

On weakly J -prime ideals of commutative rings

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Abstract. Let R be a commutative ring with identity, and $J(R)$ denotes the Jacobson radical of R . In this paper, we introduce and study the notion of weakly J -prime ideal. A proper ideal I of R is said to be weakly J -prime if for every $a, b \in R$, $0 \neq ab \in I$ implies $a \in I + J(R)$ or $b \in I$. We exhibit the relationship between the weakly J -prime ideal and the other related concepts such as J -prime ideal, weakly J -ideal, and weakly prime ideal. The main result of this work is a global characterization proving that every proper ideal of a ring R is weakly J -prime if and only if R is a quasi-local ring or isomorphic to the direct product of two fields. Furthermore, we investigate the behavior of weakly J -prime ideal property under some ring extensions such as homomorphic image, localization, direct product and power series. Finally, we provide a characterization of this property in trivial ring extension and amalgamated algebra.

Key Words: Weakly J -prime ideal, J -prime ideal, weakly prime ideal, weakly J -ideal, weakly n -ideal, weakly r -ideal, trivial ring extension, amalgamation algebra.

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1 Introduction

Throughout this paper, all rings are assumed to be commutative with a nonzero identity. We will use the standard notation $U(R)$, $\sqrt{0}$, $J(R)$, and $Z(R)$ to denote the set of units, the nilradical, the Jacobson radical, and the set of zero-divisors of a ring R , respectively. For any undefined terminology, we refer the reader to standard notations as in [10] and [18].

Classical ideals (e.g., prime, primary, maximal) form the bedrock of commutative algebra. Recent decades have witnessed a surge in the study of generalized ideals, often achieved by relaxing standard conditions to reveal deeper structural nuances. A highly successful paradigm in this area is the “weakly” generalization, which relaxes a defining property by specifically excluding the zero element. This approach was famously pioneered by Anderson and Smith in their foundational study of weakly prime ideals [5]. Recall that a proper ideal I of a ring R is weakly prime if for every $a, b \in R$, $0 \neq ab \in I$ implies $a \in I$ or $b \in I$. This approach has since been fruitfully extended to weakly primary

ideals where for every $a, b \in R$, $0 \neq ab \in I$ implies $a \in I$ or $b \in \sqrt{I}$ [9], weakly quasi-primary ideals [2], weakly n -ideals [17], and weakly r -ideals [25], see also [7, 12].

Concurrently, another active branch of literature has focused on ideals characterized by specific Jacobson radical conditions. Khashan and Bani-Ata [20] introduced J -ideals; a proper ideal I is said to be a J -ideal if for every $a, b \in R$, $ab \in I$ with $a \notin J(R)$ implies $b \in I$. This was subsequently generalized to weakly J -ideals by Khashan and Celikel [21], defined by the condition that for every $a, b \in R$, $0 \neq ab \in I$ with $a \notin J(R)$ implies $b \in I$. Very recently, to provide a natural bridge to the prime ideal hierarchy, Assalami et al. [8] introduced J -prime ideals. Recall that a proper ideal I of a ring R is said to be a J -prime ideal if for every $a, b \in R$, $ab \in I$ implies $a \in I + J(R)$ or $b \in I$.

In this paper, we introduce and study the notion of weakly J -prime ideal. A proper ideal I of R is said to be weakly J -prime if for every $a, b \in R$, $0 \neq ab \in I$ implies $a \in I + J(R)$ or $b \in I$. We exhibit the relationship between the weakly J -prime ideal and the other related concepts such as J -prime ideal, weakly J -ideal, and weakly prime ideal. The main result of this work is a global characterization proving that every proper ideal of a ring R is weakly J -prime if and only if R is a quasi-local ring or isomorphic to the direct product of two fields. Furthermore, we investigate the behavior of weakly J -prime ideal property under some ring extensions such as homomorphic image, localization, direct product and power series. Finally, we provide a characterization of this property in trivial ring extension and amalgamated algebra.

2 Main Results

Definition 2.1. Let R be a ring. A proper ideal I of R is said to be weakly J -prime if for every $a, b \in R$, $0 \neq ab \in I$ implies $a \in I + J(R)$ or $b \in I$.

Proposition 2.2. Let R be a ring. Then the following statements hold:

1. Every J -prime ideal of R is a weakly J -prime ideal.
2. Every weakly J -ideal (and hence every weakly n -ideal) of R is a weakly J -prime ideal.
3. Every weakly prime ideal of R is a weakly J -prime ideal.

Proof. 1. Clear.

2. Let I be a weakly J -ideal of R and consider $a, b \in R$ such that $0 \neq ab \in I$ and $a \notin I + J(R)$. Then we have $a \notin J(R)$. As I is a weakly J -ideal, this implies $b \in I$. Therefore, I is a weakly J -prime ideal of R . The rest follows from the fact that every weakly n -ideal is also a weakly J -ideal.
3. Let I be a weakly prime ideal of R and consider $a, b \in R$ such that $0 \neq ab \in I$ and $a \notin I + J(R)$. Since $I \subseteq I + J(R)$, it follows that $a \notin I$. As I is weakly prime, this implies $b \in I$. Therefore, I is a weakly J -prime ideal of R . □

We illustrate the relationship between the class of weakly J -prime ideals and the other related classes of ideals by the following diagram:

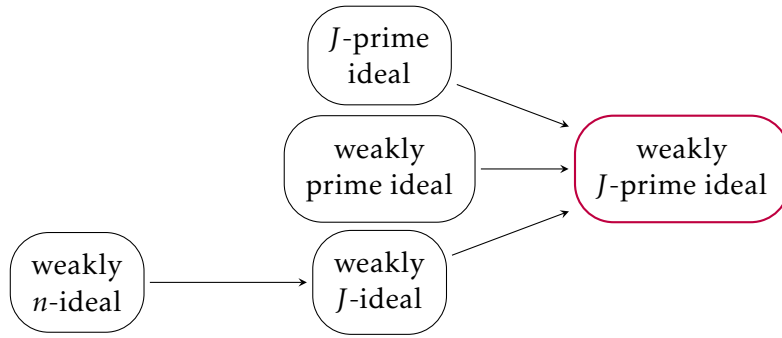


Figure 1: Weakly J -prime ideals vs other classical ideals

The following examples show that the converses of the statements in Proposition 2.2 do not hold in general.

Example 2.3 (Weakly J -prime ideal which is not J -prime).

