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Characterization of indices of quintic number fields defined by $x^5 + ax^4 + b$

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Abstract. Let K be a number field generated by a root of a monic irreducible polynomial of the form $x^5 + ax^4 + b \in \mathbb{Z}[x]$. In this paper, we compute explicitly the index $i(K)$ of the field K. We show that $i(K) \in \{1, 2, 4\}$, and we give necessary and sufficient conditions only on a and b to determine the exact value of $i(K)$. In particular, if $i(K) > 1$, then K is not monogenic.

Key Words: Index of a number field, Theorem of Ore, prime ideal factorization, common index divisor, Newton polygon, monogenity.
2020 MSC: Primary 11R04, 11R16, 11R21, 11Y40.

1 Introduction

Let α be a root of a monic irreducible polynomial $F(x)$ over $\mathbb{Q}$, $K := \mathbb{Q}(\alpha)$, and $\mathbb{Z}_K$ the ring of integers of K. A primitive element of $\mathbb{Z}_K$ is an element η verifying $K = \mathbb{Q}(\eta)$. Let $\widehat{\mathbb{Z}}_K$ be the set of all primitive elements of $\mathbb{Z}_K$. It is well known that $\mathbb{Z}_K$ is a free $\mathbb{Z}$-module. For any η in $\mathbb{Z}_K$, the index of $\mathbb{Z}[\eta]$ in $\mathbb{Z}_K$ is denoted by $(\mathbb{Z}_K : \mathbb{Z}[\eta])$. Let

$$i(K) := \gcd\{(\mathbb{Z}_K : \mathbb{Z}[\eta]) \mid \eta \in \widehat{\mathbb{Z}}_K\} \tag{1}$$

be the index of the field K. A rational prime p which divides $i(K)$ is called a prime common index divisor of K.

The investigation of indices of number fields has been an important problem in algebraic number theory since the 19th century. Many researchers treated it; especially, Dedekind [9], Hensel [25], and Hasse [23]. Let $I(x_2, ..., x_n)$ be an index form associated with an integral basis $\{1, \omega_2, ..., \omega_n\}$ of K. It is a decomposable form with coefficients in $\mathbb{Z}$. By the fundamental work of Hensel [24], a rational prime p is a common index divisor of K if and only if $I(x_2, \ldots, x_n) \equiv 0 \pmod{p}$, for every $x_2, ..., x_n$ in $\mathbb{Z}$.

In [18, 19, 21, 22], Győry made a general breakthrough by proving that an index form equation can have only finitely many integral solutions and giving effective bounds for the solutions. Also, he provided effective finiteness results for relative extensions (see [20, 21]). Evertse's and Győry's book [11] provided detailed surveys on the discriminant form and index form theory and their applications, including related Diophantine equations.

Recall that Problem 6 of Narkiewicz [28] is still an open problem looking for necessary and sufficient conditions for a field to have index 1. Likewise, Problem 22 of Narkiewicz [28] asks for an explicit formula for the highest power $\nu_p(i(K))$, of a given rational prime p dividing $i(K)$. Practically, Engstrom [10] showed in 1930 that, for number fields K of degree $n \le 7$, $\nu_p(i(K))$ is obtained from the decomposition of the ideal $p\mathbb{Z}_K$. Afterwards, Nart [29] determined $v_p(i(K))$ in totally ramified cases.

Many authors have studied the indices of different classes of number fields: for cubic number fields, see [1] by Bayad and Seddik, quartic number fields with Gaál, Pethő and Pohst [14], biquadratic number fields by Pethő and Ziegler [32], simplest quartic fields by Bayad and Seddik [2], and multiquadratic number fields by Pethő and Pohst [31]. In [3], Bayad and Seddik studied indices in simplest number fields with degrees less than 7.

Using the general approach of Győry [19], together with some computational methods, Gaál and Győry [13], Gaál and Győry and Bilu [6] described efficient algorithms applied to solve index form equations in the quintic and sextic case, respectively. For further results and efficient algorithms for different classes of number fields, see the books [11] by Evertse and Győry, and [12] by Gaál.

Actually, the interest in studying the indices of number fields defined by trinomials of the form $x^n + ax^m + b$ increases clearly. We remain for example that Davis and Spearman [8] studied the case $(n,m) = (4,1)$, Ben Yakkou studied the case $(n,m) = (7,2)$, and Ben Yakkou with Boudine [5] treated the case $(n,m) = (8,1)$.

In this paper, we consider an irreducible polynomial $F(x) = x^5 + ax^4 + b \in \mathbb{Z}[x]$, $K := \mathbb{Q}(\alpha)$, with $F(\alpha) = 0$. Using Newton polygon techniques, we solve Problems 6 and 22 of Narkiewicz [28] for these number fields. Indeed, we show that for any values of a and b, $i(K) \in \{1,2,4\}$, and we give sufficient and necessary conditions on a and b to decide whether $i(K) = 1$, $i(K) = 2$, or $i(K) = 4$. Note that in a recently published paper of Kchit [26], he provided some special cases when $i(K)$ is divisible by 2 or 3 where K is the number field defined by $x^5 + ax^4 + bx + c \in \mathbb{Z}[x]$ with $b \neq 0$. So, this class of number fields is completely different to the class considered in this paper. Furthermore, a particular advantage that distinguishes this paper is that we consider all possible cases not only some special cases. In the second section, we describe the methodology and necessary techniques that we will use to prove our results. In the third section, we show our main results and we prove them.

2 Preliminaries and methodology

In this section, we give some preliminaries and we explain the methodology.

