda(x,y)(m)=l_3(x,y,m),\quad \mu(x,y)(m)=l_2(x,m,y),\quad \text{and}\quad \rho(x,y)(m)=l_1(m,x,y).$$

Example 3.2. Any ternary Leibniz algebra is a bimodule over itself.

The lowing proposition connects bimodules over Leibniz algebras to bimodules on ternary Leibniz algebras.

Proposition 3.3. *[8] Let (M,l,r) be a bimodule over a Leibniz algebra $(L,[-,-])$. Let us define*

$$\begin{aligned}
\bar{l}_1(m,x,y) &:= r(m,[x,y]),\\
\bar{l}_2(x,m,y) &:= l(x,r(m,y)),\\
\bar{l}_3(x,y,m) &:= l(x,l(y,m)),
\end{aligned}\tag{6}$$

for all $x,y\in L$ and $m\in M$. Then $(M,\bar{l}_1,\bar{l}_2,\bar{l}_3)$ is a bimodule over the ternary Leibniz algebra $\bar{L}$ (as in Theorem 2.10).

Theorem 3.4. Let (M,l_1,l_2,l_3,R_M) be a Rota-Baxter bimodule over the Rota-Baxter ternary Leibniz algebra $(L,[-,-,-],R)$. Let us define the three linear maps l'_1,l'_2,l'_3 as follows

$$\begin{aligned}
l'_1(m,x,y) &= l_1(R_M(m),R(x),y)+l_1(m,R(x),R(y))+l_1(R_M(m),x,R(y)),\\
l'_2(x,m,y) &= l_2(R(x),R_M(m),y)+l_2(x,R_M(m),R(y))+l_2(R(x),m,R(y)),\\
l'_3(x,y,m) &= l_3(R(x),y,R_M(m))+l_3(x,R(y),R_M(m))+l_3(R(x),R(y),m).
\end{aligned}$$

Then (M,l'_1,l'_2,l'_3) is a bimodule over the ternary Leibniz algebra $L_R=(L,[-,-,-]_R)$.

Proof. We only prove the first axiom. The other being done similarly. Then, for any $x,y,z,t\in L$ and any $m \in M$, we have

$$\begin{aligned}
&l_1'\Big(l_1'(m,x,y),z,t\Big)-l_1'\Big(m,x,[y,z,t]_R\Big)-l_1'\Big(m,[x,z,t]_R,y\Big)-l_1'\Big(l_1'(m,z,t),x,y\Big)\\
&=l_1\Big(R_M l_1'(m,x,y),R(z),t\Big)+l_1\Big(l_1'(m,x,y),R(z),R(t)\Big)+l_1\Big(R_M l_1'(m,x,y),z,R(t)\Big)\\
&\quad -l_1\Big(R_M(m),R(x),[y,z,t]_R\Big)-l_1\Big(m,R(x),R([y,z,t]_R)\Big)-l_1\Big(R_M(m),x,R([y,z,t]_R)\Big)\\
&\quad -l_1\Big(R_M(m),R([x,z,t]_R),y\Big)-l_1\Big(m,R([x,z,t]_R),R(y)\Big)-l_1\Big(R_M(m),[x,z,t]_R,R(y)\Big)\\
&\quad -l_1\Big(R_M(l_1'(m,z,t)),R(x),y\Big)-l_1\Big(l_1'(m,z,t),R(x),R(y)\Big)-l_1\Big(R_M l_1'(m,z,t),x,R(y)\Big)\\
&=l_1\Big(l_1(R_M(m),R(x),R(y)),R(z),t\Big)\\
&\quad +l_1\Big(l_1(R_M(m),R(x),y)+l_1(m,R(x),R(y))+l_1(R_M(m),x,R(y)),R(z),R(t)\Big)\\
&\quad +l_1\Big(l_1(R_M(m),R(x),R(y)),z,t\Big)\\
&\quad -l_1\Big(R_M(m),R(x),[R(y),R(z),t]+[R(y),z,R(t)]+[y,R(z),R(t)]\Big)\\
&\quad -l_1\Big(m,R(x),[R(y),R(z),R(t)]\Big)-l_1\Big(R_M(m),x,[R(y),R(z),R(t)]\Big)\\
&\quad -l_1\Big(R_M(m),[R(x),R(z),R(t)],y\Big)-l_1\Big(m,[R(x),R(z),R(t)],R(y)\Big)\\
&\quad -l_1\Big(R_M(m),[R(x),R(z),t]+[R(x),z,R(t)]+[x,R(z),R(t)],R(y)\Big)\\
&\quad -l_1\Big(l_1(R_M(m),R(z),t),x,y\Big)\\
&\quad -l_1\Big(l_1(R_M(m),R(z),t)+l_1(R_M(m),z,R(t))+l_1(m,R(z),R(t)),R(x),R(y)\Big)\\
&\quad -l_1\Big(l_1(R_M(m),R(z),R(t)),x,R(y)\Big).
\end{aligned}$$

The left hand side vanishes by axiom (6), and the conclusion follows. □

Definition 3.5. Let $(L,[-,-,-])$ be a ternary Leibniz algebra and M a bimodule on L. A trilinear map $\omega : L\times L\times L \to M$ is said to be a 2-cocycle on L with values in M if the following equality holds

$$\begin{aligned}
&l_1\Big(\omega(x,y,z),t,u\Big)+\omega\Big([x,y,z],t,u\Big)\\
&=l_3\Big(x,y,\omega(z,t,u)\Big)+l_2\Big(x,\omega(y,t,u),z\Big)+l_1\Big(\omega(x,t,u),y,z\Big)\\
&\quad+\omega\Big(x,y,[z,t,u]\Big)+\omega\Big(x,[y,t,u],z\Big)+\omega\Big([x,t,u],y,z\Big).
\end{aligned}$$

for any $x,y,z,t,u \in L$.

Observe that if $\omega : L\times L\times L \to L$ is a 2-cocycle which is a ternary bracket, the cocycle condition is nothing but the compatibility condition (8).

Theorem 3.6. Let $(L,[-,-,-])$ be a ternary Leibniz algebra, (M,l_1,l_2,l_3) a bimodule on L and $\omega : L\times L\times L \to M$ a 2-cocycle on L with values in M. Then $L\oplus M$ is a ternary Leibniz algebra with the bracket

$$\begin{aligned}
[x+m_1,\,y+m_2,\,z+m_3]_\omega &= [x,y,z]+l_1(m_1,y,z)+l_2(x,m_2,z)+l_3(x,y,m_3)\\
&\quad+\omega(x,y,z). \qquad (7)
\end{aligned}$$

for all $x,y,z \in L, m_i \in M, i=1,2,3$. Moreover, the brackets $[-,-,-]_0$ and $[-,-,-]_\omega$ are compatible, where $0 : L\times L\times L \to M$ is the null 2-cocycle.

