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On Strongly Gorenstein Properties and Global Dimensions in Amalgamated Duplication

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Abstract. This paper investigates the behavior of strongly Gorenstein projective, injective, and flat modules over the amalgamated duplication $R \bowtie I$ of a commutative ring R along an ideal I. We establish conditions under which these Gorenstein homological properties are preserved or transferred from R-modules to $(R \bowtie I)$-modules via tensor products and Hom functors. Furthermore, we compare the Gorenstein global dimension (G-gldim) and the weak Gorenstein global dimension (wG-gldim) of R and $R \bowtie I$, proving inequalities and equalities when I is flat or pure. Our results generalize foundational work on trivial ring extensions and provide new tools for constructing rings with controlled homological properties.

Key Words: Pure ideal, Gorenstein global dimension, Gorenstein weak global dimension, Amalgamated duplication.
2020 MSC: 13D05, 13D02, 16E10, 16E65.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with identity, and all modules are unital.

Gorenstein homological algebra, initiated by Auslander and Bridger [2] and further developed by Enochs, Jenda, Holm, and others [16, 15, 19], provides a framework for extending classical homological invariants to non-regular rings. In this context, notions such as Gorenstein projective, injective, and flat modules play a central role, particularly in the study of Gorenstein rings and the complexity of modules with infinite classical homological dimensions. A key simplification in the theory was introduced by Bennis and Mahdou [3], who defined the class of *strongly Gorenstein modules*, which we recall below.

The *amalgamated duplication* $R \bowtie I$, introduced by D'Anna and Fontana [8, 9, 10], constructs a new ring from a commutative ring R and an ideal I as follows:

$$R \bowtie I = \{(r, r+i) \mid r \in R, i \in I\},$$

with component-wise addition and multiplication. This pullback construction generalizes trivial extensions and has proven to be a valuable tool for producing examples with specific homological properties [14, 20, 21]. While Gorenstein dimensions in trivial extensions have been studied extensively [22], their behavior in amalgamated duplications remains less understood, although some cases have been explored in [20, 21]. This paper seeks to bridge that gap by extending the results of [22] to the setting of amalgamated duplications.

We now recall some key definitions that will be used throughout our analysis.

Let R be a ring. An R-module M is said to be *Gorenstein projective* if there exists a totally acyclic complex

$$\mathbf{P}^\bullet = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$

consisting of projective modules such that the complex remains acyclic under the functor $\mathrm{Hom}_R(-,Q)$ for every projective R-module Q, and $M \cong \ker(P_0 \to P_{-1})$.

Dually, M is *Gorenstein injective* if it is the cycle of a totally acyclic complex of injective modules:

$$\mathbf{E}^\bullet = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots,$$

which remains acyclic under the functor $\mathrm{Hom}_R(I,-)$ for every injective R-module I, and $M \cong \ker(E_0 \to E_{-1})$.

An R-module M is called *Gorenstein flat* if there exists an acyclic complex of flat R-modules

$$\mathbf{F}^\bullet: \ \cdots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \cdots$$

such that $M \cong \ker(F_0 \to F_{-1})$ and the complex $\mathbf{F} \otimes_R I$ is acyclic for every injective R-module I.

According to [3], an R-module M is *strongly Gorenstein projective* if and only if there exists a short exact sequence

$$0 \to M \xrightarrow{\iota} P \xrightarrow{\pi} M \to 0,$$

where P is projective and the sequence remains exact under $\mathrm{Hom}_R(-,Q)$ for all projective R-modules Q. Likewise, M is *strongly Gorenstein injective* if there exists a short exact sequence

$$0 \to M \xrightarrow{\iota} E \xrightarrow{\pi} M \to 0,$$

where E is injective and the sequence remains exact under $\mathrm{Hom}_R(I,-)$ for all injective R-modules I.

An R-module M is called *strongly Gorenstein flat* if there exists a short exact sequence

$$0 \to M \xrightarrow{\phi} F \xrightarrow{\psi} M \to 0$$

where F is a flat R-module, and for every injective R-module I, the sequence remains exact after tensoring with I, that is,

$$0 \to M \otimes_R I \xrightarrow{\phi \otimes 1_I} F \otimes_R I \xrightarrow{\psi \otimes 1_I} M \otimes_R I \to 0$$

is exact. Equivalently, $\mathrm{Tor}_1^R(M,I) = 0$ for all injective R-modules I.

