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Intuitionistic fuzzy J-ideals in (m,n)-near rings

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Abstract. A near ring is a non-empty set A with two binary operations $+$ and $\cdot$, where $(A,+)$ forms a group, $(A,\cdot)$ is a semigroup, and at least one distributive law holds. Near-rings generalize rings by relaxing distributive properties, and (m,n)-near rings further extend this framework to multi-ary operations.

In this work, we recall the basic properties of (m,n)-near rings and their ideals, and we introduce intuitionistic fuzzy ideals in this setting. We investigate their structural properties, including the intuitionistic fuzzy nilradical and Jacobson radical, and define intuitionistic fuzzy n-ideals, J-ideals, and maximal J-ideals. Key results highlight their algebraic significance and interrelations. This study provides a unified framework for intuitionistic fuzzy structures in (m,n)-near rings and lays the foundation for future research in fuzzy algebra.

Key Words: (m,n)-near ring, Intuitionistic fuzzy J-ideal, Nilradical.
2020 MSC: Primary 46G05, 46L05; Secondary 47A30, 47B47.

1 Introduction

The notion of an (m,n)-near ring extends the classical concept of a near ring to a multi-ary framework. Such structures allow the study of generalized distributive laws and interactions between multi-parameter operations, and include ordinary near rings as particular cases. They also provide algebraic models for systems where multi-ary operations naturally arise, such as fuzzy systems, automata theory, and information processing structures.

Recent works in commutative algebra have focused on various generalizations of prime and J-type ideals; in particular, El Khalfi studied φ-(n,J)-ideals, and related notions such as φ-1-absorbing primary submodules have also been investigated, motivating further extensions to intuitionistic fuzzy J-ideals and quasi J-ideals in (m,n)-near rings [8, 16]. Quasi J-ideals of commutative rings are defined as ideals whose radical is a J-ideal, providing a natural generalization of J-ideals via the radical operation [9].

In this paper, we recall the fundamental concepts and properties of (m,n)-near rings and their ideals, and we develop the theory of intuitionistic fuzzy ideals within this framework. We study their structural aspects, including the intuitionistic fuzzy nilradical and the intuitionistic fuzzy Jacobson radical, and introduce the notions of intuitionistic fuzzy n-ideals, J-ideals, and maximal J-ideals. Our main results emphasize the algebraic significance of these concepts and clarify the relationships among them. This work establishes a coherent framework for the study of intuitionistic fuzzy structures in (m,n)-near rings and provides a basis for further developments in fuzzy algebra. Throughout this paper, we denote an i-(m,n)-near ring simply by an (m,n)-near ring. We now present some basic concepts.

A function $\mu : R \longrightarrow [0,1]$ defined on a set R is called a *fuzzy set*. Let $\mathrm{Im}(\mu)$ denote the image of μ. For a fuzzy set μ in R and any $\alpha \in [0,1]$, the subset

$$\mu_\alpha = \{x \in R \mid \mu(x) \geq \alpha\}$$

is called the α-*level subset* (or α-*cut*) of μ.

Definition 1.1 ([1]). An *intuitionistic fuzzy set* Q in a set R is defined as

$$Q = \{\langle s, \mu(s), \eta(s)\rangle \mid s \in R\},$$

where

$$\mu : R \longrightarrow [0,1], \quad \eta : R \longrightarrow [0,1], \quad \text{with } 0 \leq \mu(s) + \eta(s) \leq 1 \text{ for all } s \in R.$$

Here, $\mu(s)$ and $\eta(s)$ represent, respectively, the degrees of membership and non-membership of the element s in the set Q.

Every fuzzy set corresponds to an intuitionistic fuzzy set defined by

$$FS = \{\langle s, \mu(s), 1 - \mu(s)\rangle \mid s \in R\}.$$

Example 1.2. Let $\mathbb{N}$ be the set of natural numbers and $A = \{< \mathbb{N}, \mu(s), \eta(s) > \mid s \in X\}$ where $\mu(s) = \frac{s}{2s+2}$, $\eta(s) = \frac{1}{2s+2}$. Then A is an intuitionistic fuzzy set.

Definition 1.3. [1] Let $D = \{< s, \mu(s), \eta(s) > \mid s \in R\}$ and $E = \{< s, \mu'(s), \eta'(s) > \mid s \in R\}$ be any two intuitionistic fuzzy set of set R. Then we say that

(1) $D \subseteq E$ if and only if $\mu(s) \leq \mu'(s)$ and $\eta(s) \geq \eta'(s)$ for all $s \in R$.

(2) $D = E$ if and only if $\mu(s) = \mu'(s)$ and $\eta(s) = \eta'(s)$ for all $s \in R$.

(3) $D \cap E = \{< s, (\mu \cap \mu')(s), (\eta \cap \eta')(s) > \mid s \in R\}$, where $(\mu \cap \mu')(s) = \min\{\mu(s), \mu'(s)\} = \mu(s) \wedge \mu'(s)$ and $(\eta \cap \eta')(s) = \max\{\eta(s), \eta'(s)\} = \eta(s) \vee \eta'(s)$

(4) $D \cup E = \{< s, (\mu \cup \mu')(s), (\eta \cup \eta')(s) > \mid s \in R\}$, where $(\mu \cup \mu')(s) = \max\{\mu(s), \mu'(s)\} = \mu(s) \vee \mu'(s)$ and $(\eta \cup \eta')(s) = \min\{\eta(s), \eta'(s)\} = \eta(s) \wedge \eta'(s)$

Definition 1.4. [12] An intuitionistic fuzzy set $A = \{< s, \mu(s), \eta(s) > \mid s \in R\}$ of an m-ary group (R, f) is called an intuitionistic fuzzy subgroup of R if

(1) for all $w_1, w_2, ..., w_m \in R$, $\mu\Big(f(w_1, w_2, ..., w_m)\Big) \geq \min\{\mu(w_1), \mu(w_2), ..., \mu(w_m)\}$,

(2) for all $a_1^m, b \in R$ and $1 \leq i \leq m$ there is $x_i \in R$ so that $f(a_1^{i-1}, x_i, a_{i+1}^m) = b$ and $\mu(x_i) \geq \min\{\mu(a_1), \mu(a_2), ..., \mu(a_{i-1}),$

(3) for all $w_1, w_2, ..., w_m \in R$, $\eta\Big(f(w_1, w_2, ..., w_m)\Big) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\}$,

(4) for all $w_1^m, b \in R$ and $1 \leq i \leq m$ there is $x_i \in R$ so that $f(w_1^{i-1}, x_i, w_{i+1}^m) = b$ and $\eta(x_i) \leq \max\{\eta(w_1), \eta(w_2), ..., \eta(w_{i}$

Example 1.5. Let $\mathbb{Z}$ be the set of integer numbers and h be an m-ary operation on $\mathbb{Z}$ defined for each $o_1, o_2 ..., o_m \in \mathbb{Z}$ as follows:

$$h(o_1, o_2, ..., o_m) = o_1 + o_2 + ... + o_m.$$

Then $(\mathbb{Z},h)$ is an m-group.

$$\mu(w)=\begin{cases}0.8 & \text{if } w\in 2\mathbb{Z}\\ 0.2 & \text{if } w\notin 2\mathbb{Z},\end{cases}$$

$$\eta(w)=\begin{cases}0.1 & \text{if } w\in 2\mathbb{Z}\\ 0.7 & \text{if } w\notin 2\mathbb{Z}.\end{cases}$$

(1) for all $w_1,w_2,\dots,w_m\in\mathbb{Z}$, if $h(w_1,w_2,\dots,w_m)\notin 2\mathbb{Z}$, then there is $j\in\{1,2,\dots,m\}$ such that $w_j\notin 2\mathbb{Z}$ so $\mu\Big(h(w_1,w_2,\dots,w_m)\Big)\geq\min\{\mu(w_1),\mu(w_2),\dots,\mu(w_m)\}$,

(2) for all $a_1^m,b\in\mathbb{Z}$ and $1\leq i\leq m$ there is $x_i\in\mathbb{Z}$ so that $h(a_1^{i-1},x_i,a_{i+1}^m)=b$ and if $x_i\notin 2\mathbb{Z}$, then there is $j\in\{1,2,\dots,m\}$ that $w_j\notin 2\mathbb{Z}$ or $b\notin 2\mathbb{Z}$ so $\mu(x_i)\geq\min\{\mu(a_1),\mu(a_2),\dots,\mu(a_{i-1}),\mu(a_{i+1}),\dots,\mu(a_m),\mu(b)\}$,