Proof. We have to prove the ternary Leibniz identity for the bracket $[-,-,-]_\omega$. Indeed, for any $x,y,z \in L, m_i \in M$, $i = 1,2,3,4,5$,

$$\begin{aligned}
&[[x+m_1, y+m_2, z+m_3]_\omega, t+m_4, u+m_5]_\omega \\
&= \Big[[x,y,z] + l_1(m_1,y,z) + l_2(x,m_2,z) + l_3(x,y,m_3) + \omega(x,y,z),\ t+m_4,\ u+m_5\Big]_\omega \\
&= [[x,y,z],t,u] + l_1\Big(l_1(m_1,y,z) + l_2(x,m_2,z) + l_3(x,y,m_3) + \omega(x,y,z),\ t,\ u\Big) \\
&\quad + l_2([x,y,z],m_4,u) + l_3([x,y,z],t,m_5) + \omega([x,y,z],t,u) \\
&= [[x,y,z],t,u] \\
&\quad + l_1\Big(l_1(m_1,y,z),t,u\Big) + l_1\Big(l_2(x,m_2,z),t,u\Big) + l_1\Big(l_3(x,y,m_3),t,u\Big) + l_1\Big(\omega(x,y,z),t,u\Big) \\
&\quad + l_2([x,y,z],m_4,u) + l_3([x,y,z],t,m_5) + \omega([x,y,z],t,u).
\end{aligned}$$

In the same manner,

$$\begin{aligned}
&[[x+m_1, t+m_4, u+m_5]_\omega, y+m_2, z+m_3]_\omega \\
&= [[x,t,u],y,z] + l_1\Big(l_1(m_1,t,u),y,z\Big) \\
&+ l_1\Big(l_2(x,m_2,u),y,z\Big) + l_1\Big(l_3(x,t,m_5),y,z\Big) + l_1\Big(\omega(x,t,u),y,z\Big) \\
&+ l_2([x,t,u],m_2,z) + l_3([x,t,u],y,m_3) + \omega([x,t,u],y,z). \\
&[x+m_1,y+m_2,[z+m_3,t+m_4,u+m_5]_\omega]_\omega \\
&= [x,y,[z,t,u]] + l_1(m_1,y,[z,t,u]) + l_2(x,m_2,[z,t,u]) \\
&+ l_3(x,y,l_1(m_3,t,u)) + l_3(x,y,l_2(z,m_4,u)) + l_3(x,y,l_3(z,t,m_5) \\
&+ \omega(x,y,[z,t,u]) + l_3(x,y,\omega(z,t,u)). \\
&[x+m_1,[y+m_2,t+m_4,u+m_5]_\omega,z+m_3]_\omega \\
&= [x,[y,t,u],z] + l_1(m_1,[y,t,u],z) + l_2(x,[m_2,t,u],z) \\
&+ l_2(x,l_2(y,m_4,u),z) + l_2(x,l_3(y,t,m_5),z) + l_3(x,[y,t,u],m_3) \\
&+ l_2(x,\omega(y,t,u),z) + \omega(x,[y,t,u],z).
\end{aligned}$$

For the second part, we have, by using above result,

$$\begin{aligned}
&[[x+m_1,y+m_2,z+m_3],t+m_4,u+m_5]_\omega \\
&+[[x+m_1,y+m_2,z+m_3]_\omega,t+m_4,u+m_5] \\
&-[x+m_1,y+m_2,[z+m_3,t+m_4,u+m_5]]_\omega \\
&-[x+m_1,y+m_2,[z+m_3,t+m_4,u+m_5]_\omega] \\
&-[x+m_1,[y+m_2,t+m_4,u+m_5],z+m_3]_\omega \\
&-[x+m_1,[y+m_2,t+m_4,u+m_5]_\omega,z+m_3] \\
&-[[x+m_1,t+m_4,u+m_5],y+m_2,z+m_3]_\omega \\
&-[[x+m_1,t+m_4,u+m_5]_\omega,y+m_2,z+m_3]
\end{aligned}$$

$$
\begin{aligned}
&= [[x,y,z],t,u] + l_1(l_1(m_1,y,z),t,u) + l_1(l_2(x,m_2,z),t,u) + l_1(l_3(x,y,m_3)t,u)\\
&+l_2([x,y,z],m_4,u) + l_3([x,y,z],t,m_5) + \omega([x,y,z],t,u)\\
&+[[x,y,z],t,u] + l_1(l_1(m_1,y,z),t,u) + l_1(l_2(x,m_2,z),t,u) + l_1(l_3(x,y,m_3),t,u]\\
&+l_1(\omega(x,y,z),t,u) + l_2([x,y,z],m_4,u) + l_3([x,y,z],t,m_5)\\
&-[x,y,[z,t,u]] - l_1(m_1,y,[z,t,u]) - l_2(x,m_2,[z,t,u]) - l_3(x,y,l_1(m_3,t,u))\\
&-l_3(x,y,l_2(z,m_4,u)) - l_3(x,y,l_3(z,t,m_5)) - l_3(x,y,\omega(z,t,u))
\end{aligned}
$$

$$
\begin{aligned}
&-[x,y,[z,t,u]] - l_1(m_1,y,[z,t,u]) - l_2(x,m_2,[z,t,u]) - l_3(x,y,l_1(m_3,t,u))\\
&-l_3(x,y,l_2(z,m_4,u)) - l_3(x,y,l_3(z,t,m_5)) - \omega(x,y,[z,t,u])\\
&-[x,[y,t,u],z] - l_1(m_1,[y,t,u],z) - l_2(x,l_1(m_2,t,u),z)\\
&-l_2(x,l_2(y,m_4,u),z) - l_2(x,l_3(y,t,m_5),z) - l_3(x,[y,t,u],m_3) - \omega(x,[y,t,u],z)\\
&-[x,[y,t,u],z] - l_1(m_1,[y,t,u],z) - l_2(x,l_1(m_2,t,u),z) - l_2(x,l_2(y,m_4,u),z)\\
&-l_2(x,l_3(y,t,m_5),z) - l_3(x,[y,t,u],m_3) - l_2(x,\omega(y,t,u),z)\\
&-[[x,t,u],y,z] - l_1(l_1(m_1,t,u),y,z) - l_1(l_2(x,m_2,u),y,z) - l_1(l_3(x,t,m_5),y,z)\\
&-l_2([x,t,u],m_2,z) - l_3([x,t,u],y,m_3) - \omega([x,t,u],y,z)\\
&-[[x,t,u],y,z] - l_1(l_1(m_1,t,u),y,z) - l_1(l_2(x,m_2,u),y,z) - l_1(l_3(x,t,m_5),y,z)\\
&-l_1(\omega(x,t,u),y,z) - l_2([x,t,u],m_2,z) - l_3([x,t,u],y,m_3).
\end{aligned}
$$

The left hand side vanishes by ternary Leibniz algebra identity, ternary Leibniz algebra modules axioms and 2-cocycle condition. □

Corollary 3.7. *([8]) Let (M,l_1,l_2,l_3) be a bimodule over a ternary Leibniz algebra L. Then $L\oplus M$ is a ternary Leibniz algebra with respect to the multiplication*

$$
\begin{aligned}
[x_1+m_1,x_2+m_2,x_3+m_3]_0 \quad = \quad & [x_1,x_2,x_3] + l_1(m_1,x_2,x_3)\\
&+l_2(x_1,m_2,x_3) + l_3(x_1,x_2,m_3),
\end{aligned}
$$

for any $x_i+m_i\in L\oplus M, i=1,2,3$.

