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On complemented commutative rings

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Abstract. Let R be a commutative ring with nonzero identity. In this paper, we study the complemented elements of a ring R, the transfer of complemented property to various context of commutative ring extensions such as direct product and trivial ring extension, and we investigate the subgraph of the zero-divisor graph $\Gamma(R)$ of R induced by its complemented elements. Namely, it is proved that the graph of complemented ring is uniquely complemented.

Key Words: Complemented rings, Von Neumann regular rings, Zero-divisor graph, Total graph.
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1 Introduction

Throughout this paper all rings are commutative with nonzero unity. We denote respectively by $Z(R)$, $reg(R)$, $nilp(R)$, $idem(R)$ and $U(R)$ the set of all zero-divisors of the ring R, the set of all regular elements of R, the ideal of nilpotent elements of R and the group of units of R. The rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$, respectively.

For a commutative ring R, an element $z \in R$ is a zero-divisor if there exists a nonzero element $t \in R$ such that $tz = 0$. An element which is not a zero-divisor is called a non-zero-divisor (or a regular) element of R.

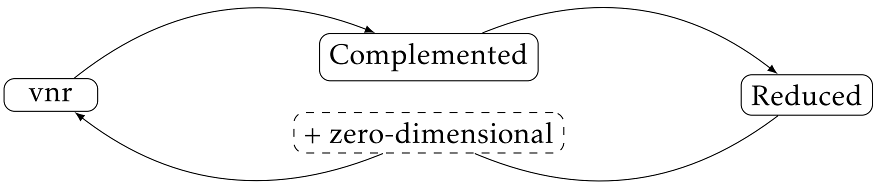
An element x of a ring R is von Neumann regular (or vnr) if there exists an element $y \in A$ such that $x^2y = x$. A ring R is von Neumann regular if every element of R is von Neumann regular. In [5], it is proved that a ring R is von Neumann regular if and only if for all $x \in R$, there exists y such that $xy = 0$ and $x + y$ is a unit in R.

In [5], von Neumann regular ring R is characterized by the following equivalent properties:
(1) R is von Neumann regular.
(2) For all $x \in R$, there exists y such that $xy = 0$ and $x + y$ is a unit.
(3) For all $x \in R$, there exists $u \in U(R)$ such that $x^2u = x$.
(4) For all $x \in R$, there exist $u \in U(R)$ and $e \in Idem(R)$ such that $x = ue$.

An element x of a ring R is complemented if there exists an element $y \in A$ such that $xy = 0$ and $x+y$ is a regular element of R. A commutative ring R is complemented (also known as quasi-regular) if every element of R is complemented. By [15, Theorem 2.5], a ring R is complemented if and only if $T(R)$ (the total ring of quotients of R) is von Neumann regular. Also, by [16, Theorem 3], complemented rings are characterized by the statement that for every element x there is a regular element r such that $rx = x^2$. It is easy to see that every von Neumann regular ring is complemented and any complemented ring is reduced. In view of [14, Theorem 3.1], R is von Neumann regular if and only if R is reduced and zero-dimensional.

The set of all von Neumann regular elements of a ring R is denoted by $vnr(R)$ and the set of all complemented elements of a ring R is denoted by $qr(R)$. It is clear that $vnr(R) \subseteq qr(R)$ and a ring R is von Neumann regular (resp., complemented) if and only if $vnr(R) = R$ (resp., $qr(R) = R$).

The following diagram always hold, where the implications cannot be reversed in general.



Recall that for a ring R and an R-module M, the idealization of M over R (also called The trivial ring extension of R by M) is the commutative ring formed from $R \times M$ by defining addition and multiplication as $(r,m)+(s,n) = (r+s, m+n)$ and $(r,m)(s,n) = (rs, rn+sm)$, respectively. A standard notation for this "idealized ring" is $R(+)M$ and some authors note it as follows: $R \propto M$ or $R \ltimes M$, see [14] for basic properties of rings resulting from the idealization construction. Several authors have studied the transfer of certain properties in trivial ring extension, for example see [7, 8, 13]. In Proposition 2.9, we examine the complemented rings in trivial ring extension.