(3) for all $w_1,w_2,\dots,w_m\in\mathbb{Z}$, if $h(w_1,w_2,\dots,w_m)\notin 2\mathbb{Z}$, then there is $j\in\{1,2,\dots,m\}$ that $w_j\notin 2\mathbb{Z}$ so $\eta\Big(h(w_1,w_2,\dots,w_m)\Big)\leq\max\{\eta(w_1),\eta(w_2),\dots,\eta(w_m)\}$,

(4) for all $w_1^m,b\in\mathbb{Z}$ and $1\leq i\leq m$ there is $x_i\in\mathbb{Z}$ so that $h(w_1^{i-1},x_i,w_{i+1}^m)=b$ and if $x_i\notin 2\mathbb{Z}$, then there is $j\in\{1,2,\dots,m\}$ that $w_j\notin 2\mathbb{Z}$ or $b\notin 2\mathbb{Z}$, so

$$\eta(x_i)\leq\max\{\eta(w_1),\eta(w_2),\dots,\eta(w_{i-1}),\eta(w_{i+1}),\dots,\eta(w_m),\eta(b)\}.$$

Therefore $A=\{<w,\mu(w),\eta(w)>\ |\ w\in\mathbb{Z}\}$ is an intuitionistic fuzzy subgroup of m-group $(\mathbb{Z},h)$.

Definition 1.6. [13] Let R be a non-empty set and f,g be m-ary and n-ary operations on S, respectively. Then (R,f,g) is called an i-(m,n)-near ring, if the following conditions hold:

(1) (R,f) is an m-ary group (not necessarily abelian);

(2) (R,g) is an n-ary semigroup;

(3) The n-ary operation g is i-distributive with respect to the m-ary operation f,

where the definition of i-distributive condition is as follows:
for every o_1^n, $w_1^m\in R$, if $i=n$, then

$$g(o_1^{n-1},f(w_1,w_2,\dots,w_m))=f(g(o_1^{n-1},w_1),g(o_1^{n-1},w_2),\dots,g(o_1^{n-1},w_m)).$$

If $i=1$, then

$$g(f(w_1,w_2,\dots,w_m),o_2^n)=f(g(w_1,o_2^n),g(w_2,o_2^n),\dots,g(w_m,o_2^n)).$$

If $1<i<n$, then

$$g(o_1^{i-1},f(w_1,w_2,\dots,w_m),o_{i+1}^n)=f(g(o_1^{i-1},w_1,o_{i+1}^n),g(o_1^{i-1},w_2,o_{i+1}^n),\dots,g(o_1^{i-1},w_m,o_{i+1}^n)).$$

If for all $i\in\{1,2,\dots,n\}$, (R,f,g) is an i-(m,n) near ring and (R,f) is commutative group, then (R,f,g) is called an (m,n)-ring. Every (m,n)-ring [7] is an (m,n)-near ring. Further background on near rings can be found in [4, 6, 10, 17], while [13, 14] provides additional information on (m,n)-near rings.

Example 1.7. Let $\mathbb{Z}$ be the set of integer numbers and h,k be m-ary and n-ary operations on $\mathbb{Z}$, respectively, which are defined below. Then $(\mathbb{Z},h,k)$ is an (m,n)-near ring. For all $o_1^m,j_1^n\in\mathbb{Z}$

$$h(o_1,o_2,\dots,o_m)=o_1+o_2+\dots+o_m,\ k(j_1,j_2,\dots,j_n)=j_n.$$

It is clear that $(\mathbb{Z},h)$ is an m-ary group. We prove that $(\mathbb{Z},k)$ is an n-ary semigroup; if $1\leq i<n$, then

$$k(j_1^{i-1}, k(j_i, j_{i+1}, ..., j_{n+i-1}), j_{n+i}, ..., j_{2n-1}) = j_{2n-1}$$

if $i = n$, then

$$k(j_1^{n-1}, k(j_n, j_{n+1}, ..., j_{2n-1})) = k(j_n, j_{n+1}, ..., j_{2n-1}) = j_{2n-1}.$$

We prove that the n-ary operation k is n-distributive with respect to the m-ary operation h.

$$k(j_1, j_2, ..., j_{n-1}, h(o_1, o_2, ..., o_m)) = h(o_1, o_2, ..., o_m),$$
$$h(k(j_1, j_2, ..., j_{n-1}, o_1), k(j_1, j_2, ..., j_{n-1}, o_2), ..., k(j_1, j_2, ..., j_{n-1}, o_m)) = h(o_1, o_2, ..., o_m)).$$

Therefore

$$k(j_1, j_2, ..., j_{n-1}, h(o_1, o_2, ..., o_m)) = h(k(j_1, j_2, ..., j_{n-1}, o_1), k(j_1, j_2, ..., j_{n-1}, o_2), ..., k(j_1, j_2, ..., j_{n-1}, o_m)).$$

If (R, f, g) is an (m, n)-near ring and (R, g) is commutative, then R is called a *commutative* (m, n)*-near ring*.

Definition 1.8. Let (R, f, g) be an (m, n)-near ring. Then a subgroup (J, f) of an m-ary group (R, f) with $g(\underbrace{J, J, ..., J}_{n}) \subseteq J$ is called an (m, n)-sub near ring of (R, f, g), it is denoted by $J \leqslant R$.

Example 1.9. In Example 1.7, $(2\mathbb{Z}, h)$ is a subgroup of $\mathbb{Z}$ and $k(\underbrace{2\mathbb{Z}, 2\mathbb{Z}, ..., 2\mathbb{Z}}_{n}) = 2\mathbb{Z} \subseteq 2\mathbb{Z}$, so $(2\mathbb{Z}, h, k)$ is an (m, n)-sub near ring of $(\mathbb{Z}, h, k)$.

For further details on fuzzy (m, n)-subnear rings, see [15]. The remainder of the paper is organized as follows. In Section 2, we introduce intuitionistic fuzzy (m, n)-subnear rings and investigate their basic properties. Section 3 is devoted to intuitionistic fuzzy J-ideals of (m, n)-near rings. We then study quasi J-ideals of (m, n)-near rings and analyze their main structural features. The paper concludes with a section of conclusions and directions for future research.

2 Intuitionistic fuzzy (m, n)-sub near ring

The theory of fuzzy sets, introduced by Zadeh, has found widespread applications across many areas of science and engineering. Among the various higher-order extensions of fuzzy sets, *intuitionistic fuzzy sets*, introduced by Atanassov [1, 3, 2], provide an effective framework for modeling uncertainty and vagueness in algebraic structures.

Definition 2.1. [12] Let (R, f, g) be an (m, n)-near ring and $A = \{< j, \mu(j), \eta(j) > \mid j \in R\}$ be an intuitionistic fuzzy set of R. Then A is called an intuitionistic fuzzy (m, n)-sub near ring of R, if

(1) for all $j_1, j_2, ..., j_m \in R$,

$$\mu\Big(f(j_1, j_2, ..., j_m)\Big) \geq \min\{\mu(j_1), \mu(j_2), ..., \mu(j_m)\},$$

(2) for all $j_1^m, b \in R$ and $1 \leq i \leq m$ there is $x_i \in R$ so that

$$f(j_1^{i-1}, x_i, j_{i+1}^m) = b,\ \mu(x_i) \geq \min\{\mu(j_1), \mu(j_2), ..., \mu(j_{i-1}), \mu(j_{i+1}), ..., \mu(j_m), \mu(b)\},$$

(3) for all $w_1, w_2, ..., w_n \in R$,

$$\mu\Big(g(w_1, w_2, ..., w_n)\Big) \geq \min\{\mu(w_1), \mu(w_2), ..., \mu(w_n)\},$$

(4) for all $l_1, l_2, \ldots, l_m \in R$,

$$\eta\Big(f(l_1, l_2, \ldots, l_m)\Big) \leq \max\{\eta(l_1), \eta(l_2), \ldots, \eta(l_m)\},$$

(5) for all $d_1^m, b \in R$ and $1 \leq i \leq m$ there is $x_i \in R$ so that

$$f(d_1^{i-1}, x_i, d_{i+1}^m) = b,\ \eta(x_i) \leq \max\{\eta(d_1), \eta(d_2), \ldots, \eta(d_{i-1}), \eta(d_{i+1}), \ldots, \eta(d_m), \eta(b)\},$$

