Moroccan Journal of Algebra
and Geometry with Applications
x (xx) (2025), 1–22

Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco **ISSN: 2820-7114**

Representations and $\mathcal{O}$-operators of Hom-(pre)-Jacobi-Jordan algebras

Sylvain Attan
Département de Mathématiques, Université d'Abomey-Calavi, 01 BP 4521, Cotonou 01, Bénin.
e-mail: *syltane2010@yahoo.fr*

Communicated by El Hassan El Kinani
(Received 03 June 2025 , Revised 12 October 2025, Accepted 18 October 2025)

Abstract. Representations and $\mathcal{O}$-operators of Hom-(pre)-Jacobi-Jordan algebras are introduced and studied. The anticommutator of a Hom-pre-Jacobi-Jordan algebra is a Hom-Jacobi-Jordan algebra and the left multiplication operator gives a representation of a Hom-Jacobi-Jordan algebra. The notion of matched pairs and Nijenhuis operators of Hom-(pre)-Jacobi-Jordan algebras are given and various relevant constructions are obtained.

Key Words: Hom-(pre)-Jacobi-Jordan algebras, matched pair, $\mathcal{O}$-operators, Nijenhuis operator.
2020 MSC: 16W10, 17A30, 17B10, 17C50.

1 Introduction

Jacobi-Jordan algebras are commutative algebras satisfying the Jacobi identity. Their first interest is motivated by the fact that they constitute an interesting sub-category of the well-referenced category of Jordan algebras which were introduced to explain some aspects of physics [17]. These algebras first appeared in [30], where an example of infinite-dimensional solvable but not nilpotent Jacobi-Jordan algebra was given. Since then, different names are used for these algebraic structures, indeed they are called for the first time mock-Lie algebras in [12], where the corresponding operad appears in the list of quadratic cyclic operads with one generator whereas the terms Jordan algebras of nil index 3 [27], Lie-Jordan algebras [24] and finally Jacobi-Jordan algebras in the recent paper [6] are used. Thanks to the approach of Eilenberg nicely described in [16], representations of Jacobi-Jordan algebras are introduced in [31] where many facts and conjectures about these algebras are made. As Pre-Lie algebras (also called left-symmetric algebras, quasi-associative algebras, Vinberg algebras and so on), left(resp. right)pre-Jacobi-Jordan algebras [11] are apparently first introduced in [3] under the name left skew-symmetric (resp. right skew-symmetric) algebras. The study of some relevant properties such as bimodules, matched pairs is considered for these algebras [11]. Moreover, a 2-dimensional classification and some double constructions of these algebras are given. Observe that there is a close relationship between pre-Jacobi-Jordan algebras and Jacobi-Jordan: a pre-Jacobi-Jordan algebra $(A,\cdot)$ gives rise to a Jacobi-Jordan algebra $(A,*)$ via the anticommutator multiplication, which is called the subadjacent Jacobi-Jordan algebra. Furthermore, for a given pre-Jacobi-Jordan algebra $(A,\cdot)$,the map $L: A \to gl(A)$, defined by $L_x y = x \cdot y$ for all $x, y \in A$, gives rise to a representation of its subadjacent pre-Jacobi-Jordan algebra.

The theory of Hom-algebras was first initiated by D. Larsson, S. D. Silvestrov and J. T. Hartwig [15],[19], [20] with the introduction of Hom-Lie algebras . It is known that any associative algebra is Lie-admissible i.e., the commutator algebra of any associative algebra is a Lie algebra. To be in adequacy with this fact in the theory of Hom-algebras, Hom-associative algebras were introduced [23] and it was shown that the commutator Hom-algebra of any Hom-associative algebra is a Hom-Lie algebra. Since then, other types of Hom-algebras such as Hom-Novikov algebras, Hom-alternative algebras, Hom-Jordan algebras or Hom-Malcev algebras are defined and discussed in [22], [28], [29].

The main objective of this paper is the study of representations of Hom-(pre)-Jacobi-Jordan algebras (see [1], [2], [8], [26] for the study of representations of other Hom-algebras) as well as the one of $\mathcal{O}$-operators, also known as relative or generalized Rota-Baxter operators on these Hom-algebras. First introduced in [4] by Baxter for associative algebras, Rota-Baxter operators have several applications in probability [4], combinatorics [7],[14], [25] and quantum field theory [10]. Rota-Baxter operators of weight 0 for Lie algebras [13] were introduced in terms of the classical Yang-Baxter equation and later on, Kupershmidt [18] defined $\mathcal{O}$-operators as generalized Rota-Baxter operators to understand classical Yang-Baxter equations and related integrable systems.

The paper is organized as follows. Section 2 is devoted to reminders of fundamental concepts. In Section 3, we introduce the notion of Hom-Jacobi-Jordan algebra, provide some properties and define the notion of a representation, $\mathcal{O}$-operators and matched pairs of a Hom-Jacobi-Jordan algebra. We prove that any Hom-Jacobi-Jordan algebra is a Hom-Jordan algebra. Moreover, we develop some constructions theorems about representations, $\mathcal{O}$-operators and matched pairs for these Hom-algebras. Finally, the last section contains many relevant results. First, we introduce Hom-pre-Jacobi-Jordan algebras and prove that the anticommutator of a Hom-pre-Jacobi-Jordan algebra is a Hom-Jacobi-Jordan algebra. Next, we introduce the notion of a representation of a left Hom-pre-Jacobi-Jordan algebra and develop some constructions theorems. The notion of matched pairs and $\mathcal{O}$-operators of such Hom-algebras have also been introduced and interesting results have been obtained.

Throughout this paper, all vector spaces and algebras are meant over a ground field $\mathbb{K}$ of characteristic 0.

2 Basic results on Hom-(pre)-Jacobi-Jordan algebras

This section is devoted to some definitions which are a very useful for next sections. Some elementary results are also proven.

Definition 2.1. *A Hom-module is a pair (A, α_M) consisting of a $\mathbb{K}$-module A and a linear self-map $\alpha_A : A \longrightarrow A$. A morphism $f : (A, \alpha_A) \longrightarrow (B, \alpha_B)$ of Hom-modules is a linear map $f : A \longrightarrow B$ such that $f \circ \alpha_A = \alpha_B \circ f$.*

Definition 2.2. *A Hom-algebra is a triple (A, μ, α) in which (A, α) is a Hom-module, $\mu : A^{\otimes 2} \longrightarrow A$ is a linear map. The Hom-algebra (A, μ, α) is said to be multiplicative if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$. A morphism $f : (A, \mu_A, \alpha_A) \longrightarrow (B, \mu_B, \alpha_B)$ of Hom-algebras is a morphism of the underlying Hom-modules such that $f \circ \mu_A = \mu_B \circ f^{\otimes 2}$.*

Definition 2.3. *Let (A, μ, α) be a Hom-algebra and $\lambda \in \mathbb{K}$. Let P be a linear map satisfying*

$$\mu(P(x), P(y)) = P(\mu(P(x), y) + \mu(x, P(y)) + \lambda\mu(x, y)), \ \forall x, y \in A \tag{1}$$

Then, P is called a Rota-Baxter operator of weight λ and (A, μ, α, P) is called a Rota-Baxter Hom-algebra of weight λ.

In the sequel, to unify our terminologies by a Rota-Baxter operator (resp. a Rota-Baxter Hom-algebra), we mean a Rota-Baxter operator (resp. a Rota-Baxter Hom-algebra) of weight $\lambda = 0$.

Definition 2.4. *[23] A Hom-associative algebra is a multiplicative Hom-algebra (A, μ, α) satisfying the Hom-associativity condition i.e.,*

$$as_\alpha(x, y, z) := \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z)) = 0 \text{ for all } x, y, z \in A \tag{2}$$

Definition 2.5. *[29] A Hom-Jordan algebra is a multiplicative Hom-algebra $(A,*,\alpha)$ such that the product "$*$" is commutative i.e., $x*y=y*x$ for all $x,y\in A$ and the following so-called Hom-Jordan identity holds*

$$as_{\alpha}(x*x,\alpha(y),\alpha(x))=0\ \forall x,y\in A \tag{3}$$

3 Hom-Jacobi-Jordan algebras

Definition 3.1. *A Hom-Jacobi-Jordan algebra is a multiplicative Hom-algebra $(A,*,\alpha)$ such that*

$$\begin{aligned}&x*y=y*x\ (\text{ commutativity})\\&J_{\alpha}(x,y,z):=\circlearrowleft_{(x,y,z)}(x*y)*\alpha(z)=0\end{aligned} \tag{4}$$

where $\circlearrowleft_{(x,y,z)}$ is the sum over cyclic permutation of x,y,z and J_{α} is called the Hom-Jacobian.

Remark 3.2. *If $\alpha=Id$ (identity map) in a Hom-Jacobi-Jordan algebra $(A,*,\alpha)$, then it reduces to a usual Jacobi-Jordan algebra $(A,*)$. It follows that the category of Hom-Jacobi-Jordan algebras contains the one of Jacobi-Jordan algebras.*

It is easy to prove the following:

Proposition 3.3. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra and β be a self-morphism of $(A,*,\alpha)$. Then $A_{\beta^n}:=(A,*_{\beta^n}:=\beta^n\circ *,\beta^n\alpha)$ is a Hom-Jacobi-Jordan algebra for each $n\in\mathbb{N}$.*

Example 3.4. *(i) Let $(A,*)$ be a Jacobi-Jordan algebra and α be a self-morphism of $(A,*)$. Then $A_{\alpha}:=(A,*_{\alpha}:=\alpha\circ *,\alpha)$ is a Hom-Jacobi-Jordan algebra.*

(ii) Consider the 4-dimensional Jacobi-Jordan algebra $\mathcal{A}:=(A,*)$ [6] where non-zero products with respect to a basis (e_1,e_2,e_3,e_4) are given by: $e_1*e_1:=e_2$; $e_1*e_4=e_4*e_1:=e_4$. Then, the linear map α defined by $\alpha(e_1):=-e_1-e_3$; $\alpha(e_2):=a_{12}e_1+e_2+2e_4$; $\alpha(e_3):=e_1+a_{23}e_2+e_3$;$\alpha(e_4):=a_{14}e_1-e_2+a_{34}e_3-e_4$ is a self-morphism of $\mathcal{A}$ for all scalars $a_{12},a_{23},a_{14},a_{34}$. Hence, $\mathcal{A}_{\alpha}:=(A,*_{\alpha}=\alpha\circ *,\alpha)$ is a Hom-Jacobi-Jordan algebra where non-zero products are: $e_1*_{\alpha}e_1:=a_{12}e_1+e_2+2e_4$; $e_1*_{\alpha}e_4=e_4*_{\alpha}e_1=:=a_{14}e_1-e_2+a_{34}e_3-e_4$.

Proposition 3.5. *Any Hom-Jacobi-Jordan algebra is a Hom-Jordan algebra.*

Proof. Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra. Then, by (4) with $z=x$ we obtain by the ommutativity of $*$

$$(x*x)*\alpha(y)+2\alpha(x)*(x*y)=0 \text{ for all } x,y\in A. \tag{5}$$

Now, if one replaces x by $\alpha(x)$ in (5), we get by the multiplicativity of α with respect to $*$

$$\alpha(x*x)*\alpha(y)+2\alpha^2(x)*(\alpha(x)*y)=0 \text{ for all } x,y\in A. \tag{6}$$

Replacing y by $x*y$ in (6) yields by the multiplicativity of α with respect to $*$

$$\begin{aligned}&\alpha(x*x)*(\alpha(x)*\alpha(y))=-2\alpha^2(x)*\Big(\alpha(x)*(x*y)\Big)=\alpha^2(x)*\Big(-2\alpha(x)*(x*y)\Big)\\&=\alpha^2(x)*\Big((x*x)*\alpha(y)\Big)\ (\text{ by (5) }).\end{aligned}$$

The Hom-Jordan identity follows by the commutativity of $*$. □

We know that the plus Hom-algebra of any Hom-associative algebra is a Hom-Jordan algebra i.e., any Hom-associative algebra is Hom-Jordan admissible [1]. In Hom-Jacobi-Jordan algebras case, this is not true, indeed:

Proposition 3.6. *Let $(A,\cdot,\alpha)$ be a Hom-associative algebra. Then $(A,\star,\alpha)$ is a Hom-Jacobi-Jordan algebra if and only if $2\circlearrowleft_{(x,y,z)}\Big((x\cdot y)\cdot\alpha(z)+(y\cdot x)\cdot\alpha(z)\Big)=0$ for all $x,y,z\in A$ where $x\star y=x\cdot y+y\cdot x$.*

Proof. By straightforward computations for all $x,y,z\in A$,

$$\begin{aligned}&\circlearrowleft_{(x,y,z)}(x\star y)\star\alpha(z)=(x\cdot y)\cdot\alpha(z)+(y\cdot x)\cdot\alpha(z)+\alpha(z)\cdot(x\cdot y)+\alpha(z)\cdot(y\cdot x)\\&+(y\cdot z)\cdot\alpha(x)+(z\cdot y)\cdot\alpha(x)+\alpha(x)\cdot(y\cdot z)+\alpha(x)\cdot(z\cdot y)+(z\cdot x)\cdot\alpha(y)+(x\cdot z)\cdot\alpha(y)\\&+\alpha(y)\cdot(z\cdot x)+\alpha(y)\cdot(x\cdot z)=2\circlearrowleft_{(x,y,z)}\Big((x\cdot y)\cdot\alpha(z)+(y\cdot x)\cdot\alpha(z)\Big)\text{ (by Hom-associativity).}\end{aligned}$$

□

Now, we give the definition of representations of a Hom-Jacobi-Jordan algebra.