Let $R = \mathbb{Z} \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Consider the ideal $I = \langle 0 \rangle \ltimes \langle (\bar{1}, \bar{0}) \rangle$ of R . Since $J(\mathbb{Z}) = \langle 0 \rangle$, we have $J(R) = J(\mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2) = \langle 0 \rangle \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Then:

1. I is a weakly J -prime ideal.
2. I is not a J -prime ideal.

Proof. (1) Since I is a weakly J -ideal of R by [21, Example 2.2], then by Proposition 2.2(2), I is a weakly J -prime ideal.

(2) Consider the elements $x = (2, (\bar{0}, \bar{0}))$ and $y = (0, (\bar{0}, \bar{1}))$ in R . Their product is

$$xy = (2 \cdot 0, 2(\bar{0}, \bar{1}) + 0(\bar{0}, \bar{0})) = (0, (\bar{0}, \bar{2})) = (0, (\bar{0}, \bar{0})) \in I.$$

Observe that $I + J(R) = \langle 0 \rangle \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$. We clearly have $x \notin I + J(R)$ and $y \notin I$. Thus, I is not a J -prime ideal. □

The following example shows that the converses of Proposition 2.2(2) and (3) are not true in general. We give an example of a weakly J -prime ideal that is neither weakly prime nor weakly J -ideal.

Example 2.4.

Let $R = \mathbb{Z}_{12}$ and consider the ideal $I = \langle \bar{4} \rangle$. Note that $J(R) = \langle \bar{6} \rangle$ and $I + J(R) = \langle \bar{4} \rangle + \langle \bar{6} \rangle = \langle \bar{2} \rangle$. Then:

1. I is a weakly J -prime ideal.
2. I is not a weakly J -ideal (and hence not a weakly n -ideal).
3. I is not a weakly prime ideal.

Proof. (1) Since I is a J -prime ideal of R by [8, Example 2.3], by Proposition 2.2(1), I is a weakly J -prime ideal.

(2) Observe that $0 \neq \bar{2} \cdot \bar{2} = \bar{4} \in I$ with $\bar{2} \notin J(R)$ but $\bar{2} \notin I$. Thus, I is not a weakly J -ideal.

(3) Observe that $0 \neq \bar{2} \cdot \bar{2} = \bar{4} \in I$ but $\bar{2} \notin I$. Thus, I is not a weakly prime ideal. □

Corollary 2.5. A proper ideal I of a ring R with $J(R) \subseteq I$ is weakly prime if and only if I is weakly J -prime.

As an application of Corollary 2.5, we provide another example of a weakly J -prime ideal that is not weakly prime. Consider the local ring $R = k[[x]]$, where k is a field. The ideal $I = (x^2)$ is weakly J -prime but not weakly prime in R .

Recall that a ring R is called semiprimitive or Jacobson semisimple if $J(R) = \langle 0 \rangle$, see [26]. To illustrate Corollary 2.5, the following example gives a class of rings in which every weakly J -prime ideal is a weakly prime.

Example 2.6. Let R be a semiprimitive ring and I an ideal of R , then I is weakly prime if and only if I is weakly J -prime.

Recall that a ring R is called présimplifiable if $Z(R) \subseteq J(R)$, see [4]. Next, we characterize rings in which $\langle 0 \rangle$ is a J -prime ideal, rings in which every r -ideal is a J -prime and rings in which every weakly J -prime ideal is a J -prime ideal.

Proposition 2.7. Let R be a ring. The following statements are equivalent:

1. R is a présimplifiable ring.
2. $\langle 0 \rangle$ is a J -prime ideal of R .
3. Every weakly J -prime ideal of R is a J -prime ideal.
4. Every weakly r -ideal of R is a J -prime ideal.

Proof. (1) \implies (3). Let R be a présimplifiable ring and let I be a weakly J -prime ideal of R . Suppose that $ab \in I$ for some $a, b \in R$. If $ab \neq 0$, then using the fact that I is weakly J -prime, it follows that $a \in I + J(R)$ or $b \in I$. Now, consider the case where $ab = 0$. If $b = 0$, then $b \in I$ is immediate. If $b \neq 0$, then a is a zero-divisor. Since R is présimplifiable, we have $a \in Z(R) \subseteq J(R) \subseteq I + J(R)$.

(3) \implies (2). Since $\langle 0 \rangle$ is weakly J -prime, hypothesis (3) implies that $\langle 0 \rangle$ is a J -prime ideal.

(1) \implies (4). Assume that R is a présimplifiable ring. Let I be a weakly r -ideal of R , and let $a, b \in R$ such that $ab \in I$. If $a \in Z(R)$, then $a \in J(R) \subseteq I + J(R)$. If $a \notin Z(R)$, then $\text{Ann}(a) = \langle 0 \rangle$. In this case, if $ab = 0$, the condition $\text{Ann}(a) = \langle 0 \rangle$ forces $b = 0 \in I$. Alternatively, if $ab \neq 0$, the fact that $0 \neq ab \in I$ and $\text{Ann}(a) = \langle 0 \rangle$ forces $b \in I$ as I is a weakly r -ideal.

(4) \implies (2). Since $\langle 0 \rangle$ is weakly r -ideal, hypothesis (4) implies that $\langle 0 \rangle$ is a J -prime ideal.

(2) \implies (1). This implication follows directly from [8, Proposition 2.6]. □

Corollary 2.8. Let R be an integral domain. Then every weakly J -prime ideal of R is a J -prime ideal.

Corollary 2.9. Let R be a quasi-local ring. Then every weakly J -prime ideal of R is a J -prime ideal.

Proposition 2.10. For a proper ideal I of R with $I \subseteq J(R)$, the following statements are equivalent:

1. I is a weakly J -prime ideal.
2. I is a weakly J -ideal.
3. Whenever $a, b \in R$ with $0 \neq ab \in I$, then $a \in J(I)$ or $b \in I$.

Proof. (1) \iff (2). Assume that I is a weakly J -prime ideal of R . Let $a, b \in R$ such that $0 \neq ab \in I$. If $a \notin J(R)$, then $a \notin J(R) = I + J(R)$ since $I \subseteq J(R)$. Because I is a weakly J -prime ideal, it follows that $b \in I$, and hence I is a weakly J -ideal. Conversely, suppose that I is a weakly J -ideal. Then by Proposition 2.2(2), I is a weakly J -prime ideal.