(6) for all $h_1, h_2, \ldots, h_n \in R$,

$$\eta\Big(g(h_1, h_2, \ldots, h_n)\Big) \leq \max\{\eta(h_1), \eta(h_2), \ldots, \eta(h_n)\}.$$

Example 2.2. In Example 1.7, consider

$$\mu(w) = \begin{cases} 0.7 & \text{if } w \in \mathbb{Z} \\ 0.2 & \text{if } w \notin \mathbb{Z}, \end{cases}$$

$$\eta(w) = \begin{cases} 0.2 & \text{if } w \in \mathbb{Z} \\ 0.9 & \text{if } w \notin \mathbb{Z}, \end{cases}$$

(1) for all $w_1, w_2, \ldots, w_m \in \mathbb{Z}$, if $h(w_1, w_2, \ldots, w_m) \notin 2\mathbb{Z}$, then there is $j \in \{1, 2, \ldots, m\}$ that $w_j \notin 2\mathbb{Z}$ so $\mu\Big(h(w_1, w_2, \ldots, w_m)\Big) \geq \min\{\mu(w_1), \mu(w_2), \ldots, \mu(w_m)\}$,

(2) for all $a_1^m, b \in \mathbb{Z}$ and $1 \leq i \leq m$ there is $x_i \in \mathbb{Z}$ so that $h(a_1^{i-1}, x_i, a_{i+1}^m) = b$ and if $x_i \notin 2\mathbb{Z}$, then there is $j \in \{1, 2, \ldots, m\}$ that $w_j \notin 2\mathbb{Z}$ or $b \notin 2\mathbb{Z}$ so $\mu(x_i) \geq \min\{\mu(a_1), \mu(a_2), \ldots, \mu(a_{i-1}), \mu(a_{i+1}), \ldots, \mu(a_m), \mu(b)\}$

(3) for all $l_1, l_2, \ldots, l_n \in \mathbb{Z}$,

$$\mu\Big(k(l_1, l_2, \ldots, l_n)\Big) = \mu(l_n) \geq \min\{\mu(l_1), \mu(l_2), \ldots, \mu(l_n)\},$$

(4) for all $w_1, w_2, \ldots, w_m \in \mathbb{Z}$, if $h(w_1, w_2, \ldots, w_m) \notin 2\mathbb{Z}$, then there is $j \in \{1, 2, \ldots, m\}$ so that $w_j \notin 2\mathbb{Z}$ so $\eta\Big(h(w_1, w_2, \ldots, w_m)\Big) \leq \max\{\eta(w_1), \eta(w_2), \ldots, \eta(w_m)\}$,

(5) for all $w_1^m, b \in \mathbb{Z}$ and $1 \leq i \leq m$ there is $x_i \in R$ so that $h(w_1^{i-1}, x_i, w_{i+1}^m) = b$ and if $x_i \notin 2\mathbb{Z}$, then there is $j \in \{1, 2, \ldots, m\}$ so that $w_j \notin 2\mathbb{Z}$ or $b \notin 2\mathbb{Z}$ so $\eta(x_i) \leq \max\{\eta(w_1), \eta(w_2), \ldots, \eta(w_{i-1}), \eta(w_{i+1}), \ldots, \eta(w_m), \eta(b)\}$.

(6) for all $l_1, l_2, \ldots, l_n \in \mathbb{Z}$,

$$\eta\Big(k(l_1, l_2, \ldots, l_n)\Big) = \eta(l_n) \leq \max\{\eta(l_1), \eta(l_2), \ldots, \eta(l_n)\},$$

so $A = \{< w, \mu(w), \eta(w) > \mid w \in \mathbb{Z}\}$ is an intuitionistic fuzzy sub near ring of (m,n)-near ring $(\mathbb{Z}, h, k)$

Definition 2.3. [15] Let A be a non-empty subset of an (m,n)-near ring (R, f, g). Then A is called an ideal of S if

(1) A is a normal subgroup of the m-ary group (R, f),

(2) for all $o_1, o_2, \ldots, o_n \in R$, $g(o_1^{i-1}, A, o_{i+1}^n) \subseteq A$,

(3) for every $r_1^{k-1}, r_{k+1}^m, w_1^{j-1}, w_{j+1}^n \in R$ and $1 \leq k \leq m$, $d \in A$, there is $o \in A$ so that

$$g(w_1^{j-1}, f(r_1^{k-1}, d, r_{k+1}^m), w_{j+1}^n)$$
$$= f(g(w_1^{j-1}, r_1, w_{j+1}^n), g(w_1^{j-1}, r_2, w_{j+1}^n), \ldots, g(w_1^{j-1}, r_{k-1}, w_{j+1}^n), o, g(w_1^{j-1}, r_{k+1},$$
$$w_{j+1}^n), \ldots, g(w_1^{j-1}, r_m, w_{j+1}^n)).$$

A is called an i-ideal of R if it satisfies conditions (1) and (2). A is called a j-ideal of R, for $j \neq i$, if it satisfies conditions (1) and (3).

Example 2.4. In Example 1.7, subset $2\mathbb{Z}$ is an n-ideal of $(\mathbb{Z},h,k)$ because $(2\mathbb{Z},h)$ is a normal subgroup of m-ary group $(\mathbb{Z},h)$, $(2\mathbb{Z},h)$ is an m-ary group and for all $s_1,s_2,...,s_{n-1}\in\mathbb{Z}$, $k(s_1^{n-1},2\mathbb{Z})=2\mathbb{Z}\subseteq 2\mathbb{Z}$

If O is an i-ideal for every $1\le i\le n$, then O is called an ideal of (m,n)-near ring (R,f,g).

Definition 2.5. [12] Let (R,f,g) be an i-(m,n)-near ring and $A=\{< x,\mu(x),\eta(x)> \mid x\in R\}$ be an intuitionistic fuzzy set of R. Then A is called an intuitionistic fuzzy ideal of R, if

(1) for all $l_1,l_2,...,l_m\in R$,

$$\mu\Big(f(l_1,l_2,...,l_m)\Big)\ge\min\{\mu(l_1),\mu(l_2),...,\mu(l_m)\},$$

(2) for all $l_1^m,b\in R$ and $1\le i\le m$ there is $x_i\in R$ so that

$$f(l_1^{i-1},x_i,l_{i+1}^m)=b,\ \mu(x_i)\ge\min\{\mu(l_1),\mu(l_2),...,\mu(l_{i-1}),\mu(l_{i+1}),...,\mu(l_m),\mu(b)\},$$

(3) for all $w_1^{i-1},w_{i+1}^m,w\in R$ and $1\le j\le m$, there is $a\in R$ that

$$f(w_1^{j-1},w,w_{j+1}^m)=f(w_1^{i-1},a,w_{i+1}^m),\ \mu(a)\ge\mu(w),$$

(4) for all $w_1,w_2,...,w_n\in R$,

$$\mu\Big(g(w_1,w_2,...,w_n)\Big)\ge\min\{\mu(w_1),\mu(w_2),...,\mu(w_n)\},$$

(5) for all $w_1^n,w\in R$ and $1\le i\le n$,

$$\mu\Big(g(w_1^{i-1},w,w_{i+1}^n)\Big)\ge\mu(w),$$

(6) for all $k\ne i$, $1\le k\le m$, $d_1^{i-1},d_{i+1}^n,w_1^m\in R$ and $w_k\in R$ there is $h_k\in R$ so that $g\Big(d_1^{i-1},f(w_1,w_2,...,w_m),d_{i+1}^n\Big)=f\Big(g(d_1^{i-1},w_1,d_{i+1}^n),g(d_1^{i-1},w_2,d_{i+1}^n),...,g(d_1^{i-1},$ $w_{k-1},d_{i+1}^n),h_k,g(d_1^{i-1},w_{k+1},d_{i+1}^n),...,g(d_1^{i-1},w_m,d_{i+1}^n)\Big)$ and $\mu(h_k)\ge\mu(w_k)$.