Definition 3.7. *A representation of a Hom-Jacobi-Jordan algebra $(A,*,\alpha)$ is a triple (V,ρ,ϕ) where V is a vector space, $\phi\in gl(V)$ and $\rho:A\to gl(V)$ is a linear map such that the following equalities hold for all $x,y\in A$:*

$$\phi\rho(x)=\rho(\alpha(x))\phi; \tag{7}$$

$$\rho(x*y)\phi=-\rho(\alpha(x))\rho(y)-\rho(\alpha(y))\rho(x)\text{ for all }x,y\in A \tag{8}$$

Remark 3.8. *If $\alpha=Id_A$, $\phi=Id_V$, then (V,ρ) is a representation of the Jacobi-Jordan algebra $(A,\cdot)$ [31].*

Hence, we get the following first example.

Example 3.9. *Let $(A,\cdot)$ be a Jacobi-Jordan algebra and (V,ρ) be a representation of $(A,\cdot)$ in the usual sense. Then (V,ρ,Id_V) is a representation of the Hom-Jacobi-Jordan algebra $\mathcal{A}:=(A,\cdot,Id_A)$.*

To give other examples of representations of Hom-Jacobi-Jordan algebras, let prove the following:

Proposition 3.10. *Let $\mathcal{A}_1:=(A_1,*_1,\alpha_1)$ and $\mathcal{A}_2:=(A_2,*_2,\alpha_2)$ be two Hom-Jacobi-Jordan algebras and $f:\mathcal{A}_1\to\mathcal{A}_2$ be a morphism of Hom-Jacobi-Jordan algebras. Then $A_2^f:=(A_2,\rho,\alpha_2)$ is a representation of $\mathcal{A}_1$ where $\rho(a)b:=f(a)*_2b$ for all $(a,b)\in A_1\times A_2$.*

Proof. First, we have for all $(x,b)\in A_1\times A_2$,

$$\alpha_2(\rho(x)b)=\alpha_2(f(x))*_2\alpha_2(b)=f(\alpha_1(x))*_2\alpha_2(b)=\rho(\alpha_1(x))\alpha_2(b),$$

i.e., $\alpha_2(\rho(x))=\rho(\alpha_1(x))\alpha_2$. Next, for all $(x,y)\in A_1^{\times 2}$ and $b\in A_2$, since f is a morphism we have:

$$\begin{aligned}&\rho(x*_1y)\alpha_2(b)=f(x*_1y)*_2\alpha_2(b)=(f(x)*_2f(y))*_2\alpha_2(b)\\&=-(f(y)*_2b)*_2\alpha_2f(x)-(b*_2f(x))*_2\alpha_2f(y)\text{ (by (4))}\\&-f(\alpha_1(x))*_2(f(y)*_2b)-f(\alpha_1(y))*_2(f(x)*_2b)\text{ (commutativity of }*_2\text{)}\\&-\rho(\alpha_1(x))\rho(y)b-\rho(\alpha_1(y))\rho(x)b.\end{aligned}$$

Hence, $\rho(x*_1y)\alpha_2=-\rho(\alpha_1(x))\rho(y)-\rho(\alpha_1(y))\rho(x)$. □

Now, using Proposition 3.10, we obtain the following example as applications.

Example 3.11.

1. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra. Define a left multiplication $L:A\to gl(A)$ by $L(x)y:=x*y$ for all $x,y\in A$. Then (A,L,α) is a representation of $(A,*,\alpha)$, called a regular representation*

2. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra and (B,α) be a Hom-ideal of $(A,*,\alpha)$. Then (B,α) inherits a structure of representation of $(A,*,\alpha)$ where $\rho(a)b := a*b$ for all $(a,b)\in A\times B$.*

The following result can be proven easily.

Proposition 3.12. *Let $\mathcal{V} := (V,\rho,\phi)$ be a representation of a Hom-Jacobi-Jordan algebra $\mathcal{A} := (A,*,\alpha)$ and β be a self-morphism of $\mathcal{A}$. Then $\mathcal{V}_{\beta^n} =: (V,\rho_{\beta^n} := \rho\beta^n,\phi)$ is a representation of $\mathcal{A}$ for each $n\in\mathbb{N}$. In particular,$\mathcal{V}_{\alpha^n} =: (V,\rho_{\alpha^n} := \rho\alpha^n,\phi)$ is a representation of $\mathcal{A}$ for each $n\in\mathbb{N}$.*

Corollary 3.13. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra. For any integer $n\in\mathbb{N}$, define $L^n : A\to gl(A)$ by*

$$L^n(x)y = \alpha^n(x)*y \text{ for all } x,y\in A.$$

Then (A,L^n,α) is a representation of the Hom-Jacobi-Jordan algebra $(A,*,\alpha)$.

Let us prove the following necessary result.

Proposition 3.14. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra. Then (V,ρ,ϕ) is a representation of $(A,*,\alpha)$ if and only if the direct sum of vector spaces $A\oplus V$, turns into a Hom-Jacobi-Jordan algebra with the multiplication and the linear map defined by*

$$(x+u)\diamond(y+v) := x*y+(\rho(x)v+\rho(y)u) \tag{9}$$
$$(\alpha\oplus\phi)(x+u) := \alpha(x)+\phi(u) \tag{10}$$

This Hom-Jacobi-Jordan algebra is called the semi-direct product of A with V and is denoted by $A\ltimes V$.

Proof. It is clear that $\diamond$ is commutative and its multiplicativity with respect to $\alpha\oplus\phi$ is equivalent to Condition (7) and the multiplicativity of $*$ with respect to α. Next, for all $x,y,z\in A$, $u,v,w\in V$, we have by straightforward computations

$$\begin{aligned}
&\circlearrowleft_{(x+u,y+v,z+w)}\Big(((x+u)\diamond(y+v))\diamond(\alpha(z)+\phi(w))\Big)\\
=&\circlearrowleft_{(x+u,y+v,z+w)}\Big((x*y)*\alpha(z)+\rho(x*y)\phi(w)+\rho(\alpha(z))\rho(x)v+\rho(\alpha(z))\rho(y)u\Big)\\
=&\circlearrowleft_{(x,y,z)}(x*y)*\alpha(z)+\Big(\rho(x*y)\phi(w)+\rho(\alpha(x))\rho(y)w+\rho(\alpha(y))\rho(x)w\Big)+\Big(\rho(y*z)\phi(u)\\
&+\rho(\alpha(y))\rho(z)u+\rho(\alpha(z))\rho(y)u\Big)+\Big(\rho(z*x)\phi(v)+\rho(\alpha(z))\rho(x)v+\rho(\alpha(x))\rho(z)v\Big).
\end{aligned}$$

Hence, (4) holds for $A\ltimes V$ if and only if (8) holds. □

Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra and (V,ρ,ϕ) be a representation of $(A,*,\alpha)$ such that ϕ is invertible. Define $\rho^* : A\to gl(V^*)$ by $\langle\rho^*(x)(\xi),u\rangle = \langle\xi,\rho(x)u\rangle$, $\forall x\in A,\ u\in v,\xi\in V^*$. Note that in general ρ^* is not a representation of A (see [5] for details). Define $\rho^* : A\to gl(V^*)$ by

$$\rho^*(x)(\xi) := \rho^*(\alpha(x))((\phi)^{-2})^*(\xi)) \text{ for all } x\in A,\xi\in V^*. \tag{11}$$

Now, we can prove

Theorem 3.15. Let (V,ρ,ϕ) be a representation of a regular Hom-Jacobi-Jordan algebra $(A,*,\alpha)$ such that ϕ is invertible. Then $\rho^* : A\to gl(V^*)$ defined above by (11) is a representation of $(A,*,\alpha)$ on V^* with respect to $(\phi^{-1})^*$.

Proof. First, by (7) and (11), we get for all $x \in A, \xi \in V^*$,

$$\rho^*(\alpha(x))((\phi^{-1})^*(\xi)) = \rho^*(\alpha^2(x))((\phi^{-3})^*(\xi)) = (\phi^{-1})^*(\rho^*(\alpha(x))(\phi^{-2})^*(\xi)) = (\phi^{-1})^*(\rho^*(x)(\xi))$$

which implies $\rho^*(\alpha(x)) \circ (\phi^{-1})^* = (\phi^{-1})^* \circ \rho^*(x)$.

On the other hand, by (8), for all $x, y \in A$, $\xi \in V^*$ and $u \in V$, we have

$$\begin{aligned}
&\langle \rho^*(x*y)((\phi^{-1})^*(\xi)), u\rangle = \langle \rho^*(\alpha(x)*\alpha(y))((\phi^{-3})^*(\xi)), u\rangle = \langle (\phi^{-3})^*(\xi), \rho(\alpha(x)*\alpha(y))(u)\rangle\\
&= \langle (\phi^{-3})^*(\xi), -\rho(\alpha^2(x))\rho(\alpha(y))(\phi^{-1}(u)) - \rho(\alpha^2(y))\rho(\alpha(x))(\phi^{-1}(u))\rangle \text{ (by (8))}\\
&= \langle (\phi^{-4})^*(\xi), -\rho(\alpha^3(x))\rho(\alpha^2(y))(u) - \rho(\alpha^3(y))\rho(\alpha^2(x))(u)\rangle\\
&= \langle -\rho^*(\alpha^2(y))\rho^*(\alpha^3(x))(\phi^{-4})^*(\xi) - \rho^*(\alpha^2(x))\rho^*(\alpha^3(y))(\phi^{-4})^*(\xi), u\rangle\\
&= \langle -\rho^*(\alpha(y))\rho^*(x)\xi - \rho^*(\alpha(x))\rho^*(y)\xi, u\rangle
\end{aligned}$$

Hence, $\rho^*(x*y)(\phi^{-1})^* = -\rho^*(\alpha(x))\rho^*(y) - \rho^*(\alpha(y))\rho^*(x)$. Therefore, ρ^* is a representation of $(A, *, \alpha)$ on V^* with respect to $(\phi^{-1})^*$. □

Corollary 3.16. *Let $(A, *, \alpha)$ be a regular Hom-Jacobi-Jordan algebra. Then $ad^* : A \longrightarrow gl(A^*)$ defined by*

$$ad^*_x\xi := ad^*_{\alpha(x)}(\alpha^{-2})^*(\xi) \text{ for all } x \in A, \xi \in A^* \tag{12}$$

is a representation of the regular Hom-Jacobi-Jordan algebra $(A, *, \alpha)$ on A^* with respect to $(\alpha^{-1})^*$, which is called the coadjoint representation.

also, we get:

Corollary 3.17. *Let $(A, *, \alpha)$ be a regular Hom-Jacobi-Jordan algebra. Then there is a natural Hom-Jacobi-Jordan algebra $(A \oplus A^*, \diamond, \alpha \oplus (\alpha^{-1})^*)$, where the product $\diamond$ is given by*

$$\begin{aligned}
&(x_1 + \xi_1) \diamond (x_2 + \xi_2) := x_1 * x_2 + ad^*_{x_1}\xi_2 + ad^*_{x_2}\xi_1\\
&= x_1 * x_2 + ad^*_{\alpha(x_1)}(\alpha^{-2})^*(\xi_2) + ad^*_{\alpha(x_2)}(\alpha^{-2})^*(\xi_1) \text{ for all } x_1, x_2 \in A, \xi_1, \xi_2 \in A^*.
\end{aligned}$$