Now we introduce compatible ternary Leibniz algebras.

Definition 3.8. A compatible ternary Leibniz algebra is a triple $(L,[-,-,-]_1,[-,-,-]_2)$ where $(L,[-,-,-]_1)$ and $(L,[-,-,-]_2)$ are two ternary Leibniz algebras such that:

$$
\begin{aligned}
&[[x,y,z]_1,t,u]_2 + [[x,y,z]_2,t,u]_1\\
&= [x,y,[z,t,u]_1]_2 + [x,y,[z,t,u]_2]_1 + [x,[y,t,u]_1,z]_2\\
&+[x,[y,t,u]_2,z]_1 + [[x,t,u]_1,y,z]_2 + [[x,t,u]_2,y,z]_1,
\end{aligned}
$$

for any $x,y,z\in L$.

Example 3.9. Any compatible ternary Leibniz algebra is a compatible bimodule over itself.

The proof of the next proposition is straightforward.

Proposition 3.10. *The triple $(L,[-,-,-]_1,[-,-,-]_2)$ is a compatible ternary Leibniz algebra if and only if the new bilinear map*

$$[-,-,-] = k_1[-,-,-]_1 + k_2[-,-,-]_2$$

defines a ternary Leibniz algebra structure on L.

Proposition 3.11. *Let $(L,[-,-,-])$ be a ternary Leibniz algebra and (M,λ,μ,ρ) be a representation of L on M. Then, the direct sum $L\oplus M$ has a Leibniz algebra structure with the product:*

$$[x+m_1,y+m_2,z+m_3]:=[x,y,z]+\rho(y,z)(m_1)+\mu(x,z)(m_2)+\lambda(x,y)(m_3),$$

for any $x,y,z\in L, m_i, i=1,2,3$.

Proof. The proof is the same as in that of Corollary 3.7. □

Definition 3.12. Let $(L,[-,-,-])$ be a ternary Leibniz algebra, (V,λ,μ,ρ) a representation of L on V and $T:V\to L$ a linear map. We say that T is an O-operator on L with respect to the representation (λ,μ,ρ) if

$$[T(u),T(v),T(w)]=T\Big(\lambda(T(u),T(v))w+\mu(T(u),T(w))v+\rho(T(v),T(w))u\Big), \tag{8}$$

for any $u,v,w\in V$.

Definition 3.13. Let T,T' be two O-operators on a ternary Leibniz algebra $(L,[-,-,-])$ with respect to a representation (V,λ,μ,ρ) A morphism from T' to T consists of a ternary Leibniz algebra morphism $f:L\to L$ and a linear map $\varphi:V\to V$ such that

$$\begin{array}{rcl} T\circ\varphi &=& f\circ T',\\ (\varphi\circ\lambda(x,y))(u) &=& \lambda(f(x),f(y))\varphi(u),\\ (\varphi\circ\mu(x,y))(u) &=& \mu(f(x),f(y))\varphi(u),\\ (\varphi\circ\rho(x,y))(u) &=& \rho(f(x),f(y))\varphi(u), \end{array} \tag{9}$$

for any $x,y\in L, u\in V$.

Theorem 3.14. Let $T:V\to L$ be an O-operator on L with respect to the representation (V,λ,ρ,μ). Let us define the bracket $[-,-,-]_T:V\times V\times V\to V$ as follows

$$[u_1,u_2,u_3]_T=\lambda(T(u_1),T(u_2))(u_3)+\mu(T(u_1),T(u_3))(u_2)+\rho(T(u_2),T(u_3))(u_1) \tag{10}$$

for any $u_1,u_2,u_3\in V$. Then $(V,[-,-,-]_T)$ is a ternary Leibniz algebra.
Let T' be another O-operator on a ternary Leibniz algebra $(L,[-,-,-])$ with respect to a representation (V,λ,μ,ρ). and (f,ϕ) a homomorphism from T' to T. Then, ϕ is a homomorphism from ternary Leibniz algebra $(V,[-,-,-]_{T'})$ to $(V,[-,-,-]_T)$.

Proof. For any $u_i\in V, i=1,2,3,4,5$, we have

$$\begin{array}{l} [[u_1,u_2,u_3]_T,u_4,u_5]_T=\\ =\lambda(T([u_1,u_2,u_3]_T),u_4)(u_5)+\mu(T([u_1,u_2,u_3]_T),T(u_5))(u_4)+\rho(T(u_4),T(u_5))([u_1,u_2,u_3]_T)\\ =\lambda(T(u_1),T(u_2),T(u_3),u_4)(u_5)+\mu(T(u_1),T(u_2),T(u_3),T(u_5))(u_4)\\ +\rho(T(u_4),T(u_5))\Big(\lambda(T(u_1),T(u_2))(u_3)+\mu(T(u_1),T(u_3))(u_2)+\rho(T(u_2),T(u_3))(u_1)\Big)\\ =\lambda(T(u_1),T(u_2),T(u_3),u_4)(u_5)+\mu(T(u_1),T(u_2),T(u_3),T(u_5))(u_4)\\ +\rho(T(u_4),T(u_5))(\lambda(T(u_1),T(u_2))(u_3))+\rho(T(u_4),T(u_5))(\mu(T(u_1),T(u_3))(u_2))\\ +\rho(T(u_4),T(u_5))(\rho(T(u_2),T(u_3))(u_1). \end{array}$$

In the same way, we get successively,

$$\begin{array}{l} [[u_1,u_4,u_5]_T,u_2,u_3]_T=\\ =\lambda(T(u_1),T(u_4),T(u_5),u_2)(u_3)+\mu(T(u_1),T(u_4),T(u_5),T(u_3))(u_2)\\ +\rho(T(u_2),T(u_3))(\lambda(T(u_1),T(u_4))(u_5))+\rho(T(u_2),T(u_3))(\mu(T(u_1),T(u_5))(u_4)) \end{array}$$

$$+\rho(T(u_2),T(u_3))(\rho(T(u_4),T(u_5))(u_1).$$
$$[u_1,[u_2,u_4,u_5]_T,u_3]_T =$$
$$\lambda(T(u_1),[T(u_2),T(u_4),T(u_5)])(u_3)+\mu(T(u_1),T(u_3))\lambda(T(u_2),T(u_4))(u_5)$$
$$+\mu(T(u_1),T(u_3))\mu(T(u_2),T(u_5))(u_4)+\mu(T(u_1),T(u_3))\rho(T(u_4),T(u_5))(u_2)$$
$$+\rho([T(u_2),T(u_4),T(u_5)],T(u_3))(u_1).$$

$$[u_1,u_2,[u_3,u_4,u_5]_T]_T =$$
$$\lambda(T(u_1),T(u_2))\lambda(T(u_3),T(u_4))(u_5)+\lambda(T(u_1),T(u_2))\mu(T(u_3),T(u_5))(u_4)$$
$$+\lambda(T(u_1),T(u_2))\rho(T(u_4),T(u_5))(u_3)+\mu(T(u_1),[T(u_3),T(u_4),T(u_5)])(u_2)$$
$$+\rho(T(u_2),[T(u_3),T(u_4),T(u_5)])(u_1).$$