The concept of the graph of the zero-divisors of a ring R was introduced by Beck in [9] and further studied by Anderson and Naseer in [1]. In their work, all elements of the ring were vertices of the graph. Then, in [3], Anderson and Livingston considered only the set of non-zero zero-divisors of R as vertices. The definition of Anderson and Livingston, which we adopt in this paper, is as follows:

The zero-divisor graph of R, denoted by $\Gamma(R)$, is the undirected graph with vertices $V(R) = Z(R) \setminus \{0\}$, and for distinct $x,\ y \in V(R)$, the vertices x and y are adjacent if and only if $xy = 0$. The graph $\Gamma(R)$ is complete if and only if $xy = 0$ for all distinct $x,\ y \in V(R)$.

We say that the graph G is connected if there is a path between any two distinct vertices of G. Recall that the distance between (connected) vertices x and y, denoted by $d(x,y)$, is the length of a shortest path connecting x and y, notice that $d(x,x) = 0$ and $d(x,y) = \infty$ if no such path exists. The diameter of a connected graph G, denoted by $diam(G)$, is the supremum of the distances between vertices. We define the girth of G, denoted by $gr(G)$, as the length of a shortest cycle in G, provided G contains a cycle, otherwise $gr(G) = \infty$. Recall that distinct vertices x and y of a graph G are orthogonal, written $x \perp y$, if x and y are adjacent and there is no vertex z of G which is adjacent to both x and y (i.e., the edge $x - y$ is not part of any triangle in G). We say that G is complemented if for each vertex x of G, there is a vertex y of G (called a complement of x) such that $x \perp y$, and G is uniquely complemented if G is complemented and whenever $x \perp y$ and $x \perp z$ in G, then y and z are adjacent to exactly the same vertices in G. By [3, Theorem 2.3], $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$. the reader is referred to [1, 2, 3, 9].

In [5], Anderson and Badawi investigate the induced subgraphs $\Gamma(vnr(R))$ and $\Gamma(idem(R))$ of the zero-divisor graph $\Gamma(R)$ of R determined by von Neumann regular and idempotent elements of $Z(R)$, respectively. Namely, they proved that:

- $\Gamma(vnr(R))$ and $\Gamma(idem(R))$ are connected.
- $diam(\Gamma(vnr(R))) \leq 3$ and $diam(\Gamma(idem(R))) \leq 3$.
- $gr(\Gamma(vnr(R))) \leq 4$ (resp., $gr(\Gamma(idem(R)) \leq 4$), if $\Gamma(vnr(R))$ (resp., $\Gamma(idem(R))$) contains a cycle.
- $\Gamma(vnr(R))$ and $\Gamma(idem(R))$ are uniquely complemented.

In [4], Anderson and Badawi introduced the total graph of a ring R, denoted by $T(\Gamma(R))$, as the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x+y \in Z(R)$. Let $reg(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $reg(R)$, let $Z(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ with vertices $Z(R)$, and let let $nilp(\Gamma(R))$ be the (induced) subgraph of $T(\Gamma(R))$ (and $Z(\Gamma(R))$) with vertices $nilp(R)$. Their study of $T(\Gamma(R))$ breaks into two cases depending on whether or not $Z(R)$ is an ideal of R. They proved that the subgraph $Z(\Gamma(R))$ of $T(\Gamma(R))$ is always connected, and $Z(\Gamma(R))$ is complete if and only if $Z(R)$ is an ideal of R. Moreover, if $Z(R)$ is an ideal of R, then $Z(\Gamma(R))$ and $reg(\Gamma(R))$ are disjoint subgraphs of $T(\Gamma(R))$, and $reg(\Gamma(R))$ is the union of disjoint subgraphs, each of which is either a complete graph or a complete bipartite graph. But, if $Z(R)$ is not an ideal of R, then the subgraphs $Z(\Gamma(R))$ and $reg(\Gamma(R))$ are never disjoint, and $T(\Gamma(R))$ is connected if and only if $Z(R) = R$.