Definition 3.18. *A matched pair of Hom-Jacobi-Jordan algebras denoted by $(A_1, A_2, \rho_1, \rho_2)$, consists of two Hom-Jacobi-Jordan algebras $\mathcal{A}_1 := (A_1, *, \alpha_1)$ and $\mathcal{A}_2 := (A_2, \bullet, \alpha_2)$ together with representations $\rho_1 : A_1 \longrightarrow gl(A_2)$ and $\rho_2 : A_2 \longrightarrow gl(A_1)$ with respect to α_2 and α_1 respectively such that for all $x, y \in A_1$, $a, b \in A_2$. the following conditions hold*

$$\begin{aligned}
&\rho_1(\alpha_1(x))(a \bullet b) + (\rho_1(x)a) \bullet \alpha_2(b) + (\rho_1(x)b) \bullet \alpha_2(a)\\
&+\rho_1(\rho_2(a)x)\alpha_2(b) + \rho_1(\rho_2(b)x)\alpha_2(a) = 0 \qquad (13)\\
&\rho_2(\alpha_2(a))(x * y) + (\rho_2(a)x) * \alpha_1(y) + (\rho_2(a)y) * \alpha_1(x)\\
&+\rho_2(\rho_1(x)a)\alpha_1(y) + \rho_2(\rho_1(y)a)\alpha_1(x) = 0 \qquad (14)
\end{aligned}$$

Theorem 3.19. Let $\mathcal{A}_1 := (A_1, *, \alpha_1)$ and $\mathcal{A}_2 := (A_2, \bullet, \alpha_2)$ be Hom-Jacobi-Jordan algebras. Then, $(A_1, A_2, \rho_1, \rho_2)$, is a matched pair of Hom-Jacobi-Jordan algebras if and only if $(A_1 \oplus A_2, \diamond, \alpha_1 \oplus \alpha_2)$ is a Hom-Jacobi-Jordan algebra where

$$\begin{aligned}
(x + b) \diamond (y + b) &:= (x * y + \rho_2(a)y + \rho_2(b)x) + (a \bullet b + \rho_1(x)b + \rho_1(y)a) \qquad (15)\\
(\alpha_1 \oplus \alpha_2)(x + a) &:= \alpha_1(x) + \alpha_2(a)
\end{aligned}$$

Proof. First, the commutativity of $\diamond$ and its multiplicativity with respect to $\alpha_1 \oplus \alpha_2$ are equivalent to the multiplicativity of $*$ and $\bullet$ with respect to α_1 and α_2 respectively and Condition (7). Next, for all $x, y, z \in A_1$ and $a, b, c \in A_2$, we compute

$$\begin{aligned}
&\circlearrowleft_{(x+a,y+b,z+c)} \Big((x+b)\diamond(y+b)\Big)\diamond(\alpha_1\oplus\alpha_2)(z+c)\\
=&\circlearrowleft_{(x+a,y+b,z+c)} \Big((x*y)*\alpha(z)+(\rho_2(a)y)*\alpha_1(z)+(\rho_2(b)x)*\alpha_1(z)+\rho_2(a\bullet b)\alpha_1(z)\\
&+\rho_2(\rho_1(x)b)\alpha_1(z)+\rho_2(\rho_1(y)a)\alpha_1(z)+\rho_2(\alpha_2(c))(x*y)+\rho_2(\alpha_2(c))\rho_2(a)y+\rho_2(\alpha_2(c))\rho_2(b)x\Big)\\
&+\circlearrowleft_{(x+a,y+b,z+c)} \Big((a\bullet b)\bullet\alpha(c)+(\rho_1(x)b)\bullet\alpha_2(c)+(\rho_1(y)a)\bullet\alpha_2(c)+\rho_1(x*y)\alpha_2(c)\\
&+\rho_1(\rho_2(a)y)\alpha_2(c)+\rho_1(\rho_2(b)x)\alpha_2(c)+\rho_1(\alpha_1(z))(a\bullet b)+\rho_1(\alpha_1(z))\rho_1(x)b+\rho_1(\alpha_1(z))\rho_1(y)a\Big)\\
=&\Big(\circlearrowleft_{(x,y,z)}(x*y)*\alpha_1(z)\Big)+\Big(\rho_2(a\bullet b)\alpha_1(z)+\rho_2(\alpha_2(a))\rho_2(b)z+\rho_2(\alpha_2(b))\rho_2(a)z\Big)\\
&+\Big(\rho_2(b\bullet c)\alpha_1(x)+\rho_2(\alpha_2(b))\rho_2(c)x+\rho_2(\alpha_2(c))\rho_2(b)x\Big)+\Big(\rho_2(c\bullet a)\alpha_1(y)+\rho_2(\alpha_2(c))\rho_2(a)y\\
&+\rho_2(\alpha_2(a))\rho_2(c)y\Big)+\Big(\rho_2(\alpha_2(c))(x*y)+(\rho_2(c)x)*\alpha_1(y)+(\rho_2(c)y)*\alpha_1(x)+\rho_2(\rho_1(x)c)\alpha_1(y)\\
&+\rho_2(\rho_1(y)c)\alpha_1(x)\Big)+\Big(\rho_2(\alpha_2(b))(z*x)+(\rho_2(b)z)*\alpha_1(x)+(\rho_2(b)x)*\alpha_1(z)+\rho_2(\rho_1(z)b)\alpha_1(x)\\
&+\rho_2(\rho_1(x)b)\alpha_1(z)\Big)+\Big(\rho_2(\alpha_2(a))(y*z)+(\rho_2(a)y)*\alpha_1(z)+(\rho_2(a)z)*\alpha_1(y)+\rho_2(\rho_1(y)a)\alpha_1(z)\\
&+\rho_2(\rho_1(z)a)\alpha_1(y)\Big)+\Big(\circlearrowleft_{(a,b,c)}(a\bullet b)\bullet\alpha_2(c)\Big)+\Big(\rho_1(x*y)\alpha_2(c)+\rho_1(\alpha_1(x))\rho_1(y)c+\rho_1(\alpha_1(y))\rho_1(x)c\Big)
\end{aligned}$$

$$\begin{aligned}
&+\Big(\rho_1(y*z)\alpha_2(a)+\rho_1(\alpha_1(y))\rho_1(z)a+\rho_1(\alpha_1(z))\rho_1(y)a\Big)+\Big(\rho_1(z*x)\alpha_2(b)+\rho_1(\alpha_1(z))\rho_1(x)b\\
&+\rho_1(\alpha_1(x))\rho_1(z)b\Big)+\Big(\rho_1(\alpha_1(z))(a\bullet b)+(\rho_1(z)a)\bullet\alpha_2(b)+(\rho_1(z)b)\bullet\alpha_2(a)+\rho_1(\rho_2(a)z)\alpha_2(b)\\
&+\rho_1(\rho_2(b)z)\alpha_2(a)\Big)+\Big(\rho_1(\alpha_1(y))(c\bullet a)+(\rho_1(y)c)\bullet\alpha_2(a)+(\rho_1(y)a)\bullet\alpha_2(c)+\rho_1(\rho_2(c)y)\alpha_2(a)\\
&+\rho_1(\rho_2(a)y)\alpha_2(c)\Big)+\Big(\rho_1(\alpha_1(x))(b\bullet c)+(\rho_1(x)b)\bullet\alpha_2(c)+(\rho_1(x)c)\bullet\alpha_2(b)+\rho_1(\rho_2(b)x)\alpha_2(c)\\
&+\rho_1(\rho_2(c)x)\alpha_2(b)\Big).
\end{aligned} \tag{16}$$

Therefore, by (4) and (8), we have

$$\begin{aligned}
&\circlearrowleft_{(x+a,y+b,z+c)} \Big((x+b)\diamond(y+b)\Big)\diamond(\alpha_1\oplus\alpha_2)(z+c)\\
=&\Big(\rho_2(\alpha_2(c))(x*y)+(\rho_2(c)x)*\alpha_1(y)+(\rho_2(c)y)*\alpha_1(x)+\rho_2(\rho_1(x)c)\alpha_1(y)+\rho_2(\rho_1(y)c)\alpha_1(x)\Big)\\
&+\Big(\rho_2(\alpha_2(b))(z*x)+(\rho_2(b)z)*\alpha_1(x)+(\rho_2(b)x)*\alpha_1(z)+\rho_2(\rho_1(z)b)\alpha_1(x)+\rho_2(\rho_1(x)b)\alpha_1(z)\Big)\\
&+\Big(\rho_2(\alpha_2(a))(y*z)+(\rho_2(a)y)*\alpha_1(z)+(\rho_2(a)z)*\alpha_1(y)+\rho_2(\rho_1(y)a)\alpha_1(z)+\rho_2(\rho_1(z)a)\alpha_1(y)\Big)\\
&+\Big(\rho_1(\alpha_1(z))(a\bullet b)+(\rho_1(z)a)\bullet\alpha_2(b)+(\rho_1(z)b)\bullet\alpha_2(a)+\rho_1(\rho_2(a)z)\alpha_2(b)+\rho_1(\rho_2(b)z)\alpha_2(a)\Big)
\end{aligned}$$

$$+\Big(\rho_1(\alpha_1(y))(c\bullet a)+(\rho_1(y)c)\bullet\alpha_2(a)+(\rho_1(y)a)\bullet\alpha_2(c)+\rho_1(\rho_2(c)y)\alpha_2(a)+\rho_1(\rho_2(a)y)\alpha_2(c)\Big)$$
$$+\Big(\rho_1(\alpha_1(x))(b\bullet c)+(\rho_1(x)b)\bullet\alpha_2(c)+(\rho_1(x)c)\bullet\alpha_2(b)+\rho_1(\rho_2(b)x)\alpha_2(c)+\rho_1(\rho_2(c)x)\alpha_2(b)\Big).$$

Hence, (4) is satisfied in $A_1\oplus A_2$ if and only if (13) and (14) hold.

□

Definition 3.20. *Let (V,ρ,ϕ) be a representation of a Hom-Jacobi-Jordan algebra $(A,*,\alpha)$. A linear operator $T:V\longrightarrow A$ is called an $\mathcal{O}$-operator of A associated to ρ if it satisfies*

$$T\phi=\alpha T \tag{17}$$
$$T(u)*T(v)=T\Big(\rho(T(u))v+\rho(T(v))u\Big)\ \textit{for all } u,v\in V \tag{18}$$

Observe that Rota-Baxter operators on Hom-Jacobi-Jordan algebras are $\mathcal{O}$-operators with respect to the regular representation.

Example 3.21. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra and (V,ρ,ϕ) be a representation of $(A,*,\alpha)$. It is easy to verify that $A\oplus V$ is a representation of $(A,*,\alpha)$ under the maps $\rho_{A\oplus V}:A\longrightarrow gl(A\oplus V)$ defined by*

$$\rho_{A\oplus V}(a)(b+v):=a*b+\rho(a)v.$$

Define the linear map $T:A\oplus V\longrightarrow A, a+v\longmapsto a$. Then T is an $\mathcal{O}$-operator on A with respect to the representation $(A\oplus V,\rho_{A\oplus V},\alpha\oplus\phi)$.

Let give another example of $\mathcal{O}$-operators of Hom-Jacobi-Jordan algebras. As Hom-associative algebras case [9], let give some characterizations of $\mathcal{O}$-operators on Hom-Jacobi-Jordan algebras.

Proposition 3.22. *A linear map $T:V\longrightarrow A$ is an $\mathcal{O}$-operator associated to a representation (V,ρ,ϕ) of a Hom-Jacobi-Jordan algebra $(A,*,\alpha)$ if and only if the graph of T,*

$$G_r(T):=\{(T(v),v),v\in V\}$$

is a subalgebra of the semi-direct product algebra $A\ltimes V$.

The following result shows that an $\mathcal{O}$-operator can be lifted up the Rota-Baxter operator.

Proposition 3.23. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra, (V,ρ,ϕ) be a representation of A and $T:V\longrightarrow A$ be a linear map. Define $\widehat{T}\in End(A\oplus V)$ by $\widehat{T}(a+v):=Tv$. Then T is an $\mathcal{O}$-operator associated to (V,ρ,ϕ) if and only if $\widehat{T}$ is a Rota-Baxter operator on $A\oplus V$.*

In order to give another characterization of $\mathcal{O}$-operators, let introduce the following:

Definition 3.24. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra. A linear map $N:A\longrightarrow A$ is said to be a Nijenhuis operator if $N\alpha=\alpha N$ and its Nijenhuis torsions vanish, i.e.,*

$$N(x)*N(y)=N(N(x)*y+x*N(y)-N(x\cdot y)),\ \textit{for all } x,y\in A,$$

Observe that the deformed multiplications $*_N:A\oplus A\longrightarrow A$ given by

$$x*_N y:=N(x)*y+x*N(y)-N(x*y),$$

gives rise to a new Hom-Jacobi-Jordan multiplication on A, and N becomes a morphism from the Hom-Jacobi-Jordan algebra $(A,*_N,\alpha)$ to the initial Hom-Jacobi-Jordan algebra $(A,*,\alpha)$.

Now, we can easily check the following result.

