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Some conditions on a non-associative algebra that imply power associativity

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Abstract. Let A be an algebra with no nonzero joint divisor of zero, containing a nonzero flexible idempotent e, and satisfying the identity $(x, x^2, x) = 0$, as well as an identity of the type $(x^p, x^q, x^r) = 0$ with $p, q, r \in \{1, 2\}$ and $p \neq r$. We show that A has a Peirce decomposition relative to e. Moreover, if in addition A satisfies the identity $(x^2, x^2, x^2) = 0$, then is a unital power-associative algebra.

Key Words: (121)-power associative, (121) and (222)-power associative, asymmetric and symmetric identities, joint divisor of zero, flexible idempotent.
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1 Introduction

Throughout this paper the algebras are nonzero and considered over any field of zero characteristic. The algebras are not assumed to be associative, finite-dimensional, or unital.

Let A be an algebra with product $(x, y) \mapsto xy$. Given elements a, b, c in A, we set $(a, b, c) := (ab)c - a(bc)$ for the associator of a, b and c; $[a, b] := ab - ba$ for the commutator of a and b; and $a \bullet b := \frac{1}{2}(ab + ba)$ for the symmetrized product of a and b.

Let a be an element in A. We will denote by L_a (respectively, R_a) the operator of left (respectively, right) multiplication by a on A. If we consider on A the symmetrized product $\bullet$, and if $L_a^\bullet$ (respectively, $R_a^\bullet$) denotes the operator of left (respectively, right) multiplication by a on $(A, \bullet)$, then it is clear that $L_a^\bullet = R_a^\bullet = \frac{1}{2}(L_a + R_a)$.

The algebra A is said to be *power-associative* whenever, for each element x of A, the subalgebra $A(x)$ of A generated by x is associative. A well-known Albert's result [1, Theorem 2] asserts that A *is power-associative if and only if* A *satisfies the identities*

$$(x, x, x) = 0 \quad \textit{and} \quad (x^2, x, x) = 0.$$

Algebras satisfying the identity $(x, x, x) = 0$ are called third-power associative algebras. It is clear that every third-power associative algebra satisfies the identity $(x^2, x^2, x^2) = 0$. Moreover, it is also well known that every third-power associative algebra satisfies the identity $(x, x^2, x) = 0$ (see, for example, [4, Lemma 2.1]).

Following [4] we say an identity $(x^p, x^q, x^r) = 0$ with $p, q, r \in \{1, 2\}$ is symmetric when $p = r$, and asymmetric otherwise. Thus the asymmetric identities are

$$(x^2, x, x) = 0, \quad (x, x, x^2) = 0, \quad (x^2, x^2, x) = 0, \quad \text{and} \quad (x, x^2, x^2) = 0.$$

Recall also that an element x of A is said to be a *divisor of zero* in A if there exists $y \in A \setminus \{0\}$ such that $xy = 0$ or $yx = 0$, and that x is said to be a *joint divisor of zero* in A if there is $y \in A \setminus \{0\}$ such that $xy = 0$ and $yx = 0$.

By definition, an *absolute-valued algebra* is an algebra A over $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$ endowed with an absolute value, i.e. a norm $\|\cdot\|$ on the vector space of A satisfying $\|xy\| = \|x\|\|y\|$ for all $x, y \in A$. Clearly absolute-valued real algebras have no nonzero divisor of zero, and hence have no nonzero joint divisor of zero. Every absolute-valued real unital algebra is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (Hamilton's quaternions) or $\mathbb{O}$ (Cayley's octonions) [11]. *These algebras are precisely the third-power associative absolute-valued real algebras satisfying an asymmetric identity* [2, Corollary 3.1].

Let A be a finite-dimensional absolute-valued algebra. If $dim(A) \leq 4$, then the two identities $(x,x,x) = 0$, $(x,x^2,x) = 0$ are equivalent [6, Proposition 3.1 and Theorem 3.24].

Let $\mathbb{A} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$. We recall that ${}^*\mathbb{A}$ and $\mathbb{A}^*$ are obtained by endowing the normed space $\mathbb{A}$ with the products $x \cdot y = x^* y$, and $x \cdot y = xy^*$, respectively, where $x \mapsto x^*$ means the standard involution. These algebras satisfy the identity $(x^2, x^2, x^2) = 0$. However, none of these algebras is third-power associative.

Cabrera and Rodríguez show the presence of a nonzero idempotent in any third-power associative absolute-valued algebra [5, Proposition 2.8.85].

The algebra A is called flexible if $(x,y,x) = 0$ for all $x, y \in A$. The absolute-valued real algebra of the pseudo-octonions $\mathbb{P}$ is flexible [8], so $\mathbb{P}$ is third-power associative, and hence $\mathbb{P}$ contains a nonzero idempotent.

We recall that an idempotent e is called central (respectively, flexible) if $[e,x] = 0$ (respectively, $(e,x,e) = 0$) for all x in A. Every central idempotent is flexible. The reciprocal is false because every nonzero idempotent of the *pseudo-octonions* algebra $\mathbb{P}$ is flexible and $\mathbb{P}$ does not contain a nonzero central idempotent.

The opposite algebra $A^{(0)}$ of A is defined by endowing the vector space of A with the product $(a,b) \mapsto ba$.

If we set $(.,.,.)^{(0)}$ to denote the associator in the algebra $A^{(0)}$, then it is clear that

$$(x,y,z)^{(0)} = -(z,y,x) \quad \text{for all} \quad x,y,z \in A. \tag{1}$$

Equality (1) implies that:

$$A \quad \text{satisfies} \quad (x^p, x^q, x^r) = 0 \Leftrightarrow A^{(0)} \quad \text{satisfies} \quad (x^r, x^q, x^p)^{(0)} = 0. \tag{2}$$

By the sake of convenience, in what follows we will say that an algebra is (121)-power associative when it verifies the identity $(x,x^2,x) = 0$, and we will say that it is (121) and (222)-power associative when it verifies the identities $(x,x^2,x) = 0$ and $(x^2,x^2,x^2) = 0$.