In [11], it is shown that $Z(R)$ is an ideal of R if and only if $Q(R)$ is a local ring, with $Q(R)$ is the classical ring of R, if and only if $Z(R)$ is a prime ideal of R and $Q(R)$ is the localisation at $Z(R)$.

In this paper, we investigate the induced subgraph $\Gamma(qr(R))$ of the zero-divisor graph $\Gamma(R)$ of R determined by complemented elements of $Z(R)$. In particular, we show that $\Gamma(qr(R))$ is connected with diameter at most 3, $\Gamma(qr(R))$ has girth at most 4 if it contains a cycle, and that $\Gamma(qr(R))$ is uniquely complemented.

2 On complemented rings

We start with the main definition:

Definition 2.1. [12, Definition 3.3] Let R be a ring. An element x of R is said to be complemented if there exists an element $y \in R$ such that $xy = 0$ and $x + y \in reg(R)$. A ring R is complemented (also known as quasi-regular) if every element is complemented.

Following Definition 2.1, it is clear that every regular element is complemented. Then we conclude easily the next result.

Proposition 2.2. *Let R be a ring. Then, R is complemented if and only if $Z(R) \subseteq qr(R)$.*

Proof. The direct implication is obvious. Conversely, one can see easily that for every regular element $r \in R$, we have $r.0 = 0$ and $r + 0 = r \in reg(R)$, then r is complemented. Moreover, by assumption, $Z(R) \subseteq qr(R)$. Hence R is complemented. □

The following proposition gives the characterization of the complemented rings.

Proposition 2.3. *Let R be a ring and $T(R)$ its ring of quotients. Then, the following statements are equivalent.*

1. *R is complemented.*
2. *$Z(R) \subseteq qr(R)$.*
3. *For all $x \in R$, there is $y \in R$ such that $xy = 0$ and $x + y \in reg(R)$.*
4. *For all $x \in R$ there is $d \in reg(R)$ such that $x^2 = dx$.*
5. *$T(R)$ is von Neumann regular.*
6. *$T(R)$ is complemented.*

Proof. (1) ⇔ (2) By Proposition 2.2.
(1) ⇔ (3) This is in view of Definition [12, Definition 3.3].
(1) ⇔ (4) ⇔ (5) ⇔ (6) By Theorem [12, Theorem 3.5]. □

Following Proposition 2.3, a ring R is complemented if and only if for every element $x \in R$ there is an element $d \in reg(R)$ such that $dx = x^2$. The proof of the next result is based on this important characterization of complemented rings.

Theorem 2.4. Let R be a ring. Then, the following statements hold.

1. $qr(R)$ is multiplicatively closed.
2. $qr(R) \cap nilp(R) = \{0\}$.
3. Let $(A_i)_{i\in I}$ be a family of commutative rings, then $qr(\prod_i A_i) = \prod_i qr(A_i)$. In particular, $\prod_i A_i$ is complemented if and only if each A_i is complemented.

Note that the proof of statement (3) of Theorem 2.4 looks like that of [12, Proposition 8] which requires the following lemma.

Lemma 2.5. *[12, Lemma 2.7] Let $\{R_i\}_{i\in I}$ be a family of rings. Then, $reg(\prod_{i\in I} R_i) = \prod_{i\in I} reg(R_i)$.*