Proposition 3.25. *Let $\mathcal{A}:=(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra and $\mathcal{V}:=(V,\rho,\phi)$ be a representation of $(A,*,\alpha)$. A linear map $T:V\longrightarrow A$ is an $\mathcal{O}$-operator associated to $\mathcal{V}$ if and only if $N_T:=\begin{pmatrix}0&T\\0&0\end{pmatrix}$: $A\oplus V\longrightarrow A\oplus V$ is a Nijenhuis operator on the Hom-Jordan algebra $A\oplus V$.*

4 Hom-pre-Jacobi-Jordan algebras

In this section, we generalize the notion of left (resp. right) pre-Jacobi-Jordan algebra first introduced in [3] as left-skew-symmetric (resp. right-skew-symmetric) algebras to the Hom case and study the relationships with Hom-Jacobi-Jordan algebras in terms of $\mathcal{O}$-operators of Hom-Jacobi-Jordan algebras.

4.1 Definition and basic properties

First let introduce the following:

Definition 4.1. *Let $(A,\cdot,\alpha)$ be a Hom-algebra. The anti-Hom-associator of $(A,\cdot,\alpha)$ is the map defined by*

$$Aasso_{\alpha}(x,y,z) := (x\cdot y)\cdot\alpha(z)+\alpha(x)\cdot(y\cdot z) \text{ for all } x,y,z\in A. \tag{19}$$

A multiplicative Hom-algebra $(A,\cdot,\alpha)$ is said to be an anti-Hom-associative algebra is $Aasso_{\alpha}(x,y,z)=0$ for all $x,y,z\in A$,

Clearly, any Hom-associative algebra is an anti-Hom-associative algebra.
Now, we give the definition of the main object of this subsection.

Definition 4.2. *A left Hom-pre-Jacobi-Jordan algebra is a multiplicative Hom-algebra $(A,\cdot,\alpha)$ satisfying*

$$Aasso_{\alpha}(x,y,z) = -Aasso_{\alpha}(y,x,z) \text{ for all } x,y,z\in A. \tag{20}$$

i.e., the anti-Hom-associator is left skew-symmetric. Actually, (20) is Equivariant to

$$(x\star y)\cdot\alpha(z) = -\alpha(x)\cdot(y\cdot z)-\alpha(y)\cdot(x\cdot z) \text{ for all } x,y,z\in A. \tag{21}$$

where $x\star y = x\cdot y = y\cdot x$ for all $x,y\in A$.

If the anti-Hom-associator is right skew-symmetric i.e.,

$$Aasso_{\alpha}(x,y,z) = -Aasso_{\alpha}(x,z,y) \tag{22}$$

or equivalently

$$\alpha(x)\cdot(y\star z) = -(x\cdot y)\cdot\alpha(z)-(x\cdot z)\cdot\alpha(y) \text{ for all } x,y,z\in A,$$

then, the multiplicative Hom-algebra is said to be a right Hom-pre-Jacobi-Jordan algebra.

If $\alpha = Id$(identity map) in (20 (resp. (22)), we obtain the identity defining the so-called left (resp. right) pre-Jacobi-Jordan algebra. Hence, any pre-Jacobi-Jordan algebra is a pre-Hom-Jacobi-Jordan algebra with Id as twisting map.

Remark 4.3. *(i) Any anti-associative algebra is a left and right Hom-pre-Jacobi-Jordan algebra.*
(ii) Observe that if $(A,\cdot,\alpha)$ is a left Hom-pre-Jacobi-Jordan algebra, then, the Hom-algebra defined on the same vector space A with "opposite" multiplication $x\perp y := y\cdot x$ is a right Hom-pre-Jacobi-Jordan algebra and vice-versa. Hence, all the statements for left Hom-pre-Jacobi-Jordan algebras have their corresponding statements for left Hom-pre-Jacobi-Jordan algebras. Thus, we will only consider the left Hom-pre-Jacobi-Jordan algebra case in this paper that we often call Hom-pre-Jacobi-Jordan algebra for short.

It is easy to prove the following.

Proposition 4.4. *Let $\mathcal{A} := (A,\mu,\alpha)$ be a Hom-pre-Jacobi-Jordan algebra and β be a morphism of $\mathcal{A}$. Then, $\mathcal{A}_{\beta^n} := (A,\mu_{\beta^n} := \beta^n\mu,\beta^n\alpha)$ is a Hom-pre-Jacobi-Jordan algebra for each $n\in\mathbb{N}$. In particular, $\mathcal{A}_{\alpha^n} := (A,\mu_{\alpha^n} := \alpha^n\mu,\alpha^{n+1})$ is a Hom-pre-Jacobi-Jordan algebra for each $n\in\mathbb{N}$.*

Example 4.5. *Let* $\mathcal{A} := (A, \mu)$ *be a pre-Jacobi-Jordan algebra and* β *be a morphism of* $\mathcal{A}$. *Then,* $\mathcal{A}_{\beta^n} := (A, \mu_{\beta^n} := \beta^n \mu, \beta^n)$ *is a Hom-pre-Jacobi-Jordan algebra for each* $n \in \mathbb{N}$.

Proposition 4.6. *Let* $(A, \cdot, \alpha)$ *be a left Hom-pre-Jacobi-Jordan algebra. Then the product given by*

$$x \star y = x \cdot y + y \cdot x \tag{23}$$

defines a Hom-Jacobi-Jordan algebra structure on A, *which is called the associated (or sub-adjacent) Hom-Jacobi-Jordan algebra of* $(A, \cdot, \alpha)$ *denoted by* A^C *and* $(A, \cdot, \alpha)$ *is also called a compatible left Hom-pre-Jacobi-Jordan algebra structure on the Hom-Jacobi-Jordan algebra* $A^C = (A, \star, \alpha)$.

Proof. For all $x, y, z \in A$, we prove (4) as follows

$$\begin{aligned} J_\alpha(x,y,z) =& \circlearrowleft_{(x,y,z)} (x \star y) \star \alpha(z) \\ =& \circlearrowleft_{(x,y,z)} \Big((x \cdot y) \cdot \alpha(z) + (y \cdot x) \cdot \alpha(z) + \alpha(z) \cdot (x \cdot y) + \alpha(z) \cdot (y \cdot x) \Big) \\ =& \circlearrowleft_{(x,y,z)} \Big(Aasso_\alpha(x,y,z) + Aasso_\alpha(y,x,z) \Big) = 0 \text{ (by (20)).} \end{aligned}$$

□

As consequence, we get

Proposition 4.7. *Let* $(A, \cdot, \alpha)$ *be a Hom-algebra. Then* $(A, \cdot, \alpha)$ *is a left Hom-pre-Jacobi-Jordan algebra if and only if* $(A, \star, \alpha)$ *defined by Eq. (23) is a Hom-Jacobi-Jordan algebra and* (A, L, α) *is a representation of* $(A, \star, \alpha)$, *where* L *denotes the left multiplication operator on* A.

Proof. Straightforward. □

Proposition 4.8. *Let* $(A, *, \alpha)$ *be a Hom-Jacobi-Jordan algebra and* (V, ρ, ϕ) *be a representation. If* T *is an* $\mathcal{O}$*-operator associated to* ρ, *then* $(V, \cdot, \phi)$ *is a left Hom-pre-Jacobi-Jordan algebra, where*

$$u \cdot v := \rho(T(u))v \text{ for } u, v \in V. \tag{24}$$

Therefore there exists an associated Hom-Jacobi-Jordan algebra structure on V *given by Eq. (23) and* T *is a homomorphism of Hom-Jacobi-Jordan algebras. Moreover,* $T(V) := \{T(v) | v \in V\} \subset A$ *is a Hom-Jacobi-Jordan subalgebra of* $(A, *, \alpha)$ *and there is an induced left Hom-pre-Jacobi-Jordan algebra structure on* $T(V)$ *given by*

$$T(u) \bullet T(v) := T(u * v) \text{ for } u, v \in V. \tag{25}$$

The corresponding associated Hom-Jacobi-Jordan algebra structure on $T(V)$ *given by Eq. (23) is just a Hom-Jacobi-Jordan subalgebra of* $(A, *, \alpha)$ *and* T *is a homomorphism of left Hom-pre-Jacobi-Jordan algebras.*

Proof. Let $u, v, w \in V$ and put $u \star v = u \cdot v + v \cdot u$. Note first that $T(u \star v) = T(u) * T(v)$. Then using (17), we compute (21) as follows

$$\begin{aligned} &(u \star v) \cdot \phi(w) = \rho(T(u) * T(v))\phi(w) - \rho(T\phi(u))\rho(tv)w - \rho(T\phi(v))\rho(tu)w \\ &= -\phi(u) \cdot (v \cdot w) - \phi(v) \cdot (u \cdot w). \end{aligned}$$

Therefore, $(V, \cdot, \phi)$ is a left Hom-pre-Jacobi-Jordan algebra. The other conclusions follow immediately. □

An obvious consequence of Proposition 4.8 is the following construction of a left Hom-pre-Jacobi-Jordan algebra in terms of a Rota-Baxter operator (of weight zero) of a Hom-Jacobi-Jordan algebra.

Corollary 4.9. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra and P be a Rota-Baxter operator (of weight zero) on A. Then there is a left Hom-pre-Jacobi-Jordan algebra structure on A given by*

$$x\cdot y := P(x)*y \text{ for all } x,y\in A. \tag{26}$$

Proof. Straightforward. □

Corollary 4.10. *Let $(A,*,\alpha)$ be a Hom-Jacobi-Jordan algebra. Then there exists a compatible left Hom-pre-Jacobi- Jordan algebra structure on A if and only if there exists an invertible $\mathcal{O}$-operator of $(A,*,\alpha)$.*

Proof. Let $(A,\cdot,\alpha)$ be a left Hom-pre-Jacobi-Jordan algebra and $(A,\star,\alpha)$ be the associated Hom-Jacobi-Jordan algebra. Then the identity map $id: A\longrightarrow A$ is an invertible $\mathcal{O}$-operator of $(A,\star,\alpha)$ associated to (A,ad,α).

Conversely, suppose that there exists an invertible $\mathcal{O}$-operator T of $(A,*,\alpha)$ associated to a representation (V,ρ,ϕ), then by Proposition 4.8, there is a left Hom-pre-Jacobi-Jordan algebra structure on $T(V)=A$ given by

$$T(u)\cdot T(v)=T(\rho(T(u))v), \text{ for all } u,v\in V.$$

If we set $T(u)=x$ and $T(v)=y$, then we obtain

$$x\cdot y=T(\rho(x)T^{-1}(y)), \text{ for all } x,y\in A..$$

It is a compatible left Hom-pre-Jacobi-Jordan algebra structure on $(A,*,\alpha)$. Indeed,

$$\begin{aligned}&x\cdot y+y\cdot x=T(\rho(x)T^{-1}(y)+\rho(y)T^{-1}(x))\\&=T(T^{-1}(x))*T(T^{-1}(y))=x*y.\end{aligned}$$

□

Proposition 4.11. *Let $T: V\longrightarrow A$ be an $\mathcal{O}$-operator on the Hom-Jacobi-Jordan algebra $(A,*,\alpha)$ with respect to the representation (V,ρ,ϕ). Let us define a map $\rho_T: V\longrightarrow gl(A)$ given by*

$$\rho_T(u)x := T(u)*x-T(\rho(x)u) \text{ for all } (u,x)\in V\times A$$

Then, the triplet (A,ρ_T,α) is a representation of the sub-adjacent Hom-Jacobi-Jordan algebra $V^c=(V,\star,\phi)$ associated with the left Hom-pre-Jacobi-Jordan algebra $(V,\cdot,\phi)$ defined in Proposition 4.8.