The *Gorenstein projective dimension* of M, denoted $\mathrm{Gpd}_R(M)$, is the smallest non-negative integer n (if such exists) such that there is an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0,$$

where each G_i is a Gorenstein projective R-module. If no such finite n exists, we write $\mathrm{Gpd}_R(M) = \infty$.

The *Gorenstein injective dimension* of M, denoted $\mathrm{Gid}_R(M)$, is the smallest non-negative integer n (if such exists) such that there is an exact sequence

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^n \to 0,$$

where each I^i is a Gorenstein injective R-module. If no such finite n exists, we write $\mathrm{Gid}_R(M) = \infty$.

The *Gorenstein flat dimension* of M, denoted $\mathrm{Gfd}_R(M)$, is the smallest non-negative integer n (if such exists) such that there is an exact sequence

$$0 \to G_0 \to G_1 \to \cdots \to G_n \to M \to 0,$$

where each G_i is a Gorenstein flat R-module. If no such finite n exists, we write $\mathrm{Gfd}_R(M) = \infty$.

The *Gorenstein global dimension* of R, denoted G-gldim(R), is defined by

$$G\text{-gldim}(R) := \sup\{\mathrm{Gpd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

The *weak Gorenstein global dimension* of R, denoted wG-gldim(R), is defined by

$$wG\text{-gldim}(R) := \sup\{\mathrm{Gfd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Our main contributions are as follows:

(i) We prove (Theorem 2.1) that if M is a strongly Gorenstein projective (resp. injective, flat) R-module, then $M \otimes_R (R \bowtie I)$ (resp. $\mathrm{Hom}_R(R \bowtie I, M)$, $M \otimes_R (R \bowtie I)$) is a strongly Gorenstein projective (resp. injective, flat) module over $R \bowtie I$, under certain conditions on $\mathrm{pd}_R(I)$, $\mathrm{fd}_R(R \bowtie I)$, and the vanishing of / groups.

(ii) When I is flat, we establish the equalities (Theorem 3.2):

$$\mathrm{Gpd}_R(M) = \mathrm{Gpd}_{R\bowtie I}(M \otimes_R (R \bowtie I)),$$

and

$$\mathrm{Gfd}_R(M) = \mathrm{Gfd}_{R\bowtie I}(M \otimes_R (R \bowtie I)).$$

For pure ideals, we show the inequalities (Corollary 3.4):

$$G\text{-gldim}(R) \leq G\text{-gldim}(R \bowtie I),$$

and

$$wG\text{-gldim}(R) \leq wG\text{-gldim}(R \bowtie I).$$

(iii) We provide explicit examples, including infinite-dimensional cases and homological properties, illustrating the sharpness of our results.

Throughout this paper, the Gorenstein dimensions follow the conventions in [3, 19].

2 Main Results

Our first main result relates the strongly Gorenstein projective, injective, and flat modules over R and those over $R \bowtie I$ as follows.

Theorem 2.1. Let I be an ideal of R, and let M be an R-module. Then:

1. (a) Assume that $\mathrm{pd}_R(I) < \infty$. If M is a strongly Gorenstein projective R-module, then $M \otimes_R (R \bowtie I)$ is a strongly Gorenstein projective $(R \bowtie I)$-module.
   (b) Conversely, suppose that I is a flat R-module. If $M \otimes_R (R \bowtie I)$ is a strongly Gorenstein projective $(R \bowtie I)$-module, then M is a strongly Gorenstein projective R-module.
2. Suppose that ${}^k_R(R \bowtie I, M) = 0$ for all $k \geq 1$ and $\mathrm{fd}_R(R \bowtie I) < \infty$. If M is a strongly Gorenstein injective R-module, then $\mathrm{Hom}_R(R \bowtie I, M)$ is a strongly Gorenstein injective $(R \bowtie I)$-module.

3. Assume that I is a flat ideal of R. If M is strongly Gorenstein flat over R, then the $(R \bowtie I)$-module $M \otimes_R (R \bowtie I)$ is strongly Gorenstein flat.

To prove Theorem 2.1, we need the following lemma.