Proof. (1) Let $x, y \in qr(R)$. Then there exist two regular elements a and b such that $ax = x^2$ and $by = y^2$. Thus $abxy = (xy)^2$. It is well known that $ab \in reg(R)$ since $reg(R)$ is multiplicatively closed. Hence $xy \in qr(R)$.
(2) Let $x \in qr(R) \cap nilp(R)$. Suppose that $x^2 = 0$. Then $dx = x^2 = 0$ for some regular element d and this forces $x = 0$.
(3) Let $k \in I$ be an arbitrary element such that $a_k \in A_k$. Consider $(x_i)_{i\in I} \in \prod_{i\in I} A_i$ such that $x_k = a_k$ if $i = k$ and $x_i = 0$ otherwise. Since $\prod_{i\in I} A_i$ is complemented, then there exists $(r_i)_{i\in I} \in reg(\prod_{i\in I} A_i)$ such that $(r_i)_{i\in I}(x_i)_{i\in I} = (r_i x_i)_{i\in I} = [(x_i)_{i\in I}]^2$. In view of Lemma 2.5, we claim that $r_i \in reg(A_i)$ for every $i \in I$. Then $r_k a_k = (a_k)^2$ with $r_k \in reg(A_k)$. Hence a_k is complemented. Conversely, let $(x_i)_{i\in I} \in A = \prod_{i\in I} A_i$. Since A_i is complemented for every $i \in I$. Then for each $i \in I$ there exists $r_i \in reg(A_i)$ such that $r_i x_i = (x_i)^2$. Hence $(r_i)_{i\in I}(x_i)_{i\in I} = (r_i x_i)_{i\in I} = [(x_i)^2_{i\in I}] = [(x_i)_{i\in I}]^2$. In virtue of Lemma 2.5, $(r_i)_{i\in I} \in reg(\prod_{i\in I} A_i)$. Therefore, $(x_i)_{i\in I}$ is complemented. The "In particular" can be deduced easily. □

It is clear that every unit is von Neumann regular element and every von Neumann regular is complemented, then we show that $qr(R)$ is a subring of R forces R to be reduced.

Theorem 2.6. Let R be a ring. If $qr(R)$ is a subring of R, then R is reduced.

Proof. Let $x \in nilp(R)$. Then $1 + x \in U(R) \subset qr(R)$, and since $qr(R)$ is closed under addition, thus $x = -1 + (1 + x) \in qr(R)$. Hence, in view of Theorem 2.4, $x \in nilp(R) \bigcap qr(R) = \{0\}$. Therefore, R is reduced. □

Recall that a ring R is potent if for all $x \in R$, $x^k = x$ for some positive integer $k > 1$. the next result show that every potent ring is complemented.

Theorem 2.7. Let R be a ring. If R is potent then R is complemented.

Proof. Let $x \in R$. Then $x^k = x$ for some integer $k > 1$, then the proof of Proposition [12, Proposition 3.1] show that $(x+1-x^{k-1})(x^{2k-3}+1-x^{k-1}) = 1$. Hence $x+1-x^{k-1} \in U(R) \subseteq reg(R)$. Furthermore, $(x+1-x^{k-1})x = x^2$ and by Theorem [12, Theorem 3.5], x is complemented. Therefore, R is complemened. □

Remark 2.8. The converse of the Theorem 2.7 does not hold. Indeed, $\mathbb{Q}$, the field of rational is complemented but it is not potent since for every $r \in \mathbb{Q} \setminus \{0,1\}$, there is no integer $k > 1$ such that $r^k = r$.

Recall that in [10], for a ring R and an ideal I, *Marco D'Anna* and *Marco Fontana* introduced in 2007 the amalgamated duplication of the ring R along the ideal I as a subring of $R \times R$ with unity $(1,1)$ and defined by $R \bowtie I = \{(x, x+i) \mid x \in R,\ i \in I\}$. Furthermore, in [6], the authors proved that, if $R \bowtie I$ is complemented, then so is R, and the equivalence holds when I contains a regular element of R. On the other hand, recall that for an R-module M, the idealization of M over R is the commutative ring formed from $R \times M$ by defining addition and multiplication as $(r,m)+(s,n) = (r+s, m+n)$ and $(r,m)(s,n) = (rs, rn+sm)$, respectively. A standard notation for this "idealized ring" is $R(+)M$ and several authors note it as follows: $R \propto M$ or $R \ltimes M$, see [14] for basic properties of rings resulting from the idealization construction. Recall that in [6], when $M \neq 0$ the idealization $R(+)M$ is never complemented, since every complemented ring is reduced and when $M \neq 0$, $R(+)M$ is never reduced. However we have the following result.