Proof. First, pick $(u,x)\in V\times A$. Then, the multiplicativity of α with respect to $*$, conditions (17) and (7) in (V,ρ,ϕ) give rise to (7) for (A,ρ_T,α) as follows

$$\alpha(\rho_T(u)x)=\alpha T(u)*\alpha(x)-\alpha T(\rho(x)u)=T\phi(u)*\alpha(x)-T(\rho(\alpha(x))\phi(u))=\rho_T(\phi(u))\alpha(x).$$

Next, let $u,v\in V,\ x\in A$. Recall that $u\star v=\rho(Tu)v+\rho(Tv)u$ and $T(u\star v)=T(u)*T(v)$. Then, by straightforward computations, we have

$$\rho_T(u\star v)\alpha(x)=(T(u)*T(v))*\alpha(x)-T\Big(\rho(\alpha(x))\rho(Tu)v\Big)-T\Big(\rho(\alpha(x))\rho(Tv)u\Big).$$

Also, we compute

$$\begin{aligned}&\rho_T(\phi(u))\rho_T(v)x=T\phi(u)*(Tv*x)-T\phi(u)*T(\rho(x)v)-T\Big(\rho(Tv*x)\phi(u)\Big)\\&+T\Big(\rho(T(\rho(x)v))\phi(u)\Big)=T\phi(u)*(Tv*x)-T\Big(\rho(T\phi(u))\rho(x)v+\rho(T(\rho(x)v))\phi(u)\Big)\\&-T\Big(\rho(T\phi(v))\rho(x)u+\rho(\alpha(x))\rho(Tv)u\Big)+T\Big(\rho(T(\rho(x)v))\phi(u)\Big)\ \text{(by (18), (17) and (8))}\\&=T\phi(u)*(Tv*x)-T\Big(\rho(T\phi(u))\rho(x)v\Big)+T\Big(\rho(T\phi(v))\rho(x)u\Big)+T\Big(\rho(\alpha(x))\rho(Tv)u\Big).\end{aligned}$$

Switching u and v in the above equation, we come to

$$\rho_T(\phi(v))\rho_T(u)x = T\phi(v)*(Tu*x) - T\Big(\rho(T\phi(v))\rho(x)u\Big) + T\Big(\rho(T\phi(u))\rho(x)v\Big) + T\Big(\rho(\alpha(x))\rho(Tu)v\Big).$$

It follows by (4) that

$$-\rho_T(\phi(u))\rho_T(v)x - \rho_T(\phi(v))\rho_T(u)x = \rho_T(u\star v)\alpha(x),$$

i.e., (8) holds in (A,ρ_T,α). □

4.2 Representations and $\mathcal{O}$-operators

This subsection is devoted to the study of representations and $\mathcal{O}$-operators of Hom-pre-Jacobi-Jordan algebras. We first give the definition of representations of pre-Jacobi-Jordan algebras and then, generalize this definition in the Hom-case. As mentioned, let first introduce the notion of a representation of a left Jacobi-Jordan algebra.

Definition 4.12. *A representation of a left pre-Jacobi-Jordan algebra $(A,\cdot)$ on a vector space V consists of a pair (ρ,λ), where $\rho,\lambda : A \longrightarrow gl(V)$ are linear maps satisfying:*

$$\rho(x\star y) = -\rho(x)\rho(y) - \rho(y)\rho(x) \tag{27}$$

$$\lambda(y)\lambda(x) + \lambda(x\cdot y) = -\lambda(y)\rho(x) - \rho(x)\lambda(y) \tag{28}$$

for all $x,y \in A$ where $x\star y := x\cdot y + y\cdot x$.

Observe that, Condition (27) means that (V,ρ) is a representation of the subadjacent Jacobi-Jordan of $(A,\cdot)$. One can easily prove that (V,ρ,λ) is a representation of a left pre-Jacobi-Jordan algebra $(A,\cdot)$ if and only if the direct sum $A\oplus V$ of vector space turns into a left pre-Jacobi-Jordan algebra under the product

$$(x+u)\diamond(y+v) := x\cdot y + (\rho(x)v + \lambda(y)u) \text{ for all } x,y \in A \text{ and } u,v \in V.$$

This left pre-Jacobi-Jordan algebra is called a semi-direct product of A and V denoted by $A \ltimes V$.

Remark 4.13. *Our definition of representations of left pre-Jacobi-Jordan algebras given above is different of those given in [11].*

Now, we can extend the notions of representations of pre-Jacobi-Jordan algebras in the Hom-case as follows:

Definition 4.14. *A representation of a left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$ on a vector space V with respect to $\phi \in gl(V)$ consists of a pair (ρ,λ), where $\rho : A \longrightarrow gl(V)$ is a representation of the subadjacent Hom-Jacobi-Jordan algebra A^C on V with respect to $\phi \in gl(V)$, and $\lambda : A \longrightarrow gl(V)$ is a linear map satisfying:*

$$\phi\lambda(x) = \lambda(\alpha(x))\phi \tag{29}$$

$$\lambda(\alpha(y))\lambda(x) + \lambda(x\cdot y)\phi = -\lambda(\alpha(y))\rho(x) - \rho(\alpha(x))\lambda(y) \tag{30}$$

Remark 4.15. *If $\alpha = Id_A$, $\phi = Id_V$, then (V,ρ,λ) is a representation of the pre-Jacobi-Jordan algebra $(A,\cdot)$ (see Definition 4.12 above).*

Hence, we obtain

Example 4.16. *Let $(A,\cdot)$ be a left pre-Jacobi-Jordan algebra and (V,ρ,λ) be a representation of $(A,\cdot)$ in the usual sense. Then (V,ρ,λ,Id_V) is a representation of the left Hom-pre-Jacobi-Jordan algebra $\mathcal{A} := (A,\cdot,Id_A)$.*

To give other examples of representations of Hom-Jacobi-Jordan algebras, let prove the following:

Proposition 4.17. *Let $\mathcal{A}_1 := (A_1,\cdot,\alpha_1)$ and $\mathcal{A}_2 := (A_2,\top,\alpha_2)$ be two left Hom-pre-Jacobi-Jordan algebras and $f : \mathcal{A}_1 \to \mathcal{A}_2$ be a morphism of left pre-Hom-Jacobi-Jordan algebras. Then $A_2^f := (A_2,\rho,\lambda,\alpha_2)$ is a representation of $\mathcal{A}_1$ where $\rho(a)b := f(a) \top b$ and $\lambda(a)b = b \top f(a)$ for all $(a,b) \in A_1 \times A_2$.*

Proof. First, it is clear that $\alpha_2\rho(x) = \rho(\alpha_1(x))\alpha_2$ and $\alpha_2\lambda(x) = \lambda(\alpha_1(x))\alpha_2$ for all $x \in A_1$. Next, set $x \star y := x\cdot y + y \cdot x$ and $b \circledast c = b \top c + c \top b$ for all $x,y \in A_1$ and $b,c \in A_2$. Then, one can show that $f(x\star y) = f(x) \circledast f(y)$. Hence, since f is a morphism, we have:

$$\begin{aligned}
&\rho(x\star y)\alpha_2(b) = f(x\star y)\alpha_2(b) = (f(x)\circledast f(y)) \top \alpha_2(b)\\
&= (f(x) \top f(y)) \top \alpha_2(b) + (f(x) \top f(y)) \top \alpha_2(b)\\
&= -\alpha_2 f(x) \top (f(y) \top b) - \alpha_2 f(y) \top (f(x) \top b) \text{ (by Condition 21 in } A_2 \text{)}\\
&= -\rho(\alpha_1(x))\rho(y)b - \rho(\alpha_1(y))\rho(x)b,
\end{aligned}$$

i.e., (8) is satisfied. Finally, using again f is a morphism, we obtain by straightforward computations

$$\begin{aligned}
&\lambda(\alpha_1(y))\lambda(x)b + \lambda(x\cdot y)\alpha_2(b)\\
&= (b \top f(x)) \top \alpha_2 f(x) + \alpha_2(b) \top (f(x) \top f(y))\\
&= -(f(x) \top b) \top \alpha_2 f(y) - \alpha_2 f(x) \top (b \top f(y)) \text{ (by Condition 20 in } A_2 \text{)}\\
&-\lambda(\alpha_1(y))\rho(x)b - \rho(\alpha_1(x))\lambda(y)v.
\end{aligned}$$

Hence, we get also (30) and the conclusion follows. □

Now, using Proposition 4.17, we obtain the following examples as applications.

Example 4.18.

1. *Let $(A,\cdot,\alpha)$ be a left Hom-pre-Jacobi-Jordan algebra. Define a left and right multiplications $L,R : A \to gl(A)$ by $L(x)y := x\cdot y$ and $R(x)y := y\cdot x$ for all $x,y \in A$. Then (A,L,R,α) is a representation of $(A,*,\alpha)$, called a regular representation*
2. *Let $(A,\cdot,\alpha)$ be a left Hom-pre-Jacobi-Jordan algebra and (B,α) be a two-sided-Hom-ideal of $(A,\cdot,\alpha)$. Then (B,α) inherits a structure of representation of $(A,\cdot,\alpha)$ where $\rho(a)b := a\cdot b$ and $\lambda(a)(b) = b\cdot a$ for all $(a,b) \in A \times B$.*

Similarly, the following result can be proven.

Proposition 4.19. *Let $\mathcal{V} := (V,\rho,\lambda,\phi)$ be a representation of a left Hom-pre-Jacobi-Jordan algebra $\mathcal{A} := (A,\cdot,\alpha)$ and β be a self-morphism of $\mathcal{A}$. Then $\mathcal{V}_{\beta^n} =: (V,\rho_{\beta^n} := \rho\beta^n,\lambda_{\beta^n} := \lambda\beta^n,\phi)$ is a representation of $\mathcal{A}$ for each $n \in \mathbb{N}$. In particular, $\mathcal{V}_{\alpha^n} =: (V,\rho_{\alpha^n} := \rho\alpha^n,\lambda_{\alpha^n} := \lambda\alpha^n,\phi)$ is a representation of $\mathcal{A}$ for each $n \in \mathbb{N}$.*

Corollary 4.20. *Let $(A,\cdot,\alpha)$ be a left Hom-pre-Jacobi-Jordan algebra. For any integer $n \in \mathbb{N}$, define $L^n, R^n : A \to gl(A)$ by*

$$L^n(x)y = \alpha^n(x)\cdot y, R^n(x)y = y\cdot \alpha^n(x) \text{ for all } x,y \in A.$$

Then (A,L^n,R^n,α) is a representation of the left Hom-pre-Jacobi-Jordan algebra $(A,*,\alpha)$.

Proposition 4.21. *Let $(A,\cdot,\alpha)$ be a left Hom-pre-Jacobi-Jordan algebra, V be a vector space, $\rho,\lambda : A \to gl(V)$ be linear maps and $\phi \in gl(V)$. Then (V,ρ,λ,ϕ) is a representation of A if and only if the direct sum $A \oplus V$ (as vector space) turns into a left Hom-pre-Jacobi-Jordan algebra (the semidirect sum) by defining the multiplication in $A \oplus V$ as*

$$(x+u)\diamond(y+v) := x\cdot y + (\rho(x)v + \lambda(y)u) \text{ for all } x,y \in A \text{ and } u,v \in V. \tag{31}$$

We denote it by $A \ltimes V$.

Proof. The multiplicativity of $\diamond$ with respect to $\alpha \oplus \phi$ is equivalent to conditions (29), (29) and the multiplicativity of $\cdot$ with respect to α. Next, for all $x,y,z \in A,\ u,v,w \in V$, we have by straightforward computations

$$\begin{aligned}&Aasso_{\alpha\oplus\phi}(x+u,y+v,z+w) = (x\cdot y)\cdot\alpha(z) + \rho(x\cdot y)\phi(w) + \lambda(\alpha(z))\rho(x)v + \lambda(\alpha(z))\lambda(y)u\\ &+\alpha(x)\cdot(y\cdot z) + \rho(\alpha(x))\rho(y)w + \rho(\alpha(x))\lambda(z)v + \lambda(y\cdot z)\phi(u) = Aasso_\alpha(x,y,z) + \Big(\rho(x\cdot y)\phi(w)\\ &+\rho(\alpha(x))\rho(y)w\Big) + \Big(\lambda(\alpha(z))\rho(x)v + \rho(\alpha(x))\lambda(z)v\Big) + \Big(\lambda(\alpha(z))\lambda(y)u + \lambda(y\cdot z)\phi(u)\Big)\end{aligned}$$

Switching $x+u$ and $y+v$ in the expression above of $Aasso_{\alpha\oplus\phi}(x+u,y+v,z+w)$, we get

$$\begin{aligned}&Aasso_{\alpha\oplus\phi}(y+v,x+u,z+w) = Aasso_\alpha(x,y,z) + \Big(\rho(x\cdot y)\phi(w) + \rho(\alpha(x))\rho(y)w\Big) + \Big(\lambda(\alpha(z))\rho(x)v\\ &+\rho(\alpha(x))\lambda(z)v\Big) + \Big(\lambda(\alpha(z))\lambda(y)u + \lambda(y\cdot z)\phi(u)\Big).\end{aligned}$$

Hence, (20) holds for $A \oplus V$ if and only if (8) (with $* = \star$) and (30) hold. □

Proposition 4.22. *Let (V,ρ,λ,ϕ) is a representation of a left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$ and $(A,\star,\alpha)$ be its associated Hom-Jacobi-Jordan algebra. Then $(V,\rho+\lambda,\phi)$ is a representation of $(A,\star,\alpha)$,*

Proof. By Proposition 4.21, $A \ltimes V$ is a left Hom-pre-Jacobi-Jordan algebra. Consider its associated Hom-Jacobi-Jordan algebra $(A\oplus V,\tilde{\diamond},\alpha+\phi)$, we have

$$\begin{aligned}(x+u)\tilde{\diamond}(y+v) &= (x+u)\diamond(y+v) + (y+v)\diamond(x+u)\\ &= x\cdot y + (\rho(x)v + \lambda(y)u) + y\cdot x + (\rho(y)u + \lambda(x)v)\\ &= x\star y + \Big((\rho+\lambda)(x)v + (\rho+\lambda)(y)u\Big)\end{aligned}$$

Thanks to Proposition 3.14, we deduce that $(V,\rho+\lambda,\phi)$ is a representation of $(A,*,\alpha)$. □

As Hom-Jacobi-Jordan algebras case, let $(A,\cdot,\alpha)$ be a regular left Hom-pre-Jacobi-Jordan algebra and (V,ρ,λ,ϕ) be a representation of $(A,\cdot,\alpha)$ such that ϕ is invertible. Define $\rho^*,\lambda^* : A \to gl(V^*)$ by $\langle\rho^*(x)(\xi),u\rangle = \langle\xi,\rho(x)u\rangle$, and $\langle\lambda^*(x)(\xi),u\rangle = \langle\xi,\lambda(x)u\rangle$, $\forall x \in A,\ u \in v, \xi \in V^*$. Define $\rho^*,\lambda^* : A \to gl(V^*)$ by

$$\rho^*(x)(\xi) := \rho^*(\alpha(x))((\phi)^{-2})^*(\xi)) \text{ for all } x \in A, \xi \in V^*. \tag{32}$$

$$\lambda^*(x)(\xi) := \lambda^*(\alpha(x))((\phi)^{-2})^*(\xi)) \text{ for all } x \in A, \xi \in V^*. \tag{33}$$

Now, we can prove

Theorem 4.23. Let (V,ρ,λ,ϕ) be a representation of a regular left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$ such that ϕ is invertible. Then $(V,\rho^*+\lambda^*,-\lambda^*,(\phi^{-1})^*)$ is a representation of $(A,\cdot,\alpha)$ called the dual representation of $(A,\cdot,\alpha)$.

