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Irrationality and transcendence of Certain Infinite Series

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Abstract. In number theory, Diophantine approximation concerns the study of how real numbers can be approximated by rational numbers. It is particularly interested in the quality of this approximation.
In this paper, we study the irrationality and transcendence of certain infinite series by applying methods from Diophantine approximation, including Roth's theorem and related results on irrationality. We establish a sufficient conditions that will assure us that a a series of positive rational terms converges to an irrational and to a transcendental number. Moreover, we gave a measure of irrationality for the series under consideration. Finally, we present an example to illustrate the applicability of our results.

Key Words: Irrationality, transcendence, measure of irrationality, infinite series.
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1 Introduction

In number theory, the Diophantine approximation studies how real numbers can be approximated by rational numbers. It is particularly interested in the quality of this approximation. A key aspect of this theory is the measurement of the distance between a real number θ and a fraction $\frac{p}{q}$, often expressed as:

$$\left|\theta - \frac{p}{q}\right|.$$

An approximation is said to be well-approximated if this distance is small compared with the size of the denominator q. In 1844, J. Liouville in [8] showed that algebraic numbers cannot be precisely approximated by rational numbers, implying that if a number can be very well-approximated, it must be transcendental. In 1955, K. Roth improved Liouville's work with his famous Theorem in [14], which can be formulated as follows.

Theorem 1.1. Let θ be a real number, δ a real number > 2, if there exists an infinity rational numbers $\dfrac{p}{q}$ with $gcd(p,q) = 1$ such that

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{q^{\delta}},$$

then θ is a transcendental number.

In addition, it is important to mention the following theorem.

Theorem 1.2. If a real number θ admits an infinite number of rational fractions $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{1+\epsilon}}$$

for some $\epsilon > 0$, then θ is irrational.

This theorem is closely related to the theory of Diophantine approximation [17]. It follows from the study of rational approximations of real numbers and establishes a link between the quality of the approximation of a number by rational ones and its nature, whether rational or irrational, see [1],[2].

Infinite series play a fundamental role in mathematics, particularly in analysis and number theory, their study allows for a deeper understanding of various mathematical and physical phenomena. One of the most interesting questions about infinite series is their relation with irrational and transcendental numbers. A real number can be expressed as the sum of an infinite series. The character of a real number (rationality, irrationality or algebraicity and transcendence) depends on the conditions of the infinite series representing the number. In this paper, we are particularly interested in numbers that can be expressed as the sum of infinite series whose general term is of the form b_n/a_n, where (a_n) and (b_n) are two sequences of positive integers, and we have introduced new criteria to demonstrate the irrationality and transcendence of these series.

In 1975, P. Erdös [5] examines the difficulty of determining whether the sum of a convergent infinite series is rational or irrational. He presents several theorems concerning the irrationality of sums of infinite series. For example, in [6], Erdös and Straus established deeper results by giving precise conditions guaranteeing the irrationality of the series $\sum 1/a_n$, where $(a_n)_n$ an increasing sequence of positive integers. In 1987, Badea established criteria for determining the irrationality of certain convergent series of rational numbers (see [4]), in this work Badea generalized earlier results of Erdös and Straus, providing specific conditions on the sequences involved to guarantee that the sum of the series is irrational. For example, he proved that if two sequences of positive integers (a_n) and (b_n) satisfy certain recurrent inequalities, then the series $\sum b_n/a_n$ is irrational.

Recent developments in studying the transcendence of rapidly converging infinite series have produced intriguing results. Nyblom's works [12], [13] provide additional insights on this topic. Hančl focused primarily on the question of irrationality and transcendence of infinite series, particularly those that converge rapidly. In collaboration with various co-authors in 2017 and 2019, he developed techniques to identify the conditions under which the sum of an infinite series is either irrational or transcendental, depending on the growth of the terms of the series (see [7],[8]).

In 2023, F. Sghiouer, A. Kacha and K. Belhroukia [18] established a new criterion ensuring the transcendence of the sum of certain infinite series, we showed that if a sequence of natural numbers (a_n) grows fast enough according to a specific condition, then the series $\sum 1/a_n$ converges to a transcendental number. These results provide a new perspective on the interplay between Diophantine approximation and the arithmetic properties of infinite series [19], [20]. Furthermore, Rucki gave two sufficient conditions under that the irrationality of the sum of the series $\sum b_n/a_n$ is proved [**?**], these conditions are more restrictive.

2 Main results

2.1 Transcendence

The first main result in our manuscript is the following theorem.

Theorem 2.1. Let (a_n) and (b_n) be sequences of positive integers and let α and r be two real number such that $\alpha > 3$ and $r > 1$.. For each $n \geq 1$, define q_n by

$$\frac{p_n}{q_n} = \sum_{k=1}^{n} \frac{b_k}{a_k}, \qquad \gcd(p_n, q_n) = 1.$$

If

$$a_{n+1} > r^{n+1} b_{n+1} > a_n^{\alpha} \quad for\ n \geq 1,$$

then the series $\sum \frac{b_n}{a_n}$ converges and its sum $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is transcendental.