The present work is inspired by [2] and [4]. We have provided a generalization of [2, Proposition 2.3] and an extension of [2, Theorem 1.1] to (121) and (222)-power associative algebras.

Section 2 is reserved for establishing the preliminary results useful for the continuation of the work.

In section 3, we give a Peirce decomposition relative to a nonzero flexible idempotent for any (121)-power associative algebra with no nonzero joint divisor of zero, and satisfying an asymmetric identity (Theorem 3.1). We also prove our main result, which reads as follows:

Theorem 1.1. *Let A be an (121) and (222)-power associative algebra with no nonzero joint divisor of zero, containing a nonzero flexible idempotent, and satisfying an asymmetric identity. Then A is a unital power-associative algebra.*

As a first consequence, we derive that if A is an algebra with no nonzero joint divisor of zero, satisfying an asymmetric identity and containing a nonzero flexible idempotent, then following assertions are equivalent:

(i) A is a third-power associative algebra,

(ii) A is an (121) and (222)-power associative algebra (Corollary 3.8).

As a second consequence, we derive that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the unique (121) and (222)-power associative absolute-valued algebras containing a nonzero flexible idempotent, and satisfying an asymmetric identity (Corollary 3.9).

2 Preliminary results

Lemma 2.1. *Let A be an (121)-power associative algebra with a nonzero flexible idempotent e. Then, for every $x \in A$, we have:*

(i) $(e,e,x)+(x,e,e)=0$.

(ii) $[e, x-2x\bullet e]=0$.

(iii) $[e,(x-2x\bullet e)e]=0$.

Proof. (i) A linearization of $(x,x^2,x)=0$ gives that

$$(x,x^2,y)+(x,2x\bullet y,x)+(y,x^2,x)=0. \tag{3}$$

Putting $x=e$, $y=x$ in (3) and keeping in mind that $(e,z,e)=0$, we have

$$0=(e,e,x)+(e,2e\bullet x,e)+(x,e,e)=(e,e,x)+(x,e,e).$$

(ii) We have

$$\begin{aligned} 0 &= (e,e,x)+(x,e,e) \\ &= (e,e,x)+(e,x,e)+(x,e,e) \\ &= ex-e(ex)+(ex)e-e(xe)+(xe)e-xe \\ &= e(x-ex-xe)-(x-ex-xe)e \\ &= e(x-2x\bullet e)-(x-2x\bullet e)e, \end{aligned}$$

which concludes the proof (ii).

(iii) Keeping in mind that e commute with $x-2x\bullet e$ and e is flexible, we have

$$\begin{aligned} \big((x-2x\bullet e)e\big)e &= \big(e(x-2x\bullet e)\big)e \\ &= e\big((x-2x\bullet e)e\big), \end{aligned}$$

and hence $[e,(x-2x\bullet e)e]=0$. ☐

Lemma 2.2. *If A is an (121)-power associative algebra with no nonzero joint divisor of zero, containing a nonzero flexible idempotent e, and satisfying an asymmetric identity, then*

$$e(x - 2e \bullet x) = (x - 2e \bullet x)e = x - 2e \bullet x \quad \textit{for every} \quad x \in A. \tag{4}$$

Proof. The first equality in (4) follows from in Lemma 2.1(ii).

Suppose that A satisfies the identity $(x^2, x, x) = 0$. Then, linearizing we obtain the identity

$$(2x \bullet y, x, x) + (x^2, y, x) + (x^2, x, y) = 0. \tag{5}$$

Taking $x = e$ and $y = x$ in (5), we obtain

$$\begin{aligned} 0 &= (2x \bullet e, e, e) + (e, x, e) + (e, e, x) \\ &= \Big((2x \bullet e)e\Big)e - (2x \bullet e)e + (ex)e - e(xe) + ex - e(ex) \\ &= e(x - xe - ex) + \Big((2x \bullet e)e - 2x \bullet e + ex\Big)e \\ &= e(x - xe - ex) + \Big((xe)e + (ex)e - ex - xe + ex\Big)e \\ &= e(x - xe - ex) + \Big((xe)e + (ex)e - xe\Big)e \\ &= e(x - xe - ex) + \Big((xe + ex - x)e\Big)e \\ &= e(x - 2x \bullet e) - \Big((x - 2x \bullet e)e\Big)e \\ &= (x - 2x \bullet e)e - \Big((x - 2x \bullet e)e\Big)e \\ &= \Big((x - 2x \bullet e) - (x - 2x \bullet e)e\Big)e. \end{aligned}$$

Now, keeping in mind Lemma 2.1(ii)-(iii), we have

$$e\Big((x - 2x \bullet e) - (x - 2x \bullet e)e\Big) = \Big((x - 2x \bullet e) - (x - 2x \bullet e)e\Big)e = 0. \tag{6}$$

Since $e \neq 0$ and A has no nonzero joint divisor of zero, the second equality in (4) follows from (6).

Suppose that A satisfies the identity $(x^2, x^2, x) = 0$. By linearizing $(x^2, x^2, x) = 0$, we obtain

$$(2x \bullet y, x^2, x) + (x^2, 2x \bullet y, x) + (x^2, x^2, y) = 0. \tag{7}$$

Putting $x = e$ and $y = x$ in (7) we have $(2e \bullet x, e, e) + (e, 2e \bullet x, e) + (e, e, x) = 0$, so $(2e \bullet x, e, e) + (e, e, x) = 0$, and hence

$$(e, e, x) = -(2e \bullet x, e, e). \tag{8}$$

Combining Lemma 2.1(i) and (8), we find $(x, e, e) = (2e \bullet x, e, e)$, and so

$$\begin{aligned} 0 &= (x - 2e \bullet x, e, e) \\ &= \Big((x - 2e \bullet x)e\Big)e - (x - 2e \bullet x)e \\ &= \Big((x - 2e \bullet x)e - (x - 2e \bullet x)\Big)e. \end{aligned}$$

Now, keeping in mind Lemma 2.1(ii)-(iii), we have

$$e\Big((x - 2x \bullet e) - (x - 2x \bullet e)e\Big) = \Big((x - 2x \bullet e) - (x - 2x \bullet e)e\Big)e = 0. \tag{9}$$

Since $e \neq 0$ and A has no nonzero joint divisor of zero, the second equality in (4) follows from (9).