Proof. First, since (V,ρ,λ,ϕ) is a representation of a left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$, thanks to Proposition 4.22, we get that $(V,\rho+\lambda,\phi)$ is a representation of the sub-adjacent Hom-Jacobi-Jordan $A^C=(A,\star,\alpha)$. Hence, by Theorem 3.15, we deduce that $(V^*,(\rho+\lambda)^*=\rho^*+\lambda^*,(\phi^{-1})^*)$ is a representation of the Hom-Jacobi-Jordan algebra A^C. Next, by (29) and (33), we get for all $x\in A,\xi\in V^*$,

$$-\lambda^*(\alpha(x))((\phi^{-1})^*(\xi))=-\lambda^*(\alpha^2(x))((\phi^{-3})^*(\xi))=(\phi^{-1})^*(-\lambda^*(\alpha(x))(\phi^{-2})^*(\xi))=(\phi^{-1})^*(-\lambda^*(x)(\xi))$$

which implies $-\lambda^*(\alpha(x))\circ(\phi^{-1})^*=(\phi^{-1})^*\circ(-\lambda^*(x))$.

Finally, using (32 and 33), we compute for all $x,y\in A$, $\xi\in V^*$ and $u\in V$:

$$\begin{aligned}
&\langle\lambda^*(x\cdot y)(\phi^{-1})^*(\xi),u\rangle=\langle\lambda^*(\alpha(x)\cdot\alpha(y))(\phi^{-3})^*(\xi),u\rangle=\langle(\phi^{-3})^*(\xi),\lambda(\alpha(x)\cdot\alpha(y))\rangle\\
&=\langle(\phi^{-3})^*(\xi),-\lambda(\alpha^2(y))\lambda(\alpha(x))\phi^{-1}(u)-\lambda(\alpha^2(y))\rho(\alpha(x))\phi^{-1}(u)-\rho(\alpha^2(x))\lambda(\alpha(y))\phi^{-1}(u)\rangle\ (\text{ by }(30)\ )\\
&=\langle(\phi^{-4})^*(\xi),-\lambda(\alpha^3(y))\lambda(\alpha^2(x))u-\lambda(\alpha^3(y))\rho(\alpha^2(x))u-\rho(\alpha^3(x))\lambda(\alpha^2(y))u\rangle\\
&=\langle-\lambda^*(\alpha^2(x))\lambda^*(\alpha^3(y))(\phi^{-4})^*(\xi)-\rho^*(\alpha^2(x))\lambda^*(\alpha^3(y))(\phi^{-4})^*(\xi)-\lambda^*(\alpha^2(y))\rho^*(\alpha^3(x))(\phi^{-4})^*(\xi),u\rangle\\
&=\langle-\lambda^*(\alpha(x))\lambda^*(y)(\xi)-\rho^*(\alpha(x))\lambda^*(y)(\xi)-\lambda^*(\alpha^2(y))\rho^*(x)(\xi),u\rangle
\end{aligned}$$

Hence, $\lambda^*(\alpha(y))\lambda^*(x)-\lambda^*(x\cdot y)(\phi^{-1})^*=\lambda^*(\alpha(y))(\rho^*+\lambda^*)(x)+(\rho^*+\lambda^*)(\alpha(x))\lambda^*(y)$. This ends the proof. ☐

Corollary 4.24. *Let $(A,\cdot,\alpha)$ be a regular left Hom-pre-Jacobi-Jordan algebra. Then $(A,L^*+R^*,-R^*,(\alpha^{-1})^*)$ is a representation of $(A,\cdot,\alpha)$ where $L^*,R^*:A\longrightarrow gl(A^*)$ are defined by*

$$\begin{aligned}
L^*_x\xi &:= L^*_{\alpha(x)}(\alpha^{-2})^*(\xi)\ \textit{for all } x\in A,\xi\in A^*,\\
R^*_x\xi &:= R^*_{\alpha(x)}(\alpha^{-2})^*(\xi)\ \textit{for all } x\in A,\xi\in A^*
\end{aligned}$$

This representation is called the coadjoint representation.

also, we get:

Corollary 4.25. *Let $(A,\cdot,\alpha)$ be a regular left Hom-pre-Jacobi-Jordan algebra. Then there is a natural left Hom-pre-Jacobi-Jordan algebra $(A\oplus A^*,\diamond,\alpha\oplus(\alpha^{-1})^*)$, where the product $\diamond$ is given by*

$$(x_1+\xi_1)\diamond(x_2+\xi_2):=x_1*x_2+L^*_{x_1}\xi_2+R^*_{x_1}\xi_2-R^*_{x_2}\xi_1\ \textit{for all } x_1,x_2\in A,\xi_1,\xi_2\in A^*.$$

Similarly as in [21], the following result can be proved.

Proposition 4.26. *Let (V,ρ,λ,ϕ) be a representation of a regular left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$ such that ϕ is invertible. Then the dual representation of $(V,\rho^*,\lambda^*,(\phi^{-1})^*)$ is (V,ρ,λ,ϕ).*

Fom Theorem 4.23 and Proposition 4.26, it follows easily

Proposition 4.27. *Let (V,ρ,λ,ϕ) be a representation of a regular left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$ such that ϕ is invertible. Then the following conditions are equivalent:*

(i) *$(V,\rho+\lambda,-\lambda,\phi)$ is a representation of the left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$,*

(ii) *$(V,\rho^*,\lambda^*,(\phi^{-1})^*)$ is a representation of of the left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$,*

(iii) *$\mu(\alpha(x))\mu(y)=\mu(\alpha(y))\mu(x)$ for all $x,y\in A$.*

Definition 4.28. *A matched pair of left Hom-pre-Jacobi-Jordan algebras denoted by* $(A_1, A_2, \rho_1, \lambda_1, \rho_2, \lambda_2)$, *consists of two left Hom-pre-Jacobi-Jordan algebras* $\mathcal{A}_1 := (A_1, \cdot, \alpha_1)$ *and* $\mathcal{A}_2 := (A_2, \top, \alpha_2)$ *together with representations* $\rho_1, \mu_1 : A_1 \to gl(A_2)$ *and* $\rho_2, \mu_2 : A_2 \to gl(A_1)$ *with respect to* α_2 *and* α_1 *respectively such that for all* $x, y, z \in A_1$, $a, b, c \in A_2$. *the following conditions hold*

$$\begin{aligned}
\rho_1(\alpha_1(x))(a \top b) &= -\rho_1(\rho_2(a)x + \lambda_2(a)x)\alpha_2(b) - (\rho_1(x)a + \lambda_1(x)a) \top \alpha_2(b) \\
&\quad -\lambda_1(\lambda_2(b)x)\alpha_2(a) - \alpha_2(a) \top (\rho_1(x)b) \qquad (34)\\
\lambda_1(\alpha_1(x))(a \circledast b) &= -\alpha_2(a) \top (\lambda_1(x)b) - \alpha_2(b) \top (\lambda_1(x)a) - \lambda_1(\rho_2(a)x)\alpha_2(b) \\
&\quad -\lambda_1(\rho_2(b)x)\alpha_2(a) \qquad (35)\\
\rho_2(\alpha_2(a))(x \cdot y) &= -\rho_2(\rho_1(x)a + \lambda_1(x)a)\alpha_1(y) - (\rho_2(a)x + \lambda_2(a)x) \cdot \alpha_1(y) \\
&\quad -\lambda_2(\lambda_1(y)a)\alpha_1(x) - \alpha_1(x) \cdot (\rho_2(a)y) \qquad (36)\\
\lambda_2(\alpha_2(a))(x \star y) &= -\alpha_1(x) \cdot (\lambda_2(a)y) - \alpha_1(y) \cdot (\lambda_2(a)x) - \lambda_2(\rho_1(x)a)\alpha_1(y) \\
&\quad -\lambda_2(\rho_1(y)a)\alpha_1(x) \qquad (37)
\end{aligned}$$

where $\star$ *is a product of the sub-adjacent Hom-Jacobi-Jordan algebra* A_1^C *and* $\circledast$ *is a product of the sub-adjacent Hom-Jacobi-Jordan algebra* A_2^C.

Theorem 4.29. Let $\mathcal{A}_1 := (A_1, \cdot, \alpha_1)$ and $\mathcal{A}_2 := (A_2, \top, \alpha_2)$ be left Hom-pre-Jacobi-Jordan algebras. Then, $(A_1, A_2, \rho_1, \lambda_1, \rho_2, \lambda_2)$, is a matched pair of left Hom-pre-Jacobi-Jordan algebras if and only if $(A_1 \oplus A_2, \diamond, \alpha_1 \oplus \alpha_2)$ is a left Hom-pre-Jacobi-Jordan algebra where

$$\begin{aligned}
(x + a) \diamond (y + b) &:= (x \cdot y + \rho_2(a)y + \lambda_2(b)x) + (a \top b + \rho_1(x)b + \lambda_1(y)a) \qquad (38)\\
(\alpha_1 \oplus \alpha_2)(x + a) &:= \alpha_1(x) + \alpha_2(a)
\end{aligned}$$

Proof. First, the commutativity of $\diamond$ and its multiplicativity with respect to $\alpha_1 \oplus \alpha_2$ is equivalent to the multiplicativity of $\cdot$ and $\top$ with respect to α_1 and α_2 respectively and Condition (7). Next, let $x, y, z \in A_1$ and $a, b, \in A_2$, then by straightforward computations

$$\begin{aligned}
&Aasso_{\alpha_1 \oplus \alpha_2}(x + a, y + b, z + c) \\
&= ((x + a) \diamond (y + b)) \diamond (\alpha_1 \oplus \alpha_2)(z + c) + (\alpha_1 \oplus \alpha_2)(x + a) \diamond ((y + b) \diamond (z + c)) \\
&= (x \cdot y) \cdot \alpha_1(z) + (\rho_2(a)y) \cdot \alpha_1(z) + (\lambda_2(b)x) \cdot \alpha_1(z) + \rho_2(a \top b)\alpha_1(z) + \rho_2(\rho_1(x)b)\alpha_1(z) \\
&+\rho_2(\lambda_1(y)a)\alpha_1(z) + \lambda_2(\alpha_2(c))(x \cdot y) + \lambda_2(\alpha_2(c))\rho_2(a)y + \lambda_2(\alpha_2(c))\lambda_2(b)x + (a \top b) \top \alpha_2(c) \\
&+(\rho_1(x)b) \top \alpha_2(c) + (\lambda_1(y)a) \top \alpha_2(c) + \rho_1(x \cdot y)\alpha_2(c) + \rho_1(\rho_2(a)y)\alpha_2(c) + \rho_1(\lambda_2(b)x)\alpha_2(c) \\
&+\lambda_1(\alpha_1(z))(a \top b) + \lambda_1(\alpha_1(z))\rho_1(x)b + \lambda_1(\alpha_1(z))\lambda_1(y)a + \alpha_1(x) \cdot (y \cdot z) + \alpha_1(x) \cdot (\rho_2(b)z) \\
&+\alpha_1(x) \cdot (\lambda_2(c)y) + \rho_2(\alpha_2(a))(y \cdot z) + \rho_2(\alpha_2(a))\rho_2(b)z + \rho_2(\alpha_2(a))\lambda_2(c)y + \lambda_2(b \top c)\alpha_1(x) \\
&+\lambda_2(\rho_1(y)c)\alpha_1(x) + \lambda_2(\lambda_1(z)b)\alpha_1(x) + \alpha_2(a) \top (b \top c) + \alpha_2(a) \top (\rho_1(y)c) + \alpha_2(a) \top (\lambda_1(z)b) \\
&+\rho_1(\alpha_1(x))(b \top c) + \rho_1(\alpha_1(x))\rho_1(y)c + \rho_1(\alpha_1(x))\lambda_1(z)b + \lambda_1(y \cdot z)\alpha_2(a) + \lambda_1(\rho_2(b)z)\alpha_2(a) \\
&+\lambda_1(\lambda_2(c)y)\alpha_2(a).
\end{aligned}$$