Throughout the paper, we consider that $F(x) = x^5 + ax^4 + b$ is a monic irreducible polynomial over $\mathbb{Q}$, $K := \mathbb{Q}(\alpha)$, where α is a root $F(x)$, and $\mathbb{Z}_K$ the ring of integers of K. Without loss of the generality, we assume that for every rational prime p, we have

$$v_p(a) < 1 \,\text{or}\, v_p(b) < 5. \tag{2}$$

To explain this, suppose that $v_p(a) \geq 1$ and $v_p(b) \geq 5$. By the euclidean division, write $v_p(b) = 5q' + r$ with $0 \leq r \leq 4$. Let

$$q = \min\{v_p(a), q'\}, \quad \theta = \frac{\alpha}{p^q}, \quad A = \frac{a}{p^q}, \quad B = \frac{b}{p^{5q}},$$

and define $G(x) = x^5 + Ax^4 + B$. Then we have:

1. $G(x)$ is irreducible over $\mathbb{Q}$ because $G(x) = \frac{1}{p^{5q}}F(p^q x)$, and $F(x)$ was assumed to be irreducible.
2. $G(\theta) = 0$ and $K = \mathbb{Q}(\alpha) = \mathbb{Q}(\theta)$.
3. The coefficients A and B of $G(x)$ satisfy the condition $v_p(A) < 1$ or $v_p(B) < 5$. Indeed, if $q = v_p(a)$, then we get $v_p(A) = 0 < 1$. Else, we have $q = q'$, then $v_p(B) = r < 5$.

Therefore, replacing $F(x)$ by $G(x)$ yields the desired claim.

Since any prime common index divisor p of K divides $(\mathbb{Z}_K : \mathbb{Z}[\eta])$ for any $\eta \in \widehat{\mathbb{Z}}_K$, p divides $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$. Further, Dedekind [9] proved that

$$\Delta(F) = (\mathbb{Z}_K : \mathbb{Z}[\alpha])^2 D_K, \tag{3}$$

where D_K is the field discriminant of K, and $\Delta(F)$ is the discriminant of $F(x)$. It follows that p^2 divides $\Delta(F)$. However, the divisibility of $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ by a prime p, is not sufficient to prove that p is a prime common index divisor. Therefore, we need other tools. Let m be a positive integer, we denote $N_p(m)$ the number of distinct monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree m, and $L_p(m)$ the number of distinct prime ideals of $\mathbb{Z}_K$, that are factors of $p\mathbb{Z}_K$ with residue degree m (if there are no prime ideals with residue degree m lying above p, we set $L_p(m) = 0$). Then, we use the following lemma (see [28, Theorem 4.34]):

Lemma 2.1. *Suppose that p is a rational prime and K is a number field. Then p is a prime common index divisor of K if and only if $L_p(m) > N_p(m)$ for some positive integer m.*

In view of the central role of this lemma in our arguments, we include its proof. We first recall the well-known theorem of Dedekind on the splitting of primes in number fields (see [7, Theorem 4.8.13] or [28, Theorem 4.33]).

Lemma 2.2 (Theorem of Dedekind)**.** *Let $K = \mathbb{Q}(\eta)$ be a number field, with $\eta \in \mathbb{Z}_K$ having minimal polynomial $F_\eta(x)$. Let p be a rational prime, and suppose $\overline{F_\eta(x)} = \overline{\phi_1(x)}^{e_1} \cdots \overline{\phi_t(x)}^{e_t}$, where $\overline{\phi_i(x)}$ are distinct monic and irreducible polynomial in $\mathbb{F}_p[x]$. Suppose that p does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\eta])$, then $p\mathbb{Z}_K = \prod_{i=1}^t \mathfrak{p}_i^{e_i}$, where $\mathfrak{p}_i = p\mathbb{Z}_K + \phi_i(\eta)\mathbb{Z}_K$ are distinct prime ideals of $\mathbb{Z}_K$. Furthermore, the residual degree f_i of $\mathfrak{p}_i$ over p equals the degree of ϕ_i.*

With the above notation, for $\eta \in \widehat{\mathbb{Z}}_K$, $N_p(F_\eta, m, p)$ denotes the number of monic distinct factors of degree m of $\overline{F_\eta(x)}$ in $\mathbb{F}_p[x]$.

Proof of Lemma 2.1.
$\Rightarrow$) Assume that p is a common index divisor. Then p divides $[\mathbb{Z}_K : \mathbb{Z}[\eta]]$ for all $\eta \in \widehat{\mathbb{Z}}_K$. By Dedekind's theorem, if $p \nmid [\mathbb{Z}_K : \mathbb{Z}[\eta]]$, then for all positive integers m, we have

$$L_p(m) = N_p(F_\eta, m, p) \leq N_p(m),$$

Since, for every $\eta \in \widehat{\mathbb{Z}}_K$, Dedekind's correspondence fails when $p \mid i(K)$, the above inequality cannot hold for all $m \in \mathbb{N}^*$. Therefore, there must exist some m for which the number of distinct prime ideals lying above p with residue degree m exceeds the number of monic irreducible polynomials of degree m in $\mathbb{F}_p$; that is,

$$\exists\, m \in \mathbb{N}^* \quad \text{such that} \quad L_p(m) > N_p(m).$$

$\Leftarrow$) Assume $L_p(m) > N_p(m)$ for some positive integer m and prove that p is a prime common index divisor of K.
We have

$$p \nmid i(K) \implies \exists \eta \in \widehat{\mathbb{Z}}_K \text{ such that } p \nmid [\mathbb{Z}_K : \mathbb{Z}[\eta]] \implies L_p(m) \leq N_p(m), \quad \forall m \in \mathbb{N}^*.$$

By contraposition, the desired fact is proved.

□

In fact, the number $N_p(m)$ is known for every rational prime p and every positive integer m (see Proposition 4.35 of [28]). So the challenge here, is the computation of $L_p(m)$ for an adequate p and m; namely, we need the factorization of $p\mathbb{Z}_K$. To get it, Ore [30] has already developed a method based on Newton polygon techniques when $F(x)$ is p-regular. So let us recall some notions of Newton polygon techniques (for more details, see [16, 17, 27, 30].