If A satisfies $(x,x,x^2)=0$ or $(x,x^2,x^2)=0$, then $A^{(0)}$ satisfies respectively $(x^2,x,x)^{(0)}=0$ or $(x^2,x^2,x)^{(0)}=0$. Since e is a nonzero flexible idempotent in $A^{(0)}$ and $A^{(0)}$ is an (121)-power associative algebra, the two paragraphs above complete the proof. □

The following result can be derived from the proof of [2, Proposition 2.3]. Nevertheless, for the sake of completeness, we give here a proof.

Lemma 2.3. *Let A be an (121)-power associative algebra with no nonzero joint divisor of zero, containing a nonzero flexible idempotent e, and satisfying the asymmetric identity $(x^2,x^2,x)=0$. Suppose in addition that*

(i) $A = A_1 \oplus A_{\frac{1}{2}}$, *where* $A_k := \{x \in A : e \bullet x = kx\}$ *for* $k = 1, \frac{1}{2}$,

(ii) $A_1 = \{x \in A : ex = xe = x\}$,

(iii) $ex^2 = x^2e = e \bullet x^2$ *and* $e(ex^2) = (x^2e)e = \frac{1}{2}(3e \bullet x^2 - x^2)$ *for every* $x \in A_k$ $(k = 1, \frac{1}{2})$.

Then $x^2 \in A_1$ *for every* $x \in A_1 \cup A_{\frac{1}{2}}$.

Proof. Suppose that A satisfies the identity $(x^2,x^2,x)=0$. Then, linearizing (7) we obtain the identity

$$\begin{aligned} &2(x^2, x\bullet y, y) + (x^2, y^2, x) + 2(x\bullet y, x^2, y) \\ &+4(x\bullet y, x\bullet y, x) + (y^2, x^2, x) = 0. \end{aligned} \tag{10}$$

Taking $x = e$ and replacing y with $x \in A_1$ in (10) we obtain

$$2(e,x,x) + (e,x^2,e) + 2(x,e,x) + 4(x,x,e) + (x^2,e,e) = 0.$$

Keeping in mind that $(e,x^2,e) = 0$ and $(x,e,x) = 0$ (by (ii)), we have

$$\begin{aligned} 0 &= 2(e,x,x) + 4(x,x,e) + (x^2,e,e) \\ &\overset{(ii)}{=} 2x^2 - 2ex^2 + 4x^2e - 4x^2 + (x^2e)e - x^2e \\ &\overset{(iii)}{=} -2x^2 + x^2e + (x^2e)e \\ &\overset{(iii)}{=} -2x^2 + e\bullet x^2 + \frac{1}{2}(3e\bullet x^2 - x^2) \\ &= \frac{5}{2}(-x^2 + e\bullet x^2). \end{aligned}$$

Therefore $e \bullet x^2 = x^2$, and so $x^2 \in A_1$.

Taking $x = e$ and replacing y with $x \in A_{\frac{1}{2}}$ in (10) we obtain

$$(e,x,x) + (e,x^2,e) + (x,e,x) + (x,x,e) + (x^2,e,e) = 0.$$

Since $(e,x^2,e)=0$, we get

$$\begin{aligned}
0 &= (e,x,x)+(x,e,x)+(x,x,e)+(x^2,e,e)\\
&= (ex)x-ex^2+(xe)x-x(ex)+x^2e-x(xe)+(x^2e)e-x^2e\\
&= (ex)x-ex^2+(xe)x-x(ex)-x(xe)+(x^2e)e\\
&= 2(e\bullet x)x-ex^2-2x(e\bullet x)+(x^2e)e\\
&= 2[e\bullet x,x]-ex^2+(x^2e)e\\
&= 2[\frac{1}{2}x,x]-ex^2+(x^2e)e\\
&= -ex^2+(x^2e)e\\
&\overset{(iii)}{=} -e\bullet x^2+\frac{1}{2}(3e\bullet x^2-x^2)\\
&= \frac{1}{2}(e\bullet x^2-x^2).
\end{aligned}$$

Therefore $e\bullet x^2=x^2$, and hence $x^2\in A_1$. □

3 Main results and consequences

The following result provides us with a *Peirce decomposition* (relative to a nonzero flexible idempotent) for any (121)-power associative algebra with no nonzero joint divisor of zero, and satisfying an asymmetric identity.

Theorem 3.1. *Let A be an (121)-power associative algebra with no nonzero joint divisor of zero, containing a nonzero flexible idempotent e, and satisfying an asymmetric identity. Then we have:*

(i) $A=A_1\oplus A_{\frac{1}{2}}$, *where* $A_k:=\{x\in A : e\bullet x=kx\}$ *for* $k=1,\frac{1}{2}$.

(ii) $A_1=\{x\in A : ex=xe=x\}$.

(iii) $A_1\bullet A_1\subseteq A_1$.

(iv) $A_{\frac{1}{2}}\bullet A_{\frac{1}{2}}\subseteq A_1$.

(v) *The projections P_k from A onto A_k ($k=1,\frac{1}{2}$) corresponding to the decomposition $A=A_1\oplus A_{\frac{1}{2}}$ are given by*

$$P_1=2L_e^\bullet-I \qquad \text{and} \qquad P_{\frac{1}{2}}=2(I-L_e^\bullet),$$

where I stands for the identity mapping on A.