(2) \iff (3). This follows directly from [21, Proposition 2.4]. □

Proposition 2.11. *Let I be a proper ideal of a ring R such that $I + J(R) \subseteq \sqrt{0}$. Then I is a weakly J -prime ideal if and only if I is a weakly n -ideal.*

Proof. Suppose that I is a weakly J -prime ideal. Let $0 \neq ab \in I$ for some $a, b \in R$. Then $a \in I + J(R) \subseteq \sqrt{0}$ or $b \in I$. Hence, I is a weakly n -ideal. The converse is true by Proposition 2.2. \square

Recall from [23] that a ring R is a UN -ring if and only if $R = U(R) \cup \sqrt{0}$ if and only if R is local with maximal ideal $\sqrt{0}$. To illustrate Proposition 2.11, the following example gives a class of rings in which every weakly J -prime ideal is a weakly n -ideal.

Example 2.12. Let R be a UN -ring and I a proper ideal of R . Then $I + J(R) \subseteq \sqrt{0}$, hence by Proposition 2.11, I is a weakly J -prime ideal if and only if I is a weakly n -ideal.

Theorem 2.13. Let I be a weakly J -prime ideal of a ring R . If I is not a J -prime ideal, then $I^2 = \langle 0 \rangle$.

Proof. Suppose that $I^2 \neq \langle 0 \rangle$. We show that I is a J -prime ideal. Let $ab \in I$ for some $a, b \in R$. If $ab \neq 0$, the weakly J -prime condition gives $a \in I + J(R)$ or $b \in I$. Now assume that $ab = 0$. If $aI \neq \langle 0 \rangle$, say $ax \neq 0$ where $x \in I$, then $0 \neq ax = a(b+x) \in I$ and so $a \in I + J(R)$ or $b+x \in I$. The fact that $x \in I$ implies $a \in I + J(R)$ or $b \in I$. Similarly, if $bI \neq 0$, we obtain the same result. Finally, assume that $aI = 0$ and $bI = 0$. Since $I^2 \neq \langle 0 \rangle$, there exist $x, y \in I$ with $xy \neq 0$, then $0 \neq xy = (a+x)(b+y) \in I$, which yields $a+x \in I + J(R)$ or $b+y \in I$. As $x, y \in I$, it follows that $a \in I + J(R)$ or $b \in I$. Thus, I is a J -prime ideal. \square

Observe that an ideal I satisfies $I^2 = \langle 0 \rangle$ need not be a weakly J -prime. For instance, the ideal $I = \langle 0 \rangle \times 9\mathbb{Z}$ of $R = \mathbb{Z} \times \mathbb{Z}$ satisfies $I^2 = \langle 0 \rangle$. But, I is not a weakly J -prime ideal of R as $(0, 0) \neq (3, 0)(0, 3) \in I$ with $(3, 0) \notin I + J(R)$ and $(0, 3) \notin I$.

Corollary 2.14. *Let I be a weakly J -prime ideal of a ring R . that is not a J -prime ideal. Then the following assertions hold:*

1. $I \subseteq \sqrt{0}$.
2. If M is an R -module and $IM = M$, then $M = 0$.

Proof. (1) By Theorem 2.13, $I^2 = \langle 0 \rangle$. Thus, every element of I is nilpotent, yielding $I \subseteq \sqrt{0}$.

(2) If $IM = M$, then $M = IM = I^2M = 0$ by Theorem 2.13. \square

In particular, if R is a reduced ring, then by Corollary 2.14, I is weakly J -prime if and only if either $I = \langle 0 \rangle$ or I is J -prime.

Theorem 2.15. Let R be a ring. The following statements are equivalent:

1. Every proper ideal of R is weakly J -prime.
2. R is a quasi-local ring, or $R = F_1 \times F_2$ where F_1 and F_2 are fields.

Proof. (1) \implies (2). Suppose that every proper ideal of R is a weakly J -prime ideal, and assume that R is not a quasi-local ring. Then R must have at least two distinct maximal ideals, say M_1 and M_2 . Choose elements $x \in M_1 \setminus M_2$ and $y \in M_2 \setminus M_1$, and consider the principal ideal $I = \langle xy \rangle$. By hypothesis, I is a weakly J -prime ideal. Suppose, for the sake of contradiction, that $xy \neq 0$. Since $0 \neq xy \in I$, the weakly J -prime condition implies that either $x \in I + J(R)$ or $y \in I$. If $x \in I + J(R)$, noting that $I = \langle xy \rangle \subseteq M_2$ and $J(R) \subseteq M_2$, we deduce $x \in M_2$, which contradicts our choice of x . If $y \in I$, then $y \in \langle xy \rangle \subseteq M_1$, contradicting our choice of y . Therefore, we must have $xy = 0$. Since this holds for all $x \in M_1 \setminus M_2$ and $y \in M_2 \setminus M_1$, it forces the product $M_1 M_2 = \langle 0 \rangle$. By the Chinese Remainder Theorem, $R \cong R/M_1 \times R/M_2$, which means $R \cong F_1 \times F_2$ where F_1 and F_2 are fields.

(2) \implies (1). First, suppose that R is a quasi-local ring. Then by [8, Corollary 2.8(2)], every proper ideal I of R is a weakly J -prime ideal. Now, suppose that $R = F_1 \times F_2$, where F_1 and F_2 are fields. The only proper ideals of R are $I_0 = \langle(0, 0)\rangle$, $I_1 = F_1 \times \langle 0 \rangle$, and $I_2 = \langle 0 \rangle \times F_2$, all of which are easily verified to be weakly J -prime ideals. Hence, in either case, every proper ideal of R is weakly J -prime. \square

We illustrate Theorem 2.15 with the following examples:

Example 2.16. 1. Let p be a prime number and $n \geq 1$ an integer. Consider the ring $R = \mathbb{Z}_{p^n}$. Since R is a quasi-local ring with the unique maximal ideal $M = \langle \bar{p} \rangle$, then every proper ideal of \mathbb{Z}_{p^n} is a weakly J -prime ideal.

2. Let p and q be two distinct prime numbers. Consider the ring $R = \mathbb{Z}_{pq}$. By the Chinese Remainder Theorem, $R \cong \mathbb{Z}_p \times \mathbb{Z}_q$. Thus, by Theorem 2.15, every proper ideal of \mathbb{Z}_{pq} is a weakly J -prime ideal.

Corollary 2.17. Let R be an integral domain. Then every proper ideal of R is a weakly J -prime ideal if and only if R is a quasi-local domain.