Let p be a rational prime, v_p the p-adic valuation on $\mathbb{Q}_p$, and $\mathbb{F}_p$ the finite field with p element. For any polynomial $F(x) = \sum_{i=0}^{r} a_i x^i \in \mathbb{Z}_p[x]$, we define its valuation to be $v_p(F(x)) = \min\{v_p(a_i) \mid 0 \le i \le r\}$. Further, let $\phi(x)$ be a monic polynomial in $\mathbb{Z}[x]$, where $\overline{\phi(x)}$ is irreducible in $\mathbb{F}_p[x]$. So, the $\phi(x)$-expansion of $F(x)$ is of the form

$$F(x) = a_n(x)\phi(x)^n + a_{n-1}(x)\phi(x)^{n-1} + \cdots + a_1(x)\phi(x) + a_0(x),$$

where $a_i(x)$ is a polynomial such that $\deg(a_i(x)) < \deg(\phi(x))$ for each $i \in \{0,1,...,n\}$.

We consider the set $\mathfrak{S}_{\{F,\phi\}} = \{(i, v_p(a_i(x))) \mid a_i(x) \neq 0 \text{ and } 0 \le i \le n\}$. Then, the lower boundary convex envelope of $\mathfrak{S}_{\{F,\phi\}}$ is called the ϕ-Newton polygon of $F(x)$, denoted by $N_\phi(F)$. It is formed on some sides S_k with different slopes $\lambda_k \in \mathbb{Q}$, then we write $N_\phi(F) = S_1 + S_2 + \cdots + S_g$, where g is the number of its sides with distinct slopes. In particular, the ϕ-principal Newton polygon of $F(x)$, denoted by $N_\phi^+(F)$, is the polygon formed by the sides of $N_\phi(F)$ which have negative slopes.

Every side S of $N_\phi^+(F)$ joins two points: 1) the beginning point $(s, v_p(a_s(x)))$, and 2) the final point $(r, v_p(a_r(x)))$, where $0 \le s < r \le n$. In this case, the length of S is $l(S) := r - s$, its height is $h(S) := v_p(a_s(x)) - v_p(a_r(x))$, its slope is $\lambda = -\frac{h(S)}{l(S)} < 0$, its degree is $d(S) := \gcd(l(S), h(S))$, and its ramification index is $e(S) = \frac{l(S)}{d(S)}$. Let $d = d(S)$ and $e = e(S)$. Then, the residual polynomial of $F(x)$, associated with S, is

$$R_\lambda(F)(y) = c_{s+de}y^d + c_{s+(d-1)e}y^{d-1} + \cdots + c_{s+e}y + c_s \in \mathbb{F}_\phi[y] := \left(\mathbb{F}_p[x]/\langle\overline{\phi(x)}\rangle\right)[y],$$

where

$$c_i = \begin{cases} 0, & \text{if } (i, v_p(a_i(x))) \text{ lies strictly above } N_\phi^+(F), \\ \dfrac{a_i(x)}{p^{v_p(a_i(x))}} \bmod (p, \phi(x)), & \text{if } (i, v_p(a_i(x))) \text{ lies on } N_\phi^+(F). \end{cases}$$

The polynomial $F(x)$ is called ϕ-regular, if for each side S_λ of $N_\phi^+(F)$, $R_\lambda(F)(y)$ is separable in $\mathbb{F}_\phi[y]$. Further, $F(x)$ is called p-regular, if it is ϕ-regular, for every monic polynomial $\phi(x) \in \mathbb{Z}[x]$ such that $\overline{\phi(x)}$ is a monic irreducible factor of $\overline{F(x)}$ in $\mathbb{F}_p[x]$.

Now, we have all necessary tools to give Theorem of Ore (see [17]).

Theorem 2.3 (Theorem of Ore). With the above notations, suppose that

- $\overline{F(x)} = \prod_{i=1}^{t} \overline{\phi_i(x)}^{l_i}$ is the factorization of $\overline{F(x)}$ into a product of powers of distinct monic irreducible polynomials in $\mathbb{F}_p[x]$,
- $N_{\phi_i}^+(F) = S_{i1} + S_{i2} + \cdots + S_{ir_i}$ and λ_{ij} is the slope of S_{ij}, for each $i \in \{1,2,\ldots,t\}$ and $j \in \{1,2,\ldots,r_i\}$,
- $R_{\lambda_{ij}}(F)(y) = \prod_{s=1}^{s_{ij}} \psi_{ijs}(y)^{n_{ijs}}$ is the factorization of $R_{\lambda_{ij}}(F)(y)$ into a product of powers of distinct irreducible polynomials in $\mathbb{F}_{\phi_i}[x]$, for each $i \in \{1,2,\ldots,t\}$ and $j \in \{1,2,\ldots,r_i\}$.

If $F(x)$ is p-regular, then

$$p\mathbb{Z}_K = \prod_{i=1}^{t}\prod_{j=1}^{r_i}\prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}},$$

where e_{ij} is the ramification index of S_{ij}, and $\mathfrak{p}_{ijs}$ is a prime ideal of $\mathbb{Z}_K$ with residue degree $f_{ijs} = \deg(\phi_i(x)) \times \deg(\psi_{ijs}(x))$.

This Theorem provides a fantastic link between Newton polygons associated with $F(x)$ and the factorization of $p\mathbb{Z}_K$. This is exactly what we need to apply Lemma 2.1, so to prove that a given rational prime p is either a common index divisor of K or not. But, notice that the application of Theorem of Ore needs that $F(x)$ should be p-regular. However, in some cases, we do not obtain the p-regularity. Therefore, we use Newton polygons of second order as introduced by Guàrdia, Montes and Nart (see [16, 17]).