Proposition 2.9. *Let R be a ring and M an R-module. Then, $R(+)M$ is complemented ring if and only if R is complemented and $M = 0$.*

Proof. Assume that the ring $R(+)M$ is complemented. Then, by Proposition [12, Proposition 3.4], $R(+)M$ is reduced and this forces $M = 0$. On the other hand, let $r \in R$, then $(r,0)$ is complemented in $R(+)M$. Thus, there is $(s,0) \in reg(\mathrm{R(+)M})$ such that $(r,0)^2 = (s,0)(r,0)$. so, $r^2 = sr$. Furthermore, $s \in reg(R)$ since $Z(\mathrm{R(+)M}) = \{(r,m) \mid r \in Z(R) \cup Z(M),\ \ m \in M\}$ one can see Lemma [5, Lemma 3.6]. Hence, R is complemented. Conversely, assume that $M = 0$ and R is complemented and let $(r,0) \in R(+)M$. Then, there exists $s \in reg(R)$ such that $r^2 = sr$. Thus, we get $(r,0)^2 = (s,0)(r,0)$, it is easy to see that $(s,0) \in reg(R(+)M)$. So $(r,0)$ is complemented and consequently, $R(+)M$ is complemented.

□

3 Zero-divisor graph and total graph of complemented rings

As in [3], the zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, is the undirected graph with vertices $V(R) = Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this section, we consider the induced subgraph $\Gamma(qr(R))$ of $\Gamma(R)$ with vertices $qr(R) \cap V(R)$. For $V(R) \neq \emptyset$ and by Proposition 2.2, we get $\Gamma(qr(R)) = \Gamma(R)$ if and only if R is a complemented ring.

We start this section with the following result.

Proposition 3.1. *Let R be a ring and let $V(R) = Z(R) \setminus \{0\}$. Then, $x \in qr(R) \cap V(R)$ if and only if there exists $y \in (qr(R) \cap V(R)) \setminus \{x\}$ such that $xy = 0$.*

Proof. Let $x \in qr(R) \cap V(R)$. Then there exists $y \in R$ such that $xy = 0$ and $x + y \in reg(R)$. We claim that $y \neq 0$, if $y = 0$ we get $x \in reg(R)$ contradiction. Moreover, $y \neq x$ since x is not a nilpotent in view of Theorem 2.4(2). Hence $y \in V(R)$. Furthermore, $x + y \in reg(R)$ then, $y \in qr(R)$, which complete the proof. □

The following result examines the diameter and the girth for the graph of complemented rings.

Theorem 3.2. Let R be a ring. Then, the following hold.

1. $\Gamma(qr(R))$ is connected with $diam(\Gamma(qr(R)) \leq 3$.
2. $gr(\Gamma(qr(R)) \leq 4$ if $\Gamma(qr(R))$ contains a cycle.

Proof. (1) Let x and y be two elements in $qr(R) \cap V(R)$ with $x \neq y$. By Proposition 3.1, there are $a,\ b \in qr(R) \cap V(R)$ such that $ax = by = 0$. If $ab \neq 0$, then $ab \in qr(R) \cap V(R)$ by Theorem 2.4(1), and x—ab—y is a path of length 2 from x to y in $\Gamma(qr(R))$. If $ab = 0$, then x—a—b—y is a path of length at most 3 from x to y in $\Gamma(qr(R))$. Consequently, $\Gamma(qr(R))$ is connected and $diam(\Gamma(qr(R)) \leq 3$.
(2) Let x—y—$c_1 - \cdots - c_n$—x be a cycle in $\Gamma(qr(R))$. If $c_1c_n = 0$, then x—y—c_1—c_n—x is a cycle of length 4 in $\Gamma(qr(R))$. Suppose that $c_1c_n \neq 0$. Then $x \neq c_1c_n$ and $y \neq c_1c_n$ since $x(c_1c_n) = y(c_1c_n) = 0$ and $x,\ y \notin nilp(R)$ by Theorem 2.4(2). Since $c_1c_n \in qr(R) \cap V(R)$ by Theorem 2.4(1), x—y—c_1c_n—x is a cycle of length 3 in $\Gamma(qr(R))$. Thus $gr(\Gamma(qr(R)) \leq 4$. □