Switching $x + a$ and $y + b$ in the above expression of $as_{\alpha_1 \oplus \alpha_2}(x + a, y + b, z + c)$, we obtain $as_{\alpha_1 \oplus \alpha_2}(y + b, x +$

$a, z+c$). Next, after rearranging terms, we obtain

$$\begin{aligned}
&Aasso_{\alpha_1\oplus\alpha_2}(x+a, y+b, z+c) + Aasso_{\alpha_1\oplus\alpha_2}(y+b, x+a, z+c)\\
&= \Big(Aasso_{\alpha_1}(x,y,z) + Aasso_{\alpha_1}(y,x,z)\Big)\\
&+\Big(Aasso_{\alpha_2}(a,b,c) + Aasso_{\alpha_2}(b,a,c)\Big) + \Big(\lambda_1(\alpha_1(z))\lambda_1(y)a + \lambda_1(y\cdot z)\alpha_2(a) + \lambda_1(\alpha_1(z))\rho_1(y)a\\
&+\rho_1(\alpha_1(y))\lambda_1(z)a\Big) + \Big(\lambda_1(\alpha_1(z))\lambda_1(x)b + \lambda_1(x\cdot z)\alpha_2(b) + \lambda_1(\alpha_1(z))\rho_1(x)b + \rho_1(\alpha_1(x))\lambda_1(z)b\Big)\\
&+\Big(\lambda_2(\alpha_2(c))\lambda_2(b)x + \lambda_2(b \top c)\alpha_1(x) + \lambda_2(\alpha_2(c))\rho_2(b)x + \rho_2(\alpha_2(b))\lambda_2(c)x\Big) + +\Big(\lambda_2(\alpha_2(c))\lambda_2(a)y\\
&+\lambda_2(a \top c)\alpha_1(y) + \lambda_2(\alpha_2(c))\rho_2(a)y + \rho_2(\alpha_2(a))\lambda_2(c)y\Big) + \Big(\rho_1(x\star y)\alpha_2(c) + \rho_1(\alpha_1(x))\rho_1(y)c
\end{aligned}$$

$$\begin{aligned}
&+\rho_1(\alpha_1(y))\rho_1(x)c\Big) + \Big(\rho_2(a\circledast b)\alpha_1(z) + \rho_2(\alpha_2(a))\rho_2(b)z + \rho_2(\alpha_2(b))\rho_2(a)z\Big)\\
&+\Big(\rho_1(\alpha_1(y))(a \top c) + \rho_1(\rho_2(a)y + \lambda_2(a)y)\alpha_2(c) + (\rho_1(y)a + \lambda_1(y)a) \top \alpha_2(c) + \lambda_1(\lambda_2(c)y)\alpha_2(a)\\
&+\alpha_2(a) \top (\rho_1(y)c)\Big) + \Big(\rho_1(\alpha_1(x))(b \top c) + \rho_1(\rho_2(b)x + \lambda_2(b)x)\alpha_2(c) + (\rho_1(x)b + \lambda_1(x)b) \top \alpha_2(c)\\
&+\lambda_1(\lambda_2(c)x)\alpha_2(b) + \alpha_2(b) \top (\rho_1(x)c)\Big) + \Big(\lambda_1(\alpha_1(z)(a\circledast b) + \alpha_2(a) \top (\lambda_1(z)b) + \alpha_2(b) \top (\lambda_1(z)a)\\
&+\lambda_1(\rho_2(a)z)\alpha_2(b) + \lambda_1(\rho_2(b)z)\alpha_2(a)\Big) + \Big(\rho_2(\alpha_2(a))(y\cdot z) + \rho_2(\rho_1(y)a + \lambda_1(y)a)\alpha_1(z)\\
&+(\rho_2(a)y + \lambda_2(a)y)\cdot\alpha_1(z) + \lambda_2(\lambda_1(z)a)\alpha_1(y) + \alpha_1(y)\cdot(\rho_2(a)z)\Big) + \Big(\rho_2(\alpha_2(b))(x\cdot z)\\
&+\rho_2(\rho_1(x)b + \lambda_1(x)b)\alpha_1(z) + (\rho_2(b)x + \lambda_2(b)x)\cdot\alpha_1(z) + \lambda_2(\lambda_1(z)b)\alpha_1(x) + \alpha_1(x)\cdot(\rho_2(b)z)\Big)\\
&+\Big(\lambda_2(\alpha_2(c))(x\star y) + \alpha_1(x)\cdot(\lambda_2(c)y) + \alpha_1(y)\cdot(\lambda_2(c)x) + \lambda_2(\rho_1(x)c)\alpha_1(y) + \lambda_2(\rho_1(y)c)\alpha_1(x)\Big).
\end{aligned}$$

Thanks to (8), (20) and (30), we obtain

$$\begin{aligned}
&Aasso_{\alpha_1\oplus\alpha_2}(x+a, y+b, z+c) + Aasso_{\alpha_1\oplus\alpha_2}(y+b, x+a, z+c)\\
&= \Big(\rho_1(\alpha_1(y))(a \top c) + \rho_1(\rho_2(a)y + \lambda_2(a)y)\alpha_2(c) + (\rho_1(y)a + \lambda_1(y)a) \top \alpha_2(c) + \lambda_1(\lambda_2(c)y)\alpha_2(a)\\
&+\alpha_2(a) \top (\rho_1(y)c)\Big) + \Big(\rho_1(\alpha_1(x))(b \top c) + \rho_1(\rho_2(b)x + \lambda_2(b)x)\alpha_2(c) + (\rho_1(x)b + \lambda_1(x)b) \top \alpha_2(c)\\
&+\lambda_1(\lambda_2(c)x)\alpha_2(b) + \alpha_2(b) \top (\rho_1(x)c)\Big) + \Big(\lambda_1(\alpha_1(z)(a\circledast b) + \alpha_2(a) \top (\lambda_1(z)b) + \alpha_2(b) \top (\lambda_1(z)a)\\
&+\lambda_1(\rho_2(a)z)\alpha_2(b) + \lambda_1(\rho_2(b)z)\alpha_2(a)\Big) + \Big(\rho_2(\alpha_2(a))(y\cdot z) + \rho_2(\rho_1(y)a + \lambda_1(y)a)\alpha_1(z)\\
&+(\rho_2(a)y + \lambda_2(a)y)\cdot\alpha_1(z) + \lambda_2(\lambda_1(z)a)\alpha_1(y) + \alpha_1(y)\cdot(\rho_2(a)z)\Big) + \Big(\rho_2(\alpha_2(b))(x\cdot z)\\
&+\rho_2(\rho_1(x)b + \lambda_1(x)b)\alpha_1(z) + (\rho_2(b)x + \lambda_2(b)x)\cdot\alpha_1(z) + \lambda_2(\lambda_1(z)b)\alpha_1(x) + \alpha_1(x)\cdot(\rho_2(b)z)\Big)\\
&+\Big(\lambda_2(\alpha_2(c))(x\star y) + \alpha_1(x)\cdot(\lambda_2(c)y) + \alpha_1(y)\cdot(\lambda_2(c)x) + \lambda_2(\rho_1(x)c)\alpha_1(y) + \lambda_2(\rho_1(y)c)\alpha_1(x)\Big).
\end{aligned}$$

We deduce that (20) holds in $A_1 \oplus A_2$ if and only (34), (35), (36) and (37) holds. ☐

Proposition 4.30. *Let $(A_1, A_2, \rho_1, \lambda_1, \rho_2, \lambda_2)$ be a matched pair of left Hom-pre-Jacobi-Jordan algebras $\mathcal{A}_1 := (A_1, \cdot, \alpha_1)$ and $\mathcal{A}_2 := (A_2, \top, \alpha_2)$. Then, $(A_1, A_2, \rho_1 + \lambda_1, \rho_2 + \lambda_2)$ is a matched pair of sub-adjacent Hom-Jacobi-Jordan algebras $A_1^C := (A_1, \star, \alpha_1)$ and $A_2^C := (A_2, \circledast, \alpha_2)$.*

Proof. Since $(A_1, \rho_2, \lambda_2, \alpha_1)$ and $(A_2, \rho_1, \lambda_1, \alpha_2)$ are representations of left Hom-pre-Jacobi-Jordan algebras $(A_2, \top, \alpha_2)$ and $(A_1, \cdot, \alpha_1)$ respectively, it follows by Proposition 4.22 that $(A_1, \mu_2 := \rho_2 + \lambda_2, \alpha_1)$ and $(A_2, \mu_1 := \rho_1 + \lambda_1, \alpha_2)$ are representations of sub-adjacent Hom-Jacobi-Jordan algebras A_2^C and A_1^C respectively. Next, let $x, y \in A_1$ and $a, b \in A_2$. Then, by straightforward computations, after rearranging terms we get:

$$\begin{aligned}
&\mu_1(\alpha_1(x))(a \circledast b) + (\mu_1(x)a) \circledast \alpha_2(b) + (\mu_1(x)b) \circledast \alpha_2(a) + \mu_1(\rho_2(a)x)\alpha_2(b) + \mu_1(\rho_2(b)x)\alpha_2(a)\\
&= \Big(\rho_1(\alpha_1(x))(a \top b) + \rho_1(\rho_2(a)x + \lambda_2(a)x)\alpha_2(b) + (\rho_1(x)a + \lambda_1(x)a) \top \alpha_2(b) + \lambda_1(\lambda_2(b)x)\alpha_2(a)\\
&+ \alpha_2(a) \top (\rho_1(x)b)\Big) + \Big(\rho_1(\alpha_1(x))(b \top a) + \rho_1(\rho_2(b)x + \lambda_2(b)x)\alpha_2(a) + (\rho_1(x)b + \lambda_1(x)b) \top \alpha_2(a)\\
&+ \lambda_1(\lambda_2(a)x)\alpha_2(b) + \alpha_2(b) \top (\rho_1(x)a)\Big) + \Big(\lambda_1(\alpha_1(x))(a \circledast b) + \alpha_2(a) \top (\lambda_1(x)b) + \alpha_2(b) \top (\lambda_1(x)a)\\
&+ \lambda_1(\rho_2(a)x)\alpha_2(b) + \lambda_1(\rho_2(b)x)\alpha_2(a)\Big) = 0 \text{ (by (34), (35)).}
\end{aligned}$$

Hence, we obtain (13). Similarly, we compute

$$\begin{aligned}
&\mu_2(\alpha_2(a))(x \star y) + (\mu_2(a)x) \star \alpha_1(y) + (\mu_2(a)y) \star \alpha_1(x) + \mu_2(\rho_1(x)a)\alpha_1(y) + \mu_2(\rho_1(y)a)\alpha_1(x)\\
&= \Big(\rho_2(\alpha_2(a))(x \cdot y) + \rho_2(\rho_1(x)a + \lambda_1(x)a)\alpha_1(y) + (\rho_2(a)x + \lambda_2(a)x) \cdot \alpha_1(y) + \lambda_2(\lambda_1(y)a)\alpha_1(x)\\
&+ \alpha_1(x) \cdot (\rho_2(a)y)\Big) + \Big(\rho_2(\alpha_2(a))(y \cdot x) + \rho_2(\rho_1(y)a + \lambda_1(y)a)\alpha_1(x) + (\rho_2(a)y + \lambda_2(a)y) \cdot \alpha_1(x)\\
&+ \lambda_2(\lambda_1(x)a)\alpha_1(y) + \alpha_1(y) \cdot (\rho_2(a)x)\Big) + \Big(\lambda_2(\alpha_2(a))(x \star y) + \alpha_1(x) \cdot (\lambda_2(a)y) + \alpha_1(y) \cdot (\lambda_2(a)x)\\
&+ \lambda_2(\rho_1(x)a)\alpha_1(y) + \lambda_2(\rho_1(y)a)\alpha_1(x)\Big) = 0 \text{ (by (36), (37)).}
\end{aligned}$$