(7) for all $l_1,l_2,...,l_m\in R$,

$$\eta\Big(f(l_1,l_2,...,l_m)\Big)\le\max\{\eta(l_1),\eta(l_2),...,\eta(l_m)\},$$

(8) for all $d_1^m,b\in R$ and $1\le i\le m$ there is $w_i\in R$ so that

$$f(d_1^{i-1},w_i,d_{i+1}^m)=b,\ \eta(w_i)\le\max\{\eta(d_1),\eta(d_2),...,\eta(d_{i-1}),\eta(d_{i+1}),...,\eta(d_m),\eta(b)\},$$

(9) for all $w_1^{i-1},w_{i+1}^m,w\in R$ and $1\le j\le m$, there is $a\in R$ that

$$f(w_1^{j-1},w,w_{j+1}^m)=f(w_1^{i-1},a,w_{i+1}^m),\ \eta(a)\le\eta(w),$$

(10) for all $w_1,w_2,...,w_n\in R$,

$$\eta\Big(g(w_1,w_2,...,w_n)\Big)\le\max\{\eta(w_1),\eta(w_2),...,\eta(w_n)\},$$

(11) for all $w_1^n,w\in R$ and $1\le i\le n$,

$$\eta\Big(g(w_1^{i-1}, w, w_{i+1}^n)\Big) \le \eta(w),$$

(12) for all $k \ne i$, $1 \le k \le m$, $d_1^{i-1}, d_{i+1}^n, w_1^m \in R$ and $w_k \in R$ there is $h_k \in R$ so that $g\Big(d_1^{i-1}, f(w_1, w_2, ..., w_m), d_{i+1}^n\Big) = f\Big(g(d_1^{i-1}, w_1, d_{i+1}^n), g(d_1^{i-1}, w_2, d_{i+1}^n), ..., g(d_1^{i-1}, w_{k-1}, d_{i+1}^n), h_k, g(d_1^{i-1}, w_{k+1}, d_{i+1}^n), ..., g(d_1^{i-1}, w_m, d_{i+1}^n)\Big)$ and $\eta(h_k) \le \eta(w_k)$.

Note that A is an intuitionistic fuzzy i-ideal of R if it satisfies $1, 2, ..., 5, 7, ..., 11$. A is an intuitionistic fuzzy j-ideal, $j \ne i$, of R if it satisfies $1, 2, 3, 4, 6, ..., 10, 12$, $1 \le i, j \le n$.

Example 2.6. In Example 1.7, consider

$$\mu(w) = \begin{cases} 0.8 & \text{if } w \in 2\mathbb{Z} \\ 0.2 & \text{if } w \notin 2\mathbb{Z}. \end{cases}$$

$$\eta(w) = \begin{cases} 0.1 & \text{if } w \in 2\mathbb{Z} \\ 0.7 & \text{if } w \notin 2\mathbb{Z}. \end{cases}$$

(1) for all $w_1, w_2, ..., w_m \in \mathbb{Z}$, if $h(w_1, w_2, ..., w_m) \notin 2\mathbb{Z}$, then there is $j \in \{1, 2, ..., m\}$ that $w_j \notin 2\mathbb{Z}$ so $\mu\Big(h(w_1, w_2, ..., w_m)\Big) \ge \min\{\mu(w_1), \mu(w_2), ..., \mu(w_m)\}$,

(2) for all $a_1^m, b \in \mathbb{Z}$ and $1 \le i \le m$ there is $x_i \in \mathbb{Z}$ so that $f(a_1^{i-1}, x_i, a_{i+1}^m) = b$ and if $x_i \notin 2\mathbb{Z}$, then there is $j \in \{1, 2, ..., m\}$ that $w_j \notin 2\mathbb{Z}$ or $b \notin 2\mathbb{Z}$ so $\mu(x_i) \ge \min\{\mu(a_1), \mu(a_2), ..., \mu(a_{i-1}), \mu(a_{i+1}), ..., \mu(a_m), \mu(b)\}$

(3) for all $w_1^{i-1}, w_{i+1}^m, w \in \mathbb{Z}$ and $1 \le j \le m$, there is $a \in \mathbb{Z}$ so that

$$h(w_1^{j-1}, w, w_{j+1}^m) = h(w_1^{i-1}, a, w_{i+1}^m), \text{ so } w = a \text{ then } \mu(a) \ge \mu(w),$$

(4) for all $l_1, l_2, ..., l_n \in \mathbb{Z}$,

$$\mu\Big(k(l_1, l_2, ..., l_n)\Big) = \mu(l_n) \ge \min\{\mu(l_1), \mu(l_2), ..., \mu(l_n)\},$$

(5) for all $w_1^n, w \in \mathbb{Z}$ and $1 \le i \le n$,

$$\mu\Big(h(w_1^{n-1}, w)\Big) = \mu(w) \ge \mu(w),$$

(7) for all $w_1, w_2, ..., w_m \in \mathbb{Z}$, if $h(w_1, w_2, ..., w_m) \notin 2\mathbb{Z}$, then there is $j \in \{1, 2, ..., m\}$ that $w_j \notin 2\mathbb{Z}$ so $\eta\Big(h(w_1, w_2, ..., w_m)\Big) \le \max\{\eta(w_1), \eta(w_2), ..., \eta(w_m)\}$,

(8) for all $w_1^m, b \in \mathbb{Z}$ and $1 \le i \le m$ there is $x_i \in \mathbb{Z}$ so that $h(w_1^{i-1}, x_i, w_{i+1}^m) = b$ and if $x_i \notin 2\mathbb{Z}$, then there is $j \in \{1, 2, ..., m\}$ that $w_j \notin 2\mathbb{Z}$ or $b \notin 2\mathbb{Z}$ so $\eta(x_i) \le \max\{\eta(w_1), \eta(w_2), ..., \eta(w_{i-1}), \eta(w_{i+1}), ..., \eta(w_m), \eta(b)\}$.

(9) for all $w_1^{i-1}, w_{i+1}^m, w \in \mathbb{Z}$ and $1 \le j \le m$, there is $a \in \mathbb{Z}$ that

$$h(w_1^{j-1}, w, w_{j+1}^m) = h(w_1^{i-1}, a, w_{i+1}^m), \text{ so } w = a \text{ then } \eta(a) \le \eta(w),$$

(10) for all $l_1, l_2, ..., l_n \in \mathbb{Z}$,

$$\eta\Big(k(l_1, l_2, ..., l_n)\Big) = \eta(l_n) \le \max\{\eta(l_1), \eta(l_2), ..., \eta(l_n)\},$$

(11) for all $w_1^n, w \in \mathbb{Z}$ and $1 \le i \le n$,

$$\eta\Big(k(w_1^{n-1}, w)\Big) = \eta(w) \le \eta(w),$$

so $A = \{< w, \mu(w), \eta(w) > \mid w \in \mathbb{Z}\}$ is an intuitionistic fuzzy n-ideal of (m,n)-near ring $(\mathbb{Z}, h, k)$.

Lemma 2.7. *Assume that* $A = \{< w, \mu(w), \eta(w) > \mid w \in R\}$ *is an intuitionistic fuzzy ideal of* (m,n)*-near ring* (R, f, g)*. Then* $\mu(w) \leq \mu(0)$ *and* $\eta(w) \geq \eta(0)$ *for all* $w \in R$*.*

Definition 2.8. Let $A_i = \{< q, \mu_i(q), \eta_i(q) > \mid q \in R\}$ be an intuitionistic fuzzy ideal of (m,n)-near ring (R, f, g) for all $1 \leq i \leq n$. Define

$$g(A_1, A_2, ..., A_n)(q) = C(q),\ C = \{< q, C_1(q), C_2(q) > \mid q \in R\}$$

$$C_1(q) = \begin{cases} \sup\limits_{q=g(q_1,q_2,...,q_n)} \{\min\{\mu_1(q_1), \mu_2(q_2), ..., \mu_n(q_n)\}\} & \text{if } \mathrm{q} = \mathrm{g}(\mathrm{q}_1, \mathrm{q}_2, ..., \mathrm{q}_\mathrm{n}) \\ 0 & \text{otherwise.} \end{cases}$$

$$C_2(q) = \begin{cases} \inf\limits_{q=g(q_1,q_2,...,q_n)} \{\max\{\eta_1(q_1), \eta_2(q_2), ..., \eta_n(q_n)\}\} & \text{if } \mathrm{q} = \mathrm{g}(\mathrm{q}_1, \mathrm{q}_2, ..., \mathrm{q}_\mathrm{n}) \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.9. Let P be an intuitionistic fuzzy ideal of an (m,n)-near ring (R, f, g).Then P is called an *intuitionistic fuzzy prime ideal* if for any intuitionistic fuzzy ideals of R $W_i = \{\langle q, \mu_i(q), \eta_i(q)\rangle \mid q \in R\}$, $i = 1, 2, \ldots, n$, the following implication holds:

$$g(W_1, W_2, \ldots, W_n) \subseteq P \quad \Longrightarrow \quad W_1 \subseteq P \text{ or } W_2 \subseteq P \text{ or } \ldots \text{ or } W_n \subseteq P.$$