Example 2.4. Consider the polynomial $F(x) = x^5 + 3x^4 + 24$.
Since $F(x)$ is a 3-Eisenstein polynomial, it is irreducible over $\mathbb{Q}$. Let $K = \mathbb{Q}(\alpha)$, where α is a root of $F(x)$. The factorization of $F(x)$ in $\mathbb{F}_2[x]$ is $\overline{F(x)} = \overline{(x+1)} \cdot \overline{x}^4$. The x-principal Newton polygon of $F(x)$, $N_x^+(F) = S$ has a single side of degree 1 joining the points $(0,3)$ and $(4,0)$ with slope $\lambda = \frac{-3}{4}$ (see Figure 1). Thus, the residual polynomial $R_\lambda(F)(y) \in \mathbb{F}_x[y] \simeq \mathbb{F}_2[y]$ is separable as it is of degree 1. It follows that the polynomial $F(x)$ is x-regular. On the other hand, $F(x)$ is $x+1$-regular, since $v_{\overline{x+1}}(\overline{F(x)}) = 1$. Hence, $F(x)$ is 2-regular. By Theorem 2.3, we have

$$2\mathbb{Z}_K = \mathfrak{p}^4\mathfrak{q} \text{ with residue degrees } f(\mathfrak{p}/2) = f(\mathfrak{q}/2) = 1,$$

and

$$v_2((\mathbb{Z}_K : \mathbb{Z}[\alpha])) = \mathrm{ind}_x(F) + \mathrm{ind}_{x+1}(F) = 3 + 0 = 3.$$

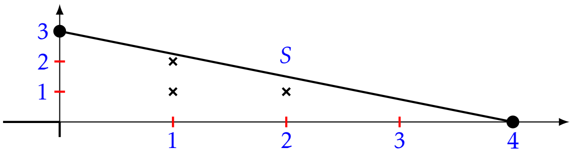

Figure 1: The x-principal Newton polygon of F with respect to v_2.

3 Main results

In this section, we give our main results and their proofs. We start by the following one.

Theorem 3.1. Let K be a number field generated by a root α of a monic irreducible polynomial $F(x) = x^5 + ax^4 + b \in \mathbb{Z}[x]$. Then the rational prime 3 is not a common index divisor of K; 3 does not divide $i(K)$.

Proof. First, we recall from Greenfield and Drucker [15] that

$$\Delta(F) = b^3(2^8a^5 + 5^5b). \tag{4}$$

By (3) and (4), if 3 is a common index divisor of K, then 9 divides $\Delta(F)$. It follows that either 3 divides b or 9 divides $2^8a^5 + 5^5b$. Also, by [7, Theorem 4.8.5], for any rational prime p, if $p\mathbb{Z}_K = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$ is the factorization of $p\mathbb{Z}_K$ into product of powers of distinct prime ideals in $\mathbb{Z}_K$ where $f(\mathfrak{p}_i/p) = [\mathbb{Z}_K/\mathfrak{p}_i : \mathbb{Z}/p\mathbb{Z}] = f_i$ is the residue degree of $\mathfrak{p}_i$ over p, then

$$\sum_{i=1}^{g} e_i f_i = 5 = \deg(K). \tag{5}$$

By (5) and Lemma 2.1, 3 divides $i(K)$ if and only if $L_3(1) \geq 4$. In this proof we distinguish three cases. [leftmargin=10pt]

- **Case 1**: suppose that 3 divides both a and b. By (2), we have $v_3(a) < 1$ or $v_3(b) < 5$. Since 3 divides a, $v_3(a) \geq 1$. It follows that $v_3(b) < 5$, that is $v_3(b) = 1, 2, 3, \text{or} 4$. In this case, we have $\overline{F(x)} = \overline{x^5}$ in $\mathbb{F}_3[x]$. Let $\phi_1(x) = x$. Then, $N^+_{\phi_1}(F)$ has only one side S_1 joining $(0, v_3(b))$ and $(5, 0)$, where its degree is $d(S_1) = 1$, its slope is $\lambda_1 = -\frac{v_3(b)}{5}$, and its residual polynomial $R_{\lambda_1}(F)(y)$ is linear. Therefore, by Theorem 2.3, there exists a unique prime ideal $\mathfrak{p}_1$ of $\mathbb{Z}_K$ with residue degree 1 such that $3\mathbb{Z}_K = \mathfrak{p}_1^5$. Thus 3 is not a common index divisor in this case.

- **Case 2**: suppose that 3 divides b and does not divide a. In this case, we have $\overline{F(x)} = \overline{x^4(x+a)}$ in $\mathbb{F}_3[x]$. Let $\phi_1(x) = x$ and $\phi_2(x) = x + a$. We get the ϕ_2-expansion of $F(x)$ as follows:

$$F(x) = \phi_2(x) - 4a\phi_2(x)^4 + 6a^2\phi_2(x) - 4a^3\phi_2(x) + a^4\phi_2(x) + b.$$

  Then $N^+_{\phi_2}(F)$ has only one side S_2 joining $(0, v_3(b)$ and $(1, 0)$, where its degree is $d(S_2) = 1$, its slope is $\lambda_2 = -v_3(b)$, and its residual polynomial $R_{\lambda_2}(F)(y)$ is linear. Therefore, $\phi_2(x)$ provides one prime ideal $\mathfrak{p}_2$ of $\mathbb{Z}_K$ with residue degree 1 such that $3\mathbb{Z}_K = \mathfrak{p}_2 \cdot \mathfrak{a}$, for an non zero ideal $\mathfrak{a}$ of $\mathbb{Z}_K$. On the other hand, $N^+_{\phi_1}(F)$ has only one side S_1 joining $(0, v_3(b))$ and $(4, 0)$, where its degree is $d(S_1) = \gcd(4, v_3(b)) \in \{1, 2, 4\}$, and its slope is $\lambda_1 = -\frac{v_3(b)}{4}$. To determine its attached residual polynomial $R_{\lambda_1}(F)(y)$, we need to distinguish three subcases:

  - **Case 2.1**: suppose that $\gcd(4, v_3(b)) = 1$; namely, $d(S_1) = 1$. In this case, $R_{\lambda_1}(F)(y)$ is linear. Therefore, $\phi_1(x)$ provides only one prime ideal $\mathfrak{p}_1$ of $\mathbb{Z}_K$ with residue degree 1. Thus, $3\mathbb{Z}_K = \mathfrak{p}_1^4 \cdot \mathfrak{p}_2$. So, $3 \nmid i(K)$.
  - **Case 2.2**: suppose that $\gcd(4, v_3(b)) = 2$; namely, $d(S_1) = 2$. Then, there exists a positive integer k and a rational integer b_3 not divisible by 3 such that $b = 3^{4k-2}b_3$. In this case, the residual polynomial $R_{\lambda_1}(F)(y)$ and the factorization of $3\mathbb{Z}_K$ (by applying Theorem 2.3) are obtained as follows:

$\overline{a}$ in $\mathbb{F}_3$	$\overline{b_3}$ in $\mathbb{F}_3$	$R_{\lambda_1}(F)(y)$	$3\mathbb{Z}_K$
$\overline{1}$	$\overline{1}$	$y^2 + \overline{1}$	$\mathfrak{q}_1^2 \cdot \mathfrak{p}_2$
$\overline{1}$	$\overline{2}$	$(y + \overline{1})(y + \overline{2})$	$\mathfrak{p}_{1,1}^2 \cdot \mathfrak{p}_{1,2}^2 \cdot \mathfrak{p}_2$
$\overline{2}$	$\overline{1}$	$\overline{2}(y + \overline{1})(y + \overline{2})$	$\mathfrak{p}_{1,1}^2 \cdot \mathfrak{p}_{1,2}^2 \cdot \mathfrak{p}_2$
$\overline{2}$	$\overline{2}$	$\overline{2}(y^2 + \overline{1})$	$\mathfrak{q}_1^2 \cdot \mathfrak{p}_2$

    where $\mathfrak{q}_1$ is a prime ideal of $\mathbb{Z}_K$ with residue degree 2, and $\mathfrak{p}_{1,1}$, $\mathfrak{p}_{1,2}$, and $\mathfrak{q}_2$ are prime ideals of $\mathbb{Z}_K$ with residue degree 1.
  - **Case 2.3**: suppose that $\gcd(4, v_3(b)) = 4$; namely, $d(S_1) = 4$. Then, there exists a positive integer k and a rational integer b_3 not divisible by 3 such that $b = 3^{4k}b_3$. In this case, the residual polynomial $R_{\lambda_1}(F)(y)$ and the factorization of $3\mathbb{Z}_K$ (by using Theorem 2.3) are obtained as follows:

$\overline{a}$ in $\mathbb{F}_3$	$\overline{b_3}$ in $\mathbb{F}_3$	$R_{\lambda_1}(F)(y)$	$3\mathbb{Z}_K$
$\overline{1}$	$\overline{1}$	$(y^2 + \overline{2}y + \overline{2})(y^2 + y + \overline{2})$	$\mathfrak{q}_1 \cdot \mathfrak{q}_2 \cdot \mathfrak{p}_2$
$\overline{1}$	$\overline{2}$	$(y^2 + \overline{1})(y + \overline{1})(y + \overline{2})$	$\mathfrak{q}_1 \cdot \mathfrak{p}_{1,1} \cdot \mathfrak{p}_{1,2} \cdot \mathfrak{p}_2$
$\overline{2}$	$\overline{1}$	$\overline{2}(y^2 + \overline{1})(y + \overline{1})(y + \overline{2})$	$\mathfrak{q}_1 \cdot \mathfrak{p}_{1,1} \cdot \mathfrak{p}_{1,2} \cdot \mathfrak{p}_2$
$\overline{2}$	$\overline{2}$	$\overline{2}(y^2 + \overline{2}y + \overline{2})(y^2 + y + \overline{2})$	$\mathfrak{q}_1 \cdot \mathfrak{q}_2 \cdot \mathfrak{p}_2$

    where $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are prime ideals of $\mathbb{Z}_K$ with residue degree 2, and $\mathfrak{p}_{1,1}, \mathfrak{p}_{1,2}$, and $\mathfrak{p}_2$ are prime ideals of $\mathbb{Z}_K$ with residue degree 1.

- **Case 3**: suppose that 3 divides neither b nor a. Then 3 divides $2^8a^5+5^5b$; namely, $a^2+2b \equiv 0 \pmod 3$. It follows that $(\overline{a},\overline{b}) \in \{(\overline{1},\overline{1}),(\overline{2},\overline{1})\}$ in $(\mathbb{F}_3)^2$. We distinguish two subcases:
  - **Case 3.1**: suppose that $a \equiv 1 \pmod 3$ and $b \equiv 1 \pmod 3$. Here, we get that $\overline{F(x)} = \overline{(x-1)^2(x^3-x+1)}$ in $\mathbb{F}_3[x]$. Let $\phi_1(x)=x-1$ and $\phi_2(x)=x^3-x+1$. The $\phi_2(x)$-expansion of $F(x)$ is given as follows:

    $$F(x)=(x^2+ax+1)\phi_2(x)+(a-1)x^2+(1-a)x+b-1.$$

    The principal Newton polygon $N^+_{\phi_2}(F)$ has only one side of degree 1. Therefore, $\phi_2(x)$ provides one prime ideal $\mathfrak{q}_2$ of $\mathbb{Z}_K$ with residue degree 3 such that $3\mathbb{Z}_K=\mathfrak{q}_2\cdot\mathfrak{a}$, for an non zero ideal $\mathfrak{a}$ of $\mathbb{Z}_K$. Since $\deg(K)=5$, by Equation (5) and Lemma 2.1, 3 does not divide $i(K)$.
  - **Case 3.2**: suppose that $a \equiv 2 \pmod 3$ and $b \equiv 1 \pmod 3$. In this case, we get that $\overline{F(x)} = \overline{x^5+2x^4+1}$, which is irreducible in $\mathbb{F}_3[x]$. Then $3\mathbb{Z}_K$ is a prime ideal of $\mathbb{Z}_K$ with residue degree 5.

The number of monic irreducible polynomials of degree 1 in $\mathbb{F}_3[x]$ is $N_3(1)=3$. However, in all possible cases, the number of distinct prime ideals of $\mathbb{Z}_K$ lying above 3 with residue degree 1 satisfies $L_3(1) \le 3 = N_3(1)$. Therefore, by Lemma 2.1, 3 is not a common index divisor of K. □

Żyliński [33] (see also Proposition 4.36 of [28]) proved that every prime common index divisor p of K verifies $p<n$, where n is the degree of K. This shows that every prime $p \ge 5$ is not a prime common index divisor of K. It follows by Theorem 3.1 that the unique possible prime divisor of $i(K)$ is 2. In the following result, we give explicitely the value of $i(K)$ in each case.