Proof. It follows from (4) that $L_e(I-2L_e^\bullet)=R_e(I-2L_e^\bullet)=I-2L_e^\bullet$, hence $L_e^\bullet(I-2L_e^\bullet)=I-2L_e^\bullet$, and so

$$2(L_e^\bullet-I)(L_e^\bullet-\frac{1}{2}I)=-(L_e^\bullet-I)(I-2L_e^\bullet)=-L_e^\bullet(I-2L_e^\bullet)+I-2L_e^\bullet=0.$$

Therefore $(L_e^\bullet-I)(L_e^\bullet-\frac{1}{2}I)=0$, and hence assertions (i) and (v) follow from [5, Proposition 1.3.3], which also gives that $A_0:=\{x\in A : e\bullet x=0\}=0$.

For $x\in A_1$ we have the equality $x=2e\bullet x-x$, so by Lemma 2.1(ii), $[e,x]=[e,2e\bullet x-x]=0$, and hence $ex=xe=e\bullet x=x$, and assertion (ii) follows.

In order to prove (iii) and (iv), we will first prove that

$$[x^2, e] = 0 \quad \text{for every} \quad x \in A_k \ \ (k = 1, \frac{1}{2}). \tag{11}$$

A linearization of (3) gives that

$$(x, 2x \bullet y, y) + (x, y^2, x) + (y, x^2, y) + (y, 2x \bullet y, x) = 0. \tag{12}$$

By setting $x = e$ in (12) and keeping in mind that e is flexible, we have

$$(e, 2e \bullet y, y) + (y, e, y) + (y, 2e \bullet y, e) = 0. \tag{13}$$

By putting $y = x \in A_1$ in (13), we have

$$\begin{aligned} 0 &= (e, 2x \bullet e, x) + (x, e, x) + (x, 2x \bullet e, e) \\ &= 2(e, x, x) + (x, e, x) + 2(x, x, e) \\ &= 2(ex)x - 2ex^2 + (xe)x - x(ex) + 2x^2e - 2x(xe) \\ &= 2x^2 - 2ex^2 + x^2 - x^2 + 2x^2e - 2x^2 \\ &= 2(x^2e - ex^2), \end{aligned}$$

and so $[x^2, e] = 0$ for every $x \in A_1$.

Taking $y = x \in A_{\frac{1}{2}}$ in (13) and keeping in mind that $ez = z - ze$ for every $z \in A_{\frac{1}{2}}$, we get

$$\begin{aligned} 0 &= (e, 2x \bullet e, x) + (x, e, x) + (x, 2x \bullet e, e) \\ &= (e, x, x) + (x, e, x) + (x, x, e) \\ &= (ex)x - ex^2 + (xe)x - x(ex) + x^2e - x(xe) \\ &= (x - xe)x - ex^2 + (xe)x - x(x - xe) + x^2e - x(xe) \\ &= x^2 - (xe)x - ex^2 + (xe)x - x^2 + x(xe) + x^2e - x(xe) \\ &= -ex^2 + x^2e, \end{aligned}$$

and hence $[x^2, e] = 0$ for every $x \in A_{\frac{1}{2}}$. Thus (11) is proved.

We realize that

$$x^2e = ex^2 = e \bullet x^2 \quad \text{for every} \quad x \in A_k \ \ (k = 1, \frac{1}{2}). \tag{14}$$

On the other hand, keeping in mind (4) and (14), we obtain for every $x \in A_k$ $(k = 1, \frac{1}{2})$ that

$$e(ex^2) = e(e \bullet x^2) = \frac{1}{2}e\Big(x^2 - (x^2 - 2e \bullet x^2)\Big) = \frac{1}{2}ex^2 - \frac{1}{2}(x^2 - 2e \bullet x^2)$$

$$= \frac{1}{2}e \bullet x^2 - \frac{1}{2}x^2 + e \bullet x^2 = \frac{1}{2}(3e \bullet x^2 - x^2),$$

and

$$(x^2e)e = (e \bullet x^2)e = \frac{1}{2}\Big(x^2 - (x^2 - 2e \bullet x^2)\Big)e = \frac{1}{2}x^2e - \frac{1}{2}(x^2 - 2e \bullet x^2)$$

$$= \frac{1}{2}e \bullet x^2 - \frac{1}{2}x^2 + e \bullet x^2 = \frac{1}{2}(3e \bullet x^2 - x^2).$$

Therefore

$$e(ex^2) = (x^2e)e = \frac{1}{2}(3e \bullet x^2 - x^2) \quad \text{for every} \quad x \in A_k \ \ (k = 1, \frac{1}{2}), \tag{15}$$

and hence, at this point, we know that A satisfies assumptions (i), (ii) and (iii) in Lemma 2.3. Now, we claim that the assumption that A satisfies an asymmetric identity yields $x^2 \in A_1$ for all $x \in A_1 \cup A_{\frac{1}{2}}$. It is clear that the claim is true when A satisfies the identity $(x^2, x^2, x) = 0$ because of Lemma 2.3.

Suppose that A satisfies the identity $(x^2, x, x) = 0$. Then, linearizing (5) we obtain the identity

$$(y^2, x, x) + 2(x \bullet y, y, x) + 2(x \bullet y, x, y) + (x^2, y, y) = 0. \tag{16}$$

Taking $y = e$ in (16), we have

$$(e, x, x) + 2(x \bullet e, e, x) + 2(x \bullet e, x, e) + (x^2, e, e) = 0. \tag{17}$$

For $x \in A_1$ in (17) we obtain

$$(e, x, x) + 2(x, e, x) + 2(x, x, e) + (x^2, e, e) = 0.$$

Note that, by (ii), $(x, e, x) = 0$, and hence

$$\begin{aligned} 0 &= (e, x, x) + 2(x, x, e) + (x^2, e, e) \\ &= (ex)x - ex^2 + 2x^2e - 2x(xe) + (x^2e)e - x^2e \\ &= x^2 - ex^2 + 2x^2e - 2x^2 + (x^2e)e - x^2e \\ &\overset{(11)}{=} -x^2 + (x^2e)e \\ &\overset{(15)}{=} -x^2 + \frac{1}{2}(3e \bullet x^2 - x^2) \\ &= \frac{3}{2}(e \bullet x^2 - x^2). \end{aligned}$$

Therefore $e \bullet x^2 = x^2$, and hence $x^2 \in A_1$.