In [11], it is proved that a fuzzy maximal ideal μ of R cannot be defined as a fuzzy ideal $A \neq \lambda_R$ such that, for any fuzzy ideal η of R satisfying $\mu \subset \eta \subseteq \lambda_R$, we have $\eta = \lambda_R$.

$$\lambda(R)(q) = \begin{cases} 1 & \text{if } \mathrm{q} \in \mathrm{R} \\ 0 & \text{otherwise.} \end{cases}$$

$$\lambda''(R)(q) = \begin{cases} 0 & \text{if } \mathrm{q} \in \mathrm{R} \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.10. [11] Let μ be a fuzzy subset of (m,n)-near ring (R, f, g). Then we define

$$\mu_* = \{q \in R \mid \mu(q) = \mu(0)\}.$$

$$\chi(R) = \{< q, \lambda_R(q), \lambda''_R(q) > \mid q \in R\}.$$

If μ is a fuzzy i-ideal of R, then μ_* is an i-ideal of R.

Example 2.11. In Example 1.2, $\mu_* = \{q \in R \mid \mu(q) = 0\} = \{0\}$, $\eta_* = \{q \in R \mid \eta(q) = \eta(0) = \frac{1}{0+2} = \frac{1}{2}\} = \{0\}$.

Example 2.12. In Example 2.6, $\mu_* = \{q \in \mathbb{Z} \mid \mu(q) = \mu(0)\} = \{q \in \mathbb{Z} \mid \mu(q) = 0.8\} = 2\mathbb{Z}$, $\eta_* = \{q \in \mathbb{Z} \mid \eta(q) = \eta(0)\} = \{q \in \mathbb{Z} \mid \eta(q) = 0.1\} = 2\mathbb{Z}$

Lemma 2.13. [11] Suppose that μ and η are fuzzy i-ideals of (m,n)-near ring (R, f, g). Then $\mu_* \bigcap \eta_* \subseteq (\mu \bigcap \eta)_*$.

Proof. Assume that $x \in \mu_* \bigcap \eta_*$, then $\mu(x) = \mu(0)$ and $\eta(x) = \eta(0)$, this gives that $(\mu \bigcap \eta)(x) = \min\{\mu(x), \eta(x)\} = \min\{\mu(0), \eta(0)\} = (\mu \bigcap \eta)(0)$, thus $x \in (\mu \bigcap \eta)_*$, this implies that $\mu_* \bigcap \eta_* \subseteq (\mu \bigcap \eta)_*$. □

Definition 2.14. Let μ be a fuzzy i-ideal of (m,n)-near ring (R, f, g). Then μ is called a fuzzy maximal i-ideal of R if

(1) μ is not constant,

(2) for any fuzzy i-ideal η of R, if $\mu \subseteq \eta$ then either $\mu_* = \eta_*$ or $\eta = \lambda_R$.

Let μ be a fuzzy ideal of (m,n)-near ring (R,f,g). Then μ is called a fuzzy maximal ideal of R if μ is not constant and for any fuzzy ideal η of R, if $\mu \subset \eta$ then either $\mu_* = \eta_*$ or $\eta = \lambda_R$.

Definition 2.15. Let $A = \{< q, \mu(q), \eta(q) > \mid q \in R\}$ be an intuitionistic fuzzy i-ideal of (m,n)-near ring (R,f,g). Then A is called intuitionistic fuzzy maximal i-ideal of R if

(1) μ and η are not constant,

(2) for any fuzzy ideal $B = \{< q, \mu_1(q), \eta_1(q) > \mid q \in R\}$ of R, if $\mu \subseteq \mu_1$ then either $\mu_* = \mu_{1*}$ or $\mu_1 = \lambda_R$ and if $\eta \subseteq \eta_1$ then either $\eta_* = \eta_{1*}$ or $\eta_1 = \lambda''_R$.

Let $A = \{< q, \mu(q), \eta(q) > \mid q \in R\}$ be an intuitionistic fuzzy ideal of (m,n)-near ring (R,f,g). Then A is called an intuitionistic fuzzy maximal ideal of R if μ and η are not constant and for any fuzzy ideal $B = \{< q, \mu_1(q), \eta_1(q) > \mid q \in R\}$ of R, if $\mu \subseteq \mu_1$ then either $\mu_* = \mu_{1*}$ or $\mu_1 = \lambda_R$ and if $\eta \subseteq \eta_1$ then either $\eta_* = \eta_{1*}$ or $\eta_1 = \lambda_R$.

Definition 2.16. Let (R,f,g) be an (m,n)-near ring. The intersection of all the fuzzy maximal ideals of S is called the Jacobson radical of R and denoted by $Jac(R)$.

Definition 2.17. Let A be an fuzzy ideal of (m,n)-near ring (R,f,g) and

$$\phi(A) = \{B \mid B \text{ is a fuzzy ideal of } R \text{ such that } g(B^{(k+1)n-k}) \subseteq A \text{ for } k \in \mathbb{N}\}.$$

The fuzzy subset $N(A) = \sup\limits_{B \in \phi(A)} \{B\}$ is called the nilradical of A.

Definition 2.18. Let $A = \{< q, \mu(q), \eta(q) > \mid q \in R\}$ be an intuitionistic fuzzy ideal of (m,n)-near ring (R,f,g) and

$$\phi(\mu) = \{B \mid B \text{ is a fuzzy ideal of R such that } \mathrm{g}(\mathrm{B}^{(k+1)n-k}) \subseteq \mu \text{ for } \mathrm{k} \in \mathbb{N}\},$$
$$\phi(\eta) = \{A \mid A \text{ is a fuzzy ideal of R such that } \mathrm{g}(\mathrm{A}^{(k+1)n-k}) \subseteq \eta \text{ for } \mathrm{k} \in \mathbb{N}\}.$$

The fuzzy subset $N(A) = \{< q, \sup\limits_{B \in \phi(\mu)} B(q), \inf\limits_{A \in \phi(\eta)} A(q) > \mid q \in R\}$ is called the nilradical of A

Definition 2.19. Let $A = \{< q, \mu(q), \eta(q) > \mid q \in R\}$ be an intuitionistic fuzzy ideal of (m,n)-near ring (R,f,g). Then A is called an intuitionistic fuzzy N-ideal of R if for all intuitionistic fuzzy ideals $W_i = \{< q, \mu_i(q), \eta_i(q) > \mid q \in R\}$, $i \in \{1,2,...,n\}$, $g(W_1, W_2, ..., W_n) \subseteq A$ and $W_1^{i-1}, W_{i+1}^n \not\subseteq N(R)$ implies $W_i \subseteq A$.

3 Intuitionistic fuzzy J-ideals of (m,n)-near rings

We begin with the following definition:

Definition 3.1. Suppose that (R,f,g) is an (m,n)-near ring and $A = \{\langle q, \mu(q), \eta(q)\rangle \mid q \in R\}$ is an intuitionistic fuzzy ideal of R. Then A is called an *intuitionistic fuzzy n-ideal* if the following holds:
For all intuitionistic fuzzy ideals $W_i = \{\langle q, \mu_i(q), \eta_i(q)\rangle \mid q \in R\}$, $i \in \{1,2,\ldots,n\}$, if $g(W_1, W_2, \ldots, W_n) \subseteq A$ and $\mathrm{Ann}(W_j) = 0$ for all $j \neq i$, then $W_i \subseteq A$, where $\mathrm{Ann}(S) = \{W_1^{j-1}, W_{j+1}^n \subseteq R \mid g(W_1^{j-1}, S, W_{j+1}^n) = 0,\ 1 \le j \le n\}$.

