Moroccan Journal of Algebra and Geometry with Applications
x (xx) (2026), 1–11

Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco ISSN: 2820-7114

On 1-absorbing prime and weakly 1-absorbing prime subsemimodules

Mohammad Adarbeh[1] and Mohammad Saleh[2]

[1] *Department of Mathematics, Birzeit University, Birzeit, Ramallah, Palestine.*
e-mail: madarbeh@birzeit.edu

[2] *Department of Mathematics, Birzeit University, Birzeit, Ramallah, Palestine.*
e-mail: msaleh@birzeit.edu

Communicated by Jean-Luc Chabert

(Received 31 December 2025, Revised 10 April 2026, Accepted 13 April 2026)

Abstract. In this paper, we introduce the concepts of 1-absorbing prime and weakly 1-absorbing prime subsemimodules over commutative semirings. Let S be a commutative semiring with $1 \neq 0$ and M an S-semimodule. A proper subsemimodule N of M is called 1-absorbing prime (weakly 1-absorbing prime) if, for all nonunits $a, b \in S$ and $m \in M$, $abm \in N$ ($0 \neq abm \in N$) implies $ab \in (N :_S M)$ or $m \in N$. We study many properties of these concepts. For example, we show that a proper subsemimodule N of M is 1-absorbing prime if and only if for all proper ideals I, J of S and subsemimodule K of M with $IJK \subseteq N$, either $IJ \subseteq (N :_S M)$ or $K \subseteq N$. Also, we prove that a proper subtractive subsemimodule N of M is weakly 1-absorbing prime if and only if for all proper ideals I, J of S and subsemimodule K of M with $0 \neq IJK \subseteq N$, either $IJ \subseteq (N :_S M)$ or $K \subseteq N$.

Key Words: Semirings, Semimodules, 1-absorbing prime subsemimodules, Weakly 1-absorbing prime subsemimodules.
2020 MSC: Primary 13Cxx, 16Y60; Secondary 13A15, 16D80.

1 Introduction

Semirings are important algebraic structures with numerous applications in science and engineering. The concept of a semiring was first introduced by Vandiver [24] in 1934. Throughout, S denotes a commutative semiring with nonzero identity and M denotes a unital S-semimodule, unless otherwise stated. Let N be a subsemimodule of M. N is called subtractive if, whenever $x, x+y \in N$, then $y \in N$ for all $x, y \in M$. The set $\{s \in S \mid sM \subseteq N\}$ forms an ideal of S and is denoted by $(N :_S M)$. In particular, the ideal $(0 :_S M)$ is called the annihilator of M and is denoted by $\mathrm{Ann}(M)$. For more details on semirings, we may refer to [5, 16, 19, 24].

Let R be a commutative ring with $1 \neq 0$ and I a proper ideal of R. I is called prime (weakly prime) if, for all $a, b \in R$, $ab \in I$ ($0 \neq ab \in I$) implies $a \in I$ or $b \in I$. Badawi [8] introduced the concept of 2-absorbing ideals of commutative rings as a generalization of prime ideals. I is called 2-absorbing if, for all $a, b, c \in R$, $abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. Then Badawi and Yousefian [9] introduced the concept of weakly 2-absorbing ideals of commutative rings as a generalization of weakly prime ideals. I is called weakly 2-absorbing if, for all $a, b, c \in R$, $0 \neq abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. Afterwards, Yassine et al. [25] (Koç et al. [17]) introduced the concept of 1-absorbing prime (weakly 1-absorbing prime) ideals of commutative rings as another generalization of prime (weakly prime) ideals. I is called 1-absorbing prime (weakly 1-absorbing prime) if, for all nonunits $a, b, c \in R$, $abc \in I$ ($0 \neq abc \in I$) implies $ab \in I$ or $c \in I$. Recently, Ugurlu [23] introduced the concept of 1-absorbing prime submodules of modules over commutative rings. A proper submodule N of an R-module M is called 1-absorbing prime if, for all nonunits $a, b \in R$ and $m \in M$, $abm \in N$ implies $ab \in (N :_R M)$ or $m \in N$. For more details on the absorbing like-properties in commutative (semi)rings, we may refer to [1, 3, 8, 9, 10, 11, 12, 15, 17, 21, 22, 25].

In this paper, we define and study the concepts of 1-absorbing prime and weakly 1-absorbing prime subsemimodules of semimodules over commutative semirings. Let S be a commutative semiring, M an S-semimodule, and N a proper subsemimodule of M. N is called 1-absorbing prime (weakly 1-absorbing prime) if, for all nonunits $a, b \in S$ and $m \in M$, $abm \in N$ $(0 \neq abm \in N)$ implies $ab \in (N :_S M)$ or $m \in N$.

In Section 2, we investigate some properties of 1-absorbing prime subsemimodules. For example, we prove that if N is a 1-absorbing prime subsemimodule of M, then $(N :_S M)$ is a 1-absorbing prime ideal of S. The converse of the last fact is not true in general (see Example 2.7). However, the converse is true if M is an MC multiplication S-semimodule (see Corollary 2.15 (1)). In Theorem 2.4, we show that if M has a 1-absorbing prime subsemimodule that is not prime, then S is a local semiring. In Theorem 2.8, we give a characterization of 1-absorbing prime subsemimodules.

In Section 3, we study some properties of weakly 1-absorbing prime subsemimodules. Firstly, we will provide an example (Example 3.2) of a weakly 1-absorbing prime subsemimodule that is not 1-absorbing prime. In Theorem 3.4, we give some characterizations of weakly 1-absorbing prime subtractive subsemimodules. In Theorem 3.14, we prove that if S is a local semiring and N is a weakly 1-absorbing prime subtractive subsemimodule of M that is not 1-absorbing prime, then $(N :_S M)^2 N = 0$.

2 1-absorbing prime subsemimodules

Henceforth, S will always denote a commutative semiring, M an S-semimodule, and N a proper subsemimodule of M.

Definition 2.1. (1) N is called prime if for all $a \in S$ and $m \in M$, $am \in N$ implies $a \in (N :_S M)$ or $m \in N$ [7].

(2) N is called 1-absorbing prime if for all nonunits $a, b \in S$ and $m \in M$, $abm \in N$ implies $ab \in (N :_S M)$ or $m \in N$.