To prove Theorem 2.1, we need some preliminary lemmas.

Lemma 2.2. *For each $n \geq 1$, let*

$$\frac{p_n}{q_n} = \sum_{k=1}^{n} \frac{b_k}{a_k}, \qquad \gcd(p_n, q_n) = 1.$$

Then

$$q_n \leq a_1 a_2 \cdots a_n. \tag{1}$$

Proof. Since $\gcd(p_n, q_n) = 1$, the denominator q_n is the least common denominator of the fractions $\frac{b_1}{a_1}, \frac{b_2}{a_2}, \ldots, \frac{b_n}{a_n}$. Hence q_m cannot exceed the product $a_1 a_2 \cdots a_n$. Therefore, $q_n \leq a_1 a_2 \cdots a_n$. □

Lemma 2.3. *Let (a_n) be a sequence of positive integers, and let $\alpha > 1$ be real. If $a_{n+1} > a_n^{\alpha}$ for all $n \geq 1$, then*

$$\lim_{n \to \infty} \frac{(a_1 a_2 \cdots a_n)^{\alpha - 1}}{a_{n+1}} = 0.$$

Proof. See pp.2-3 of [18]. □

Proof. The hypothesis $a_n > r^n b_n$ implies that $\frac{b_n}{a_n} < (\frac{1}{r})^n$. Since $\frac{1}{r} < 1$, we deduce that the series $\sum b_n/a_n$ converges. Set $\Gamma = \sum_{n=1}^{\infty} b_n/a_n$ and $p_m/q_m = \sum_{n=1}^{m} b_n/a_n$. Clearly

$$q_m^{\alpha-1} \left| \Gamma - \frac{p_m}{q_m} \right| = \sum_{n=m+1}^{\infty} q_m^{\alpha-1} \frac{b_n}{a_n}.$$

Using (1), we obtain

$$\begin{aligned} q_m^{\alpha-1} \left| \Gamma - \frac{p_m}{q_m} \right| &\leq (a_1 a_2 \ldots a_m)^{\alpha-1} \sum_{n=m+1}^{\infty} \frac{b_n}{a_n} \\ &\leq c_m \sum_{n=m+1}^{\infty} \frac{b_n a_{m+1}}{a_n} \\ &\leq c_m \left(1 + \sum_{k=1}^{\infty} \frac{b_{m+k+1}}{a_{m+k+1}} a_{m+1} \right) \end{aligned}$$

where $c_m = \dfrac{(a_1 a_2 ... a_m)^{\alpha-1}}{a_{m+1}}$. As

$$a_{m+1} < a_{m+2}^{\frac{1}{\alpha}} < .. < a_{m+k+1}^{(\frac{1}{\alpha})^k}$$

we get

$$q_m^{\alpha-1}\left|\Gamma - \frac{p_m}{q_m}\right| < c_m\left(1 + \sum_{k=1}^{\infty} \frac{b_{m+k+1}}{(a_{m+k+1})^{1-(\frac{1}{\alpha})^k}}\right).$$

Then, for sufficiently large k, it follows that

$$\left|\Gamma - \frac{p_m}{q_m}\right| < \frac{c_m}{q_m^{\alpha-1}}(1+\Gamma).$$

By virtue of Lemma 2.3 which states that $\lim_{m\to\infty} c_m = 0$, there exists a positive integer N such that, for all $m \geq N$, we have

$$\left|\Gamma - \frac{p_m}{q_m}\right| < \frac{1}{q_m^{\alpha-1}}.$$

Since $\alpha > 3$, then Γ is transcendental thanks to Theorem 1.1. □

2.2 Irrationality and measure of irrationality

The second main result is the following theorem.

Theorem 2.4. Let (a_n) and (b_n) be sequences of positive integers, and let $\alpha > 2$, $r > 1$ be two real numbers. Let q_n be as in Theorem 2.1. If

$$a_{n+1} > r^{n+1} b_{n+1} > a_n^{\alpha} \quad forall \quad n \geq 1,$$

then the series $\sum \frac{b_n}{a_n}$ converges and $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is irrational.

Proof. The argument is essentially the same as in Theorem 2.1, except that here we assume $\alpha > 2$ instead of $\alpha > 3$. By applying Theorem 1.2, we conclude that the series $\sum \frac{b_n}{a_n}$ converges to an irrational number. □

Having shown that the series converges to an irrational number under the hypotheses of Theorem 2.4, we now investigate how well this number can be approximated by rationals, by determining its irrationality measure. To this end, we first recall the definition of the irrationality measure.