Then, we get also (14). □

For the next result, we suppose that there is a left Hom-pre-Jacobi-Jordan algebra structure on A^* and we denote $\mathcal{L}$ and $\mathcal{R}$ the corresponding operations. Now, we can prove

Proposition 4.31. *Let $(A, \cdot, \alpha)$ and $(A^*, \top, (\alpha^{-1})^*)$ be two left Hom-pre-Jacobi-Jordan algebras. Then, $(A'(A^*)^C, L^*, \mathcal{L}^*)$ is a matched pair of Hom-Jacobi Jordan algebras if and only if $(A, A^*, L^* + R^*, -R^*, \mathcal{L}^* + \mathcal{R}^*, -\mathcal{R}^*)$ is a matched pair of left pre-Hom-Jacobi-Jordan algebras.*

Definition 4.32. *Let (V, ρ, λ, ϕ) is a representation of a left Hom-pre-Jacobi-Jordan algebra $(A, \cdot, \alpha)$. A linear map $T : V \to A$ is called an $\mathcal{O}$-operator of $(A, \cdot, \alpha)$ associated to (V, ρ, λ, ϕ) if*

$$T\phi = \alpha T \tag{39}$$

$$T(u) \cdot T(v) = T\Big(\rho(T(u))v + \lambda(T(v)u\Big) \textit{for all } u, v \in V \tag{40}$$

Example 4.33. *Let $(A, \cdot, \alpha)$ be a left Hom-pre-Jacobi-Jordan algebra and (V, ρ, λ, ϕ) be a representation of $(A, \cdot, \alpha)$. It is easy to verify that $A \oplus V$ is a representation of $(A, \cdot, \alpha)$ under the maps $\rho_{A \oplus V} : A \to gl(A \oplus V)$ defined by*

$$\rho_{A \oplus V}(a)(b + v) := a \cdot b + \rho(a)v.$$

Define the linear map $T : A \oplus V \to A, a + v \mapsto a$. Then T is an $\mathcal{O}$-operator of A associated to the representation $(A \oplus V, \rho_{A \oplus V}, \alpha \oplus \phi)$.

Let give another example of $\mathcal{O}$-operators of left Hom-pre-Jacobi-Jordan algebras. It is easy to prove:

Proposition 4.34. *If T is an $\mathcal{O}$-operator of a left Hom-pre-Jacobi-Jordan $(A,\cdot,\alpha)$ associated to a representation (V,ρ,λ,ϕ) then T is an $\mathcal{O}$-operator of its associated Hom-Jacobi-Jordan $(A,\star,\alpha)$ associated to the representation $(V,\rho+\lambda,\phi)$.*

As Hom-associative algebras case [9], let give some characterizations of $\mathcal{O}$-operators on left Hom-pre-Jacobi-Jordan algebras.

Proposition 4.35. *A linear map $T : V \to A$ is an $\mathcal{O}$-operator associated to a representation (V,ρ,λ,ϕ) of a left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$ if and only if the graph of T,*

$$G_r(T) := \{(T(v),v), v \in V\}$$

is a subalgebra of the semi-direct product algebra $A \ltimes V$.

The following result shows that an $\mathcal{O}$-operator can be lifted up the Rota-Baxter operator.

Proposition 4.36. *Let $(A,\cdot,\alpha)$ be a left Hom-pre-Jacobi-Jordan algebra, (V,ρ,λ,ϕ) be a representation of A and $T : V \to A$ be a linear map. Define $\widehat{T} \in End(A\oplus V)$ by $\widehat{T}(a+v) := T(v)$. Then T is an $\mathcal{O}$-operator associated to (V,ρ,ϕ) if and only if $\widehat{T}$ is a Rota-Baxter operator on $A\oplus V$.*

In the sequel, let give some results about $\mathcal{O}$-operators.

Proposition 4.37. *Let T be an $\mathcal{O}$-operator on a left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$ with respect to a representation (V,ρ,λ,ϕ). If define a map $\diamond$ by*

$$u \top v := \rho(T(u))v + \lambda(T(v))u \text{ for all } (u,v) \in V^{\times 2}$$

then, $(V,\top,\phi)$ is a left Hom-pre-Jacobi-Jordan algebra. Moreover, T is a morphism from the left Hom-pre-Jacobi-Jordan algebra $(V,\top,\phi)$ to the initial left Hom-pre-Jacobi-Jordan algebra $(A,\cdot,\alpha)$.

Proof. For all $u,v,w \in V$, rearranging terms after straightforward computations, we obtain

$$\begin{aligned}
&Aasso_\phi(u,v,w) + Aasso_\phi(v,u,w)\\
&= (u \top v) \top \phi(w) + \phi(u) \top (v \top w) + (v \top u) \top \phi(w) + \phi(v) \top (u \top w)\\
&= \Big(\rho(T(u)\cdot T(v))\phi(w) + \rho(T(v)\cdot T(u))\phi(w) + \rho(T(\phi(u)))\rho(T(v))w + \rho(T(\phi(v)))\rho(T(u))w\Big)\\
&+\Big(\lambda(T(\phi(w)))\lambda(T(v))u + \lambda(T(v)\cdot T(w))\phi(u) + \lambda(T(\phi(w)))\rho(T(v))u + \rho(T(\phi(v)))\lambda(T(w))u\Big)\\
&+\Big(\lambda(T(\phi(w)))\lambda(T(u))v + \lambda(T(u)\cdot T(w))\phi(v) + \lambda(T(\phi(w)))\rho(T(u))v + \rho(T(\phi(u)))\lambda(T(w))v\Big)\\
&= 0 \text{ (by (39), (8) and (30).}
\end{aligned}$$

Hence, $(V,\top,\phi)$ is a left Hom-pre-Jacobi-Jordan algebra. The second assertion follows from (40) and the definition of $\top$. □

In order to give another characterization of $\mathcal{O}$-operators, let introduce the following:

Definition 4.38. *Let $(A,\cdot,\alpha)$ be a left Hom-pre-Jacobi-Jordan algebra. A linear map $N : A \to A$ is said to be a Nijenhuis operator if $N\alpha = \alpha N$ and its Nijenhuis torsions vanish, i.e.,*

$$N(x)\cdot N(y) = N(N(x)\cdot y + x\cdot N(y) - N(x\cdot y)), \text{ for all } x,y \in A,$$

Observe that the deformed multiplications $*_N : A \oplus A \to A$ *given by*

$$x \cdot_N y := N(x) \cdot y + x \cdot N(y) - N(x \cdot y),$$

gives rise to a new left Hom-pre-Jacobi-Jordan multiplication on A*, and* N *becomes a morphism from the left Hom-pre-Jacobi-Jordan algebra* $(A, \cdot_N, \alpha)$ *to the initial left Hom-pre-Jacobi-Jordan algebra* $(A, \cdot, \alpha)$*.*

Now, we can easily check the following result.

Proposition 4.39. *Let* $\mathcal{A} := (A, \cdot, \alpha)$ *be a left Hom-pre-Jacobi-Jordan algebra and* $\mathcal{V} := (V, \rho, \lambda, \phi)$ *be a representation of* $(A, \cdot, \alpha)$*. A linear map* $T : V \to A$ *is an* $\mathcal{O}$*-operator on* $\mathcal{A}$ *with respect to the* $\mathcal{V}$ *if and only if* $N_T := \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} : A \oplus V \to A \oplus V$ *is a Nijenhuis operator on the left Hom-pre-Jacobi-Jordan algebra* $A \oplus V$*.*

Proposition 4.40. *If* N *is a Nijenhuis operator on a left Hom-pre-Jacobi-Jordan algebra* $(A, \cdot, \alpha)$*, then* N *is a Nijenhuis operator on the sub-adjacent Hom-Jacobi-Jordan algebra* A^C*.*

Proof. Since N is a Nijenhuis operator on $(A, \cdot, \alpha)$, we have $N\alpha = \alpha N$. For all $x, y \in A$, by Definition 4.38, we obtain

$$\begin{aligned} N(x) \star N(y) &= N(x) \cdot N(y) + N(y) \cdot N(x) \\ &= N(N(x) \cdot y + x \cdot N(y) - N(x \cdot y) + (N(y) \cdot x + y \cdot N(x) - N(y \cdot x)) \\ &= N(N(x) \star y + y \star N(x) - N(x \star y)). \end{aligned}$$

□

References

[1] S. Attan, H. Hounnon, B. Kpamègan, Hom-Jordan and Hom- alternative bimodules, Extracta Math. 35(2020), 69–97.

[2] S. Attan, H. Hounnon, B. Kpamègan, Representations and formal deformations of Hom-Leibniz algebras, Asian-European Journal of Mathematics (2020), 2150013.

[3] Amir Baklouti and Saïd Benayadi, Symplectic Jacobi-Jordan algebras, Linear and Multilinear Algebra. (2019), DOI: 10.1080/03081087.2019.1626334

[4] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, Pacific J. Math. 10(1960), 731–742.

[5] S. Benayadi, A. Makhlouf, Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms. J Geom Phys. 76(2014), 38–60.

[6] D. Burde, A. Fialowski, Jacobi-Jordan algebras, Linear Algebra Appl. 459(2014), 586–594.

[7] P. Cartier, On the structure of free Baxter algebras, Adv. Math. 9(1972), 253–265.

[8] Y. S. Cheng and Y. C. Su, (Co)homology and universal central extensions of Hom-Leibniz algebras, Acta Math. Sin. (Engl. Ser.) 27(2011), 813–830.

[9] T. Chtioui, S. Mabrouk, A. Makhlouf, Cohomology and deformations of $\mathcal{O}$-operators on Hom-associative algebras. J. Algebra. 604 (2022), 727–759.

[10] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, Commun. Math. Phys. 210 (2000), 249–273.

[11] C. Essossolim Haliya, G. D. Houndedji, On a pre-Jacobi-Jordan algebra: relevant properties and double construction, arXiv: 2007.06736v1.

[12] E. Getzler and M. Kapranov, Cyclic operads and cyclic homology, Geometry, Topology and Physics for Raoul Bott (ed. S.-T. Yau), International Press, 1995, 167-201.

[13] L. Guo, An Introduction to Rota-Baxter Algebra, Surveys of Modern Mathematics Vol. 4 (International Press; Higher Education Press, Somerville, MC; Beijing, 2012).

[14] L. Guo and W. Keigher, Baxter algebras and shuffle products, Adv. Math. 150(2000), 117–149.

[15] J. T. Hartwig, D. Larsson and S. D. Silvestrov, Deformations of Lie algebras using σ-derivations, J. Algebra. 292 (2006), 314–361.

[16] N. Jacobson, Structure and Representations of Jordan Algebras, AMS, 1968.

[17] P. Jordan, J. von Newmann, E. Wigner, On algebraic generalization of the quantum machanical formaism, Ann Math. 35(1934), 29–64.

[18] B. A. Kupershmidt, What a classical r-matrix really is, J. Nonlinear Math. Phys. 6(1999), 448–488.

[19] D. Larsson, S.D. Silvestrov, Quasi-Hom-Lie algebras, central extensions and 2-cocycle-like identities, J. Algebra. 288(2005), 321–344.

[20] D. Larsson, S.D. Silvestrov, Quasi-Lie algebras, in "Noncommutative Geometry and Representation Theory in Mathematical Physics", Contemp. Math., 391, Amer. Math. Soc., Providence, RI, 2005, 241–248.

[21] S. Liu, L. Song, T. Rong, Representations and cohomologies of regular Hom-pre-Lie algebras, J. Algebra Appl. 19(2020), 2050149.

[22] A. Makhlouf, Hom-alternative algebras and Hom-Jordan algebras, Int. Electron. J. Alg. 8(2010), 177–190.

[23] A. Makhlouf, S. D. Silvestrov, Hom-algebra structures, J. Gen. Lie Theory Appl. 2(2008), 51–64.

[24] S. Okubo and N. Kamiya, Jordan-Lie super algebra and Jordan-Lie triple system, J. Algebra. 198 (1997), 388–411.

[25] G. C. Rota, Baxter algebras and combinatorial identities, I, II, Bull. Am. Math. Soc. 75(1969), 330–334.

[26] Y. Sheng, Representation of Hom-Lie algebras, Algebr. Represent. Theory. 15(2012), 1081–1098.

[27] S. Walcher, On algebras of rank three, Comm. Algebra. 27(1999), 3401–3438.

[28] D. Yau, Hom-Novikov algebras, J. Phys. A. 44(2011), 085202.

[29] D. Yau, Hom-Maltsev, Hom-alternative, and Hom-Jordan algebras, Int. Electron. J. Algebra. 11 (2012), 177–217.

[30] K. A. Zhevlakov, Solvability and nilpotence of Jordan rings, Algebra i Logika, 5(1966), 37–58 (in Russian).

[31] P. Zumanovich, Special and exceptional Mock-Lie algebras, Linear Algebra and its Applications. 518(2017), 79–96.