Example 3.2. Let $A = \{< q, \mu(q), \eta(q) > \mid x \in \mathbb{Z}\}$, $W_i = \{< q, \mu_i(q), \eta_i(q) > \mid q \in R\}$ for $i \in \{1,2,...,n\}$ be the intuitionistic fuzzy ideals of $(\mathbb{Z}, h, k)$ in Example 1.7, $Ann(W_j) = 0$ for $1 \le j \neq i \le n$ and $g(W_1, W_2, ..., W_n) \subseteq A$. In this case $g(W_1, W_2, ..., W_n) = W_n \subseteq A$ so $W_n \subseteq A$. Hence A is an intuitionistic fuzzy n-ideal of (m,n)-near ring $(\mathbb{Z}, h, k)$.

Definition 3.3. Let $A = \{\langle q, \mu(q), \eta(q)\rangle \mid q \in R\}$ be an intuitionistic fuzzy ideal of an (m,n)-near ring (R, f, g). Then A is called an *intuitionistic fuzzy J-ideal* if for all intuitionistic fuzzy ideals $W_i = \{\langle q, \mu_i(q), \eta_i(q)\rangle \mid q \in R\}$, $\quad i = 1, 2, \ldots, n$, of R, the following statement holds:

If $g(W_1, W_2, \ldots, W_n) \subseteq A$ and $W_1^{i-1}, W_{i+1}^n \not\subseteq J(R)$, then $W_i \subseteq A$.

Theorem 3.4. If $A = \{< q, \mu(q), \eta(q) > \ \mid q \in R\}$ is an intuitionistic fuzzy J-ideal of (m,n)-near ring (R, f, g) and $A \neq \chi(R)$, then $A \subseteq J(R)$.

Proof. Assume that A is an intuitionistic fuzzy J-ideal, but $A \not\subseteq J(R)$. Suppose that $4g(A^{(i-1)}, \chi(R), A^{(n-i)}) \subseteq A$ implies $\chi(R) \subseteq A$, which would mean $A = \chi(R)$. This is a contradiction. Therefore, we must have $A \subseteq J(R)$. □

Definition 3.5. Let O be an intuitionistic fuzzy ideal of an (m,n)-near ring (R, f, g). For $O \neq R$, define

$(O : R) = \{H \mid g(R^{(i-1)}, H, R^{(n-i)}) \subseteq O$, H is an intuitionistic fuzzy subset of R$\}$.

$(O : R) = \{H \mid g(W_1^{i-1}, H, W_{i+1}^n) \subseteq O$, $\mathrm{H}, \mathrm{W}_1^{i-1}, \mathrm{W}_{i+1}^{n}$ are intuitionistic fuzzy subsets of R$\}$

$(O : X_1^{i-1}, X_{i+1}^n) = \{A \mid g(X_1^{i-1}, A, X_{i+1}^n) \subseteq O$, A is an intuitionistic fuzzy subset of R$\}$.

Theorem 3.6. Let (R, f, g) be a commutative (m,n)-near ring and O be an intuitionistic fuzzy ideal of R. Then the following statements are equivalent:

(1) O is an intuitionistic fuzzy J-ideal of R.

(2) $O = (O : A_1^{i-1}, A_{i+1}^n)$ for all $A_1^{i-1}, A_{i+1}^n \not\subseteq J(R)$.

(3) $(O : A_1^{i-1}, A_{i+1}^n) \subseteq J(R)$ for every $A_1^{i-1}, A_{i+1}^n \not\subseteq O$.

Proof. (1) $\Rightarrow$ (2) Suppose that O is an intuitionistic fuzzy J-ideal of R. For all A_1^{i-1}, A_{i+1}^n that are intuitionistic fuzzy subsets of R, the inclusion $O \subseteq (O : A_1^{i-1}, A_{i+1}^n)$ always holds. Let $A_1^{i-1}, A_{i+1}^n \not\subseteq J(R)$ and $B \subseteq (O : A_1^{i-1}, A_{i+1}^n)$, then $g(A_1^{i-1}, B, A_{i+1}^n) \subseteq O$. Because O is a J-ideal, $B \subseteq O$ then $O = g(A_1^{i-1}, O, A_{i+1}^n)$.

(2) $\Rightarrow$ (1) Suppose that O, A_1^{i-1}, A_{i+1}^n are intuitionistic fuzzy ideals of (m,n)-near ring (R, f, g). If $g(A_1, A_2, ..., A_n) \subseteq O$ and $A_1^{i-1}, A_{i+1}^n \not\subseteq J(R)$, then by (2), $A_i \subseteq (O : A_1^{i-1}, A_{i+1}^n) = O$ so O is a J-ideal of R.

(1) $\Rightarrow$ (3) Assume that $A_1^{i-1}, A_{i+1}^n \not\subseteq O$ and $B = (O : A_1^{i-1}, A_{i+1}^n)$ then $g(A_1^{i-1}, B, A_{i+1}^n) \subseteq O$. Because O is a J-ideal, so $B \subseteq J(R)$.

(3) $\Rightarrow$ (1) Suppose that $W_1, W_2, ..., W_n$ are intuitionistic fuzzy ideals of (m,n)-near ring (R, f, g) so that $g(W_1, W_2, ..., W_n) \subseteq O$ with $W_1^{i-1}, W_{i+1}^n \not\subseteq J(R)$. If $W_1, W_2, ..., W_n \not\subseteq O$, then by (3), for all $j \neq i$, $W_j \subseteq (O : W_1^{j-1}, W_{j+1}^n) \subseteq J(R)$, a contradiction. Therefore, $W_i \subseteq O$ and O is a J-ideal of R. □

Lemma 3.7. *Assume that (R, f, g) is a commutative (m,n)-near ring and G is an intuitionistic fuzzy subset of R and $G \neq \chi(R)$. In this case if O is an intuitionistic fuzzy J-ideal of R with $G \not\subseteq O$, then $(O : G)$ is an intuitionistic fuzzy J-ideal of R.*

Proof. If $(O : G) = \chi(R)$, then $g(G^{(i-1)}, \chi(R), G^{(n-i)}) \subseteq O$. This implies that $G \subseteq O$ is a contradiction, so $(O : G) \neq \chi(R)$ is an intuitionistic fuzzy ideal in R. Assume that $g(W_1, W_2, ..., W_n) \subseteq (O : G)$ and $W_1^{i-1}, W_{i+1}^n \not\subseteq J(R)$. Then we have $g(S_1^{i-1}, g(W_1, W_2, ..., W_n), S_{i+1}^n) \subseteq O$ for every intuitionistic fuzzy subsets S_1^{i-1}, S_{i+1}^n of G. Thus

$g(S_1^{i-1}, g(W_1, W_2, ..., W_n), S_{i+1}^n) = g(W_1^{i-1}, g(S_1^{i-1}, W_i, S_{i+1}^n), W_{i+1}^n) \subseteq O$.

Since O is an intuitionistic fuzzy J-ideal of R, it follows that $g(S_1^{i-1}, W_i, S_{i+1}^n) \subseteq O$ therefore $W_i \subseteq (O : G)$. □

Definition 3.8. Let $A = \{< x, \mu(x), \eta(x) > \ \mid x \in X\}$ be an intuitionistic fuzzy J-ideal of (m,n)-near ring (R, f, g). Then A is called intuitionistic fuzzy maximal J-ideal of R if

(1) μ and η are not constant,

(2) for any fuzzy J-ideal $B = \{< x, \mu_1(x), \eta_1(x) > \mid x \in R\}$ of R, if $A \subseteq B$ then $A_* = B_*$ ($\mu_* = \mu_{1*}$ and $\eta_* = \eta_{1*}$) or $B = \chi(R)$ ($\mu_1 = \lambda_R$ and $\eta_1 = \lambda''_R$).

Theorem 3.9. Suppose that O is an intuitionistic fuzzy maximal J-ideal of (m,n)-near ring (R,f,g), in this case O is an intuitionistic fuzzy prime ideal. If $O = J(R)$, then the converse is true.

Proof. Assume that O is an intuitionistic fuzzy maximal J-ideal and $A_1, A_2, ..., A_n$ are intuitionistic fuzzy ideals of R so that $g(A_1, A_2, ..., A_n) \subseteq O$ and $A_1^{i-1}, A_{i+1}^n \nsubseteq O$. Thus $(O : A_1^{i-1}, A_{i+1}^n)$ is an intuitionistic fuzzy J-ideal by Lemma 3.7 and $O \subseteq (O : A_1^{i-1}, A_{i+1}^n)$ and by maximality of O, $A_i \subseteq (O : A_1^{i-1}, A_{i+1}^n) = O$ therefore O is an intuitionistic fuzzy prime ideal of R.