Definition 2.5. Let θ be an irrational number. We say that $\mu > 0$ is an irrationality measure of θ if there exists a real constantr $C > 0$ and a positive integer $q_0 \geq 1$ such that, for every rational number p/q with $q \geq q_0$, one has

$$\left|\beta - \frac{p}{q}\right| \geq \frac{C}{q^{\mu}}.$$

Theorem 2.6. Let (a_n) and (b_n) be sequences of positive integers such that the terms a_n are pairwise coprime. Let α, α' and r be real numbers with $\alpha' > \alpha > 2$ and $r > 1$. Assume that for all $n \geq 1$,

$$a_n^{\alpha'} > a_{n+1} > r^{n+1} b_{n+1} > a_n^{\alpha}.$$

Then, a measure of irrationality of $\Gamma = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is given by

$$\mu(\alpha, \alpha') = min\left\{\frac{\alpha'}{\alpha-2}, \frac{\alpha-1}{\alpha-2}\right\} = \frac{\alpha-1}{\alpha-2}.$$

Proof. According to the first part, there exists a positive integer N such that, for all $m \geq N$, we have

$$\left|\Gamma - \frac{p_m}{q_m}\right| < \frac{1}{q_m^{\alpha-1}}. \tag{2}$$

Let $(p,q) \in \mathbb{N} \times \mathbb{N}^*$ such that

$$q > \frac{1}{2|q_N\Gamma - p_N|}$$

which is équivalent to

$$\frac{1}{2q} < |q_N\Gamma - p_N|. \tag{3}$$

We have

$$q_N p - p_N q = q_N(p - q\Gamma) + q(q_N\Gamma - p_N).$$

In order to establish a lower bound for $\left|\Gamma - \frac{p}{q}\right|$, we proceed by distinguishing two cases.

First case: $q_N p - p_N q = 0.$

So we have

$$\left|\Gamma - \frac{p}{q}\right| = \left|\Gamma - \frac{p_N}{q_N}\right|.$$

Using the hypothesis

$$a_n^{\alpha} < r^{n+1} b_{n+1} < a_{n+1} < a_n^{\alpha'},$$

we obtain

$$\left|\Gamma - \frac{p_N}{q_N}\right| = \sum_{k=N+1}^{\infty} \frac{b_k}{a_k} > \frac{b_{N+1}}{a_{N+1}} > \frac{1}{a_N^{\alpha'}}.$$

Therefore, it yields that

$$\left|\Gamma - \frac{p_N}{q_N}\right| > \frac{1}{a_N^{\alpha'}}. \tag{4}$$

Since the integers (a_n) are coprime in pairs, we have

$$q_N = a_1 a_2 \ldots a_N,$$

we then get

$$\frac{1}{q_N} = \frac{1}{a_1 a_2 \ldots a_N} < \frac{1}{a_N}.$$

So, the inequality (4) becomes

$$\left|\Gamma - \frac{p_N}{q_N}\right| > \frac{1}{q_N^{\alpha'}}. \tag{5}$$

From (2) and (3), we obtain

$$\frac{1}{2qq_N} < \left|\Gamma - \frac{p_N}{q_N}\right| < \frac{1}{q_N^{\alpha-1}},$$

which implies that

$$q_N^{\alpha-1} < 2qq_N.$$

Hence, it follows that

$$q_N < (2q)^{1/(\alpha-2)}.$$

Finally, the relationship (5) becomes

$$\left|\Gamma - \frac{p_N}{q_N}\right| > \frac{1}{(2q)^{\frac{\alpha'}{\alpha-2}}}.$$

Second case: $q_N p - p_N q \neq 0$. we have

$$\begin{aligned}|q_N p - p_N q| &= |q_N(p - q\Gamma) + q(q_N\Gamma - p_N)| \\ &\leq |q_N(p - q\Gamma)| + |q(q_N\Gamma - p_N)|.\end{aligned}$$

Therefore, we found

$$\begin{aligned}qq_N\left|\Gamma - \frac{p}{q}\right| &\geq |q_N p - p_N q| - qq_N\left|\Gamma - \frac{p_N}{q_N}\right| \\ &\geq 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

Then, we get

$$\left|\Gamma - \frac{p}{q}\right| \geq \frac{1}{2qq_N}. \tag{6}$$

From the inequality

$$q_N < (2q)^{1/(\alpha-2)},$$

we deduce that

$$\left|\Gamma - \frac{p}{q}\right| > \frac{1}{(2q)^{\left(1+\frac{1}{\alpha-2}\right)}}.$$

This completes the proof of Theorem 2.5. □

Example 2.7. We consider the following sequences to illustrate the application of our main result. Let (a_n) and (b_n) be defined as follows:

$$\begin{cases} a_1 = b_1 = 1 \\ a_2 = 15, b_2 = 3, \quad r = 2 \end{cases} \text{and} \begin{cases} a_n = 3^{\frac{(2n)!}{2^n}} \\ b_n = 3^{\frac{(2n)!}{2^{n+1}}} \end{cases} n \geq 3.$$

We verify that the conditions of Theorem 2.1 with $\alpha = 3$ and Theorem 2.4 with $\alpha = 4$ are satified. So, it is suffisiant that

$$\ln a_{n+1} > \ln(2^{n+1}b_{n+1}) \text{ and } \ln(2^{n+1}b_{n+1}) > \ln a_n^3.$$

Which is verified because, we can remark that for all $n \geq 3$, $b_n = a_n^{1/2}$ so $a_{n+1}^{1/2} > 2^{n+1}$.

It follows that the corresponding series $\sum_n \frac{b_n}{a_n}$ converges to an irrational number when $\alpha = 3$ and to a transcendental number when $\alpha = 4$.

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