Lemma 2.2. *Let R be a commutative ring and I an ideal of R. If I is flat as an R-module, then the* amalgamated duplication *ring $R \bowtie I$ is faithfully flat over R.*

Proof. Note that $R \bowtie I \cong R \oplus I$ as R-modules. So, we get the result. □

Proof of Theorem 2.1. (1) (a) Suppose M is strongly Gorenstein projective over R. Then there exists an exact sequence of R-modules

$$0 \to M \to P \to M \to 0 \qquad (\bullet_1)$$

where P is projective by [3, Proposition 2.9]. Since $R \bowtie I \cong R \oplus I$ as an R-module and $\mathrm{pd}_R(I) < \infty$, we have $\mathrm{pd}_R(R \bowtie I) < \infty$. From $(\bullet_1)$, it follows that ${}_i^R(M, R \bowtie I) = 0$ for all $i \geq 1$. Thus, the sequence

$$0 \to M \otimes_R (R \bowtie I) \to P \otimes_R (R \bowtie I) \to M \otimes_R (R \bowtie I) \to 0$$

is exact. Moreover, the $(R \bowtie I)$-module $P \otimes_R (R \bowtie I)$ is projective. For any projective $(R \bowtie I)$-module Q, we have $\mathrm{pd}_R(Q) < \infty$ by a well-known result due to Kaplansky that $\mathrm{pd}_R M \leqslant \mathrm{pd}_S M + \mathrm{pd}_R S$ where $\varphi : R \to S$ is a ring homomorphism. Since M is strongly Gorenstein projective, ${}^1_{R \bowtie I}(M \otimes_R (R \bowtie I), Q) \cong {}^1_R$ $(M, Q) = 0$ by [5, Proposition 4.1.3, p. 118]. Thus, $M \otimes_R (R \bowtie I)$ is strongly Gorenstein projective by [3, Proposition 2.9].

(b) If I is flat over R, then $R \bowtie I$ is faithfully flat over R (Lemma 2.2). Suppose $M \otimes_R (R \bowtie I)$ is a strongly Gorenstein projective $(R \bowtie I)$-module. Then there exists an exact sequence

$$0 \to M \otimes_R (R \bowtie I) \to F \to M \otimes_R (R \bowtie I) \to 0 \qquad (\bullet_2)$$

where $F = (R \bowtie I)^{(J)}$ is free. The sequence $(\bullet_2)$ is equivalent to

$$0 \to M \otimes_R (R \bowtie I) \to R^{(J)} \otimes_R (R \bowtie I) \to M \otimes_R (R \bowtie I) \to 0.$$

Since $R \bowtie I$ is faithfully flat, the sequence

$$0 \to M \to R^{(J)} \to M \to 0$$

is exact over R. Let P be projective over R. Then $P \otimes_R (R \bowtie I)$ is projective over $R \bowtie I$, and

$${}^k_R(M, P \otimes_R (R \bowtie I)) \cong {}^k_{R \bowtie I} (M \otimes_R (R \bowtie I), P \otimes_R (R \bowtie I)) = 0.$$

Since ${}_i^R(M, R \bowtie I) = 0$ for $i \geqslant 1$, we have

$${}^k_R(M, P \otimes_R (R \bowtie I)) \cong {}^k_R (M, P) \oplus {}^k_R (M, P \otimes_R I) = 0,$$

so ${}^k_R(M, P) = 0$. Thus, M is a strongly Gorenstein projective R-module.

(2) Assume that M is strongly Gorenstein injective over R. There exists an exact sequence

$$0 \to M \to E \to M \to 0$$

where E is injective over R. Since ${}^k_R(R \bowtie I, M) = 0$ for all $k \geq 1$, the sequence

$$0 \to \mathrm{Hom}_R(R \bowtie I, M) \to \mathrm{Hom}_R(R \bowtie I, E) \to \mathrm{Hom}_R(R \bowtie I, M) \to 0$$

is exact. The $(R \bowtie I)$-module $\mathrm{Hom}_R(R \bowtie I, E)$ is injective because for any $(R \bowtie I)$-module Q, by [5, Proposition 4.1.4, p. 118], we have

$$ {}^{i}_{R\bowtie I}(Q, \mathrm{Hom}_R(R \bowtie I, E)) \cong^{i}_{R} (Q, E) = 0. $$

Moreover, for any injective $(R \bowtie I)$-module J, since $\mathrm{fd}_R(R \bowtie I) < \infty$, we get that $\mathrm{id}_R(J) < \infty$.Also, by [5, Proposition 4.1.4, p. 118],

$$ {}^{i}_{R\bowtie I}(J, \mathrm{Hom}_R(R \bowtie I, M)) \cong^{i}_{R} (J, M) = 0. $$

Therefore, $\mathrm{Hom}_R(R \bowtie I, M)$ is strongly Gorenstein injective over $R \bowtie I$.