(3) Let I be a proper ideal of S. Then I is a prime (1-absorbing prime) ideal of S if it is a prime (1-absorbing prime) subsemimodule of the S-semimodule S.

Recall that a semiring S is called local if it has exactly one maximal ideal. Equivalently, the sum of two nonunits of S is a nonunit of S [18].

Example 2.2. Let S be a local semiring with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = 0$. Let M be any S-semimodule. Then every proper subsemimodule of M is 1-absorbing prime. To see this, let N be any proper subsemimodule of M and suppose that $abx \in N$ for some nonunits $a, b \in S$ and some $x \in M$. Then $a, b \in \mathfrak{m}$ and so $ab \in \mathfrak{m}^2 = 0$. Hence $ab \in (N :_S M)$. Thus N is 1-absorbing prime.

Remark 2.3. Every prime subsemimodule N of M is 1-absorbing prime.

Proof. Suppose that $abm \in N$ for some nonunits $a, b \in S$ and $m \in M$. Since N is a prime subsemimodule, either $ab \in (N :_S M)$ or $m \in N$. Hence N is 1-absorbing prime. □

Theorem 2.4. If M has a 1-absorbing prime subsemimodule that is not prime, then S is a local semiring.

Proof. Suppose that N is a 1-absorbing prime subsemimodule that is not prime. Then there exists a nonunit $s \in S$; and $m \in M$ such that $sm \in N$ but $s \notin (N :_S M)$ and $m \notin N$. Let $t_1, t_2 \in S$ be nonunits. Since N is 1-absorbing prime, $t_i sm \in N$, and $m \notin N$, then $t_i s \in (N :_S M)$ for $i = 1, 2$. It follows that

$(t_1+t_2)s \in (N :_S M)$. If t_1+t_2 is a unit in S, then $s = (t_1+t_2)^{-1}(t_1+t_2)s \in (N :_S M)$, a contradiction. So t_1+t_2 is a nonunit in S. Hence, the sum of two nonunits in S is again a nonunit in S. Thus S is a local semiring. □

Corollary 2.5. *Let S be a non-local semiring. Then N is 1-absorbing prime if and only if it is prime.*

Proposition 2.6. *If N is a 1-absorbing prime subsemimodule of M, then $(N :_S M)$ is a 1-absorbing prime ideal of S.*

Proof. Assume that $abc \in (N :_S M)$, where a,b,c are nonunits of S, and suppose $ab \notin (N :_S M)$. We prove that $c \in (N :_S M)$. Let $m \in M$. Since $abc \in (N :_S M)$, then $abcM \subseteq N$. Hence $abcm \in N$. But since N is 1-absorbing prime and $ab \notin (N :_S M)$, so $cm \in N$. Since m was arbitrary, then $c \in (N :_S M)$. Therefore, $(N :_S M)$ is a 1-absorbing prime ideal of S. □

The converse of Proposition 2.6 is not true in general, as shown in the following example.

Example 2.7. Let $S = \mathbb{Z}^{\circ}$, the semiring of non-negative integers with the usual addition and multiplication, $M = \mathbb{Z}^{\circ} \times \mathbb{Z}^{\circ}$, and $N = 2\mathbb{Z}^{\circ} \times 0$. Since $(N :_S M) = \{0\}$ is a prime ideal of S, then it is a 1-absorbing prime ideal of S. However, N is not 1-absorbing prime subsemimodule of M since $2 \cdot 2 \cdot (1,0) = (4,0) \in N$ but $4 \notin (N :_S M)$ and $(1,0) \notin N$.

Recall that the residual of a subsemimodule N by a subset J of S is the set $(N :_M J) = \{m \in M : Jm \subseteq N\}$. The following theorem gives a characterization of 1-absorbing prime subsemimodules.

Theorem 2.8. The following statements are equivalent:

(1) N is a 1-absorbing prime subsemimodule of M.

(2) For all nonunits $a,b \in S$ with $ab \notin (N :_S M)$, $(N :_M ab) \subseteq N$.

(3) For all nonunits $a,b \in S$ and subsemimodule K of M with $abK \subseteq N$, either $ab \in (N :_S M)$ or $K \subseteq N$.

(4) For all proper ideals I,J of S and subsemimodule K of M with $IJK \subseteq N$, either $IJ \subseteq (N :_S M)$ or $K \subseteq N$.

Proof. (1) ⇒ (2). Let $a,b \in S$ be nonunits and suppose that $ab \notin (N :_S M)$. Take $m \in (N :_M ab)$. Then $abm \in N$. But since N is 1-absorbing prime and $ab \notin (N :_S M)$, we have $m \in N$. Thus $(N :_M ab) \subseteq N$.

(2) ⇒ (3). Assume that $abK \subseteq N$ for some nonunits $a,b \in S$ and some subsemimodule K of M. If $ab \notin (N :_S M)$, then by (2), $(N :_M ab) \subseteq N$. So $K \subseteq (N :_M ab) \subseteq N$.

(3) ⇒ (4). Let I,J be proper ideals of S and K be a subsemimodule of M such that $IJK \subseteq N$. Suppose that $IJ \nsubseteq (N :_S M)$. Then there exist nonunits $a \in I$ and $b \in J$ such that $ab \notin (N :_S M)$. Since $abK \subseteq IJK \subseteq N$, then by (3) and since $ab \notin (N :_S M)$, we have $K \subseteq N$.

(4) ⇒ (1). Assume that $abm \in N$, where $a,b \in S$ are nonunits and $m \in M$. Take $I = Sa$, $J = Sb$, and $K = Sm$. Then we have $IJK = (Sa)(Sb)(Sm) = Sabm \subseteq N$. So by (4), $IJ \subseteq (N :_S M)$ or $K \subseteq N$. Hence $ab \in (Sa)(Sb) = IJ \subseteq (N :_S M)$ or $m \in Sm = K \subseteq N$. Therefore, N is a 1-absorbing prime subsemimodule of M. □

Definition 2.9. [20]

(1) M is called a multiplication S-semimodule if for each subsemimodule N of M, there is an ideal I of S such that $N = IM$.

(2) M is called multiplicatively cancellative (abbreviated as MC) if for any $s,s' \in S$ and $0 \neq x \in M$, $sx = s'x$ implies $s = s'$.