The next example illustrates Theorem 3.2.

Example 3.3. Let $R = \mathbb{Z}/30\mathbb{Z}$ be the ring of integers modulo 30. Then.

1. $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$.
2. $gr(\Gamma(R)) \leq 4$.

Proof. By Proposition [6, Proposition 2.7], R is complemented; since $30 = 2\times3\times5$ is square free. Thus, by Proposition 2.2, $\Gamma(qr(R)) = \Gamma(R)$. Then, by Theorem 3.2(1), $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$. Moreover, we can see easily that $\Gamma(R)$ contains a cycle (for instance, 5—6—10—12—5 is a cycle in $\Gamma(R)$). Thus, by Theorem 3.2(2), $gr(\Gamma(R)) \leq 4$. □

Remark 3.4. In view of Corollary [2, Corollary 3.10], $\Gamma(R)$ is uniquely complemented if and only if $T(R)$ (the total ring of quotients of R) is von Neumann regular or $\Gamma(R)$ is star graph. In particular, if $T(R)$ is von Neumann regular, then $\Gamma(R)$ is uniquely complemented. In virtue of Theorem [15, Theorem 2.5], a ring R is complemented if and only if $T(R)$ is von Neumann regular. Hence, $\Gamma(R)$ is uniquely complemented when R is complemented.

Recall that distinct vertices x and y of a graph G are orthogonal, written $x \perp y$, if x and y are adjacent and there is no vertex z of G which is adjacent to both x and y. We say that G is complemented if for each vertex x of G, there is a vertex y of G (called a complement of x) such that $x \perp y$; and G is uniquely complemented if G is complemented and whenever $x \perp y$ and $x \perp z$ in G, then y and z are adjacent to exactly the same vertices in G. Now to the next result.

Theorem 3.5. Let R be a ring. Then $\Gamma(qr(R))$ is uniquely complemented.

Proof. We first show that $\Gamma(qr(R))$ is complemented. Let $x \in qr(R) \cap V(R)$. Then there is $y \in (qr(R) \cap V(R)) \setminus \{x\}$ such that $xy = 0$ and $x + y = r \in reg(R)$. Suppose that $z \in qr(R) \cap V(R)$ is adjacent to both x and y in $\Gamma(qr(R))$. Then $z \neq 0$ and $xz = yz = 0$. Hence, $rz = (x+y)z = xz + yz = 0$, this forces $z = 0$, a contradiction. Therefore, $x \perp y$ and $\Gamma(qr(R))$ is complemented. We next show that $\Gamma(qr(R))$ is uniquely complemented. Indeed, assume $a \perp x$ and $a \perp y$ in $\Gamma(qr(R))$. Without lose of generality, it is sufficient to prove that, if $d \in (qr(R) \cap V(R)) \setminus \{a, x, y\}$ with $dx = 0$ then $dy = 0$. Suppose that $dy \neq 0$. Then $(dy)a = d(ya) = 0$, $(dy)x = y(dx) = 0$ and $dy \in (qr(R) \cap V(R)) \setminus \{a, x\}$ since $qr(R)$ is multiplicatively closed and $a,\ x \notin nilp(R)$. But this is a contradiction with the fact that $a \perp x$ in $\Gamma(qr(R))$. Therefore, $\Gamma(qr(R))$ is uniquely complemented. □

When the ring R is complemented we obtain the following result.