(3) Suppose that M is a strongly Gorenstein flat R-module. By [3, Proposition 3.6], there exists an exact sequence

$$ 0 \to M \to F \to M \to 0 $$

where F is a flat R-module and ${}^{k}_{R}(M, E) = 0$ for any injective R-module E and all $k > 0$. Since I is flat, $R \bowtie I$ is faithfully flat over R by Lemma 2.2, so ${}^{i}_{R}(R \bowtie I, M) = 0$ for all $i > 0$, and by [5, Proposition 4.1.1, p. 117],

$$ {}^{i}_{R\bowtie I}(M \otimes_R (R \bowtie I), E) \cong^{i}_{R} (M, E) = 0. $$

Thus, the sequence

$$ 0 \to M \otimes_R (R \bowtie I) \to F \otimes_R (R \bowtie I) \to M \otimes_R (R \bowtie I) \to 0 $$

is exact, and the $(R \bowtie I)$-module $F \otimes_R (R \bowtie I)$ is flat. Moreover, by [5, Proposition 4.1.2, p. 117], for any injective $(R \bowtie I)$-module E', we have

$$ {}^{i}_{R\bowtie I}(M \otimes_R (R \bowtie I), E') \cong^{i}_{R} (M, E') = 0. $$

Therefore, $M \otimes_R (R \bowtie I)$ is strongly Gorenstein flat over $R \bowtie I$. □

Corollary 2.3. *Let R be an integral domain and let $I \neq 0$ be a principal ideal of R. Then, for any strongly Gorenstein R-module M, the module $M \otimes_R (R \bowtie I)$ is a strongly Gorenstein projective $(R \bowtie I)$-module.*

Proof. Let M be a strongly Gorenstein R-module. Since R is an integral domain, every non-zero element $x \in R$ is regular. Hence, the ideal I, being principal over an integral domain, is free and therefore projective. Thus, $\mathrm{pd}_R(I) = 0 < \infty$. By Theorem 2.1, it follows that $M \otimes_R (R \bowtie I)$ is a strongly Gorenstein projective $(R \bowtie I)$-module. □

Corollary 2.4. *Let M be an R-module. Then:*

1. *Suppose that $\mathrm{pd}_R(I) < \infty$. If M is a Gorenstein projective R-module, then $M \otimes_R (R \bowtie I)$ is a Gorenstein projective $(R \bowtie I)$-module.*
2. *Suppose that ${}^{k}_{R}(R \bowtie I, M) = 0$ for all $k \geq 1$ and $\mathrm{fd}_R(R \bowtie I) < \infty$. If M is a Gorenstein injective R-module, then the $(R \bowtie I)$-module $\mathrm{Hom}_R(R \bowtie I, M)$ is Gorenstein injective.*

Proof. By [3, Theorem 2.7], every Gorenstein projective (resp. injective) module is a direct summand of a strongly Gorenstein projective (resp. injective) module. Since we have natural isomorphisms for any modules over the same base ring R:

$$ \mathrm{Hom}_R(P \oplus Q, N) \cong \mathrm{Hom}_R(P, N) \oplus \mathrm{Hom}_R(Q, N), $$
$$ (P \oplus Q) \otimes_R N \cong (P \otimes_R N) \oplus (Q \otimes_R N), $$

we can apply Theorem 2.1 to the strongly Gorenstein cases and then extend the result to direct summands. Indeed, if M is a Gorenstein injective R-module, then there exists a strongly Gorenstein injective R-module N such that $M \oplus T = N$ for some T (by [3, Theorem 2.7]). Note that $\mathrm{Ext}_R^k(R \bowtie I, M) = 0$ and $\mathrm{Ext}_R^k(R \bowtie I, T) = 0$ because T is also Gorenstein injective, being a direct summand of a strongly Gorenstein injective module; hence $\mathrm{Ext}_R^k(R \bowtie I, N) = 0$. The same reasoning applies to the projective case. Therefore, the conclusions follow. □

Example 2.5. Consider the ring $\mathbb{Z}$ of integers, which is an integral domain, and let n and m be two integers. The $\mathbb{Z}$-module $n\mathbb{Z}$ is projective, being free, and similarly, $m\mathbb{Z}$ is a strongly Gorenstein projective $\mathbb{Z}$-module since it is projective. Thus, by Theorem 2.1(1), the $(\mathbb{Z} \bowtie n\mathbb{Z})$-module $m\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z} \bowtie n\mathbb{Z})$ is strongly Gorenstein projective.