(3) Let $\mathfrak{m}$ be a maximal ideal of S. M is called $\mathfrak{m}$-cyclic if there exist $s \in S$, $q \in \mathfrak{m}$, and $x \in M$ such that $s + q = 1$ and $sM \subseteq Sx$.

Lemma 2.10. *[20, Theorem 4] Let M be a multiplication S-semimodule. Then for every maximal ideal $\mathfrak{m}$ of S, either $M = \{x \in M \mid x = qx \text{ for some } q \in \mathfrak{m}\}$ or M is $\mathfrak{m}$-cyclic.*

Theorem 2.11. Let M be an MC multiplication S-semimodule and I a 1-absorbing prime ideal of S. For all nonunits $a, b \in S$ and $x \in M$, if $abx \in IM$, then $ab \in I$ or $x \in IM$.

Proof. Let $a, b \in S$ be nonunits and let $x \in M$. If $x = 0$, the result is obviously true. So assume that $x \neq 0$. Suppose that $abx \in IM$ and $ab \notin I$. Let $J = \{s \in S \mid sx \in IM\}$. If $J \neq S$, then there is a maximal ideal $\mathfrak{m}$ of S such that $J \subseteq \mathfrak{m}$. Since M is MC, then $\{z \in M \mid z = qz \text{ for some } q \in \mathfrak{m}\} = \{0\}$. Since $x \in M \setminus \{0\}$ and M is a multiplication S-semimodule, then by Lemma 2.10, M is $\mathfrak{m}$-cyclic. So there exist $s \in S$, $p \in \mathfrak{m}$, and $y \in M$ such that $s + p = 1$ and $sM \subseteq Sy$. It follows that $sx = ty$ for some $t \in S$ which implies $sabx = taby \in IM$ since $abx \in IM$. So we have $s^2abx \in sIM = IsM \subseteq ISy = Iy$. Hence $s^2abx = iy$ for some $i \in I$. But then $staby = s^2abx = iy$. Note that $y \neq 0$, for if $y = 0$, then $sx = ty = 0$ and so $x = sx + px = px$ but M is MC implies $1 = p \in \mathfrak{m}$, a contradiction. Again, since M is MC, $stab = i \in I$. If st is a unit in S, then $ab \in I$, a contradiction. If st is a nonunit in S, then since $stab \in I$, $ab \notin I$, and I is a 1-absorbing prime ideal of S, we have $st \in I$. So $s^2x = sty \in IM$ and hence $s^2 \in J \subseteq \mathfrak{m}$. Since $\mathfrak{m}$ is a prime ideal, $s \in \mathfrak{m}$. But then $1 = s + p \in \mathfrak{m}$, a contradiction. Thus $J = S$ and so $1 \in J$, that is, $x \in IM$. Therefore, $ab \in I$ or $x \in IM$. □

Example 2.12. Let $S = \mathbb{Z}^\circ$, $M = -2\mathbb{Z}^\circ$, and $I = 3\mathbb{Z}^\circ$. Then M is a cyclic S-semimodule. Clearly, M is an MC. Also, since every cyclic S-semimodule is a multiplication S-semimodule [20], then M is a multiplication S-semimodule. Now, since I is a prime ideal of S, then by Remark 2.3, I is a 1-absorbing prime ideal of S. Let $a, b \in S$ be nonunits and $x \in M$. Suppose that $abx \in IM = -6\mathbb{Z}^\circ$. Write $x = -2n$ and $abx = -6m$, where $n, m \in \mathbb{Z}^\circ$. Then $-2abn = -6m$ and so $abn = 3m \in I$. Since I is prime, either $ab \in I$ or $n \in I$. But if $n \in I$, then $x = -2n \in -2I = -6\mathbb{Z}^\circ = IM$. Hence, for all nonunits $a, b \in S$ and $x \in M$, if $abx \in IM$, then $ab \in I$ or $x \in IM$.

Theorem 2.13. Let M be an MC multiplication S-semimodule and I be a proper ideal of S. Then the following statements are equivalent:

(1) IM is a 1-absorbing prime subsemimodule of M.

(2) I is a 1-absorbing prime ideal of S.

Proof. First, note that since M is an MC multiplication S-semimodule, then by [20, Theorem 9], $IM \neq M$.
(1) $\Rightarrow$ (2). Suppose that $abc \in I$ for some nonunits $a, b, c \in S$. Then $abcM \subseteq IM$. By (1) and Theorem 2.8, either $ab \in (IM :_S M)$ or $cM \subseteq IM$. So $abM \subseteq IM$ or $cM \subseteq IM$. Then $SabM \subseteq IM$ or $ScM \subseteq IM$. It follows from [20, Theorem 9] that $ab \in Sab \subseteq I$ or $c \in Sc \subseteq I$. Thus (2) holds.
(2) $\Rightarrow$ (1). Suppose that $abx \in IM$ for some nonunits $a, b \in S$ and $x \in M$. Then by Theorem 2.11, we have $ab \in I$ or $x \in IM$. But if $ab \in I$, then $abM \subseteq IM$, that is, $ab \in (IM :_S M)$. Hence $ab \in (IM :_S M)$ or $x \in IM$. Therefore, (1) holds. □

Example 2.14. Let S, M, and I be as in Example 2.12. Then by Example 2.12, M is an MC multiplication S-semimodule and I is a 1-absorbing prime ideal of S. Thus, by Theorem 2.13, $IM = -6\mathbb{Z}^\circ$ is a 1-absorbing prime subsemimodule of M.

Recall that if M is a multiplication S-semimodule, then for each subsemimodule N of M, $N = (N :_S M)M$ [20]. Part (1) of the following corollary proves that the converse of Proposition 2.6 is true under the condition that "M is an MC multiplication S-semimodule".

Corollary 2.15. *Let M be an MC multiplication S-semimodule. Then*

(1) N is a 1-absorbing prime subsemimodule of M if and only if $(N:_S M)$ is a 1-absorbing prime ideal of S.

(2) N is a 1-absorbing prime subsemimodule of M if and only if $N = IM$ for some 1-absorbing prime ideal I of S.

Lemma 2.16. *Let M_1, M_2 be S-semimodules and N_1, N_2 be subsemimodules of M_1, M_2, respectively. Let $f : M_1 \to M_2$ be a semimodule homomorphism.*

(1) $(N_2 :_S M_2) \subseteq (f^{-1}(N_2) :_S M_1)$.