Corollary 2.18. Let R be a Von Neumann regular ring. Then every proper ideal of R is a weakly J -prime ideal if and only if R is a field or $R \cong F_1 \times F_2$ where F_1 and F_2 are fields. In particular, if R is a Boolean ring, every proper ideal is a weakly J -prime ideal if and only if $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proposition 2.19. Let R be a ring. Then the following statements are equivalent:

1. R is Noetherian.
2. Every weakly J -prime ideal of R is finitely generated.

Proof. Assume that R is Noetherian. This is trivial, as every ideal is finitely generated. Conversely, assume that every weakly J -prime ideal of R is finitely generated. By Proposition 2.2, every prime ideal of R is a weakly J -prime ideal. Assume that every weakly J -prime ideal is finitely generated. In particular, every prime ideal is finitely generated. Then the result follows from [22, Theorem 2.8.6]. \square

It is well known that a proper ideal I of R is weakly prime if and only if $(I : a) = I \cup (0 : a)$ for all $a \in R \setminus I$, see [5, Theorem 3]. We now extend this fact to weakly J -prime ideals.

Theorem 2.20. Let R be a ring and I a proper ideal of R . The following statements are equivalent:

1. I is a weakly J -prime ideal.
2. $(I : a) = I \cup (0 : a)$ for every $a \in R \setminus (I + J(R))$.
3. $(I : a) = I$ or $(I : a) = (0 : a)$ for every $a \in R \setminus (I + J(R))$.
4. For any ideals J, K of R , $0 \neq JK \subseteq I$ with $J \not\subseteq I + J(R)$ implies $K \subseteq I$.
5. $(I : b) \subseteq (I + J(R)) \cup (0 : b)$ for every $b \in R \setminus I$.

Proof. (1) \implies (2). Let $a \in R \setminus (I + J(R))$. The inclusion $I \cup (0 : a) \subseteq (I : a)$ is trivial. Let $x \in (I : a)$. Then $ax \in I$. If $ax \neq 0$, since I is a weakly J -prime ideal and $a \notin I + J(R)$, then $x \in I$. If $ax = 0$, then $x \in (0 : a)$. Therefore, $(I : a) \subseteq I \cup (0 : a)$, yielding the equality.

(2) \implies (3). Let $a \in R \setminus (I + J(R))$. Since $(I : a) = I \cup (0 : a)$ and it is well known that no ideal is the union of two ideals, it follows that $(I : a) = I$ or $(I : a) = (0 : a)$.

(3) \implies (1). Let $a, b \in R$ such that $0 \neq ab \in I$. Assume that $a \notin I + J(R)$. We must show that $b \in I$. From assumption, since $a \notin I + J(R)$, we have either $(I : a) = I$ or $(I : a) = (0 : a)$. Using the fact that

$ab \in I$, we know $b \in (I : a)$. Furthermore, since $ab \neq 0$, it is clear that $b \notin (0 : a)$. Then $(I : a) = I$. Therefore, $b \in I$, making I , a weakly J -prime ideal.

(1) \implies (4). Assume that $K \not\subseteq I$. For any $z \in J \setminus (I + J(R))$, we claim $zK = \langle 0 \rangle$. Indeed, if $zk \neq 0$ for some $k \in K$, then $0 \neq zk \in I$ implies $k \in I$. For any $k' \in K$ with $zk' = 0$, $z(k + k') = zk \neq 0$ implies $k + k' \in I$, forcing $k' \in I$. Thus $K \subseteq I$, a contradiction. Hence $zK = \langle 0 \rangle$. Now, fix $x \in J \setminus (I + J(R))$, so $xK = \langle 0 \rangle$. Since $JK \neq \langle 0 \rangle$, there exists $y \in J$ with $yK \neq \langle 0 \rangle$. By our claim, $y \in I + J(R)$. Consequently, $x + y \in J \setminus (I + J(R))$ and $(x + y)K = xK + yK = yK \neq \langle 0 \rangle$, contradicting the claim. Therefore, $K \subseteq I$.

(4) \implies (1). Let $a, b \in R$ such that $0 \neq ab \in I$. Apply (4) by taking the principal ideals $J = \langle a \rangle$ and $K = \langle b \rangle$. If $a \notin I + J(R)$, then $J \not\subseteq I + J(R)$, which implies $K = \langle b \rangle \subseteq I$, and so $b \in I$. Thus, I is a weakly J -prime ideal.

(1) \implies (5). Let $b \in R \setminus I$ and pick $x \in (I : b)$. Then $xb \in I$. If $xb = 0$, then $x \in (0 : b)$. If $xb \neq 0$, the weakly J -prime condition and the fact that $b \notin I$ forces $x \in I + J(R)$. Thus, $(I : b) \subseteq (I + J(R)) \cup (0 : b)$.

(5) \implies (1). Let $0 \neq ab \in I$ for some $a, b \in R$. Assume $b \notin I$. Then $a \in (I : b)$. From assumption, it follows that $a \in I + J(R)$ as $ab \neq 0$. Therefore, I is a weakly J -prime ideal. \square

Proposition 2.21. *Let S be a non-empty subset of R . If I and $(0 : S)$ are weakly J -prime ideals of R where $S \not\subseteq I$, then $(I : S)$ is a weakly J -prime ideal.*

Proof. Since $S \not\subseteq I$, $(I : S)$ is a proper ideal. Let $0 \neq ab \in (I : S)$ with $a \notin (I : S) + J(R)$. Since $I \subseteq (I : S)$ and $(0 : S) \subseteq (I : S)$, we have $a \notin I + J(R)$ and $a \notin (0 : S) + J(R)$. The condition $ab \in (I : S)$ implies $bS \subseteq (I : a)$. Since I is weakly J -prime and $a \notin I + J(R)$, Theorem 2.20(3) yields $(I : a) = I$ or $(I : a) = (0 : a)$. If $(I : a) = I$, then $bS \subseteq I$, so $b \in (I : S)$. If $(I : a) = (0 : a)$, then $abS = \langle 0 \rangle$, meaning $0 \neq ab \in (0 : S)$. Since $(0 : S)$ is weakly J -prime and $a \notin (0 : S) + J(R)$, it follows that $b \in (0 : S) \subseteq (I : S)$. Hence, $(I : S)$ is a weakly J -prime ideal. \square

Proposition 2.22. *Let R be a ring and let K, I_1, I_2, I be ideals of R . Then the following assertions hold:*

1. *If I_1 and I_2 are weakly J -prime ideals of R with $\langle 0 \rangle \neq I_1K = I_2K$ and $K \not\subseteq I_i + J(R)$ for all $i \in \{1, 2\}$, then $I_1 = I_2$.*
2. *If I is an ideal such that IK is a weakly J -prime ideal with $IK \neq \langle 0 \rangle$ and $K \not\subseteq I + J(R)$, then $IK = I$.*

Proof. (1) Since $0 \neq I_2K = I_1K \subseteq I_1$ and I_1 is a weakly J -prime ideal with $K \not\subseteq I_1 + J(R)$, it follows from Theorem 2.20(4) that $I_2 \subseteq I_1$. By symmetry, we also have $I_1 \subseteq I_2$, which implies $I_1 = I_2$.