Example 2.6. Let $R = \mathbb{Z}$, $I = n\mathbb{Z}$ for an integer $n \geq 2$, and $M = \mathbb{Q}$ (the field of rational numbers). The amalgamated duplication

$$R \bowtie I = \mathbb{Z} \bowtie n\mathbb{Z} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid b - a \in n\mathbb{Z}\}$$

is isomorphic to $\mathbb{Z} \oplus n\mathbb{Z}$ as a $\mathbb{Z}$-module via $(a,b) \mapsto (a, b-a)$. Note that

$$\mathrm{fd}_{\mathbb{Z}}(\mathbb{Z} \bowtie n\mathbb{Z}) = \mathrm{fd}_{\mathbb{Z}}(\mathbb{Z} \oplus n\mathbb{Z}) = \sup\{\mathrm{fd}_{\mathbb{Z}}(\mathbb{Z}), \mathrm{fd}_{\mathbb{Z}}(n\mathbb{Z})\}.$$

Since $\mathrm{fd}_{\mathbb{Z}}(\mathbb{Z}) = 0$ and $\mathrm{fd}_{\mathbb{Z}}(n\mathbb{Z}) = 0$, it follows that $\mathrm{fd}_{\mathbb{Z}}(\mathbb{Z} \bowtie n\mathbb{Z}) = 0 < \infty$.

For all $k \geq 1$,

$$\begin{aligned} {}^k_{\mathbb{Z}}(\mathbb{Z} \bowtie n\mathbb{Z}, \mathbb{Q}) &\cong^k_{\mathbb{Z}} (\mathbb{Z} \oplus n\mathbb{Z}, \mathbb{Q}) \\ &\cong^k_{\mathbb{Z}} (\mathbb{Z}, \mathbb{Q}) \oplus^k_{\mathbb{Z}} (n\mathbb{Z}, \mathbb{Q}) \\ &= 0, \quad \text{since } \mathbb{Q} \text{ is an injective } \mathbb{Z}\text{-module.} \end{aligned}$$

Moreover, since $\mathbb{Q}$ is injective over $\mathbb{Z}$, it is also strongly Gorenstein injective.

Hence, by Theorem 2.1(2), $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \bowtie n\mathbb{Z}, \mathbb{Q})$ is strongly Gorenstein injective over $\mathbb{Z} \bowtie n\mathbb{Z}$.

Example 2.7. Let k be a field and set $R := k[X,Y]$, and let $I := (X)$ be the ideal of R generated by X. Since I is free as an R-module, it is flat. Thus, for any strongly Gorenstein flat R-module N, the $(R \bowtie I)$-module $N \otimes_R (R \bowtie I)$ is strongly Gorenstein flat by Theorem 2.1(3).

3 Gorenstein global (weak) dimensions in an amalgamated duplication

Next, we compare the Gorenstein homological dimensions of R and $R \bowtie I$.

We begin by bounding the Gorenstein projective and injective dimensions over R by those over $R \bowtie I$.

Theorem 3.1. Let I be an ideal of R, and let M be an R-module. Then we have the following inequalities under the corresponding assumptions:

1. Assume that ${}^R_k(M, R \bowtie I) = 0$ for all $k \geq 1$, and that $\mathrm{Gpd}_R(M) < \infty$. Then

$$\mathrm{Gpd}_R(M) \leq \mathrm{Gpd}_{R \bowtie I}(M \otimes_R (R \bowtie I)).$$

2. Assume that ${}^k_R(R \bowtie I, M) = 0$ for all $k \geq 1$, and that $\mathrm{Gid}_R(M) < \infty$. Then

$$\mathrm{Gid}_R(M) \leq \mathrm{Gid}_{R \bowtie I}(\mathrm{Hom}_R(R \bowtie I, M)).$$