(2) If f is onto, then $(N_1 :_S M_1) \subseteq (f(N_1) :_S M_2)$.

Proof. (1) Let $s \in (N_2 :_S M_2)$ and let $m_1 \in M_1$, then $sM_2 \subseteq N_2$. Now $f(sm_1) = sf(m_1) \in sM_2 \subseteq N_2$. So $sm_1 \in f^{-1}(N_2)$. Hence $sM_1 \subseteq f^{-1}(N_2)$. That is, $s \in (f^{-1}(N_2) :_S M_1)$.
(2) Let $s \in (N_1 :_S M_1)$ and let $m_2 \in M_2$. So there exists $m_1 \in M_1$ such that $m_2 = f(m_1)$ since f is onto. But $sM_1 \subseteq N_1$, so $sm_1 \in N_1$. Hence $sm_2 = f(sm_1) \in f(N_1)$. Thus $sM_2 \subseteq f(N_1)$. This means that $s \in (f(N_1) :_S M_2)$. □

Recall that N is called a strong subsemimodule of M if for each $x \in N$, there exists $y \in N$ such that $x + y = 0$ [14].

Proposition 2.17. *Let M_1, M_2 be S-semimodules and $f : M_1 \to M_2$ be a semimodule homomorphism.*

(1) If N_2 is a 1-absorbing prime subsemimodule of M_2 and $Im(f) \nsubseteq N_2$, then $f^{-1}(N_2)$ is a 1-absorbing prime subsemimodule of M_1.

(2) If f is onto, N_1 is a 1-absorbing prime subtractive strong subsemimodule of M_1, and $Ker(f) \subseteq N_1$, then $f(N_1)$ is a 1-absorbing prime subsemimodule of M_2.

Proof. (1) First, since $\mathrm{Im}(f) \nsubseteq N_2$, $f^{-1}(N_2) \neq M_1$. Suppose that $abm_1 \in f^{-1}(N_2)$, where a, b are nonunits of S and $m_1 \in M_1$. Then $abf(m_1) = f(abm_1) \in N_2$. Since N_2 is a 1-absorbing prime subsemimodule of M_2, we have $ab \in (N_2 :_S M_2)$ or $f(m_1) \in N_2$. But $(N_2 :_S M_2) \subseteq (f^{-1}(N_2) :_S M_1)$ by Lemma 2.16 part (1). It follows that $ab \in (f^{-1}(N_2) :_S M_1)$ or $m_1 \in f^{-1}(N_2)$. Thus $f^{-1}(N_2)$ is a 1-absorbing prime subsemimodule of M_1.
(2) Suppose that $abm_2 \in f(N_1)$, where a, b are nonunits of S and $m_2 \in M_2$. Then there exists $m_1 \in M_1$ such that $f(m_1) = m_2$ since f is onto. So $f(abm_1) = abm_2 \in f(N_1)$. We claim that $abm_1 \in N_1$. Since $f(abm_1) \in f(N_1)$, there exists $n_1 \in N_1$ such that $f(abm_1) = f(n_1)$. Since N_1 is a strong subsemimodule, there exists $n_1' \in N_1$ such that $n_1 + n_1' = 0$. Then $f(abm_1 + n_1') = 0$. Hence $abm_1 + n_1' \in \mathrm{Ker}(f) \subseteq N_1$ but N_1 is subtractive and $n_1' \in N_1$, so $abm_1 \in N_1$. Since N_1 is a 1-absorbing prime subsemimodule of M_1, either $ab \in (N_1 :_S M_1)$ or $m_1 \in N_1$. But $(N_1 :_S M_1) \subseteq (f(N_1) :_S M_2)$ by Lemma 2.16 part (2). So $ab \in (f(N_1) :_S M_2)$ or $m_2 = f(m_1) \in f(N_1)$. Thus $f(N_1)$ is a 1-absorbing prime subsemimodule of M_2. □

Recall that a subset T of S is called multiplicatively closed if $1 \in T$ and $xy \in T$ for all $x, y \in T$. The localization of M at the multiplicatively closed subset T of S is defined as follows: First, define an equivalence relation $\sim$ on $M \times T$ by $(m, t) \sim (m', t')$ if and only if $stm' = st'm$ for some $s \in T$. Let $\frac{m}{t}$ denote the equivalence class of $(m, t) \in M \times T$ and let $T^{-1}M$ denote the set of all equivalence classes of $M \times T$. Define addition on $T^{-1}M$ by $\frac{m}{t} + \frac{m'}{t'} = \frac{t'm + tm'}{tt'}$. For $\frac{s}{t} \in T^{-1}S$ and $\frac{m}{t'} \in T^{-1}M$, define $\frac{s}{t} \cdot \frac{m}{t'} = \frac{sm}{tt'}$. Then $T^{-1}M$ is an $T^{-1}S$-semimodule [13].

Proposition 2.18. *Let T be a nonempty multiplicatively closed subset of S. If $T^{-1}N \neq T^{-1}M$ and N is 1-absorbing prime in M, then $T^{-1}N$ is 1-absorbing prime in $T^{-1}M$.*

Proof. Suppose that $\frac{a}{t}\frac{b}{t'}\frac{m}{t''} \in T^{-1}N$ for some nonunits $\frac{a}{t}, \frac{b}{t'} \in T^{-1}S$ and $\frac{m}{t''} \in T^{-1}M$. So $\frac{abm}{tt't''} \in T^{-1}N$. This implies that $\frac{abm}{tt't''} = \frac{n}{s}$ for some $n \in N$ and $s \in T$. So there exists $v \in T$ such that $vsabm = vtt't''n$. It follows that $abvsm = vsabm \in N$. But since $\frac{a}{t}$ and $\frac{b}{t'}$ are nonunits in $T^{-1}S$, a and b are nonunits in S. Since N is 1-absorbing prime in M, we have $ab \in (N :_S M)$ or $vsm \in N$. Thus $\frac{a}{t}\frac{b}{t'} = \frac{ab}{tt'} \in T^{-1}(N :_S M) \subseteq (T^{-1}N :_{T^{-1}S} T^{-1}M)$ or $\frac{m}{t''} = \frac{vsm}{vst''} \in T^{-1}N$. Therefore, $T^{-1}N$ is 1-absorbing prime in $T^{-1}M$. □

3 Weakly 1-absorbing prime subsemimodules

Definition 3.1. (1) N is called a weakly 1-absorbing prime subsemimodule of M if for all nonunits $a, b \in S$ and $m \in M$, $0 \neq abm \in N$ implies $ab \in (N :_S M)$ or $m \in N$.