(2) The inclusion $IK \subseteq I$ is trivial. For the reverse inclusion, using the fact that $K \not\subseteq I + J(R)$ and $IK \subseteq I$, it follows that $K \not\subseteq IK + J(R)$. Furthermore, we have $\langle 0 \rangle \neq IK \subseteq IK$. Since IK is a weakly J -prime ideal and $K \not\subseteq IK + J(R)$, Theorem 2.20(4) implies that $I \subseteq IK$. Therefore, $I = IK$. \square

It is well established in the literature that the intersection of a nonempty family of weakly J -ideals (resp., weakly n -ideals) is again a weakly J -ideal (resp., weakly n -ideal) [21, Proposition 2.11], [17, Proposition 3]. However, the weakly J -prime ideal property is not stable by the intersection property, as shown in the following example.

Example 2.23. Let $R = \mathbb{Z}$, the prime ideals $I_1 = \langle 2 \rangle$ and $I_2 = \langle 3 \rangle$ are weakly J -prime, but $I_1 \cap I_2 = \langle 6 \rangle$ is not. Indeed, $0 \neq 2 \cdot 3 \in \langle 6 \rangle$, yet $2, 3 \notin \langle 6 \rangle$. Since $J(\mathbb{Z}) = 0$, it follows that $2 \notin \langle 6 \rangle + J(\mathbb{Z})$ and $3 \notin \langle 6 \rangle$, showing that the ideal $\langle 6 \rangle$ fails to be weakly J -prime.

Recall that two ideals I and J in a ring R are said to be coprime (or comaximal) if their sum is the entire ring R , see [24].

Proposition 2.24. *Let R be a ring and let I, J be coprime proper ideals of R . If $I \cap J$ is a weakly J -prime ideal and $(I \cap J)^2 \neq \langle 0 \rangle$, then I and J are J -prime ideals.*

Proof. Suppose that $(I \cap J)^2 \neq \langle 0 \rangle$. Since $I \cap J$ is a weakly J -prime ideal, it follows from Theorem 2.13 that $I \cap J$ is a J -prime ideal. Given that I and J are coprime and $I \cap J$ is J -prime, [8, Proposition 2.15] directly implies that both I and J are J -prime ideals. \square

The following example shows the necessity of the condition $(I \cap J)^2 \neq \langle 0 \rangle$ in Proposition 2.24.

Example 2.25. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_6$ and consider the proper ideals $I = \mathbb{Z}_2 \times \langle \bar{0} \rangle$ and $J = \langle \bar{0} \rangle \times \mathbb{Z}_6$. Since $I + J = R$, the ideals I and J are coprime and $I \cap J = \langle (\bar{0}, \bar{0}) \rangle$ is a weakly J -prime ideal. Here, $(I \cap J)^2 = \langle (\bar{0}, \bar{0}) \rangle$, yet I is not a J -prime ideal. Indeed, since $J(R) = \langle (\bar{0}, \bar{0}) \rangle$, we have $I + J(R) = I$. For $a = (\bar{1}, \bar{2})$ and $b = (\bar{1}, \bar{3})$ in R , we have $0 \neq ab = (\bar{1}, \bar{0}) \in I$, but $a \notin I + J(R)$ and $b \notin I$. This highlights the necessity of the condition $(I \cap J)^2 \neq \langle 0 \rangle$.

Theorem 2.26. Let $R = R_1 \times R_2$ where R_1 and R_2 are rings, and let I be a non-zero proper ideal of R . Then the following assertions are equivalent:

1. I is a weakly J -prime ideal of R .
2. $I = I_1 \times I_2$ where I_1 is a J -prime ideal of R_1 and $I_2 = R_2$, or $I_1 = R_1$ and I_2 is a J -prime ideal of R_2 .
3. I is a J -prime ideal of R .

Proof. (1) \implies (3). Let $I = I_1 \times I_2$ be a non-zero weakly J -prime ideal of R . Assume by the way of a contradiction that I_1 and I_2 are proper ideals in R_1 and R_2 , respectively. Then $I_2 + J(R_2)$ is a proper ideal of R_2 . Choose $0 \neq (a, b) \in I$. We have $(0, 0) \neq (a, 1)(1, b) = (a, b) \in I$. Then $(a, 1) \in I + J(R)$ or $(1, b) \in I$, which is a contradiction. Thus, we may assume without loss of generality that $I_1 \neq R_1$ and $I_2 = R_2$, which implies $I^2 \neq \langle 0, 0 \rangle$. By Theorem 2.13, since I is weakly J -prime and $I^2 \neq \langle 0, 0 \rangle$, I must be a J -prime ideal of R .

(3) \implies (1). This holds by Proposition 2.2(1).

(3) \iff (2). This follows directly from [8, Theorem 2.21]. \square

Theorem 2.26 establishes that for non-zero proper ideals in a product of two rings, the concepts of weakly J -prime and strictly J -prime coincide entirely. The following example shows that this equivalence breaks down for the zero ideal, highlighting the necessity of the non-zero hypothesis.

Example 2.27. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_9$. Then $J(R) = \langle \bar{2} \rangle \times \langle \bar{3} \rangle$. Consider $I = \langle \bar{0} \rangle \times \langle \bar{0} \rangle$ which is weakly J -prime. However, I is not a J -prime ideal. Consider the elements $a = (\bar{2}, \bar{1})$ and $b = (\bar{2}, \bar{0})$ in R . We have $ab = (\bar{0}, \bar{0}) \in I$. Yet, $a \notin I + J(R)$ and $b \notin I$.

We now generalize Theorem 2.26 to a finite product of n rings, maintaining the same three-way equivalence.

Proposition 2.28. Let R_1, R_2, \dots, R_n be rings, where $n \geq 2$, and

$$R = R_1 \times R_2 \times \cdots \times R_n \text{ and } I = I_1 \times I_2 \times \cdots \times I_n,$$

where I is a non-zero proper ideal of R . Then the following statements are equivalent:

1. I is a weakly J -prime ideal of R .
2. I_j is a J -prime ideal of R_j for some $j \in \{1, 2, \dots, n\}$ and $I_i = R_i$ for every $i \neq j$.
3. I is a J -prime ideal of R .