Proof. (1) Suppose that ${}^{k}_{R}(M,R\bowtie I)=0$ for all $k\geq 1$, and that $\mathrm{Gpd}_R(M)<\infty$. By [5, Proposition 4.1.3, p. 118], for any R-module P and every integer $k\geq 1$, we have

$${}^{k}_{R}(M,P\otimes_R(R\bowtie I))\cong^{k}_{R\bowtie I}(M\otimes_R(R\bowtie I),P\otimes_R(R\bowtie I)).$$

Assume that $\mathrm{Gpd}_{R\bowtie I}(M\otimes_R(R\bowtie I))\leq p$ for some integer $p\geq 0$. Let P be an arbitrary projective R-module. Then $P\otimes_R(R\bowtie I)$ is a projective $(R\bowtie I)$-module. Therefore, by [19, Theorem 2.20], for all $i>p$ we have

$$0=^{i}_{R\bowtie I}(M\otimes_R(R\bowtie I),P\otimes_R(R\bowtie I))\cong^{i}_{R}(M,P\otimes_R(R\bowtie I)).$$

Note that

$${}^{i}_{R}(M,P\otimes_R(R\bowtie I))\cong^{i}_{R}(M,P)\oplus^{i}_{R}(M,P\otimes_R I).$$

Hence, both summands must vanish, so in particular ${}^{i}_{R}(M,P)=0$ for every projective R-module P and for all $i>p$. Therefore,

$$\mathrm{Gpd}_R(M)\leq p.$$

Thus,

$$\mathrm{Gpd}_R(M)\leq\mathrm{Gpd}_{R\bowtie I}(M\otimes_R(R\bowtie I)).$$

(2) Assume that ${}^{k}_{R}(R\bowtie I,M)=0$ for all $k\geq 1$, and that $\mathrm{Gid}_R(M)<\infty$. By [5, Proposition 4.1.2, p. 117], for any $(R\bowtie I)$-module A and all integers $n\geq 0$, we have

$${}^{n}_{R}(A,M)\cong^{n}_{R\bowtie I}(A,\mathrm{Hom}_R(R\bowtie I,M)).$$

Assume that $\mathrm{Gid}_{R\bowtie I}(\mathrm{Hom}_R(R\bowtie I,M))\leq q$ for some integer $q\geq 0$. Let E be an arbitrary injective R-module. Then $\mathrm{Hom}_R(R\bowtie I,E)$ is an injective $(R\bowtie I)$-module by Hom–tensor adjointness, so

$${}^{i}_{R\bowtie I}(\mathrm{Hom}_R(R\bowtie I,E),\mathrm{Hom}_R(R\bowtie I,M))=0$$

for all $i>q$ by [19, Theorem 2.22]. By the above isomorphism with $A=\mathrm{Hom}_R(R\bowtie I,E)$, we get

$${}^{i}_{R}(\mathrm{Hom}_R(R\bowtie I,E),M)=0\quad\text{for all } i>q.$$

Since $R\bowtie I\cong R\oplus I$ as R-modules, we have

$$\mathrm{Hom}_R(R\bowtie I,E)\cong E\oplus\mathrm{Hom}_R(I,E),$$

and hence,

$${}^{i}_{R}(E\oplus\mathrm{Hom}_R(I,E),M)\cong^{i}_{R}(E,M)\oplus^{i}_{R}(\mathrm{Hom}_R(I,E),M)=0$$

for all $i>q$. Therefore, ${}^{i}_{R}(E,M)=0$ for every injective R-module E, which implies (by [19, Theorem 2.22]) that

$$\mathrm{Gid}_R(M)\leq q.$$

Thus,

$$\mathrm{Gid}_R(M)\leq\mathrm{Gid}_{R\bowtie I}(\mathrm{Hom}_R(R\bowtie I,M)).$$ □

Now we seek two equalities for Gorenstein flat and Gorenstein projective dimensions under the flatness of the ideal I. Similar equalities to those in Theorem 3.2 were established in [6] for the flat and projective dimensions.