(2) Let I be a proper ideal of S. Then I is a weakly 1-absorbing prime ideal of S if it is a weakly 1-absorbing prime subsemimodule of the S-semimodule S.

Clearly, from Definitions 2.1 and 3.1, every 1-absorbing prime subsemimodule is weakly 1-absorbing prime, but the converse is not true in general, as shown in the following example.

Example 3.2. Let $S = \mathbb{Z}^\circ$, $M = \mathbb{Z}^\circ \times \mathbb{Z}_8$, and $N = \{(0, \overline{0})\}$. Then clearly, by definition, N is a weakly 1-absorbing prime subsemimodule of M. However, N is not 1–absorbing prime subsemimodule of M since $2 \cdot 2 \cdot (0, \overline{2}) = (0, \overline{0}) \in N$ but $2 \cdot 2 = 4 \notin (N :_S M)$ and $(0, \overline{2}) \notin N$.

Lemma 3.3. *Let N be a subsemimodule of M. If $N = N_1 \cup N_2$, where N_1, N_2 are subtractive subsemimodules of M, then $N = N_1$ or $N = N_2$.*

Proof. Assume that $N \neq N_1$ and $N \neq N_2$. Take $x_1 \in N \setminus N_1$ and $x_2 \in N \setminus N_2$. Hence $x_1 \in N_2$ and $x_2 \in N_1$. Now $x_1 + x_2 \in N = N_1 \cup N_2$. If $x_1 + x_2 \in N_1$, then since $x_2 \in N_1$ and N_1 is subtractive, we have $x_1 \in N_1$, a contradiction. Similarly, if $x_1 + x_2 \in N_2$, then since $x_1 \in N_2$ and N_2 is subtractive, we have $x_2 \in N_2$, a contradiction. Thus $N = N_1$ or $N = N_2$. □

The following theorem gives some characterizations of weakly 1-absorbing prime subtractive subsemimodules.

Theorem 3.4. Suppose that N is subtractive. Then the following statements are equivalent:

(1) N is a weakly 1-absorbing prime subsemimodule of M.

(2) For all nonunits $a, b \in S$ with $ab \notin (N :_S M)$, $(N :_M ab) = (0 :_M ab) \cup N$.

(3) For all nonunits $a, b \in S$ with $ab \notin (N :_S M)$, $(N :_M ab) = (0 :_M ab)$ or $(N :_M ab) = N$.

(4) For all nonunits $a, b \in S$ and subsemimodule K of M with $0 \neq abK \subseteq N$, then $ab \in (N :_S M)$ or $K \subseteq N$.

(5) For all nonunit $a \in S$, proper ideal J of S, and subsemimodule K of M with $0 \neq aJK \subseteq N$, either $aJ \subseteq (N :_S M)$ or $K \subseteq N$.

(6) For all proper ideals I, J of S, and subsemimodule K of M with $0 \neq IJK \subseteq N$, either $IJ \subseteq (N :_S M)$ or $K \subseteq N$.

Proof. (1) ⇒ (2). Let $a,b \in S$ be nonunits such that $ab \notin (N :_S M)$. Let $m \in (N :_M ab)$ and suppose $m \notin (0 :_M ab)$. Then $0 \neq abm \in N$. But N is weakly 1-absorbing prime subsemimodule and $ab \notin (N :_S M)$, so $m \in N$. Thus $(N :_M ab) \subseteq (0 :_M ab) \cup N$. The reverse inclusion is always true.

(2) ⇒ (3). This implication follows from Lemma 3.3 and the fact that the subsemimodules $(0 :_M ab)$ and N are subtractive.

(3) ⇒ (4). Assume that $0 \neq abK \subseteq N$ for some nonunits $a,b \in S$ and some subsemimodule K of M and assume that $ab \notin (N :_S M)$. Since $abK \subseteq N$, $K \subseteq (N :_M ab)$. So by (3), $K \subseteq (0 :_M ab)$ or $K \subseteq N$. But $K \nsubseteq (0 :_M ab)$ since $abK \neq 0$. Thus $K \subseteq N$.

(4) ⇒ (5). Suppose that $0 \neq aJK \subseteq N$, where $a \in S$ is a nonunit, J a proper ideal of S, and K a subsemimodule of M. Suppose that $aJ \nsubseteq (N :_S M)$ and $K \nsubseteq N$. Then $ab \notin (N :_S M)$ for some $b \in J$. Since $aJK \neq 0$, there exists $c \in J$ such that $acK \neq 0$. Since J is proper, $b, c, b+c$ are nonunits. We claim that $abK = 0$. If $abK \neq 0$, then $0 \neq abK \subseteq aJK \subseteq N$. So by (4) and since $ab \notin (N :_S M)$, we have $K \subseteq N$, a contradiction. Thus $abK = 0$. Now $0 \neq acK = a(b+c)K \subseteq aJK \subseteq N$. Again by (4) and since $K \nsubseteq N$, we have $a(b+c) \in (N :_S M)$. But since $0 \neq acK \subseteq aJK \subseteq N$ and $K \nsubseteq N$, then $ac \in (N :_S M)$. But $a(b+c) \in (N :_S M)$ and $(N :_S M)$ is subtractive, so $ab \in (N :_S M)$, a contradiction.

(5) ⇒ (6). Suppose that $0 \neq IJK \subseteq N$ for some proper ideals I, J of S and some subsemimodule K of M. Suppose that $IJ \nsubseteq (N :_S M)$ and $K \nsubseteq N$. Then $aJ \nsubseteq (N :_S M)$ for some $a \in I$. If $aJK \neq 0$, then $0 \neq aJK \subseteq IJK \subseteq N$. So by (5) and since $aJ \nsubseteq (N :_S M)$, we have $K \subseteq N$, a contradiction. Thus $aJK = 0$. Now since $IJK \neq 0$, there exists $b \in I$ such that $bJK \neq 0$. So $0 \neq bJK \subseteq IJK \subseteq N$. By (5) and since $K \nsubseteq N$, we have $bJ \subseteq (N :_S M)$. But $0 \neq bJK = (a+b)JK \subseteq IJK \subseteq N$ and $K \nsubseteq N$, then again by (5), $(a+b)J \subseteq (N :_S M)$. Since $(N :_S M)$ is subtractive, $aJ \subseteq (N :_S M)$, a contradiction.