Proof. (1) \implies (2): We proceed by induction on n . For $n = 2$, the claim follows from Theorem 2.26. Assume the claim holds for all $k \leq n - 1$. For n , define

$$I' = I_1 \times I_2 \times \cdots \times I_{n-1} \quad \text{and} \quad R' = R_1 \times R_2 \times \cdots \times R_{n-1}.$$

Then, by Theorem 2.26, the fact that the ideal $I = I' \times I_n$ is weakly J -prime in $R = R' \times R_n$ implies that either I' is J -prime in R' and $I_n = R_n$, or $I' = R'$ and I_n is J -prime in R_n . The induction hypothesis completes the proof.

(2) \implies (3). This follows directly from [8, Proposition 2.23].

(3) \implies (1). This holds by Proposition 2.2(1). \square

To understand the behavior of weakly J -prime ideals under ring homomorphisms, we use the following result stating that if $f : R_1 \rightarrow R_2$ is a surjective ring homomorphism, then $f(J(R_1)) \subseteq J(R_2)$ (see, for example, [20, Lemma 2.22]).

Proposition 2.29. *Let $f : R_1 \rightarrow R_2$ be a ring homomorphism. Then the following statements hold:*

1. *If f is a monomorphism and I_2 is a weakly J -prime ideal of R_2 such that $f^{-1}(I_2 + J(R_2)) \subseteq f^{-1}(I_2) + J(R_1)$, then $f^{-1}(I_2)$ is a weakly J -prime ideal of R_1 .*
2. *If f is an epimorphism and I_1 is a weakly J -prime ideal of R_1 with $\text{Ker } f \subseteq I_1$, then $f(I_1)$ is a weakly J -prime ideal of R_2 .*

Proof. (1) Let $a, b \in R_1$ such that $0 \neq ab \in f^{-1}(I_2)$. Since f is a monomorphism, $\text{Ker } f = \langle 0 \rangle$, which implies $0 \neq f(ab) = f(a)f(b) \in I_2$. Hence, $f(b) \in I_2$ or $f(a) \in I_2 + J(R_2)$. If $f(b) \in I_2$, then $b \in f^{-1}(I_2)$. If $f(a) \in I_2 + J(R_2)$, then $a \in f^{-1}(I_2 + J(R_2)) \subseteq f^{-1}(I_2) + J(R_1)$. Thus, $f^{-1}(I_2)$ is a weakly J -prime ideal of R_1 .

(2) Let $c, d \in R_2$ such that $0 \neq cd \in f(I_1)$. Since f is surjective, there exist $x, y \in R_1$ such that $f(x) = c$ and $f(y) = d$. Thus, $0 \neq f(xy) \in f(I_1)$. Using the fact that $\text{Ker } f \subseteq I_1$, it follows that $0 \neq xy \in I_1$. Since I_1 is a weakly J -prime ideal, we have $x \in I_1 + J(R_1)$ or $y \in I_1$. If $y \in I_1$, then $d = f(y) \in f(I_1)$. If $x \in I_1 + J(R_1)$, then $x = i + j$ for some $i \in I_1$ and $j \in J(R_1)$. By [20, Lemma 2.22], $f(J(R_1)) \subseteq J(R_2)$, which implies $c = f(x) = f(i) + f(j) \in f(I_1) + J(R_2)$. Consequently, $f(I_1)$ is a weakly J -prime ideal of R_2 . \square

The following example shows that the additional condition $f^{-1}(I_2 + J(R_2)) \subseteq f^{-1}(I_2) + J(R_1)$ in Proposition 2.29(1) is necessary. A monomorphism alone is insufficient to guarantee that the inverse image of a weakly J -prime ideal remains weakly J -prime.

Example 2.30. Let $R_1 = \mathbb{Z}$, $R_2 = \mathbb{Z}_{(p)}$ for some prime p , and let $f : \mathbb{Z} \rightarrow \mathbb{Z}_{(p)}$ be the natural monomorphism. Note that $J(R_1) = \langle 0 \rangle$. Since R_2 is a quasi-local ring with unique maximal ideal $p\mathbb{Z}_{(p)}$, we have $J(R_2) = p\mathbb{Z}_{(p)}$. Consider the ideal $I_2 = p^2\mathbb{Z}_{(p)}$ of R_2 . By Theorem 2.15, I_2 is a weakly J -prime ideal. However, $I_1 = f^{-1}(I_2) = p^2\mathbb{Z}$ is not a weakly J -prime ideal of R_1 . Indeed, $0 \neq p \cdot p \in I_1$, but $p \notin I_1 + J(R_1) = p^2\mathbb{Z}$. This shows that the condition in Proposition 2.29(1) is necessary, as:

$$f^{-1}(I_2 + J(R_2)) = f^{-1}(p\mathbb{Z}_{(p)}) = p\mathbb{Z},$$

$$f^{-1}(I_2) + J(R_1) = p^2\mathbb{Z} + \langle 0 \rangle = p^2\mathbb{Z},$$

and clearly $p\mathbb{Z} \not\subseteq p^2\mathbb{Z}$.

As an immediate consequence of Proposition 2.29(2) applied to the natural epimorphism, we obtain the following explicit result for quotient rings.

Corollary 2.31. *Suppose that I is a weakly J -prime ideal of R containing an ideal K of R . Then I/K is a weakly J -prime ideal of R/K .*

Let $S^{-1}R$ denote the localization of R at a multiplicatively closed subset S of R . If I is an ideal of R , then $S^{-1}I = \{\frac{a}{s} : s \in S, a \in I\}$ is an ideal of $S^{-1}R$. In the following, the notation $Z_I(R)$ denotes the set of $\{r \in R \mid rs \in I \text{ for some } s \in R \setminus I\}$.

Proposition 2.32. *Let R be a ring and let S be a multiplicatively closed subset of R such that $J(S^{-1}R) = S^{-1}J(R)$.*

1. *If I is a weakly J -prime ideal of R with $I \cap S = \emptyset$, then $S^{-1}I$ is a weakly J -prime ideal of $S^{-1}R$.*
2. *If $S^{-1}I$ is a weakly J -prime ideal of $S^{-1}R$ and $S \cap Z(R) = S \cap Z_I(R) = S \cap Z_{I+J(R)}(R) = \emptyset$, then I is a weakly J -prime ideal of R .*

Proof. (1) Let $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ such that $0 \neq \frac{a}{s} \frac{b}{t} \in S^{-1}I$. Then there exists $u \in S$ such that $0 \neq uab \in I$. Since I is weakly J -prime, either $ua \in I + J(R)$ or $b \in I$. If $ua \in I + J(R)$, then $\frac{a}{s} = \frac{ua}{us} \in S^{-1}(I + J(R)) = S^{-1}I + S^{-1}J(R) = S^{-1}I + J(S^{-1}R)$. If $b \in I$, then $\frac{b}{t} \in S^{-1}I$. Consequently, $S^{-1}I$ is weakly J -prime.