Using axioms in Definition 3.1, we get the conclusion of the first part. The second part comes from the definitions of the O-operator T and the bracket $[-,-,-]_T$. □

Taking $\rho(y,z)x=\mu(x,z)y=\lambda(x,y)z=[x,y,z]$, we obtain the lowing consequence.

Corollary 3.15. *([8]) Given a Rota-Baxter operator $R: L\to L$ of weight $\lambda=0$ on a ternary Leibniz algebra L, we can make L into another ternary Leibniz algebra with the bracket*

$$\{x,y,z\}=[R(x),R(y),z]+[R(x),y,R(z)]+[x,R(y),R(z)], \tag{11}$$

for any $x,y,z\in L$.

Now we discuss on the compatibility of representations on compatible ternary Leibniz algebras.

Definition 3.16. Let $(L,[-,-,-])$ be a ternary Leibniz algebra,
$(V,\lambda_i,\mu_i,\rho_i), i=1,2$ be two representations of L on V. We say that the representations $(V,\lambda_1,\mu_1,\rho_1)$ and $(V,\lambda_2,\mu_2,\rho_2)$ are compatible if $(V,\lambda_1+\lambda_2,\mu_1+\mu_2,\rho_1+\rho_2)$ is also a representation of L on V. In other words, the representations $(V,\lambda_1,\mu_1,\rho_1)$ and $(V,\lambda_2,\mu_2,\rho_2)$ are compatible if, for any $x,y,z,t\in L$, we have

$$\rho_1(z,t)\lambda_2(x,y)+\rho_2(z,t)\lambda_1(x,y)-\lambda_1(x,y)\rho_2(z,t)-\lambda_2(x,y)\rho_1(z,t)=0$$
$$\rho_1(z,t)\mu_2(x,y)+\rho_2(z,t)\mu_1(x,y)-\mu_1(x,y)\rho_2(z,t)-\mu_2(x,y)\rho_1(z,t)=0$$
$$\rho_1(z,t)\rho_2(x,y)+\rho_2(z,t)\rho_1(x,y)-\rho_1(x,y)\rho_2(z,t)-\rho_2(x,y)\rho_1(z,t)=0,$$
$$\lambda_1(x,y)\lambda_2(z,t)+\lambda_2(x,y)\lambda_1(z,t)+\mu_1(x,z)\lambda_2(y,t)+\mu_2(x,z)\lambda_1(y,t)+$$
$$+\rho_1(y,z)\lambda_2(x,t)+\rho_2(y,z)\lambda_1(x,t)=0,$$
$$\lambda_1(x,y)\mu_2(z,t)+\lambda_2(x,y)\mu_1(z,t)+\mu_1(x,z)\mu_2(y,t)+\mu_2(x,z)\mu_1(y,t)+$$
$$+\rho_1(y,z)\mu_2(x,t)+\rho^2(y,z)\mu_1(x,t)=0.$$

Theorem 3.17. Let $(L,[-,-,-]_1,[-,-,-]_2)$ be a compatible ternary Leibniz algebra, V be a vector space, $\lambda_i: L\times L\to End(V), \mu_i: L\otimes L\to End(V)$, and $\rho_i: L\otimes L\to End(V)$ be six $(i=1,2,3)$ bilinear maps. Then, for any $x,y,z\in L, m_j\in V, i=1,2, j=1,2,3$, the direct sum $L\oplus V$ carries a compatible ternary Leibniz structure with

$$[x+m_1,y+m_2,z+m_3]_i=[x,y,z]_i+\lambda_i(x,y)(m_3)+\mu_i(x,z)(m_2)+\rho_i(y,z)(m_1), \tag{12}$$

if and only if $(V,\lambda_i,\mu_i,\rho_i), i=1,2$ are compatible representations of L on V.

Proof. It comes from direct calculation. □

4 Formal deformation of order 2 with representation

In this section, we study formal deformations of ternary Leibniz algebras with a representation and introduce the notion of a Nijenhuis pair, which gives a trivial deformation of ternary Leibniz algebra with a representation.

Definition 4.1. Let $(L,[-,-,-])$ be a ternary Leibniz algebra, V a linear vector space, $\lambda,\mu,\rho : L\times L \to End(V)$ a representation of L on V. Let $\omega^i : L\times L \to L$, $\omega^i_\lambda, \omega^i_\mu, \omega^i_\rho : L\times L \to End(V), i=1,2$ be bilinear maps. Consider a s-parametrized family bracket operations and linear maps

$$\begin{array}{rcl} [x,y,z]_s &=& [x,y,z]+s\omega^1(x,y,z)+s^2\omega^2(x,y,z)\\ \lambda_s(x,y) &=& \lambda(x,y)+s\omega^1_\lambda(x,y,z)+s^2\omega^2_\lambda(x,y)\\ \mu_s(x,y) &=& \mu(x,y)+s\omega^1_\mu(x,y,z)+s^2\omega^2_\mu(x,y)\\ \rho_s(x,y) &=& \rho(x,y)+s\omega^1_\rho(x,y,z)+s^2\omega^2_\rho(x,y), \end{array} \tag{13}$$

for any $x,y\in L$. If $(L,[-,-,-]_s)$ are ternary Leibniz algebras and $(\lambda^i_s,\mu^i_s,\rho^i_s)$ are representations of $(L,[-,-,-]_s)$ on V, we say that $(\omega^i,\omega^i_\lambda,\omega^i_\mu,\omega^i_\rho)$ generates a one-parameter formal deformation of the ternary Leibniz algebra $(L,[-,-,-])$ with the representation (V,λ,μ,ρ).

We denote a one-parameter formal deformation of a ternary Leibniz algebra $(L,[-,-,-])$ with the representation (V,λ,μ,ρ) by $(L,[-,-,-]_s,\lambda_s,\mu_s,\rho_s)$.