Theorem 3.2. Let I be a non-zero flat ideal of a ring R. For any R-module M, we have:

(1) If R is coherent and $\mathrm{Gfd}_R(M)<\infty$, then

$$\mathrm{Gfd}_R(M)=\mathrm{Gfd}_{R\bowtie I}(M\otimes_R(R\bowtie I)).$$

(2) If $\mathrm{pd}_R(R\bowtie I)<\infty$ and $\mathrm{Gpd}_R(M)<\infty$, then

$$\mathrm{Gpd}_R(M)=\mathrm{Gpd}_{R\bowtie I}(M\otimes_R(R\bowtie I)).$$

Lemma 3.3. *[7, Proposition 2.14] Let S be an R-algebra of finite projective dimension. For every Gorenstein projective R-module M, the S-module $S\otimes_R M$ is Gorenstein projective.*

Proof. (1) Since I is flat, $R\bowtie I$ is faithfully flat over R by Lemma 2.2.

First, suppose $\mathrm{Gfd}_R(M)\le n$. By [19, Proposition 3.10] we have $\mathrm{Gfd}_{R\bowtie I}(M\otimes_R(R\bowtie I))\le \mathrm{Gfd}_R(M)\le n$.

Conversely, suppose $\mathrm{Gfd}_{R\bowtie I}(M\otimes_R(R\bowtie I))\le n$. Since $R\bowtie I$ is coherent and $\mathrm{Gfd}_{R\bowtie I}(M\otimes_R(R\bowtie I))\le n$, we get that for every injective $(R\bowtie I)$-module N, ${}^{R\bowtie I}_{k}(N,M\otimes_R(R\bowtie I))=0$ for all $k>n$. In particular, for any injective R-module E, the $(R\bowtie I)$-module $\mathrm{Hom}_R(R\bowtie I,E)$ is injective, so

$${}^{R\bowtie I}_{k}(\mathrm{Hom}_R(R\bowtie I,E),M\otimes_R(R\bowtie I))=0\quad\text{for all } k>n.$$

By [5, Proposition 4.1.2, p. 117], we have

$${}^{R}_{k}(E,M)\cong{}^{R\bowtie I}_{k}(\mathrm{Hom}_R(R\bowtie I,E),M\otimes_R(R\bowtie I))=0.$$

Therefore, $\mathrm{Gfd}_R(M)\le n$ by [19, Theorem 3.14].

(2) The proof is similar to (1) but uses [19, Theorem 3.10] . Since $\mathrm{pd}_R(R\bowtie I)<\infty$, for every Gorenstein projective R-module M, the $(R\bowtie I)$-module $M\otimes_R(R\bowtie I)$ is Gorenstein projective by *Lemma*3.3. The converse direction follows similarly using the fact that for any projective R-module P, the $(R\bowtie I)$-module $P\otimes_R(R\bowtie I)$ is projective and [5, Proposition 4.1.3, p. 118] instead of [5, Proposition 4.1.2, p. 117]. □

Corollary 3.4. *Let R be a ring and let I be a pure ideal of R. Then:*

1. *If R is coherent, then wG-$\mathrm{gldim}(R)\le wG$-$\mathrm{gldim}(R\bowtie I)$.*
2. *If $\mathrm{pd}_R(R\bowtie I)<\infty$, then G-$\mathrm{gldim}(R)\le G$-$\mathrm{gldim}(R\bowtie I)$.*

Proof. Let M be an R-module, and let I be a pure ideal of R. Since pure ideals are flat ([17, Lemma VI.9.1 (ii)]), under the stated assumptions we may apply Theorem 3.2 to obtain

$$\mathrm{Gfd}_R(M)\le \mathrm{Gfd}_{R\bowtie I}(M\otimes_R(R\bowtie I)).$$

and

$$\mathrm{Gpd}_R(M)\le \mathrm{Gpd}_{R\bowtie I}(M\otimes_R(R\bowtie I))$$

Taking the supremum over all R-modules M, we obtain the stated inequalities for the weak Gorenstein global dimensions and Gorenstein global dimensions. □

Example 3.5. Let $n\ge 1$ be an integer and set $R=\prod_{i=1}^{n}\mathbb{Z}$, and take I to be the ideal of R generated by the idempotent element $(1,0,0,\dots)$. Consider the amalgamated duplication ring $R\bowtie I$. Set $S=(R\bowtie I)[X_1,\dots,X_m]$ to be the polynomial ring in m indeterminates over $R\bowtie I$. Then

$$G\text{-}\mathrm{gldim}(S)\ge m+1.$$

Proof. Note that I is pure because it is generated by an idempotent. Moreover, by [4, Theorem 3.1], we have