(2) Let $a, b \in R$ such that $0 \neq ab \in I$. Since $S \cap Z(R) = \emptyset$, then $0 \neq \frac{a}{1} \frac{b}{1} \in S^{-1}I$. The fact that $S^{-1}I$ is weakly J -prime, implies $\frac{a}{1} \in S^{-1}I + J(S^{-1}R) = S^{-1}(I + J(R))$ or $\frac{b}{1} \in S^{-1}I$. If $\frac{a}{1} \in S^{-1}(I + J(R))$, there exists $u \in S$ with $ua \in I + J(R)$, which implies $a \in I + J(R)$ as $S \cap Z_{I+J(R)}(R) = \emptyset$. Similarly, if $\frac{b}{1} \in S^{-1}I$, there exists $v \in S$ with $vb \in I$, which implies $b \in I$ since $S \cap Z_I(R) = \emptyset$. Thus, I is weakly J -prime. \square

The following example shows the necessity of the zero-divisor assumption in Proposition 2.32(2).

Example 2.33. Let $R = \mathbb{Z}_{24}$ and $S = \{\bar{1}, \bar{3}, \bar{9}\}$. Note that $S^{-1}R \cong \mathbb{Z}_8$, $J(R) = \langle \bar{6} \rangle$, and $S^{-1}J(R) = J(S^{-1}R) = \langle \bar{2} \rangle$. For the ideal $I = \langle \bar{12} \rangle$, we have $I + J(R) = \langle \bar{6} \rangle$. Observe that I is not weakly J -prime in R since $\bar{0} \neq \bar{3} \cdot \bar{4} = \bar{12} \in I$, yet $\bar{3} \notin I + J(R)$ and $\bar{4} \notin I$. Conversely, $S^{-1}I = \langle \bar{4} \rangle$ is a weakly J -prime ideal of \mathbb{Z}_8 by Example 2.16. This failure to pull back the property arises because $S \cap Z_I(R) \neq \emptyset$.

In referring to [24], the Jacobson radical of the power series ring $R[[x]]$, where R is commutative with identity, is $J(R[[x]]) = J(R) + xR[[x]]$. Also, it is well known that $J(R[x]) = \sqrt{0_{R[x]}} = \sqrt{0}[X]$, see [10, page 11, Exercises (2)–(4)].

Proposition 2.34. *Let R be a ring and I a proper ideal of R . Then the following statements hold:*

1. *If $I[[x]]$ is a weakly J -prime ideal of $R[[x]]$, then I is a weakly J -prime ideal of R .*
2. *If I is a weakly J -prime ideal of R such that $I^2 \neq \langle 0 \rangle$, then $I[[x]]$ is a weakly J -prime ideal of $R[[x]]$.*

Proof. (1) Suppose that $I[[x]]$ is weakly J -prime in $R[[x]]$. Let $a, b \in R$ with $0 \neq ab \in I$ and assume $a \notin I + J(R)$. Consider a and b as constant power series in $R[[x]]$, we have $0 \neq ab \in I[[x]]$ and $a \notin I[[x]] + J(R[[x]])$. Then $b \in I[[x]]$, thus $b \in I$, since $I[[x]]$ is a weakly J -prime ideal in $R[[x]]$. Hence, I is weakly J -prime in R .

(2) Assume that I is a weakly J -prime ideal of R and $I^2 \neq \langle 0 \rangle$. By Theorem 2.13, I is a J -prime ideal of R . Hence, by [8, Proposition 2.27], $I[[x]]$ is a J -prime ideal of $R[[x]]$. Consequently, $I[[x]]$ is a weakly J -prime ideal of $R[[x]]$. \square

Next, we investigate the conditions under which weakly J -prime ideals align with weakly primary ideals. Recall that a proper ideal I is weakly primary if $0 \neq ab \in I$ implies $a \in \sqrt{I}$ or $b \in I$.

Proposition 2.35. *For any proper ideal I of a ring R , the following statements hold:*

1. *If I is a weakly J -prime ideal, then $\sqrt{I} \subseteq I + J(R)$.*
2. *If I is a weakly J -prime ideal and $J(R) \subseteq \sqrt{I}$, then I is a weakly primary ideal.*

Proof. (1) Assume that I is a weakly J -prime ideal of R and let $a \in \sqrt{I}$. If $a \in \sqrt{0}$, then $a \in J(R) \subseteq I + J(R)$ and the result holds. Now, suppose $a \notin \sqrt{0}$ and let n be the smallest positive integer such that $a^n \in I$. If $n = 1$, then $a \in I \subseteq I + J(R)$. If $n \geq 2$, then $0 \neq a \cdot a^{n-1} \in I$. Since I is weakly J -prime, either $a \in I + J(R)$ or $a^{n-1} \in I$. By the minimality of n , we must have $a \in I + J(R)$, which implies $\sqrt{I} \subseteq I + J(R)$.

(2) Suppose that I is a weakly J -prime ideal such that $J(R) \subseteq \sqrt{I}$. Let $a, b \in R$ with $0 \neq ab \in I$. Since I is weakly J -prime, it follows that $a \in I + J(R)$ or $b \in I$. Since $J(R) \subseteq \sqrt{I}$, then $I + J(R) \subseteq \sqrt{I}$, which implies that $a \in \sqrt{I}$ or $b \in I$. Thus, I is a weakly primary ideal of R . \square

3 Weakly J -prime ideals in trivial ring extension and amalgamated algebra

Let M be a unital R -module. The trivial extension $R \ltimes M = \{(a, m) : a \in R, m \in M\}$ is a commutative ring equipped with componentwise addition and the multiplication [18, 19]: $(a, m_1)(b, m_2) = (ab, am_2 + bm_1)$.