Corollary 4.2. *Let $(L,[-,-,-])$ be a ternary Leibniz algebra. The quintuple $(L,[-,-,-]_s,\lambda_s,\mu_s,\rho_s)$ is a one-parameter formal deformation of a ternary Leibniz algebra L with the representation (V,λ,μ,ρ) if and only if the following set of equations holds :*

$$\begin{array}{l} [\omega^1(x,y,z),t,u]+\omega^1([x,y,z],t,u)=\\ =[x,y,\omega^1(z,t,u)]+\omega^1(x,y[z,t,u])+[x,\omega^1(y,t,u),z]+\omega^1(x,[y,t,u],z)\\ +[\omega^1(x,t,u),y,z]+\omega^1([x,t,u],y,z), \end{array} \tag{14}$$

$$\begin{array}{l} \omega^1(\omega^1(x,y,z),t,u)-\omega^1(x,y,\omega^1(z,t,u))-\omega^1(x,\omega^1(y,t,u),z)-\omega^1(\omega^1(x,t,u),y,z)\\ =-[\omega^2(x,y,z),t,u]-\omega^2([x,y,z],t,u)+[x,y,\omega^2(z,t,u)]+\omega^2(x,y,[z,t,u])\\ +[x,\omega^2(y,t,u),z]+\omega^2(x,[y,t,u],z)+[\omega^2(x,t,u),y,z]+\omega^2([x,t,u],y,z) \end{array} \tag{15}$$

$$\begin{array}{l} \omega^1(\omega^2(x,y,z),t,u)+\omega^2(\omega^1(x,y,z),t,u)=\\ =\omega^1(x,y,\omega^2(z,t,u))+\omega^2(x,y,\omega^1(z,t,u))+\omega^1(x,\omega^2(y,t,u),z)+\omega^2(x,\omega^1(y,t,u),z)\\ +\omega^1(\omega^2(x,t,u),y,z)+\omega^2(\omega^1(x,t,u),y,z)\\ \omega^2(\omega^2(x,y,z),t,u)=\omega^2(x,y,\omega^2(z,t,u))+\omega^2(x,\omega^2(y,t,u),z)+\omega^2(\omega^2(x,t,u),y,z). \end{array} \tag{16}$$

$$\begin{array}{l} \lambda(\omega^1(x,y,z),t)+\omega^1_\lambda([x,y,z],t)=\\ =\lambda(x,y)\omega^1_\lambda(z,t)+\omega^1_\lambda(x,y)\lambda(z,t)+\mu(x,z)\omega^1_\lambda(y,t)+\omega^1_\mu(x,z)\lambda(y,t)\\ +\omega^1_\rho(y,z)\lambda(x,t)+\rho(y,z)\omega^1_\lambda(x,t), \end{array} \tag{17}$$

$$\begin{array}{l} \omega^1_\rho(z,t)\mu(x,y)+\rho(z,t)\omega^1_\mu(x,y)=\mu(x,\omega^1(y,z,t))+\omega^1_\mu(x,[y,z,t])+\\ +\mu(x,y)\omega^1_\rho(z,t)+\omega^1_\mu(x,y)\rho(z,t)+\omega^1_\mu([x,z,t],y)+\mu(\omega^1(x,z,t),y),\\ \rho(z,t)\omega^1_\lambda(x,y)+\omega^1_\rho(z,t)\lambda(x,y)=\lambda(x,y)\omega^1_\rho(z,t)+\omega^1_\lambda(x,y)\rho(z,t)+\lambda(x,\omega^1(y,z,t))\\ +\omega^1_\lambda(x,[y,z,t])+\lambda(\omega^1(x,z,t),y)+\omega^1_\lambda([x,z,t],y)\\ \mu(\omega^1(x,y,z),t)+\omega^1_\mu([x,y,z],t)=\lambda(x,y)\omega^1_\mu(z,t)+\omega^1_\lambda(x,y)\mu(z,t)\\ +\mu(x,z)\omega^1_\mu(y,t)+\omega^1_\mu(x,z)\mu(y,t)+\rho(x,z)\omega^1_\mu(x,t)+\omega^1_\rho(x,z)\mu(x,t)\\ \rho(z,t)\omega^1_\rho(x,y)+\omega^1_\rho(z,t)\rho(x,y)=\rho(x,\omega^1(y,z,t))+\omega_\rho(x,[y,z,t]) \end{array}$$

$$+\rho(\omega^1(x,z,t),y)+\omega^1_\rho([x,z,t],t)+\rho(x,y)\omega^1_\rho(z,t)+\omega_\rho(x,y)\rho(z,t). \tag{18}$$