For $x \in A_{\frac{1}{2}}$ in (17) we obtain

$$(e, x, x) + (x, e, x) + (x, x, e) + (x^2, e, e) = 0,$$

and hence

$$\begin{aligned} 0 &= (ex)x - ex^2 + (xe)x - x(ex) + x^2e - x(xe) + (x^2e)e - x^2e \\ &= (ex)x - ex^2 + (xe)x - x(ex) - x(xe) + (x^2e)e \\ &= 2(e \bullet x)x - ex^2 - 2x(e \bullet x) + (x^2e)e \\ &\overset{(14)}{=} 2[e \bullet x, x] - e \bullet x^2 + (x^2e)e \\ &\overset{(15)}{=} 2[e \bullet x, x] - e \bullet x^2 + \frac{1}{2}(3e \bullet x^2 - x^2) \\ &= -e \bullet x^2 + \frac{1}{2}(3e \bullet x^2 - x^2) \\ &= \frac{1}{2}(e \bullet x^2 - x^2). \end{aligned}$$

Therefore $e \bullet x^2 = x^2$, and hence $x^2 \in A_1$. Thus the claim is true when A satisfies the identity $(x^2, x, x) = 0$.

Suppose now that A satisfies $(x, x, x^2) = 0$ or $(x, x^2, x^2) = 0$. Then the opposite algebra $A^{(0)}$ satisfies respectively $(x^2, x, x)^{(0)} = 0$ or $(x^2, x^2, x)^{(0)} = 0$, and so, taking into account that $A_1 = A_1^{(0)}$ and $A_{\frac{1}{2}} = A_{\frac{1}{2}}^{(0)}$, it follows from the paragraphs above that $x^2 \in A_1$ for every $x \in A_1 \cup A_{\frac{1}{2}}$.

Now that we know that the claim is true in all cases, (iii) and (iv) follow from the equality $x \bullet y = \frac{1}{4}\big((x+y)^2 - (x-y)^2\big)$. □

Lemma 3.2. *If A is an (121)-power associative algebra with no nonzero joint divisor of zero, containing a nonzero flexible idempotent e, and satisfying an asymmetric identity, then*

$$(xx^2)e + 2(x,x,x) - e(x^2x) = 0 \quad \textit{for every} \quad x \in A_{\frac{1}{2}}. \tag{18}$$

Proof. Indeed, by taking $x \in A_{\frac{1}{2}}$ and $y = e$ in (3), we have that

$$\begin{aligned} 0 &= (x,x^2,e) + (x, 2x \bullet e, x) + (e, x^2, x) \\ &= (x,x^2,e) + (x,x,x) + (e,x^2,x) \\ &= (xx^2)e - x(x^2e) + (x,x,x) + (ex^2)x - e(x^2x) \\ &\overset{3.1(iv)}{=} (xx^2)e - xx^2 + (x,x,x) + x^2x - e(x^2x) \\ &= (xx^2)e + 2(x,x,x) - e(x^2x), \end{aligned}$$

as desired. □

Lemma 3.3. *If A is an (121) and (222)-power associative algebra with no nonzero divisor of zero, containing a nonzero flexible idempotent e, and satisfying an asymmetric identity, then*

$$(x^2x)e = e(xx^2) \quad \textit{for every} \quad x \in A_{\frac{1}{2}}. \tag{19}$$

Proof. By linearizing three times $(x^2,x^2,x^2) = 0$, we obtain respectively

$$2(x^2,x^2,x \bullet y) + 2(x^2, x \bullet y, x^2) + 2(x \bullet y, x^2, x^2) = 0, \tag{20}$$

$$\begin{aligned} &(x^2,x^2,y^2) + 4(x^2, x \bullet y, x \bullet y) + (x^2,y^2,x^2) + 4(x \bullet y, x^2, x \bullet y) \\ &+4(x \bullet y, x \bullet y, x^2) + (y^2,x^2,x^2) = 0, \end{aligned} \tag{21}$$

and

$$\begin{aligned} &2(x \bullet y, x^2, y^2) + 2(x^2, x \bullet y, y^2) + 2(x^2, y^2, x \bullet y) + 2(x \bullet y, y^2, x^2) \\ &+2(y^2, x \bullet y, x^2) + 2(y^2, x^2, x \bullet y) + 8(x \bullet y, x \bullet y, x \bullet y) = 0. \end{aligned} \tag{22}$$

By setting $x \in A_{\frac{1}{2}}$ and $y = e$ in (22) and (3) and keeping in mind that $2x \bullet e = x$, we get respectively

$$\begin{aligned} &(x,x^2,e) + (x^2,x,e) + (x^2,e,x) + (x,e,x^2) \\ &+(e,x,x^2) + (e,x^2,x) + (x,x,x) = 0, \end{aligned} \tag{23}$$

and

$$(x,x^2,e) + (x,x,x) + (e,x^2,x) = 0. \tag{24}$$

By subtracting (24) from (23) we obtain

$$(x^2,x,e) + (x^2,e,x) + (x,e,x^2) + (e,x,x^2) = 0, \tag{25}$$

so

$$\begin{aligned} 0 &= (x^2,x,e) + (x^2,e,x) + (x,e,x^2) + (e,x,x^2) \\ &= (x^2x)e - x^2(xe) + (x^2e)x - x^2(ex) + (xe)x^2 \\ &\quad -x(ex^2) + (ex)x^2 - e(xx^2) \\ &= (x^2x)e - x^2(xe+ex) + x^2x + (xe+ex)x^2 - xx^2 - e(xx^2) \\ &= (x^2x)e - x^2x + x^2x + xx^2 - xx^2 - e(xx^2) \\ &= (x^2x)e - e(xx^2). \end{aligned}$$