If I is an ideal of R and N is a submodule of M , then $I \ltimes N$ is an ideal of $R \ltimes M$ if and only if $IM \subseteq N$. Such ideals $I \ltimes N$ are referred as homogeneous ideals. In [6, Theorem 3.2], it was established that $J(R \ltimes M) = J(R) \ltimes M$. Furthermore, it was shown that all prime ideals of $R \ltimes M$ are of the form $I = I_1 \ltimes M$, where I_1 is a prime ideal of R . For a submodule N of M and an element $a \in R$, we denote the submodule $\{m \in M : am \in N\}$ by $(N :_M a)$. Now, we are ready to determine weakly J -prime homogeneous ideal of the trivial extension $R \ltimes M$.

Theorem 3.1. Let M be an R -module, I an ideal of R , and N a submodule of M with $IM \subseteq N$. Then the following statements hold:

1. If $I \ltimes N$ is a weakly J -prime ideal of $R \ltimes M$, then I is a weakly J -prime ideal of R . Furthermore, for every $a \notin I + J(R)$, any $m \in (N :_M a)$ with $am \neq 0_M$ satisfies $m \in N$.
2. Assume that $(I \ltimes N)^2 \neq \langle 0_{R \ltimes M} \rangle$. Then $I \ltimes N$ is a weakly J -prime ideal of $R \ltimes M$ if and only if I is a J -prime ideal of R and $(N :_M a) = N$ for every $a \notin I + J(R)$.

Proof. (1) Assume that $I \ltimes N$ is a weakly J -prime. Let $a, b \in R$ with $0 \neq ab \in I$, we have $0_{R \ltimes M} \neq (a, 0_M)(b, 0_M) = (ab, 0_M) \in I \ltimes N$. Since $I \ltimes N$ is weakly J -prime, we have $(a, 0_M) \in (I \ltimes N) + J(R \ltimes M) = (I + J(R)) \ltimes M$ or $(b, 0_M) \in I \ltimes N$. Therefore, $a \in I + J(R)$ or $b \in I$, respectively. Hence, I is a weakly J -prime ideal of R . Now, let $a \notin I + J(R)$ and take $m \in (N :_M a)$ with $am \neq 0$. Then $0_{R \ltimes M} \neq (a, 0_M)(0, m) = (0, am) \in (I \ltimes N)$. Since $I \ltimes N$ is weakly J -prime and $(a, 0_M) \notin (I \ltimes N) + J(R \ltimes M)$, it follows that $(0, m) \in I \ltimes N$, and thus $m \in N$.

(2) Suppose that $(I \ltimes N)^2 \neq 0$. By Theorem 2.13, $I \ltimes N$ is a J -prime ideal of $R \ltimes M$, and the result follows from [8, Theorem 3.1]. \square

As an immediate consequence of setting $N = M$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. Let M be an R -module and I a proper ideal of R such that $I^2 \neq \langle 0 \rangle$. Then $I \ltimes M$ is a weakly J -prime ideal of $R \ltimes M$ if and only if I is a J -prime ideal of R .

Example 3.3. Let (R, \mathfrak{m}) be a quasi-local ring, M an R -module, and N a submodule of M . For any proper ideal I of R such that $IM \subseteq N$, then every ideal of $R \ltimes M$ is a weakly J -prime ideal.

Let R and S be two commutative rings, $f : R \rightarrow S$ be a ring homomorphism, and J be an ideal of the ring S . The amalgamation of R with S along J with respect to f , denoted by $R \bowtie^f J$, is the subring of $R \times S$ given by $R \bowtie^f J = \{(r, f(r) + j) : r \in R, j \in J\}$, see [13, 16].

This construction is a generalization of the amalgamated duplication of a ring along an ideal. Recall that for an ideal J of a ring R , the amalgamated duplication of R along J , denoted by $R \bowtie J$, is the subring of $R \times R$ given by $R \bowtie J = \{(r, r + j) : r \in R, j \in J\}$, see [15].

In addition, if I is an ideal of R , then $I \bowtie^f J = \{(i, f(i) + j) : i \in I, j \in J\}$ is an ideal of $R \bowtie^f J$. Moreover, by Lemma 2.15 of [11], we have

$$J(R \bowtie^f J) = \{(r, f(r) + j) : r \in J(R), f(r) + j \in J(f(R) + J)\}.$$

While from [1, Lemma 2.2], if $J \subseteq J(S)$, then $J(R \bowtie^f J) = J(R) \bowtie^f J$.

Theorem 3.4. Let $f : R \rightarrow S$ be a ring homomorphism, J an ideal of S , and I a proper ideal of R . Then the following statements hold:

1. If $I \bowtie^f J$ is a weakly J -prime ideal of $R \bowtie^f J$, then I is a weakly J -prime ideal of R .
2. If I is a weakly J -prime ideal of R with $I^2 \neq \langle 0 \rangle$, and $J \subseteq J(S)$, then $I \bowtie^f J$ is a weakly J -prime ideal of $R \bowtie^f J$.

Proof. (1) Assume that $I \bowtie^f J$ is a weakly J -prime ideal of $R \bowtie^f J$. Let $a, b \in R$ such that $0 \neq ab \in I$. Then $(0, 0) \neq (a, f(a))(b, f(b)) \in I \bowtie^f J$ for some $(a, f(a)), (b, f(b)) \in R \bowtie^f J$. Hence, $(a, f(a)) \in I \bowtie^f J + J(R \bowtie^f J)$ or $(b, f(b)) \in I \bowtie^f J$ thus $a \in I + J(R)$ or $b \in I$. Therefore, I is a weakly J -prime ideal of R . (2) Assume that I is a weakly J -prime ideal of R with $I^2 \neq \langle 0 \rangle$, and $J \subseteq J(S)$. Then by Theorem 2.13, I is a J -prime ideal of R . From [8, Theorem 4.1], $I \bowtie^f J$ is a J -prime ideal of $R \bowtie^f J$. Finally, $I \bowtie^f J$ is a weakly J -prime ideal. \square

Let R be a ring and I be an ideal of R . The amalgamated duplication of R along I , denoted by $R \bowtie I$, is the subring of $R \times R$ given by $R \bowtie I = \{(r, r + j) \mid r \in R, j \in I\}$. The next corollary is an immediate consequence of Theorem 3.4.

Corollary 3.5. Let R be a ring, J an ideal of R , and I a proper ideal of R . Then the following statements hold:

1. If $I \bowtie J$ is a weakly J -prime ideal of $R \bowtie J$, then I is a weakly J -prime ideal of R .
2. If I is a weakly J -prime ideal of R with $I^2 \neq \langle 0 \rangle$ and $J \subseteq J(R)$, then $I \bowtie J$ is a weakly J -prime ideal of $R \bowtie J$.

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