Now, assume that $O = J(R)$ is an intuitionistic fuzzy prime ideal of R and $A_1, A_2, ..., A_n$ are intuitionistic fuzzy ideals of R so that $g(A_1, A_2, ..., A_n) \subseteq O$ and $A_1^{i-1}, A_{i+1}^n \nsubseteq J(R)$. Thus $A_i \subseteq J(R) = O$ and hence O is an intuitionistic fuzzy J-ideal. Moreover, O is an intuitionistic fuzzy maximal ideal by Theorem 3.4. □

Definition 3.10. Let (R,f,g) be an (m,n)-near ring and P be an intuitionistic fuzzy ideal of R. The intersection of all intuitionistic fuzzy maximal ideals of R containing P is called the Jacobson radical of P (denoted by $J(P)$).

The following properties can be easily verified for any intuitionistic fuzzy ideals O and Y of an (m,n)-near ring (R,f,g):

(1) $O \subseteq Y$ implies that $J(O) \subseteq J(Y)$.

(2) $J(R) \subseteq J(O)$.

(3) $J(J(O)) = J(O)$.

Definition 3.11. Let O be an intuitionistic fuzzy ideal of an (m,n)-near ring (R,f,g). Then O is called an *intuitionistic fuzzy J-primary ideal* if, for any intuitionistic fuzzy subsets $O_1, O_2, \ldots, O_n$ of R, the inclusion $g(O_1, O_2, \ldots, O_n) \subseteq O$ implies that either for all i, $O_i \subseteq O$ or there exists $l \in \{1, 2, \ldots, n\}$, such that $O_l \subseteq J(O)$.

Theorem 3.12. Assume that O is an intuitionistic fuzzy ideal of an (m,n)-near ring (R,f,g) so that $O \subseteq J(R)$. In this case O is an intuitionistic fuzzy J-ideal if and only if O is an intuitionistic fuzzy J-primary.

Proof. $\Rightarrow$ Suppose that R is an (m,n)-near ring, $O \neq \chi(R)$ is an intuitionistic fuzzy ideal of R, O is an intuitionistic fuzzy J-ideal of R and $A_1, A_2, ..., A_n$ are intuitionistic fuzzy subsets of R so that $g(A_1, A_2, ..., A_n) \subseteq O$ with $A_1^{i-1}, A_{i+1}^n \nsubseteq J(O)$. We have $J(R) \subseteq J(O)$, hence $A_1^{i-1}, A_{i+1}^n \nsubseteq J(R)$. O is an intuitionistic fuzzy J-ideal implies that $A_i \subseteq O$ thus, O is an intuitionistic fuzzy J-primary ideal of R.

$\Leftarrow$ Assume that O is an intuitionistic fuzzy J-primary ideal of R and $g(A_1, A_2, ..., A_n) \subseteq O$ with $A_1^{i-1}, A_{i+1}^n \nsubseteq J(R)$. $O \subseteq J(R)$ implies that $J(O) \subseteq J(J(R)) = J(R)$. Hence $A_1^{i-1}, A_{i+1}^n \nsubseteq J(O)$ then $A_i \subseteq O$ because O is an intuitionistic fuzzy J-primary. Thus, O is an intuitionistic fuzzy J-ideal. □

Theorem 3.13. Suppose that (R,f,g) is an (m,n)-near ring and O is an intuitionistic fuzzy ideal of R with $O \nsubseteq J(R)$. Then the following hold.

(1) If I_1 and I_2 are intuitionistic fuzzy J-ideals of R with $g(O^{(i-1)}, I_1, O^{(n-i)}) = g(O^{(i-1)}, I_2, O^{(n-i)})$, then $I_1 = I_2$.

(2) If X is an intuitionistic fuzzy ideal so that $g(O^{(i-1)}, X, O^{(n-i)})$ is an intuitionistic fuzzy J-ideal, then

$$g(O^{(i-1)}, X, O^{(n-i)}) = X.$$

Proof. (1) $g(O^{(i-1)}, I_2, O^{(n-i)}) = g(O^{(i-1)}, I_1, O^{(n-i)}) \subseteq I_1$. Since I_2 is an intuitionistic fuzzy J-ideal, we have $I_2 \subseteq I_1$ by Theorem 3.6. Similarly, since I_1 is an intuitionistic fuzzy J-ideal, we have $I_1 \subseteq I_2$ and the equality holds.

(2) Since $g(O^{(i-1)}, X, O^{(n-i)})$ is an intuitionistic fuzzy J-ideal, $g(O^{(i-1)}, X, O^{(n-i)}) \subseteq g(O^{(i-1)}, X, O^{(n-i)})$ and $O \not\subseteq J(H)$, so $X \subseteq g(O^{(i-1)}, X, O^{(n-i)})$. The other inclusion is trivial. □

Lemma 3.14. *Suppose that (R, f, g) and (X, f', g') are two (m,n)-near rings. If l is an (m,n)-near ring epimorphism from R onto X, then $l(J(R)) \subseteq J(X)$.*

Theorem 3.15. Suppose that $l : (R, f, g) \to (X, f', g')$ is an (m,n)-near ring epimorphism, in this case the following hold.

(1) If I is an intuitionistic fuzzy J-ideal of R with $\ker(l) \subseteq I$, then $l(I)$ is an intuitionistic fuzzy J-ideal of X.

(2) If Y is an intuitionistic fuzzy J-ideal of X with $\ker(l) \subseteq J(R)$, then $l^{-1}(Y)$ is an intuitionistic fuzzy J-ideal of R.

Proof. (1) Assume that $O_1, O_2, ..., O_n \subseteq X$ so that $g'(O_1, O_2, ..., O_n) \subseteq l(I)$ and $O_1^{i-1}, O_{i+1}^n \not\subseteq J(X)$. Because l is an epimorphism, so there are $A_1, A_2, ..., A_n \subseteq R$ so that $O_i = l(A_i)$ for all $i \in \{1, 2, ..., n\}$. Thus $g'(O_1, O_2, ..., O_n) = l(g(A_1, A_2, ..., A_n)) \subseteq l(I)$. Since $ker(l) \subseteq I$, we conclude that $g(A_1, A_2, ..., A_n) \subseteq I$. Also, note that $A_1^{i-1}, A_{i+1}^n \not\subseteq J(R)$ because otherwise if $A_1^{i-1}, A_{i+1}^n \subseteq J(R)$, then $l(A_j) = O_j \subseteq J(X)$ for $j \in \{1, ..., i-1, i+1, ..., n\}$ by Lemma 3.14 which is a contradiction. Because I is a J-ideal of R, so $A_i \subseteq I$ and therefore $O_i = l(A_i) \subseteq l(I)$.

(2) Assume that $O_1, O_2, ..., O_n \subseteq R$ where $g(O_1, O_2, ..., O_n) \subseteq l^{-1}(Y)$ and $O_1^{i-1}, O_{i+1}^n \not\subseteq J(R)$. Thus $g'(l(O_1), l(O_2), ..., l(O_n)) = l(g(O_1, O_2, ..., O_n)) \subseteq Y$. We prove that $l(O_1), ..., l(O_{i-1}), l(O_{i+1}), ..., l(O_n) \not\subseteq J(R)$. Assume that $l(O_1), ..., l(O_{i-1}), l(O_{i+1}), ..., l(O_n) \subseteq J(R)$ and M is a maximal ideal of R. Hence $l(M)$ is a maximal ideal of X because $\ker(l) \subseteq J(R) \subseteq M$, then $l(O_1), ..., l(O_{i-1}), l(O_{i+1}), ..., l(O_n) \subseteq l(M)$ and thus $O_1^{i-1}, O_{i+1}^n \subseteq M$ as $\ker(l) \subseteq M$. Hence $O_1^{i-1}, O_{i+1}^n \subseteq J(R)$ which is a contradiction. Because Y is a J-ideal, $l(O_i) \subseteq Y$ therefore $O_i \subseteq l^{-1}(Y)$, it follows that $l^{-1}(Y)$ is a J-ideal of R. □

For a family of fuzzy sets $\{\mu_i | \ i \in \Delta\}$ in an (m,n)-near ring R, the union $\bigvee_{i \in \Delta} \mu_i$ of $\{\mu_i \mid i \in \Delta\}$ is defined by

$$(\bigvee_{i \in \Delta} \mu_i)(w) = \sup\{\mu_i(w) \mid i \in \Delta\}, \text{ for all } w \in R.$$

For a family of fuzzy sets $\{\mu_i | \ i \in \Delta\}$ in an (m,n)-near ring R, the intersection $\bigwedge_{i \in \Delta} \mu_i$ of $\{\mu_i \mid i \in \Delta\}$ is defined by

$$(\bigwedge_{i \in \Delta} \mu_i)(w) = \inf\{\mu_i(w) \mid i \in \Delta\}, \text{ for all } w \in R.$$

Theorem 3.16. Suppose that (R, f, g) is an (m,n)-near ring and $\{I_j : j \in \delta\}$ is a nonempty family of intuitionistic fuzzy J-ideals of R. In this case $\bigwedge_{j \in \delta} I_j$ is an intuitionistic fuzzy J-ideal of R.