□

Lemma 3.4. *If A is an (121)-power associative algebra with no nonzero divisor of zero, containing a nonzero flexible idempotent e, and satisfying the identity $(x^2, x, x) = 0$, then*

$$(x,x,x) + (x^2x)e = 0 \quad \text{for every} \quad x \in A_{\frac{1}{2}}. \tag{26}$$

Proof. Suppose that A satisfies the asymmetric identity $(x^2, x, x) = 0$. Taking $y = e$ and $x \in A_{\frac{1}{2}}$ in (5) and keeping in mind that $x^2 \in A_1$ and $x \bullet e = \frac{1}{2}x$ we obtain

$$\begin{aligned} 0 &= (2x \bullet e, x, x) + (x^2, e, x) + (x^2, x, e) \\ &= (x,x,x) + (x^2e)x - x^2(ex) + (x^2x)e - x^2(xe) \\ &= (x,x,x) + x^2x - x^2(ex + xe) + (x^2x)e \\ &= (x,x,x) + x^2x - x^2x + (x^2x)e \\ &= (x,x,x) + (x^2x)e. \end{aligned}$$

□

Proposition 3.5. *Let A be an (121) and (222)-power associative algebra with no nonzero joint divisor of zero, and satisfying the identity $(x^2, x, x) = 0$ or $(x, x, x^2) = 0$. If A contains a nonzero flexible idempotent e, then A is a unital power-associative algebra.*

Proof. Suppose that A satisfies the asymmetric identity $(x^2, x, x) = 0$. Let $x \in A_{\frac{1}{2}}$. By combining (19) and (26) we obtain

$$(x,x,x) + e(xx^2) = 0. \tag{27}$$

On the other hand, setting $x = e$ and $y = xx^2$ in (3), we have

$$\begin{aligned} 0 &= (e,e,xx^2) + (e, 2e \bullet (xx^2), e) + (xx^2, e, e) \\ &= (e,e,xx^2) + (xx^2, e, e) \\ &= e(xx^2) - e\Big(e(xx^2)\Big) + \Big((xx^2)e\Big)e - (xx^2)e \\ &\overset{(18)}{=} e(xx^2) - e\Big(e(xx^2)\Big) - 2(x,x,x)e + e(x^2x)e - (xx^2)e \\ &\overset{(27)}{=} -(x,x,x) + e(x,x,x) - 2(x,x,x)e + e(x^2x)e - (xx^2)e \\ &\overset{(26)}{=} -(x,x,x) + e(x,x,x) - 2(x,x,x)e - e(x,x,x) - (xx^2)e \\ &= -(x,x,x) - 2(x,x,x)e - (xx^2)e \\ &= -(x,x,x) - 2(x^2x)e + 2(xx^2)e - (xx^2)e \\ &= -(x,x,x) - 2(x^2x)e + (xx^2)e \\ &\overset{(26)}{=} -(x,x,x) + 2(x,x,x) + (xx^2)e, \end{aligned}$$

and so

$$(x,x,x) + (xx^2)e = 0. \tag{28}$$

By combining (26), (27) and (28), we obtain

$$(xx^2)e = e(xx^2) = (x^2x)e. \tag{29}$$

By subtracting (28) from (18) we have

$$(x,x,x) - e(x^2x) = 0, \tag{30}$$

and (26) implies that

$$(x^2x)e = -e(x^2x). \tag{31}$$

On the other hand, setting $x = e$ and $y = x^2x$ in (3) and keeping in mind that $(x^2x)e = -e(x^2x)$, we have

$$\begin{aligned}
0 &= (e,e,x^2x) + (e, 2e \bullet (x^2x), e) + (x^2x, e, e) \\
&= (e,e,x^2x) + (x^2x, e, e) \\
&= e\big(e(x^2x)\big) - e(x^2x) + \big((x^2x)e\big)e - (x^2x)e \\
&\overset{(31)}{=} e\big(e(x^2x)\big) + \big((x^2x)e\big)e \\
&\overset{(30)}{=} e(x,x,x) + \big((x^2x)e\big)e \\
&\overset{(26)}{=} e(x,x,x) - (x,x,x)e \\
&= e(x^2x) - e(xx^2) - (x^2x)e + (xx^2)e \\
&\overset{(29)}{=} e(x^2x) - (x^2x)e \\
&\overset{(31)}{=} -2(x^2x)e.
\end{aligned}$$

Keeping in mind (29) and (31), we obtain $(xx^2)e = e(xx^2) = 0$ and $(x^2x)e = e(x^2x) = 0$. Since $e \neq 0$ and A has no nonzero joint divisor of zero, we have $x^2x = xx^2 = 0$, so $x = 0$, and hence $A_{\frac{1}{2}} = 0$. We realize that $A = A_1$ and e is a nonzero central idempotent. Therefore, by [4, Theorem 1.2 or Corollary 3.1], A is a unital power-associative algebra.