$$\begin{aligned}
&\lambda(\omega(x,y,z),t)+\omega^2_\lambda([x,y,z],t)-\lambda(x,y)\omega^2_\lambda(z,t)-\omega^2_\lambda(x,y)\lambda(z,t)-\mu(x,z)\omega^2_\lambda(y,t)\\
&-\omega^2_\mu(x,z)\lambda(y,t)-\rho(y,z)\omega^2_\lambda(x,t)-\omega^2_\rho(y,z)\lambda(x,t)=\omega^1_\lambda(x,y)\omega^1_\lambda(z,t)\\
&+\omega^1_\mu(x,z)\omega^1_\lambda(y,t)+\omega^1_\rho(y,z)\omega^1_\lambda(x,t)-\omega^1_\lambda(\omega(x,y,z),t),
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\rho(z,t)\omega^2_\mu(x,y)+\omega^2_\rho(z,t)\mu(x,y)-\mu(x,y)\omega^2_\rho(z,t)-\mu(x,\omega^2(y,z,t))-\omega^2_\mu(x,[y,z,t])\\
&-\omega^2_\mu(x,y)\rho(z,t)-\mu(\omega^2(x,z,t),y)-\omega^2_\mu([x,z,t],y)=\omega^1_\mu(x,\omega^1(y,z,t))\\
&+\omega^1_\mu(x,y)\omega^1_\rho(z,t)+\omega^1_\mu(\omega^1(x,z,t),y)-\omega^1_\rho(z,t)\omega^1_\mu(x,y)\\
&\rho(z,t)\omega^2_\lambda(x,y)+\omega^2_\rho(z,t)\lambda(x,y)-\lambda(x,y)\omega^2_\rho(z,t)-\omega^2_\lambda(x,y)\rho(z,t)-\lambda(x,\omega^2(y,z,t))\\
&-\omega^2_\lambda(x,[y,z,t])-\lambda(\omega^2(x,z,t),y)-\omega^2_\lambda([x,z,t],y)=\omega^1_\lambda(x,\omega^1(y,z,t))\\
&+\omega^1_\lambda(\omega^1(x,z,t),y)+\omega^1_\lambda(x,y)\omega^1_\rho(z,t)-\omega^1_\rho(z,t)\omega^1_\lambda(x,y)\\
&\mu(\omega^2(x,y,z),t)+\omega^2_\mu([x,y,z],t)-\lambda(x,y)\omega^2_\mu(z,t)-\omega^2_\lambda(x,y)\mu(z,t)-\mu(x,z)\omega^2_\mu(y,t)\\
&-\omega^2_\mu(x,z)\mu(y,t)-\rho(x,z)\omega^2_\mu(x,t)-\omega^2_\rho(x,z)\mu(x,t)=\omega^1_\lambda(x,y)\omega^1_\mu(z,t)\\
&+\omega^1_\mu(x,z)\omega^1_\mu(y,t)+\omega^1_\rho(x,z)\omega^1_\mu(x,t)-\omega^1_\mu(\omega^1(x,y,z),t)\\
&\rho(z,t)\omega^2_\rho(x,y)+\omega^2_\rho(z,t)\rho(x,y)-\rho(x,\omega^2(y,z,t))-\omega^2_\rho(x,[y,z,t])-\rho(\omega^2(x,z,t),y)\\
&-\omega^2_\rho([x,z,t],y)-\rho(x,y)\omega^2_\rho(z,t)-\omega^2_\rho(x,y)\rho(z,t)=\omega^1_\rho(x,\omega^1(y,z,t))\\
&+\omega^1_\rho(\omega^1(x,z,t),y)+\omega^1_\rho(x,y)\omega^1_\rho(z,t)-\omega^1_\rho(z,t)\omega^1_\rho(x,y).
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\omega^1_\lambda(\omega^2(x,y,z),t)+\omega^2_\lambda(\omega^1(x,y,z),t)=\omega^1_\lambda(x,y)\omega^2_\lambda(z,t)+\omega^2_\lambda(x,y)\omega^1_\lambda(z,t)\\
&+\omega^1_\mu(x,z)\omega^2_\lambda(y,t)+\omega^2_\mu(x,z)\omega^1_\lambda(y,t)+\omega^1_\rho(y,z)\omega^2_\lambda(x,t)+\omega^2_\rho(y,z)\omega^1_\lambda(x,t)
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\omega^1_\rho(z,t)\omega^2_\mu(x,y)+\omega^2_\rho(z,t)\omega^1_\mu(x,y)=\omega^1_\mu(x,\omega(y,z,t))+\omega^2_\mu(x,\omega^1(x,z,t))\\
&+\omega^1_\mu(x,y)\omega^2_\rho(z,t)+\omega^2_\mu(x,y)\omega^1_\rho(z,t)+\omega^1_\mu(\omega^2(x,z,t),y)+\omega^2_\mu(\omega^1(x,z,t),y)\\
&\omega^1_\rho(z,t)\omega^2_\lambda(x,y)+\omega^2_\rho(z,t)\omega^1_\lambda(x,y)=\omega^1_\lambda(x,y)\omega^2_\rho(z,t)-\omega^2_\lambda(x,y)\omega^1_\rho(z,t)\\
&+\omega^1_\lambda(x,\omega^2(y,z,t))+\omega^2_\lambda(x,\omega^1(y,z,t))+\omega^1_\lambda(\omega^2(x,z,t),y)+\omega^2_\lambda(\omega^1(x,z,t),y)
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\omega^1_\mu(\omega^2(x,y,z),t)+\omega^2_\mu(\omega^1(x,y,z),t)=\omega^1_\lambda(x,y)\omega^2_\mu(z,t)+\omega^2_\lambda(x,y)\omega^1_\mu(z,t)\\
&+\omega^1_\mu(x,z)\omega^2_\mu(y,t)+\omega^2_\mu(x,z)\omega^1_\mu(y,t)+\omega^1_\rho(x,z)\omega^2_\mu(x,t)+\omega^2_\rho(x,z)\omega^1_\mu(x,t)\\
&\omega^1_\rho(z,t)\omega^2_\rho(x,y)+\omega^2_\rho(z,t)\omega^1_\rho(x,y)=\omega^1_\rho(x,\omega^2(y,z,t))+\omega^2_\rho(x,\omega^1(y,z,t),y)\\
&+\omega^1_\rho(\omega^2(x,z,t),y)+\omega^2_\rho(\omega^1(x,z,t),y)+\omega^1_\rho(x,y)\omega^2_\rho(z,t)+\omega^2_\rho(x,y)\omega^1_\rho(z,t),
\end{aligned} \tag{23}$$

for any $x,y,z,t,u\in L$.

Proof. It follows from direct computation by writing the ternary Leibniz algebra identity for $[-,-,-]_s$ and the representation axioms for
$([-,-,-]_s,\lambda_s,\mu_s,\rho_s)$. □

We have the following observations.

Remark 4.3. 1) Equality (14) means that ω^1 is a 2-cocycle on $(L,[-,-,-])$ with values in L.

2) The left hand side of (15) is the condition for ω^1 to be a ternary Leibniz algebra bracket while the right hand side is the compatibility condition of $[-,-,-]$ and ω^2.

3) Equality (16), is the compatibility condition of the brackets ω^1 and ω^2 or ω^1 is a 2-cocycle on (L,ω^2) with values in L.

4) Equality (16) means that ω^2 is a ternary Leibniz algebra bracket.

Corollary 4.4. *Suppose that ω^1 is a ternary Leibniz algebra bracket on L and that $(L,[-,-,-]_s,\lambda_s,\mu_s,\rho_s)$ is a one-parameter formal deformation of a ternary Leibniz algebra $(L,[-,-,-])$ with the representation (V,λ,μ,ρ). Then*

1) the brackets $[-,-,-]$ and ω^1 are compatible ternary algebra brackets,

2) the brackets $[-,-,-]$ and ω^2 are compatible ternary algebra brackets,

3) The brackets ω^1 and ω^2 are compatible ternary algebra brackets,

4) the quadruple $(\lambda+\omega^1_\lambda,\mu+\omega^1_\mu,\rho+\omega^1_\rho)$ is a representation of $(L,[-,-,-]+\omega^1)$ on V.

5) the quadruple $(\lambda+\omega^2_\lambda,\mu+\omega^2_\mu,\rho+\omega^2_\rho)$ is a representation of $(L,[-,-,-]+\omega^2)$ on V.

6) the quadruple $(\omega^1_\lambda+\omega^2_\lambda,\omega^1_\mu+\omega^2_\mu,\omega^1_\rho+\omega^2_\rho)$ is a representation of $(L,\omega^1+\omega^2)$ on V.

Proof. 1) It comes from equality (14).

2) It comes from equalities (15) and (16).

3) It comes from equalities (16) and (16).

4) It comes from equalities(17)-(18).

5) It comes from equalities (19)-(20).

6) It comes from equalities (21)-(23).

□

Corollary 4.5. *The quintuple $(L,[-,-,-]_s,\lambda_s,\mu_s,\rho_s)$ is a linear deformation $(\omega^2=\omega^2_\lambda=\omega^2_\mu=\omega^2_\rho=0)$ of a ternary Leibniz algebra $(L,[-,-,-])$ with the representation (V,λ,μ,ρ) if and only if:*

1) ω^1 is a ternary Leibniz algebra bracket.

2) $[-,-,-]$ and ω^1 are compatible ternary Leibniz algebra brackets.

3) $(\omega^1_\lambda,\omega^1_\mu,\omega^1_\rho)$ is a representation of (L,ω^1) on V.