Corollary 3.6. *Let R be a ring. If R is complemented then $\Gamma(R)$ is uniquely complemented.*

Proof. If R is complemented then $qr(R) = R$ and so $\Gamma(R) = \Gamma(qr(R))$. The result follows by Theorem 3.5. □

Notice that, if R is complemented then $T(R)$, its total ring of quotients, is von Neumann regular and therefore in view of Corollary [2, Corollary 3.10], $\Gamma(R)$ is uniquely complemented.

We close this section by rewriting [2, Theorem 3.5] and [15, Theorem 2.5] to get a nice result for reduced rings, which shows that there is an interplay between ring-theoretic properties and graph-theoretic properties.

Theorem 3.7. For a reduced ring R, the following are equivalent.

1. R is complemented.
2. $Z(R) \subseteq qr(R)$.
3. $T(R)$ is von Neumann regular.
4. $\Gamma(R)$ is complemented.
5. $\Gamma(R)$ is uniquely complemented.

Proof. ($1 \Leftrightarrow 2$) by Proposition 2.2, ($1 \Leftrightarrow 3$) This holds by Theorem [15, Theorem 2.5], and ($3 \Leftrightarrow 4 \Leftrightarrow 5$) hold by Theorem [2, Theorem 3.5]. □

The following example illustrates Theorem 3.7.

Example 3.8. Let $R = \mathbb{Z}/30\mathbb{Z}$ be as in the Example 3.3. It is proved that the ring $R = \mathbb{Z}/30\mathbb{Z}$ is complemented. Then by Theorem 3.7, $\Gamma(R)$ is uniquely complemented.

We conclude this section with the following question, which we will address in a future work:

Question. *Can we characterize the rings R for which the graph $\Gamma(qr(R))$ is connected (resp., complete)?*

In what follows we investigate the total graph of complemented rings. Recall that *Anderson* and *Badawi* introduced in [4], the total graph of a ring R, denoted by $T(\Gamma(R))$, as the (undirected) graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in Z(R)$. In this section we examine the total graph of complemented rings. Recall that a ring R is complemented if and only if every zero-divisor is complemented as in Proposition 2.2. First, we will rewrite part (1) of Theorem [4, Theorem 2.8], for the subgraph $qr(\Gamma(R))$ of $T(\Gamma(R))$ with vertices $qr(R)$.

Proposition 3.9. *Let R be a ring such that $Z(R)$ is an ideal of R. Let G be an induced subgraph of $qr(\Gamma(R))$, and let x and y be distinct vertices of G that are connected by a path in G. Then there is a path in G of length at most 2 between x and y.*

Proof. (1) It suffices to show that, if x_1, x_2, x_3 and x_4 are distinct vertices of G and if x_1—x_2—x_3—x_4 is a path from x_1 to x_4, then x_1 and x_4 are adjacent. Indeed, $x_1 + x_2$, $x_2 + x_3$, $x_3 + x_4 \in Z(R)$. Then $x_1 + x_4 = (x_1 + x_2) - (x_2 + x_3) + (x_3 + x_4) \in Z(R)$ since $Z(R)$ is an ideal of R. Then x_1 and x_4 are adjacent, It follows that if two vertices x and y in $qr(R)$ are connected then the path between x and y is of length at most 2. □

Proposition 3.10. *Let R be a ring such that $Z(R)$ is an ideal of R. Let x and y be distinct complemented elements of R that are connected by a path. If $x + y \notin Z(R)$, then x—$(-x)$—y and x—$(-y)$—y are paths of length 2 between x and y in $qr(\Gamma(R))$.*

Before proving the Proposition 3.10, we first establish the following lemma.