(6) ⇒ (1). Assume that $0 \neq abm \in N$ for some nonunits $a,b \in S$ and some $m \in M$. Take $I = Sa$, $J = Sb$, and $K = Sm$. Then $0 \neq abm \in IJK = Sabm \subseteq N$. Thus by (6), $ab \in IJ \subseteq (N :_S M)$ or $m \in K \subseteq N$. Thus N is a weakly 1-absorbing prime subsemimodule of M. □

Theorem 3.5. Let M be an MC multiplication S-semimodule and I a weakly 1-absorbing prime ideal of S. For all nonunits $a,b \in S$ and $x \in M$, if $0 \neq abx \in IM$, then $ab \in I$ or $x \in IM$.

Proof. Let $a,b \in S$ be nonunits and $x \in M$. Suppose that $0 \neq abx \in IM$ and $ab \notin I$. Let $J = \{s \in S \mid sx \in IM\}$. If $J \neq S$, then there is a maximal ideal $\mathfrak{m}$ of S such that $J \subseteq \mathfrak{m}$. Since M is an MC multiplication S-semimodule, then as in the proof of Theorem 2.11, M is $\mathfrak{m}$-cyclic. So there exist $s \in S$, $p \in \mathfrak{m}$, and $y \in M$ such that $s + p = 1$ and $sM \subseteq Sy$. It follows that $sx = ty$ for some $t \in S$. Again, by the proof of Theorem 2.11, we have $y \neq 0$ and $staby = iy$ for some $i \in I$. Since M is MC, $stab = i \in I$. If $i = 0$, then $staby = 0$ implies $staby + ptaby = taby$ and so $ptaby = taby$. If $taby \neq 0$, then since M is MC, we have $p = 1 \in \mathfrak{m}$, a contradiction. So $taby = 0$. But then $sabx = taby = 0$. Since $abx \neq 0$ and M is MC, so $s = 0$ and hence $p = 1 \in \mathfrak{m}$, again a contradiction. Thus $i \neq 0$. So $0 \neq i = stab \in I$. Since I is a weakly 1-absorbing prime ideal of S, then by a similar argument to the proof of Theorem 2.11, we obtain $x \in IM$. □

Recall that M is called a faithful S-semimodule if $\mathrm{Ann}(M) = 0$ [20].

Theorem 3.6. Let M be an MC multiplication S-semimodule and I be a proper ideal of S such that IM is subtractive. Then the following statements are equivalent:

(1) IM is a weakly 1-absorbing prime subsemimodule of M.

(2) I is a weakly 1-absorbing prime ideal of S.

Proof. First, since M is an MC multiplication S-semimodule, then by [20, Theorem 9], $IM \neq M$.
(1) ⇒ (2). Suppose that $0 \neq abc \in I$ for some nonunits $a,b,c \in S$. Since M is MC, M is faithful [20]. So $\mathrm{Ann}(M) = 0$. If $abcM = 0$, then $abc \in \mathrm{Ann}(M) = 0$, a contradiction. So $0 \neq abcM \subseteq IM$. Since IM is subtractive, then by (1) and Theorem 3.4, $ab \in (IM :_S M)$ or $cM \subseteq IM$. So $abM \subseteq IM$ or $cM \subseteq IM$.

Then by [20, Theorem 9], we have $ab \in I$ or $c \in I$. Therefore, (2) holds.
(2) $\Rightarrow$ (1). Suppose that $0 \neq abx \in IM$ for some nonunits $a, b \in S$ and $x \in M$. By Theorem 3.5, we have $ab \in I$ or $x \in IM$. So $abM \subseteq IM$ or $x \in IM$. Hence $ab \in (IM :_S M)$ or $x \in IM$. Therefore, (1) holds. □

Corollary 3.7. *Suppose that M is an MC multiplication S-semimodule and N is subtractive. Then*

(1) N is a weakly 1-absorbing prime subsemimodule of M if and only if $(N :_S M)$ is a weakly 1-absorbing prime ideal of S.

(2) N is a weakly 1-absorbing prime subsemimodule of M if and only if $N = IM$ for some weakly 1-absorbing prime ideal I of S.

Proposition 3.8. *Let T be a nonempty multiplicatively closed subset of S. If $T^{-1}N \neq T^{-1}M$ and N is a weakly 1-absorbing prime in M, then $T^{-1}N$ is a weakly 1-absorbing prime in $T^{-1}M$.*

Proof. Suppose that $0 \neq \frac{a}{t}\frac{b}{t'}\frac{m}{t''} \in T^{-1}N$ for some nonunits $\frac{a}{t}, \frac{b}{t'} \in T^{-1}S$ and $\frac{m}{t''} \in T^{-1}M$. So $0 \neq \frac{abm}{tt't''} \in T^{-1}N$. As in the proof of Proposition 2.18, $uabm \in N$ for some $u \in T$. Since $\frac{abm}{tt't''} \neq 0$, then $uabm \neq 0$. So we have $0 \neq uabm \in N$. But since $\frac{a}{t}$ and $\frac{b}{t'}$ are nonunits in $T^{-1}S$, a and b are nonunits in S. By hypothesis, $ab \in (N :_S M)$ or $um \in N$. Thus $\frac{a}{t}\frac{b}{t'} = \frac{ab}{tt'} \in T^{-1}(N :_S M) \subseteq (T^{-1}N :_{T^{-1}S} T^{-1}M)$ or $\frac{m}{t''} = \frac{um}{ut''} \in T^{-1}N$. Therefore, $T^{-1}N$ is a weakly 1-absorbing prime in $T^{-1}M$. □

Proposition 3.9. *Suppose that $x \in M$, $(0 :_S x) \subseteq (Sx :_S M)$, and Sx is subtractive. Then Sx is a weakly 1-absorbing prime subsemimodule of M if and only if it is 1-absorbing prime.*

Proof. Suppose that Sx is a weakly 1-absorbing prime subsemimodule of M and suppose $abm \in Sx$ for some nonunits $a, b \in S$ and some $m \in M$. If $abm \neq 0$, then either $ab \in (Sx :_S M)$ or $m \in Sx$ since Sx is weakly 1-absorbing prime. If $abm = 0$, then $ab(x+m) = abx \in Sx$. Either $ab(x+m) \neq 0$ or $ab(x+m) = 0$. If $ab(x+m) \neq 0$, then since Sx is weakly 1-absorbing prime, we have either $ab \in (Sx :_S M)$ or $x+m \in Sx$. But Sx is subtractive, so $ab \in (Sx :_S M)$ or $m \in Sx$. If $ab(x+m) = 0$, then $abx = 0$ (since $abm = 0$). So $ab \in (0 :_S x) \subseteq (Sx :_S M)$. Therefore, Sx is 1-absorbing prime. The converse is clear. □

Definition 3.10. Let N be a weakly 1-absorbing prime subsemimodule of M. Let $a, b \in S$ be nonunits and $m \in M$. We say that (a, b, m) is a triple-zero of N if $abm = 0$, $ab \notin (N :_S M)$, and $m \notin N$.