4) $(\lambda+\omega^1_\lambda,\mu+\omega^1_\mu,\rho+\omega^1_\rho)$ is a representation of $(L,[-,-,-]+\omega^1)$ on V.

Now we have the following definitions.

Definition 4.6. A linear map $N:L\to L$ on a ternary Leibniz algebra is called a **Nijenhuis operator** if

$$\begin{aligned}&[N(x),N(y),N(z)]=\\&\Big([N(x),N(y),z]+[N(x),y,N(z)]+[x,N(y),N(z)]\Big)\\&-N^2\Big([N(x),y,z]+[x,N(y),z]+[x,y,N(z)]\Big)+N^3([x,y,z])\Big),\end{aligned}$$

for any $x,y,z\in L$.

Definition 4.7. i) Two one-parameter formal deformations of order 2 $(L,[-,-,-]_s,\lambda_s,\mu_s,\rho_s)$ and $(L,[-,-,-]'_s,\lambda'_s,\mu'_s,\rho'_s)$ of a ternary Leibniz algebra $(L,[-,-,-])$ with the representation (V,λ,μ,ρ) are equivalent if there exists an isomorphism (Id_L+sN, Id_V+sT) from $(L,[-,-,-]_s,\lambda_s,\mu_s,\rho_s)$ to $(L,[-,-,-]'_s,\lambda'_s,\mu'_s,\rho'_s)$ i.e. for any $x,y\in L$,

$$\begin{aligned}(Id_L+sN)[x,y,z]'_s &= [(Id_L+sN)(x),(Id_L+sN)(y),(Id_L+sN)(z)]_s,\\ (Id_V+sT)\lambda'_s(x,y) &= \lambda_s((Id_L+sN)(x),(Id_L+sN)(y))\circ(Id_V+sT),\\ (Id_V+sT)\mu'_s(x,y) &= \mu_s((Id_L+sN)(x),(Id_L+sN)(y))\circ(Id_V+sT),\\ (Id_V+sT)\rho'_s(x,y) &= \rho_s((Id_L+sN)(x),(Id_L+sN)(y))\circ(Id_V+sT),\end{aligned} \tag{24}$$

ii) A one-parameter formal deformation of order 2 of a Leibniz algebra $(L,[-,-,-])$ with representation (V,λ,μ,ρ) is said to be trivial if it is equivalent to $(L,[-,-,-],\lambda,\mu,\rho)$.

Definition 4.8. A pair (N,T), where $N\in End(L)$ and $T\in End(V)$, is called a Nijenhuis pair on a ternary Leibniz algebra $(L,[-,-,-])$ with a representation (V,λ,μ,ρ) if N is a Nijenhuis operator on the ternary Leibniz algebra L and T satisfies the following conditions hold:

$$\begin{aligned}\lambda(N(x),N(y))T &= T\Big(\lambda(N(x),N(y))+\lambda(N(x),y)T+\lambda(x,N(y))T\Big)\\ &\quad -T^2\Big(\lambda(x,y)T+\lambda(N(x),y)+\lambda(x,N(y))\Big)\\ &\quad +T^3\lambda(x,y),\\ \mu(N(x),N(y))T &= T\Big(\mu(N(x),N(y))+\mu(N(x),y)T+\mu(x,N(y))T\Big)\\ &\quad -T^2\Big(\mu(x,y)T+\mu(N(x),y)+\mu(x,N(y))\Big)\\ &\quad +T^3\mu(x,y),\\ \rho(N(x),N(y))T &= T\rho(\rho(N(x),N(y))+\rho(N(x),y)T+\rho(x,N(y))T\Big)\\ &\quad -T^2\Big(\rho(x,y)T+\rho(N(x),y)+\rho(x,N(y))\Big)\\ &\quad +T^3\rho(x,y),\end{aligned}$$

for any $x,y\in L$.

Theorem 4.9. Let (N,T) be a Nijenhuis pair on a ternary Leibniz algebra $(L,[-,-,-])$ with a representation (V,λ,μ,ρ). Then, a deformation of $(L,[-,-,-],\lambda,\mu,\rho)$ can be obtained by putting

$$\begin{aligned}\omega^1(x,y,z) &= [N(x),y,z]+[x,N(y),z]+[x,y,N(z)],\\ \omega^2(x,y,z) &= [N(x),N(y),z]+[N(x),y,N(z)]+[x,N(y),N(z)]\\ &\quad -N([N(x),y,z]+[x,N(y),z]+[x,y,N(z)]),\\ \omega^1_\lambda(x,y) &= \lambda(x,y)T+\lambda(N(x),y)+\lambda(x,N(y))-T\lambda(x,y),\\ \omega^2_\lambda(x,y) &= \lambda(N(x),N(y))+\lambda(N(x),y)T+\lambda(x,N(y))T-T\omega^1_\lambda(x,y),\\ \omega^1_\mu(x,y) &= \mu(x,y)T+\mu(N(x),y)+\mu(x,N(y))-T\mu(x,y),\\ \omega^2_\mu(x,y) &= \mu(N(x),N(y))+\mu(N(x),y)T+\mu(x,N(y))T-T\omega^1_\mu(x,y),\\ \omega^1_\rho(x,y) &= \rho(x,y)T+\rho(N(x),y)+\rho(x,N(y))-T\rho(x,y),\\ \omega^2_\rho(x,y) &= \rho(N(x),N(y))+\rho(N(x),y)T+\rho(x,N(y))T-T\omega^1_\rho(x,y),\end{aligned}$$

Furthermore, this deformation is trivial.

Proof. It follows from a straightforward computation. □

Definition 4.10. A pair (N,T), where $N \in End(L)$ and $T \in End(V)$, is dual-Nijenhuis pair on a ternary Leibniz algebra $(L,[-,-,-])$ with a representation (V,λ,μ,ρ) if N is a Nijenhuis operator on the ternary Leibniz algebra L and T satisfies the following conditions:

$$\begin{array}{rcl}\lambda(x,y)T^3 &=& T\lambda(N(x),N(y)) \\ && -\Big(\lambda(N(x),N(y)) + T\lambda(N(x),y) + T\lambda(x,N(y))\Big)T \\ && +\Big(T\lambda(x,y) + \lambda(N(x),y) + \lambda(x,N(y))\Big)T^2, \\ \mu(x,y)T^3 &=& T\mu(N(x),N(y)) \\ && -\Big(\mu(N(x),N(y)) + T\mu(N(x),y) + T\mu(x,N(y))\Big)T \\ && +\Big(T\mu(x,y) + \mu(N(x),y) + \mu(x,N(y))\Big)T^2, \\ \rho(x,y)T^3 &=& T\rho(N(x),N(y)) \\ && -\Big(\rho(N(x),N(y)) + T\rho(N(x),y) + T\rho(x,N(y))\Big)T \\ && +\Big(T\rho(x,y) + \rho(N(x),y) + \rho(x,N(y))\Big)T^2,\end{array}$$

for any $x,y \in L$.