Proof. Assume that $W_1, W_2, ..., W_n$ are intuitionistic fuzzy subsets of R so that $g(W_1, W_2, ..., W_n) \subseteq \bigwedge_{j \in \delta} I_j$ and $W_1^{i-1}, W_{i+1}^n \not\subseteq J(R)$. Then $g(W_1, W_2, ..., W_n) \subseteq I_j$ for all $j \in \delta$ since I_j is an intuitionistic fuzzy J-ideal for all $j \in \delta$ we conclude that $W_i \subseteq I_j$ for all $j \in \delta$ and thus $W_i \subseteq \bigwedge_{j \in \delta} I_j$, therefore, $\bigwedge_{j \in \delta} I_j$ is an intuitionistic fuzzy J-ideal of R. □

4 Quasi J-ideals of (m,n)-near rings

Suppose that (R,f,g) is an (m,n)-near ring. Let Δ denote the family of all fuzzy maximal ideals η of R such that $\mu \subseteq \eta$.

Definition 4.1. Suppose that (R,f,g) is an (m,n)-near ring and μ is a fuzzy ideal of R. In this case the fuzzy radical of R, denoted by $\sqrt{\mu}$, is defined as follows:

$$\sqrt{\mu} = \bigwedge\{\eta \mid \eta \in \Delta(\mu)\}.$$

Definition 4.2. Suppose that (R,h,k) is a commutative (m,n)-near ring. In this case an intuitionistic fuzzy ideal $D \neq \chi(S)$ of R is named an intuitionistic fuzzy quasi J-ideal if $\sqrt{D}$ is an intuitionistic fuzzy J-ideal.

Theorem 4.3. Suppose that (R,f,g) is an (m,n)-near ring and H is an intuitionistic fuzzy quasi J-ideal of R. In this case If A_1^{i-1}, A_{i+1}^n and Y are intuitionistic fuzzy ideals of R with $g(A_1^{i-1}, Y, A_{i+1}^n) \subseteq H$, then $A_1^{i-1}, A_{i+1}^n \subseteq J(R)$ or $Y \subseteq \sqrt{H}$.

Proof. Suppose that H is an intuitionistic fuzzy quasi J-ideal of R, $g(A_1^{i-1}, Y, A_{i+1}^n) \subseteq H$ and $A_1^{i-1}, A_{i+1}^n \not\subseteq J(R)$. Since $\sqrt{H}$ is an intuitionistic fuzzy J-ideal, $\sqrt{H} = (\sqrt{H} : A_1^{i-1}, A_{i+1}^n)$ by Theorem 3.6, we have $Y \subseteq (H : A_1^{i-1}, A_{i+1}^n) \subseteq (\sqrt{H} : A_1^{i-1}, A_{i+1}^n) = \sqrt{H}$. □

Assume that O is an intuitionistic fuzzy ideal of an (m,n)-near ring (R,f,g). We denote by $J(O)$ the intersection of all intuitionistic fuzzy maximal ideals of R that contain O.

Theorem 4.4. Assume that Q is an intuitionistic fuzzy ideal of an (m,n)-near ring (R,f,g). In this case the following statements are equivalent:

(1) Q is an intuitionistic fuzzy quasi J-ideal of R.

(2) $Q \subseteq J(R)$ and if $A_1, A_2, ..., A_n \subseteq R$ with $g(A_1, A_2, ..., A_n) \subseteq Q$, then $A_j \subseteq J(Q)$ for $j \in \{1, 2, ..., i-1, i+1, ..., n\}$ or $A_i \subseteq \sqrt{Q}$.

Proof. (1) $\Longrightarrow$ (2) Assume that Q is an intuitionistic fuzzy quasi J-ideal of R. Because $\sqrt{Q}$ is an intuitionistic fuzzy J-ideal, thus $Q \subseteq \sqrt{Q} \subseteq J(R)$ by Theorem 3.4, so (2) obtained because $J(R) \subseteq J(Q)$.

(2) $\Longrightarrow$ (1) Assume that $g(A_1, A_2, ..., A_n) \subseteq Q$ and $A_1^{i-1}, A_{i+1}^n \not\subseteq J(R)$. Because $Q \subseteq J(R)$, hence $J(Q) \subseteq J(J(R)) = J(R)$ thus $A_1^{i-1}, A_{i+1}^n \not\subseteq J(Q)$. Therefore, $A_i \subseteq \sqrt{Q}$ and Q is a quasi J-ideal of R. □

Suppose that (R,f,g) is an (m,n)-near ring, and let $L(R)$ denote the set of all intuitionistic fuzzy ideals of R. D. Zhao [19] introduced the concept of expansions of ideals of the ring R.

Definition 4.5. Suppose that (R,f,g) is an (m,n)-near ring. A function $\delta : L(R) \longrightarrow L(R)$ named an intuitionistic fuzzy ideal expansion if the following conditions are satisfied for any intuitionistic fuzzy ideals W and Q of R :

(1) $W \subseteq \delta(W)$,

(2) If $W \subseteq Q$, then $\delta(W) \subseteq \delta(Q)$.

Example 4.6. Suppose that (R,f,g) is an (m,n)-near ring. In this case a function $\delta_1 : L(R) \longrightarrow L(R)$ defined by $\delta_1(W) = \sqrt{W}$ is an ideal expansion of an (m,n)-near ring R.

For an ideal expansion δ defined on a ring R, the class of δ-n-ideals has been recently defined and studied [5].

Definition 4.7. Suppose that (R, f, g) is an (m,n)-near ring and $\delta : L(R) \longrightarrow L(R)$ is an intuitionistic fuzzy ideal expansion. In this case an intuitionistic fuzzy ideal $W \neq \chi(R)$ of R is named an intuitionistic fuzzy δ-n-ideal if $A_1, A_2, ..., A_n \subseteq R$ with $g(A_1, A_2, ..., A_n) \subseteq W$ and $Ann(W_j) = 0$ for $j = 1, ..., i-1, i+1, ..., n$ or $A_i \subseteq \delta(W)$.

If $\delta = \delta_1$ defined by $\delta_1(W) = \sqrt{W}$ for all $W \in L(R)$, then W is named an intuitionistic fuzzy quasi n-ideal.

For example, every intuitionistic fuzzy n-ideal is also an intuitionistic fuzzy δ-n-ideal.

5 Conclusions

This paper investigates (m,n)-near rings from the standpoint of intuitionistic fuzzy algebra. After recalling the fundamental definitions and structural properties of (m,n)-near rings and their ideals, intuitionistic fuzzy ideals are introduced and their basic behavior is analyzed. Special attention is devoted to radical–type notions, including the intuitionistic fuzzy nilradical and the intuitionistic fuzzy Jacobson radical. Furthermore, several new classes of ideals are defined and studied, such as intuitionistic fuzzy n-ideals, intuitionistic fuzzy J-ideals, and maximal intuitionistic fuzzy J-ideals, together with their relationships and characterizations.

Future research directions. Possible directions for further work include:

- Investigating fundamental relations and quotient constructions preserving intuitionistic fuzzy properties, see [18].
- Developing categorical links between intuitionistic fuzzy algebraic systems and hyperstructures.
- Analyzing topological structures induced by intuitionistic fuzzy spectra of (m,n)-near rings, see [18].
- Exploring applications of intuitionistic fuzzy radicals to uncertainty modelling in algebraic information systems.

This research provides a foundation for expanding the interplay between fuzzy algebra, hyperstructures, and near-ring theory, opening avenues for both theoretical developments and applied mathematical contexts.

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