Example 3.11. Let $S = \mathbb{Z}^{\circ}$, $M = \mathbb{Z}_{20}$, and $N = \{\overline{0}\}$. Then $(2, 2, \overline{5})$ is a triple-zero of N since $2 \cdot 2 \cdot \overline{5} = \overline{0}$, $2 \cdot 2 = 4 \notin (N :_S M)$, and $\overline{5} \notin N$.

Remark 3.12. If N is a weakly 1-absorbing prime subsemimodule of M that is not 1-absorbing prime, then N has a triple-zero (a, b, m) for some nonunits $a, b \in S$ and $m \in M$.

Theorem 3.13. Let S be a local semiring. If N is a weakly 1-absorbing prime subtractive subsemimodule of M and (a, b, m) is a triple-zero of N for some nonunits $a, b \in S$ and $m \in M$, then

(1) $abN = a(N :_S M)m = b(N :_S M)m = 0$.

(2) $a(N :_S M)N = b(N :_S M)N = (N :_S M)^2 m = 0$.

Proof. First, note that since (a, b, m) is a triple-zero of L, then $abm = 0$, $ab \notin (N :_S M)$, and $m \notin N$.
(1) Suppose that $abN \neq 0$. So there exists $n \in N$ such that $abn \neq 0$. Now, since $abm = 0$, we have $ab(m+n) = abm + abn = abn \neq 0$. So $0 \neq ab(m+n) \in N$ but since N is a weakly 1-absorbing prime subsemimodule, so $ab \in (N :_S M)$ or $m + n \in N$. But N is subtractive, so $ab \in (N :_S M)$ or $m \in N$, a contradiction since (a, b, m) is a triple-zero of N. Hence $abN = 0$. Next, assume that $a(N :_S M)m \neq 0$. Then there exists $t \in (N :_S M)$ such that $atm \neq 0$. So $a(t+b)m = atm + abm = atm + 0 = atm \neq 0$. So $0 \neq a(t+b)m = atm \in atM \subseteq aN \subseteq N$. Since S is a local semiring and t, b are nonunits, then

$t+b$ is a nonunit. But N is a weakly 1-absorbing prime subsemimodule, so $a(t+b) \in (N :_S M)$ or $m \in N$. Since N is subtractive, then by [6, Lemma 3 (i)], $(N :_S M)$ is subtractive but $at \in (N :_S M)$ and $at + ab \in (N :_S M)$, so $ab \in (N :_S M)$. Hence $ab \in (N :_S M)$ or $m \in N$, a contradiction since (a,b,m) is a triple-zero of N. Thus $a(L :_S M)m = 0$. Similarly, we have $b(L :_S M)m = 0$.

(2) Suppose that $a(N :_S M)N \neq 0$. Then there exist $t \in (N :_S M)$ and $n \in N$ such that $atn \neq 0$. By (1), $abN = a(N :_S M)m = 0$, so $abn = atm = 0$. Thus $a(b+t)(m+n) = abm+abn+atm+atn = atn \neq 0$. We have $0 \neq a(b+t)(m+n) = atn \in N$. But $b+t$ is a nonunit (since S is local) and N is weakly 1-absorbing prime, so $a(b+t) \in (N :_S M)$ or $m+n \in N$. But N is subtractive implies $ab \in (N :_S M)$ or $m \in N$, a contradiction since (a,b,m) is a triple-zero of N. Thus $a(N :_S M)N = 0$. Similarly, we have $b(N :_S M)N = 0$. Finally, we show that $(N :_S M)^2 m = 0$. Suppose that $(N :_S M)^2 m \neq 0$. Then there exist $s,t \in (N :_S M)$ such that $stm \neq 0$. By above, we have $(a+s)(b+t)m = stm \neq 0$ and $0 \neq (a+s)(b+t)m \in N$. Since S is a local semiring, we have $a+s$ and $b+t$ are nonunits. Hence $(a+s)(b+t) \in (N :_S M)$ or $m \in N$. Since $(N :_S M)$ is subtractive, we have $ab \in (N :_S M)$ or $m \in N$, a contradiction since (a,b,m) is a triple-zero of N. Therefore, $(N :_S M)^2 m = 0$. □

Theorem 3.14. Let S be a local semiring. If N is a weakly 1-absorbing prime subtractive subsemimodule of M that is not 1-absorbing prime, then $(N :_S M)^2 N = 0$.

Proof. Assume that N is a weakly 1-absorbing prime subtractive subsemimodule of M that is not 1-absorbing prime. So by Remark 3.12, N has a triple-zero (a,b,m) for some nonunits $a,b \in S$ and $m \in M$. Assume that $(N :_S M)^2 N \neq 0$. Then $stn \neq 0$ for some $s,t \in (N :_S M)$ and $n \in N$. From Theorem 3.13, we have $(a+s)(b+t)(m+n) = stn \neq 0$. So $0 \neq (a+s)(b+t)(m+n) = stn \in N$. But since N is weakly 1-absorbing prime, then $(a+s)(b+t) \in (N :_S M)$ or $m+n \in N$ but N is subtractive implies $ab \in (N :_S M)$ or $m \in N$, a contradiction since (a,b,m) is a triple-zero of N. Thus $(N :_S M)^2 N = 0$. □

Corollary 3.15. *Let S be a local semiring. If N is a weakly 1-absorbing prime subtractive subsemimodule of M that is not 1-absorbing prime, then $(N :_S M)^3 \subseteq Ann(M)$.*

Proof. Since $(N :_S M)M \subseteq N$, then $(N :_S M)^3 M = (N :_S M)^2 (N :_S M)M \subseteq (N :_S M)^2 N$. But $(N :_S M)^2 N = 0$ by Theorem 3.14. So $(N :_S M)^3 M = 0$. Thus $(N :_S M)^3 \subseteq \mathrm{Ann}(M)$. □

As in [4], if N_1 and N_2 are subsemimodules of a multiplication S-semimodule M with $N_1 = I_1 M$ and $N_2 = I_2 M$ for some ideals I_1 and I_2 of S, then the product of N_1 and N_2, denoted by $N_1 N_2$, is defined by $N_1 N_2 = I_1 I_2 M$.