Theorem 3.2. Let $K=\mathbb{Q}(\alpha)$ and $F(x)$ be as in Theorem 3.1 Then, Table 1 gives the exact value of $i(K)$.

Conditions	$i(K)$
$a \equiv 1 \pmod 2$, $b=2^{4k-2}b_2$, for some positive integer $k>1$ and an odd integer b_2 verifying $b_2-a \equiv 0 \pmod 8$	2
$a \equiv 1 \pmod 2$, $b=2^{4k}b_2$, for some positive integer k and an odd integer b_2 verifying $3 \le v_2(a+b_2-2^k) \le 2v_2(5\times 2^k-4a)-2$, where $v_2(a+b_2-2^k)$ is an even integer.	2
$a \equiv 1 \pmod 2$, $b=2^{4k}b_2$, for some positive integer k and an odd integer b_2 verifying $v_2(a+b_2-2^k) \ge \max(3, 2v_2(5\times 2^k-4a))$	4
Otherwise	1

Table 1: The value of $i(K)$

Proof. By (3) and (4), if 2 is a common index divisor of K, then 2 divides b. Also, by (5) and Lemma 2.1, 2 divides $i(K)$ if and only if either $L_2(1) \ge 3$ or $L_2(2)=2$. We distinguish two cases:

[leftmargin=10pt]

- **Case 1:** suppose that 2 divides b, and 2 does not divide a. In this case, we get that $\overline{F(x)} = \overline{x^5+x^4=x^4(x+1)}$ in $\mathbb{F}_2[x]$. Let $\phi_1(x)=x$ and $\phi_2(x)=x-1$. The $\phi_2(x)$-expansion of $F(x)$ is given as follows:

  $$F(x)=\phi_2(x)^5+(5+a)\phi_2(x)^4+(10+4a)\phi_2(x)^3+(10+6a)\phi_2(x)^2+(5+4a)\phi_2+(1+a+b).$$

  Then, $N^+_{\phi_2}(F)$ has only one side of degree 1. Thus, by Theorem 2.3, there exists a prime ideal $\mathfrak{p}_2$ of $\mathbb{Z}_K$ with residue degree 1 such that $2\mathbb{Z}_K=\mathfrak{p}_2\cdot\mathfrak{a}$, for an non zero ideal $\mathfrak{a}$ of $\mathbb{Z}_K$. On the other hand, notice that $N_{\phi_1}^{+}(F)$ has only one side S_1, joining $(0, v_2(b))$ and $(4, 0)$. Then $l(S_1) = 4$, $h(S_1) = v_2(b)$, and $d(S_1) = \gcd(4, v_2(b))$. It follows that $d(S_1) \in \{1, 2, 4\}$. So we distinguish it into three cases:

- **Case 1.1:** suppose that $d(S_1) = 1$. Then, Theorem 2.3 of Ore shows that ϕ_1 provides a unique prime ideal $\mathfrak{p}_1$ of $\mathbb{Z}_K$ with residue degree 1 such that $2\mathbb{Z}_K = \mathfrak{p}_1^4 \cdot \mathfrak{p}_2$.
- **Case 1.2:** suppose that $d(S_1) = 2$. Then $v_2(b) = 4k-2$ and $b = 2^{4k-2}b_2$, for some positive integer k and an odd rational integer b_2. Then, $\lambda_1 = -\frac{2k-1}{2}$ and $R_{\lambda_1}(F)(y) = y^2 + 1 = (y + \overline{1})^2$. We analyse the Newton polygon of second order. Let $\Phi(x) = x^2 + 2^k x + 2^{2k-1}$ and the valuation of second order defined by (see [17, Proposition 2.7]): $V(\sum_{i=0}^{n} a_i x^i) = 2\min\{v_2(a_i) + i \times \frac{2k-1}{2}\}$. We have $V(\Phi(x)) = 4k-2$ and $V(x) = 2k-1$. Further, we have the Φ-expansion of $F(x)$ as follows

  $$F(x) = (x + a - 2^{k+1})\Phi(x)^2 + [2^{k+1}(2^k - a)x + 2^{3k}]\Phi(x) - 2^{4k-2}x + 2^{4k-2}(b_2 - a).$$

  So we obtain the following:

  $$\begin{cases} V((x + a - 2^{k+1})\Phi(x)^2) = 8k-4, \\ V([2^{k+1}(2^k - a)x + 2^{3k}]\Phi(x)) = 8k-2, \\ V(-2^{4k-2}x + 2^{4k-2}(b_2 - a)) = \min\{10k-5, 8k-4+2v_2(b_2-a)\}. \end{cases}$$

  Then, we get in the following table, the second order Newton polygon $N_{\Phi}^2(F)$, the associated residual polynomials $R^{(2)}(F)(y)$ in second order, and by applying [17, Theorems 3.1 and 3.4], we obtain the factorization of $2\mathbb{Z}_K$.

condition on k	condition on a and b_2	$N_{\Phi}^2(F)$	$R^{(2)}(F)(y)$	$2\mathbb{Z}_K$
$k = 1$		1 side	$y + 1$	$\mathfrak{p}_{1,1}^4 \cdot \mathfrak{p}_2$
$k > 1$	$v_2(b_2 - a) > 2$	2 sides	$y + 1$ for each side	$\mathfrak{p}_{1,1}^2 \cdot \mathfrak{p}_{1,2}^2 \cdot \mathfrak{p}_2$
$k > 1$	$v_2(b_2 - a) \leq 2$	1 side	$y^2 + y + 1$	$\mathfrak{q}^2 \cdot \mathfrak{p}_2$