Finally, suppose that A satisfies the asymmetric identity $(x,x,x^2) = 0$. So $A^{(0)}$ is an (121) and (222)-power associative algebra with no nonzero joint divisor of zero and $A^{(0)}$ satisfies $(x^2,x,x)^{(0)} = 0$. Since e is a nonzero flexible idempotent in $A^{(0)}$, the paragraphs above prove that $A^{(0)}$ is a unital power-associative algebra, and hence A is a unital power-associative algebra. □

Lemma 3.6. *If A is an (121)-power associative algebra with no nonzero joint divisor of zero, containing a nonzero flexible idempotent e, and satisfying the identity $(x^2,x^2,x) = 0$, then*

$$(x^2x)e = 0 \quad \textit{for every} \quad x \in A_{\frac{1}{2}}. \tag{32}$$

As a consequence,

$$\big(x^2x - e(x^2x)\big)e = 0 \quad \textit{for every} \quad x \in A_{\frac{1}{2}}. \tag{33}$$

Proof. Let $x \in A_{\frac{1}{2}}$. Taking $y = e$ in (10) and keeping in mind that $x \bullet e = \frac{1}{2}x$ and $x^2 \in A_1$, we get

$$\begin{aligned}
0 &= 2(x^2, x\bullet e, e) + (x^2, e, x) + 2(x\bullet e, x^2, e) + 4(x\bullet e, x\bullet e, x) + (e, x^2, x) \\
&= (x^2,x,e) + (x^2,e,x) + (x,x^2,e) + (x,x,x) + (e,x^2,x) \\
&= (x^2x)e - x^2(xe) + (x^2e)x - x^2(ex) + (xx^2)e - x(x^2e) \\
&\quad +(x,x,x) + (ex^2)x - e(x^2x) \\
&= (x^2x)e - x^2(xe+ex) + x^2x + (xx^2)e - xx^2 + (x,x,x) + x^2x - e(x^2x) \\
&= (x^2x)e - x^2x + x^2x + (xx^2)e + (x,x,x) + (x,x,x) - e(x^2x),
\end{aligned}$$

and so

$$(x^2x)e + (xx^2)e + 2(x,x,x) - e(x^2x) = 0. \tag{34}$$

By subtracting (18) from (34) we obtain $(x^2x)e = 0$. As a consequence, we see that

$$\Big(x^2x - e(x^2x)\Big)e = (x^2x)e - (e, x^2x, e) - e\Big((x^2x)e\Big) = 0.$$

□

Proposition 3.7. *Let A be an (121) and (222)-power associative algebra with no nonzero joint divisor of zero, and satisfying the identity $(x^2, x^2, x) = 0$ or $(x, x^2, x^2) = 0$. If A contains a nonzero flexible idempotent e, then A is a unital power-associative algebra.*

Proof. Suppose that A satisfies the asymmetric identity $(x^2, x^2, x) = 0$. Let $x \in A_{\frac{1}{2}}$.

Putting $x = e$ and $y = x^2x$ in (3) and keeping in mind the equality (32), we have

$$\begin{aligned} 0 &= (e, e, x^2x) + \Big(e, 2e \bullet (x^2x), e\Big) + (x^2x, e, e) \\ &= (e, e, x^2x) + (x^2x, e, e) \\ &= e(x^2x) - e\Big(e(x^2x)\Big) + \Big((x^2x)e\Big)e - (x^2x)e \\ &= e(x^2x) - e\Big(e(x^2x)\Big). \end{aligned}$$

We realize that $e\Big((x^2x) - e(x^2x)\Big) = 0$. This equality together with (33) yields

$$e\Big(x^2x - e(x^2x)\Big) = \Big(x^2x - e(x^2x)\Big)e = 0,$$

so $x^2x - e(x^2x) = 0$ because $e \neq 0$ and A has no nonzero joint divisor of zero, and hence $x^2x = e(x^2x)$.

Keeping in mind (25) and (32), we have

$$\begin{aligned} 0 &= (x^2, x, e) + (x^2, e, x) + (x, e, x^2) + (e, x, x^2) \\ &= (x^2x)e - x^2(xe) + (x^2e)x - x^2(ex) + (xe)x^2 \\ &\quad -x(ex^2) + (ex)x^2 - e(xx^2) \\ &= -x^2(xe) + x^2x - x^2(ex) + (xe)x^2 - xx^2 + (ex)x^2 - e(xx^2) \\ &= -x^2(xe + ex) + x^2x + (xe + ex)x^2 - xx^2 - e(xx^2) \\ &= -x^2(2x \bullet e) + x^2x + (2x \bullet e)x^2 - xx^2 - e(xx^2) \\ &= -x^2x + x^2x + xx^2 - xx^2 - e(xx^2) \\ &= -e(xx^2). \end{aligned}$$

Multiplying (18) on the left by e, and keeping in mind that $x^2x = e(x^2x)$ and $e(xx^2) = 0$, we obtain

$$\begin{aligned} 0 &= e(xx^2)e + 2e(x, x, x) - e\Big(e(x^2x)\Big) \\ &= \Big(e(xx^2)\Big)e + 2e(x, x, x) - x^2x \\ &= 2e(x, x, x) - x^2x \\ &= 2e(x^2x) - 2e(xx^2) - x^2x \\ &= 2x^2x - x^2x \\ &= x^2x. \end{aligned}$$

Putting $y = e$ in (7) and keeping in mind that $x^2 \in A_1$, we get

$$\begin{aligned} 0 &= (2x \bullet e, x^2, x) + (x^2, 2x \bullet e, x) + (x^2, x^2, e) \\ &= (x, x^2, x) + (x^2, x, x) + (x^2, x^2, e) \\ &= (x^2, x, x) + (x^2, x^2, e) \\ &= (x^2, x, x) + (x^2)^2 e - x^2(x^2 e) \\ &= (x^2, x, x) + (x^2)^2 - (x^2)^2 \quad \text{because} \quad (x^2)^2 \in A_1 \\ &= (x^2, x, x) \\ &= (x^2 x)x - (x^2)^2 \\ &= -(x^2)^2 \quad \text{because} \quad x^2 x = 0. \end{aligned}$$

Since A has no nonzero joint divisor of zero and $(x^2)^2 = 0$, so $x^2 = 0$. This implies also that $x = 0$, and hence $A_{\frac{1}{2}} = 0$. We deduce that $A = A_1$ and e is a nonzero central idempotent in A. The result follows from of [4, Theorem 1.2 or Corollary 3.1].