Lemma 3.11. *Let R be a ring such that $Z(R)$ is an ideal of R. Let x and y be distinct elements of R that are connected by a path. If $x+y \notin Z(R)$, then x and y are necessarily regular elements in R.*

Proof. Assume that x and y are distinct elements of R that are connected by a path with $x+y \notin Z(R)$. It is clear that x and y are not both zero-divisors, since $Z(R)$ is an ideal of R. Furthermore, without lose of generality, suppose that $x \in reg(R)$ and $y \in Z(R)$. Then in light of Proposition 3.9, there is some $z \in R$ such that x—z—y is a path between x and y of length 2. Then, two cases to distinguish:
Case1: $z \in Z(R)$, then $x+z \in Z(R)$ and since $Z(R)$ is an ideal of R, then $x \in Z(R)$ absurd.
Case2: $z \in reg(R)$, then $z+y \in Z(R)$ and since $Z(R)$ is an ideal of R, then $z \in Z(R)$ absurd.
Consequently, x and y are necessarily regular elements. □

Proof. (*of Proposition 3.10*) Assume that x and y are distinct elements of $qr(R)$ that are connected by a path with $x+y \notin Z(R)$. In view of Lemma 3.11, we get x and $y \in reg(R)$ and by Theorem [4, Theorem 2.8 (2)], the result follows. □

Lemma 3.11 ensures that, for a commutative ring R with an identity $1 \neq 0$, if $Z(R)$ is an ideal of R then $qr(\Gamma(R))$ is never connected. The next example illustrates this fact.

Example 3.12. Let $R := \mathbb{Z}/p^n\mathbb{Z}$ with p is a prime integer and $n \geq 1$ is an integer. Then $qr(\Gamma(R))$ is never connected.

Proof. In virtue of Example [4, Example 2.7 (a)], in this case $Z(R)$ is an ideal of R. Then in view of Lemma 3.11, $qr(\Gamma(R))$ is never connected, since there is no path between a zero-divisor element and a regular element in $qr(R)$. □

It is observed in [4], that $T(\Gamma(R))$ is never connected when $Z(R)$ is an ideal of R. However, in the next result, we will give some criteria for when two elements x and y are connected.

Theorem 3.13. Let R be a ring such that $Z(R)$ is an ideal of R and x, $y \in qr(R)$. Then the following statements are equivalent.

1. x and y are connected by a path.
2. Either $x+y \in Z(R)$ or $x-y \in Z(R)$ for all $x, y \in qr(R)$.
3. Either $x+y \in Z(R)$ or $x+2y \in Z(R)$ for all $x, y \in qr(R)$. In particular, either $2x \in Z(R)$ or $3x \in Z(R)$ (but not both) for all $x \in qr(R)$.

Proof. (1) ⇒ (2) Suppose that x and y are connected by a path. If $x = y$, then $x - y \in Z(R)$. So, assume that $x \neq y$. If $x+y \notin Z(R)$, then by Lemma 3.11, x and $y \in reg(R)$ and by Theorem [4, Theorem 2.8 (2)], x—$(-y)$—y is a path between x and y. Hence $x - y \in Z(R)$.
(2) ⇒ (3) Let x, $y \in R$ such that $x+y \notin Z(R)$. Then x, $y \in reg(R)$. Since $(x+y)-y = x \notin Z(R)$, thus $(x+y)+y = x+2y \in Z(R)$ by hypothesis. In particular, if $x \in reg(R)$, then either $2x \in Z(R)$ or $3x \in Z(R)$ and not both $2x$ and $3x \in Z(R)$ since $3x - 2x = x \in Z(R)$ and $x \in reg(R)$ a contradiction.
(3) ⇒ (1) Let x, $y \in R$ be distinct elements such that $x+y \notin Z(R)$. Then, by Lemma 3.11, x and $y \in reg(R)$. Then, $x+2y \in Z(R)$ by hypothesis. Since $Z(R)$ is an ideal of R and $x+2y \in Z(R)$, we get $2y \notin Z(R)$. Thus $3y \in Z(R)$ by hypothesis. Since $x+y \notin Z(R)$ and $3y \in Z(R)$, we get $x \neq 2y$ and thus x—$(2y)$—y is a path from x to y. □

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