  where $\mathfrak{p}_{1,1}$ and $\mathfrak{p}_{1,2}$ are two prime ideals of $\mathbb{Z}_K$ with residue degree 1, and $\mathfrak{q}$ is a prime ideal of $\mathbb{Z}_K$ with residue degree 2.
- **Case 1.3:** suppose that $d(S) = 4$. Then $v_2(b) = 4k$, for a positive integer k. Let $\psi_1(x) = x + 2^k$. Recall that for any rational prime p, Ore's Theorem (Theorem 2.3) does not depend on the monic irreducible liftings of the monic irreducible factors of $F(x)$ modulo p. So, let us replace $\phi_1(x)$ by $\psi_1(x)$. Then, the ψ_1-expansion of $F(x)$ is given as follows:

  $$\begin{aligned} F(x) &= \psi_1(x)^5 + (a - 5 \times 2^k)\psi_1(x)^4 + 2^{k+2}(5 \times 2^{k-1} - a)\psi_1(x)^3 + 2^{2k+1}(3a - 5 \times 2^k)\psi_1(x)^2 \\ &+ 2^{3k}(5 \times 2^k - 4a)\psi_1(x) + (2^{4k}a - 2^{5k} + b). \end{aligned}$$

  So, we obtain
  $$\begin{cases} v_2(a - 5 \times 2^k) = 0, \\ v_2(2^{k+2}(5 \times 2^{k-1} - a)) \geq k+2, \\ v_2(2^{2k+1}(3a - 5 \times 2^k)) = 2k+1, \\ v_2(2^{3k}(5 \times 2^k - 4a)) = 3k + v_2(5 \times 2^k - 4a), \\ v_2(2^{4k}a - 2^{5k} + b) = 4k + v_2(a + b_2 - 2^k). \end{cases}$$

  If $v_2(a + b_2 - 2^k) = 1$, then $N_{\psi_1}^{+}(F(x))$ has only one side S of degree 1. Thus, $\psi_1(x)$ provides only one prime ideal $\mathfrak{p}_1$ of $\mathbb{Z}_K$ with residue degree 1 such that $2\mathbb{Z}_K = \mathfrak{p}_1^4 \cdot \mathfrak{p}_2$. As well, if $v_2(a + b_2 - 2^k) = 2$, then $N_{\psi_1}^{+}(F)$ has only one side of degree 2, with a separable residual polynomial y^2+y+1. Thus, $\psi_1(x)$ provides only one prime ideal $\mathfrak{q}_1$ of $\mathbb{Z}_K$ with residue degree 2 such that $2\mathbb{Z}_K = \mathfrak{q}_1^2 \cdot \mathfrak{p}_2$. Now suppose that $v_2(a+b_2-2^k) \geq 3$, then we distinguish three subcases:

- ⋆ **Case 1.3.1:** suppose that $v_2(a+b_2-2^k) \geq 2v_2(5\times 2^k-4a)$. Then $N^+_{\psi_1}(F)$ has three distinct sides of degree 1 each. Thus $\psi_1(x)$ provides 3 prime ideals $\mathfrak{p}_{1,1}$, $\mathfrak{p}_{1,2}$, and $\mathfrak{p}_{1,3}$ of $\mathbb{Z}_K$ with residue degree 1, such that $2\mathbb{Z}_K = \mathfrak{p}_{1,1}^2 \cdot \mathfrak{p}_{1,2} \cdot \mathfrak{p}_{1,3} \cdot \mathfrak{p}_2$.
- ⋆ **Case 1.3.2:** suppose that $3 \leq v_2(a+b_2-2^k) = 2v_2(5\times 2^k-4a)-1$. Then $N^+_{\psi_1}(F)$ has two sides S_1 and S_2, such that $d(S_1)=1$ and $d(S_2)=2$, with separable residual polynomials $x+1$ and x^2+x+1. Thus, $2\mathbb{Z}_K = \mathfrak{q}_1 \cdot \mathfrak{p}_1^2 \cdot \mathfrak{p}_2$, where $\mathfrak{q}_1$ is a prime ideal of $\mathbb{Z}_K$ with residue degree 2, and $\mathfrak{p}_1$ is a prime ideal of $\mathbb{Z}_K$ with residue degree 1.
- ⋆ **Case 1.3.3:** suppose that $3 \leq v_2(a+b_2-2^k) \leq 2v_2(5\times 2^k-4a)-2$, where $v_2(a+b_2-2^k)$ is an even integer. Equivalently, $k=2$ and there exists a positive integer ℓ, such that

  $$\begin{cases} v_2(a+b_2-4) = 2\ell, \\ 2 \leq \ell \leq v_2(5-a). \end{cases}$$

  Then $N^+_{\psi_1}(F)$ has two sides of degree 1. Thus, there exist two prime ideals $\mathfrak{p}_{1,1}$ and $\mathfrak{p}_{1,2}$ of $\mathbb{Z}_K$ with residue degree 1 such that $2\mathbb{Z}_K = \mathfrak{p}_{1,1}^2 \cdot \mathfrak{p}_{1,2}^2 \cdot \mathfrak{p}_2$.

Case 2: suppose that 2 divides both a and b. This case is similar to **Case 1** in the proof of Theorem 3.1. So, we have $3\mathbb{Z}_K = \mathfrak{p}_1^5$, where $\mathfrak{p}_1$ is a prime ideal of $\mathbb{Z}_K$ with residue degree 1.

In all cases, we have $L_2(2) \leq 1$, and the condition $L_2(1) \geq 3$ holds only in the following cases:

1. **Case 1.2**, when $k>1$ and $v_2(b_2-a) > 2$. In this case, we have $2\mathbb{Z}_K = \mathfrak{p}_1^2 \cdot \mathfrak{p}_2^2 \cdot \mathfrak{p}_3$, with $f_i = 1$ for $i = 1,2,3$. By Engstrom's table (see [10, Page 234]), we see that $v_2(i(K)) = 1$.
2. **Case 1.3.1**. In this case, we have $2\mathbb{Z}_K = \mathfrak{p}_1^2 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3 \cdot \mathfrak{p}_4$, with $f_i = 1$ for $i = 1,2,3,4$. By the above-mentioned Engstrom's table, we get $v_2(i(K)) = 2$.
3. **Case 1.3.3**. Here, the form of $2\mathbb{Z}_K$ factorization is similar to **Case 1.2** above. Hence , $v_2(i(K)) = 1$.