Finally, suppose that A satisfies the asymmetric identity $(x, x^2, x^2) = 0$. Then $A^{(0)}$ satisfies $(x^2, x^2, x)^{(0)} = 0$. Since e is a nonzero flexible idempotent in $A^{(0)}$ and $A^{(0)}$ is an (121) and (222)-power associative algebra with no nonzero joint divisor of zero, the paragraphs above prove that $A^{(0)}$ is a unital power-associative algebra, and hence A is a unital power-associative algebra. □

As a result, combining Propositions 3.5 and 3.7 we obtain Theorem 1.1.

The first consequence follows from of Theorem 1.1 and [2, Theorem 1.1].

Corollary 3.8. *Let A be an algebra with no nonzero joint divisor of zero and satisfying an asymmetric identity. If A contains a nonzero flexible idempotent, then following assertions are equivalent:*

(i) *A is third-power associative,*

(ii) *A is (121) and (222)-power associative.*

Under these conditions A is a unital power-associative algebra.

The algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are precisely the only power-associative absolute-valued real algebras [7] (see also [5, Proposition 2.6.27]). Combining this result and Theorem 1.1 we obtain the following.

Corollary 3.9. *Let A be an (121) and (222)-power associative absolute-valued real algebra satisfying an asymmetric identity. If A contains a nonzero flexible idempotent, then A is power-associative, and hence A is isomorphic to* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ *or* $\mathbb{O}$.

An algebra A is called *algebraic* if, for every $x \in A$, the subalgebra $A(x)$ of A generated by x is finite-dimensional. The algebra A is called *quadratic* if it is unital and, for every $x \in A$, x^2 lies in the linear hull of $\{\mathbf{1}, x\}$.

In [5, Proposition 2.5.10], Cabrera and Rodríguez show that every algebraic power-associative real algebra with no nonzero joint divisor of zero is a quadratic algebra. Combining this result or [4, Corollary 3.4], and Theorem 1.1 we obtain the following.

Corollary 3.10. *Let A be an algebraic (121) and (222)-power associative real algebra with no nonzero divisor of zero and satisfying an asymmetric identity. If A contains a nonzero flexible idempotent, then A is a quadratic algebra.*

A *normed algebra* is an algebra A over $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$ endowed with a norm $\|.\|$ such that $\|xy\| \leq \|x\|\,\|y\|$ for all $x, y \in A$. Let $S(A) = \{x \in A : \|x\| = 1\}$ be the unit-sphere of the normed algebra $(A, \|.\|)$. An element a of A is said to be a *joint topological divisor of zero* if there exists a sequence $(x_n)_{n\geq 0}$ in $S(A)$ such that

$$\lim_{n\to\infty} ax_n = \lim_{n\to\infty} x_n a = 0.$$

Clearly every joint divisor of zero in A is a joint topological divisor of zero in A. By [4, Corollary 3.5], *every normed (121)-power associative real algebra with no nonzero joint topological divisor of zero, satisfying an asymmetric identity, and containing a nonzero central idempotent, is a quadratic algebra*. Combining this result with Theorem 1.1 we obtain the following.

Corollary 3.11. *Let A be a normed (121) and (222)-power associative real algebra with no nonzero topological divisor of zero and satisfying an asymmetric identity. If A contains a nonzero flexible idempotent, then A is a quadratic algebra.*

Recall that an algebra A is said to be a *quasi-division algebra* in the sense of [9] (see [5, Definition 2.5.35 and Theorem 2.7.7]) if, for every nonzero element x of A, at least one of the operators L_x, R_x is bijective. Clearly quasi-division algebras have no nonzero joint divisor of zero. By [4, Corollary 3.6], *every complete normed third-power associative quasi-division real algebra, satisfying an asymmetric identity, and containing a nonzero central idempotent, is a quadratic algebra*. Combining this result with Theorem 1.1 we obtain the following.

Corollary 3.12. *Let A be a complete normed (121) and (222)-power associative quasi-division real algebra satisfying an asymmetric identity. If A contains a nonzero flexible idempotent, then A is a quadratic algebra.*

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References

[1] A. A. Albert, On the power-associativity of rings, *Summa Brasil. Math.* 2 (1948), 21–33.

[2] M. Amar, M. Cabrera and A. Diouf, Third-power associative algebras which satisfy an asymmetric identity, have no nonzero joint divisor of zero, and contain a nonzero idempotent, *Commun. Algebra* 53 (2025), 3974–3983.

[3] S. Attan, Representations and O-operators of Hom-(pre)-Jocobi-Jordan algebras, Moroccan J. Algebra Geom. Appl. (2025), to appear.

[4] M. Cabrera and A. Diouf. On power-associativity of algebras with no nonzero joint divisor of zero and containing a nonzero central idempotent, *Commun. Algebra* 52 (2024), 295–304.

[5] M. Cabrera and Á. Rodríguez, *Non-associative normed algebras. Volume 1: The Vidav-Palmer and Gelfand-Naimark Theorems*, Encyclopedia Math. Appl. 154, Cambridge University Press, 2014.

[6] A. Chandid, M. I. Ramírez and A. Rochdi, On finite-dimensional absolute-valued algebras satisfying $(x^p, x^q, x^r) = 0$, *Commun. Algebra* 40 (2012), 1525–1546.

[7] M. L. El-Mallah and A. Micali, Sur les algèbres normées sans diviseurs topologiques de zéro. *Bol. Soc. Mat. Mexicana* 25 (1980), 23–28.

[8] S. Okubo, Pseudo-quaternion and pseudo-octonion algebras, *Hadronic J.* 1 (1978), 1250–1278.

[9] Á. Rodríguez, Continuity of homomorphisms into normed algebras without topological divisors of zero. *Rev. R. Acad. Cienc. Exactas Fis. Nat. (Esp.)* 94 (2000), 505–514.

[10] S. Souleymane, T. Joseph and C. André, Evolution algebras satisfying a train identity of degree 2 and exponent $m > 3$, Moroccan J. Algebra Geom. Appl. 4(2) (2025), 275–291.

[11] K. Urbanik and F. B. Wright, Absolute-valued algebras, *Proc. Amer. Math. Soc.* 11 (1960), 861–866.