Corollary 3.16. *Let S be a local semiring and M a multiplication S-semimodule. If N is a weakly 1-absorbing prime subtractive subsemimodule of M that is not 1-absorbing prime, then $N^3 = 0$.*

Proof. Since M is a multiplication S-semimodule, $N = (N :_S M)M$. Thus

$$\begin{aligned} N^3 &= (N :_S M)^3 M \\ &= (N :_S M)^2 (N :_S M)M \\ &= (N :_S M)^2 N \\ &= 0 \qquad \text{by Theorem 3.14.} \end{aligned}$$

Therefore, $N^3 = 0$. □

Theorem 3.17. Let S be a local semiring and M an MC multiplication S-semimodule. Let I be a proper ideal of S such that $I^3 \neq 0$ and IM is subtractive. Then the following statements are equivalent:

(1) IM is a weakly 1-absorbing prime subsemimodule of M.

(2) IM is a 1-absorbing prime subsemimodule of M.

(3) I is a 1-absorbing prime ideal of S.

(4) I is a weakly 1-absorbing prime ideal of S.

Proof. (1) ⇒ (2). Since $I^3 \neq 0$, and M is faithful (since M is MC), then $I^3M \neq 0$ and so $(IM)^3 \neq 0$. So by (1) and Corollary 3.16, (2) holds.
(2) ⇒ (3). Follows from Theorem 2.13.
(3) ⇒ (4). Clear.
(4) ⇒ (1). Follows from Theorem 3.6. □

Conflict of interest: The authors declare that they have no conflict of interest.

Acknowledgments: The authors would like to thank the editor and referees for their valuable comments that improved the quality of the paper.

References

[1] M. Adarbeh and M. Saleh, On 2-absorbing ideals of noncommutative semirings, J. Algebra Appl. 24(12) (2025), 2550278 (10 pages).

[2] D. F. Anderson, E. Smith, Weakly prime ideals, Houst. J. Math. 29(4) (2003), 831–840.

[3] M. Assalami, A. Kharbouch, and H. Kim, On S-(m,n)-absorbing ideals and S-(m,n)-absorbing prime ideals of commutative rings, Moroccan J. Algebra, Geom. Appl. (2025), In press.

[4] S. E. Atani and S. K. G. Ghaleh, On multiplication modules, In International Mathematical Forum 1(24) (2006), 1175-1180.

[5] R. E. Atani, The ideal theory in quotients of commutative semirings, Glas. Math. 42(62) (2007), 301-308.

[6] R. E. Atani and S. E. Atani, On subsemimodules of semimodules, Bul. Acad. Ştiinţe Repub. Mold. Mat. 2(63) (2010), 20–30.

[7] R. E. Atani, On prime subsemimodules of semimodules, Int. J. Algebra 4(26) (2010), 1299–1306

[8] A. Badawi, On 2-absorbing ideals in commutative rings, Bull. Aust. Math. Soc. 75 (2007) 417–429.

[9] A. Badawi and D. A. Yousefian, On weakly 2-absorbing ideals of commutative rings, Houst. J. Math. 39 (2013), 441–452.

[10] A. Badawi and E. Y. Celikel, On 1-absorbing primary ideals of commutative rings, J. Algebra Appl. 19 (6) (2020), 2050111, 12 pp.

[11] A. Badawi and E. Y. Celikel, On weakly 1-absorbing primary ideals of commutative rings, Algebra Colloq. 29(2) (2022), 189–202.

[12] H. Behzadipour and P. Nasehpour, On 2-absorbing ideals of commutative semirings, J. Algebra Appl. 19(02) (2020), 2050034, 12 pp.

[13] R. P. Deore, Characterizations of semimodules, Southeast Asian Bull. Math. 36 (2012), 187–196.

[14] M. K. Dubey and P. Sarohe, On prime and primary avoidance theorem for subsemimodules, Quasigroups Relat. Syst. 25 (2017), 221-229.

[15] A. El Khalfi, A note on S-AB-(m,n)-absorbing ideals, Moroccan J. Algebra, Geom. Appl. (2026), In press.

[16] J. S. Golan, Semirings and their applications, Kluwer Academic Publishers, Dordrecht, 1999.

[17] S. Koç, U. Tekir, and E. Yıldız, On weakly 1-absorbing prime ideals, Ric. Mat. 72(2), (2023), 723-738.

[18] P. Nasehpour, Valuation semirings, J. Algebra Appl. 16(11) (2018), 1850073.

[19] P. Nasehpour, Some remarks on semirings and their ideals, Asian-Eur. J. Math. 12 (2020), 2050002 (14 pages).

[20] R. R. Nazari and S. Ghalandarzadeh, Multiplication semimodules, Acta Univ. Sapientiae Math. 11(1) (2019), 172–185

[21] S. Payrovi and S. Babaei, On the 2-absorbing ideals in commutative rings, Bull. Malays. Math. Soc. 23 (2013), 1511–1526.

[22] M. Saleh and I. Murra, On weakly 1-absorbing primary ideals of commutative semirings, Commun. Adv. Math. Sci. (2022), 199-208.

[23] A. E. Ugurlu, On 1-absorbing prime submodules, Moroccan J. Algebra Geom. Appl. 2(1) (2023), 162-173.

[24] H. S. Vandiver, Note on a simple type of algebra in which the cancellation law of addition does not hold, Bull. Amer. Math. Soc. 40(3) (1934), 916-920.

[25] A. Yassine, M. J. Nikmehra, and R. Nikandish, On 1-absorbing prime ideals of commutative rings, J. Algebra Appl. 20(10) (2021), 2150175.