Let $\lambda^*,\mu^*,\rho^* : L \times L \to End(V^*)$ be the dual representation of λ,μ,ρ defined by

$$\begin{array}{rcl}\langle\lambda^*(x,y)\alpha,v\rangle &=& -\langle\alpha,\lambda(x,y)v\rangle, \\ \langle\mu^*(x,y)\alpha,v\rangle &=& -\langle\alpha,\mu(x,y)v\rangle, \\ \langle\rho^*(x,y)\alpha,v\rangle &=& -\langle\alpha,\rho(x,y)v\rangle,\end{array}$$

$\forall x,y \in L, v \in V, \alpha \in V^*$.

Proposition 4.11. *(N,T) is a Nijenhuis pair on a ternary Leibniz algebra $(L,[-,-,-])$ with a representation (V,λ,μ,ρ) if and only if (N,T^*) is a dual-Nijenhuis pair on the ternary Leibniz algebra $(L,[-,-,-])$ with the representation $(V^*,\lambda^*,\mu^*,\rho^*)$.*

Proof. It comes from a direct computation by transposition. □

Proposition 4.12. *Let (N,T) be a Nijenhuis on a ternary Leibniz algebra $(L,[-,-,-])$ with a representation (V,λ,μ,ρ). Then, $N+T$ is a Nijenhuis operator on the ternary Leibniz algebra $L\oplus V$ with the multiplication defined by*

$$[x+m_1, y+m_2, y+m_3] := [x,y,z] + \rho(y,z)(m_1) + \mu(x,z)(m_2) + \lambda(x,y)(m_3),$$

for all $x,y,z \in L, m_i, i = 1,2,3$.

Proof. For any $x,y,z \in L, m,n,p \in V$ and by Definition 4.6 and Definition 4.8, we have:

$$\begin{array}{l}\{(N+T)(x+m),(N+T)(y+n),(N+T)(z+p)\} = \\ = \{(N(x)+T(m),N(y)+T(n),N(z)+T(p)\} \\ = [N(x),N(y),N(z)] + \rho(N(y),N(z))(T(m)) + \mu(N(x),N(z))(T(n)) \\ +\lambda(N(x),N(y))(T(p)) \\ = N\Big([N(x),N(y),z] + [x,N(y),N(z)] + [N(x),y,N(z)]\Big) \\ -N^2\Big([N(x),y,z] + [x,N(y),z] + [x,y,N(z)]\Big) + N^3([x,y,z]) \\ +T\Big(\rho(N(y),N(z))(m) + \rho(N(y),z)(T(m)) + \rho(y,N(z))(T(m))\Big)\end{array}$$

$$
\begin{aligned}
&-T^2\Big(\rho(y,z)(T(m))+\rho(N(y),z)+\rho(y,N(z))\Big)\\
&+T^3\rho(y,z)(m)+T\Big(\mu(N(x),N(z))(n)+\mu(N(x),z)(T(n))+\mu(x,N(z))(T(n))\Big)\\
&-T^2\Big(\mu(x,z)(T(n))+\mu(N(x),z)(n)+\mu(x,N(z))(n)\Big)+T^3\mu(x,z)(n)\\
&+T\Big(\lambda(N(x),N(y))(p)+\lambda(N(x),y)T(p)+\lambda(x,N(y))(T(p))\Big)\\
&-T^2\Big(\lambda(x,y)(T(p))+\lambda(N(x),y)(p)+\lambda(x,N(y))(p)\Big)+T^3\lambda(x,y)(p).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&(N+T)\Big(\{(N+T)(x+m),(N+T)(y+n),z+p\}\\
&+\{(N+T)(x+m),y+n,(N+T)(z+p)\}\\
&+\{x+m,(N+T)(y+n),(N+T)(z+p)\}\Big)\\
&=(N+T)\Big(\{(N(x)+T(m),N(y)+T(n),z+p\}\\
&+\{(N(x)+T(m),y+n,N(z)+T(p)\}\\
&+\{x+m,N(y)+T(n),N(z)+T(p)\}\Big)\\
&=(N+T)\Big([N(x),N(y),z]+\rho(N(y),z)(T(m))\\
&+\mu(N(x),z)(T(n))+\lambda(N(x),N(y))(p)\\
&+[N(x),y,N(z)]+\rho(y,N(z))(T(m))+\mu(N(x),N(z))(n)+\lambda(N(x),y)(T(p))\\
&+[x,N(y),N(z)]+\rho(N(y),N(z))(m)+\mu(x,N(z))(T(n))+\lambda(x,N(y))(T(p))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&=N\Big([N(x),N(y),z]+[N(x),y,N(z)]+[x,N(y),N(z)]\Big)\\
&+T\Big(\lambda(N(x),N(y))(p)+\lambda(N(x),y)(T(p))+\lambda(x,N(y))(T(p))\Big)\\
&+T\Big(\mu(N(x),z)(T(n))+\mu(N(x),N(z))(n)+\mu(x,N(z))(T(n))\Big)\\
&+T\Big(\rho(N(y),z)(T(m))+\rho(y,N(z))(T(m))++\rho(N(y),N(z))(m)\Big).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&(N+T)^2\Big(\{(N+T)(x+m),y+n,z+p\}+\{x+m,(N+T)(y+n),z+p\}\\
&+\{x+m,y+n,(N+T)(z+p)\}\Big)=\\
&=N^2\Big([N(x),y,z]+[x,N(y),z]+[x,y,N(z)]\Big)\\
&+T^2\Big(\lambda(N(x),y)(p)+\lambda(x,N(y))(T(p))+\lambda(x,y)(T(p))\Big)\\
&+T^2\Big(\mu(N(x),z)(n)+\mu(x,z)(T(n))+\mu(x,N(z))(n)\Big)\\
&+T^2\Big(\rho(y,z)(T(m))+\rho(N(y),z)(m)+\rho(y,N(z))(m)\Big).
\end{aligned}
$$

and

$$
(N+T)^3(\{x+m,y+n,z+p\})=
$$

$$= N^3([x,y,z]) + T^3\Big(\lambda(x,y)(p) + \mu(x,z)(n) + \rho(y,z)(m)\Big).$$

It follows that

$$\begin{aligned}
&\{(N+T)(x+m),(N+T)(y+n),(N+T)(z+p)\} = \\
&= (N+T)\Big(\{(N+T)(x+m),(N+T)(y+n),z+p\} \\
&+\{(N+T)(x+m),y+n,(N+T)(z+p)\} \\
&+\{x+m,(N+T)(y+n),(N+T)(z+p)\}\Big) \\
&-(N+T)^2\Big(\{(N+T)(x+m),y+n,z+p\} \\
&+\{x+m,(N+T)(y+n),z+p\} + \{x+m,y+n,(N+T)(z+p)\}\Big) \\
&+(N+T)^3(\{x+m,y+n,z+p\}).
\end{aligned}$$

This ends the proof. □

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