$$G\text{-}\mathrm{gldim}\left(\prod_{i=1}^{n}\mathbb{Z}\right)=1,$$

since it is well known that G-$\mathrm{gldim}(\mathbb{Z})=1$. Moreover, R is coherent as a finite product of Noetherian rings. Thus, applying Theorem 3.2, we conclude that G-$\mathrm{gldim}(R\bowtie I)\ge 1$. Therefore, by [4, Theorem 2.1] implies that

$$G\text{-}\mathrm{gldim}(S)\ge m+1.$$ □

References

[1] M. El-arabi and Y. Najem, *On purity with respect to modules of flat dimension at most one*, Moroccan J. Algebra Geom. Appl. 2(2) (2023), 291-297.

[2] M. Auslander and M. Bridger, *Stable Module Theory*, Mem. Amer. Math. Soc. **94**, Amer. Math. Soc., Providence, RI, 1969.

[3] D. Bennis and N. Mahdou, *Strongly Gorenstein projective, injective, and flat modules*, J. Pure Appl. Algebra **210** (2007), 437–445.

[4] D. Bennis and N. Mahdou, *Global Gorenstein dimensions of polynomial rings and of direct product of rings*, Houston J. Math. **34**(1) (2008), 101–110.

[5] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, Princeton, 1956.

[6] M. Chhiti and N. Mahdou, *Homological dimensions of the amalgamated duplication of a ring along a pure ideal*, Afr. Diaspora J. Math. **10**(1) (2010), 1–6.

[7] L. W. Christensen, H. B. Foxby, H. Holm, Beyond totally reflexive modules and back, In: Commutative Algebra. Springer, New York, NY, (2011).

[8] M. D'Anna, *A construction of Gorenstein rings*, J. Algebra **306**(2) (2006), 507–519.

[9] M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. **45**(2) (2007), 241–252.

[10] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6**(3) (2007), 443–459.

[11] M. D'Anna, C.A. Finocchiaro and M. Fontana, *New algebraic properties of an amalgamated algebra along an ideal*, Commun. Algebra **44**(5) (2016), 1836–1851.

[12] Y. El Haddaoui, H. Kim and N. Mahdou, *On ϕ-Gorenstein homological dimensions*, Moroccan J. Algebra Geom. Appl. **1**(1) (2025), 1–22.

[13] Y. El Haddaoui, *On ϕ-Prufer and ϕ-Bezout rings in amalgamation algebra along an ideal*, Moroccan J. Algebra Geom. Appl. **3**(1) (2024), 178–186.

[14] A. El Khalfi, H. Kim and N. Mahdou, *Amalgamation extension in commutative ring theory: a survey*, Moroccan J. Algebra Geom. Appl. **1**(1) (2022), 139–182.

[15] E.E. Enochs and O.M.G. Jenda, *Relative Homological Algebra*, De Gruyter Expositions in Mathematics **30**, Walter de Gruyter, Berlin, 2000.

[16] E.E. Enochs, O.M.G. Jenda and B. Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan **10** (1993), 1–9.

[17] L. Fuchs and L. Salce, *Modules over Non-Noetherian Domains*, Mathematical Surveys and Monographs **84**, Providence, RI: American Mathematical Society 2001.

[18] S. Glaz, *Commutative Coherent Rings*, Lecture Notes in Mathematics **1371**, Springer-Verlag, Berlin, 1989.

[19] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), 167–193.

[20] N. Mahdou and M. Tamekkante, *On an amalgamated duplication of a ring along an ideal which are quasi-Frobenius*, Beitr. Algebra Geom. **52** (2011), 343–347.

[21] N. Mahdou and M. Tamekkante, *Gorenstein global dimension of an amalgamated duplication of a coherent ring along an ideal*, Mediterr. J. Math. **8**(3) (2011), 293–305.

[22] N. Mahdou and K. Ouarghi, *Gorenstein dimensions in trivial ring extensions*, in: Commutative Algebra and Applications, Walter de Gruyter, 2009, 293–302.

[23] N. Mahdou and M. Tamekkante, *On (weak) Gorenstein global dimensions*, Acta Math. Univ. Comenianae **82** (2013), 285–296.

[24] M. Salimi, E. Tavasoli and S. Yassemi, *The amalgamated duplication of a ring along a semidualizing ideal*, Rend. Sem. Mat. Univ. Padova **129** (2013), 115–127.