□

Corollary 3.3. *Let $K = \mathbb{Q}(\alpha)$ and $F(x)$ be as in Theorem 3.1. Suppose that one of the following statements holds:*

- $a \equiv 1 \pmod 2$, $b = 2^{4k-2}b_2$, *for some positive integer $k>1$ and an odd rational integer b_2 verifying* $b_2 - a \equiv 0 \pmod 8$.
- $a \equiv 1 \pmod 2$, $b = 2^{4k}b_2$, *for some positive integer k and an odd rational integer b_2 verifying* $3 \leq v_2(a+b_2-2^k) \leq 2v_2(5\times 2^k-4a)-2$, *where $v_2(a+b_2-2^k)$ is an even integer.*
- $a \equiv 1 \pmod 2$, $b = 2^{4k}b_2$, *for some positive integer k and an odd rational integer b_2 verifying* $v_2(a+b_2-2^k) \geq \max(3, 2v_2(5\times 2^k-4a))$.

Then, K is non-monogenic.

Proof. It is sufficient to notice that $i(K) > 1$ in these cases. □

Examples 3.4. *To illustrate our results, we present some numerical examples. We consider irreducible polynomials $F(x)$ of the form x^5+ax^4+b and number fields K generated by a root of $F(x)$. We then provide explicit examples of such number fields whose indices are 1, 2, or 4.*

1. *Let* $F(x) = x^5 + 2^u p^w x^4 + 2^v p$*, where* u, v*, and* w *are positive integers and* p *is an odd rational prime. Then* $F(x)$ *is irreducible over* $\mathbb{Q}$*, as it satisfies the Eisenstein criterion at* p*. Moreover, by Theorems 3.1 and 3.2, we have* $i(K) = 1$*. For example, if* $F(x) = x^5 + 36x^4 + 6, x^5 + 628864x^4 + 4352,$ *or* $x^5 + 7023616x^4x^4 + 622593$*, then* $i(K) = 1$*.*

2. *Let* $F(x) = x^5 + 17x^4 + 1088$*. Then* $F(x)$ *is irreducible over* $\mathbb{Q}$ *since it is a 17-Eisenstein polynomial. According to the first condition in Table 1 of Theorem 3.2, we have* $a = 17$*,* $b_2 = 17$*,* $k = 2$*, and* $b = 1088$*. It follows that* $i(K) = 2$*. Consequently,* K *is non- monogenic.*

3. *Let* $F(x) = x^5 + ax^4 + 256b_2$ *be an irreducible polynomial over* $\mathbb{Q}$*, with* $(\overline{a}, \overline{b_2}) \in \{(\overline{19}, \overline{1}), (\overline{17}, \overline{3}), (\overline{15}, \overline{5}), (\overline{13}, \overline{7}), (\overline{11}, \overline{9}), (\overline{9}, \overline{11}), (\overline{7}, \overline{13}), (\overline{5}, \overline{15}), (\overline{3}, \overline{17}), (\overline{1}, \overline{19}), (\overline{31}, \overline{21}), (\overline{29}, \overline{23}), (\overline{27}, \overline{25}), (\overline{25}, \overline{27}), (\overline{23}, \overline{29}), (\overline{21}, \overline{31})\}$ *in* $(\mathbb{F}_{32})^2$*. These families satisfy Condition 2 in Table 1 of Theorem 3.2 for* $k = 2$*. It follows that* $i(K) = 2$*. Hence,* K *is non-monogenic.*

   In particular, when $(\overline{a}, \overline{b_2}) = (\overline{19}, \overline{1})$ *in* $(\mathbb{F}_{32})^2$*, the index of the number fields defined by the following polynomials equals 2:*

   $$x^5 + 627x^4 + 8448, \qquad x^5 + 1235x^4 + 16640 \qquad x^5 + 19922963x^4 + 268435712.$$

   Also, using the Eisenstein criterion together with Dirichlet's theorem on arithmetic progressions, we see that the following infinite families of polynomials

   $$F_h(x) = x^5 + (19+32h)p^u x^4 + 256p, F_u(x) = x^5 + (19+32h)p^u x^4 + 256p, F_p(x) = x^5 + (19+32h)p^u x^4 + 256p,$$

   where p *is a prime satisfying* $p \equiv 1 \pmod{32}$*,* u *is a positive integer, and* $h \in \mathbb{Z}$*, are irreducible over* $\mathbb{Q}$*. Moreover, the number fields generated by the roots of these polynomials have index 2.*

4. *Let* $F(x) = x^5 + ax^4 + 16b_2$ *be an irredutible polynomial over* $\mathbb{Q}$*, with* $(\overline{a}, \overline{b_2}) \in \{(\overline{1}, \overline{1}), (\overline{5}, \overline{5}), (\overline{3}, \overline{7}), (\overline{7}, \overline{3})\}$ *in* $(\mathbb{F}_8)^2$*. These families verify Condition 3 of Table 1 in Theorem 3.2 for* $k = 1$*, and thus* $i(K) = 4$*. In particular,* K *is non-monogenic.*
   For example, let p *be an odd rational prime with* $p \equiv 1 \pmod{8}$*, and let* h *and* k *be rational integers divisible by* p*. Consider the following infinite families of irreducible polynomials*

   $$F_{h,k}(x) = x^5 + (1+8k)p^u x^4 + 16p(1+8h),$$

   where u *is a positive integer. Here, we have* $a = (1+8k)p^u$ *and* $b_2 = p(1+8h)$ *satisfy* $(\overline{a}, \overline{b_2}) = (\overline{1}, \overline{1})$ *in* $(\mathbb{F}_8)^2$*. For each pair* (h, k)*, let* $K_{h,k} = \mathbb{Q}(\alpha_{h,k})$*, where* $\alpha_{h,k}$ *is a root of* $F_{h,k}(x)$*. Then* $i(K_{h,k}) = 4$*.*

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