

Duality pairs over polynomial rings

An Fu and Wenhui Zhang*

School of Mathematics and Statistics Northwest Normal University, Lanzhou 730070, China

e-mail: fuan1880@163.com

**e-mail: zhangwh@nwnu.edu.cn*

Communicated by Lixin Mao

(Received 18 December 2025, Revised 10 February 2026, Accepted 14 February 2026)

Abstract. We investigate duality pairs over polynomial rings using an equivalent category of the module category over the polynomial rings. As an application, Gorenstein projective modules relative to complete duality pairs are obtained.

Key Words: Polynomial ring, Duality pair, Gorenstein projective module.

2020 MSC: Primary 16D50, 16D90, 18G25.

1 Introduction

Throughout this paper, R is a nonzero associative ring with identity and all modules are unitary. We write $R\text{-Mod}$ (respectively, $\text{Mod-}R$) for the category of left (respectively, right) R -modules. For any left R -module M , the character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+ . $R[x]$ always means the polynomial ring over R in one indeterminate x , where x commutes with the coefficients in R .

The notion of duality pairs was introduced by Holm and Jørgensen [6]. Recall that a duality pair over R is a pair $(\mathcal{L}, \mathcal{A})$, where \mathcal{L} is a class of left R -modules and \mathcal{A} is a class of right R -modules, such that (1) $L \in \mathcal{L}$ if and only if $L^+ \in \mathcal{A}$, (2) \mathcal{A} is closed under direct summands and finite direct sums. Since then, duality pairs have found many applications in the study of cotorsion pairs, approximation theory, etc. Let $(\mathcal{L}, \mathcal{A})$ be a duality pair. Then \mathcal{L} is closed under pure submodules, pure quotients, and pure extensions. Recently, Mao explored duality pairs over formal triangular matrix rings [11]. Ma et al. further generalized duality pairs to Morita context rings with zero bimodule homomorphisms and constructed Gorenstein projective modules relative to complete duality pairs over them [7].

Polynomial rings and modules over them have been extensively studied (see e.g. [8-10, 12]). Recently, Mao gave an equivalent categories of the module categories over the polynomial rings $R[x]$ [9] and investigated Gorenstein projective, injective, flat modules and dimensions over $R[x]$ using the equivalent categories of the module categories over $R[x]$ [8]. We recall some basic facts about modules over polynomial rings. Let Ξ be the category: whose objects are couples (Y, f) with Y a left R -module and $f \in \text{Hom}_R(Y, Y)$, and a morphism from (Y, f) to (M, g) is a morphism $\gamma \in \text{Hom}_R(Y, M)$

This work was financially supported by NSFC (12461003). The authors wish to express their gratitude to the referee for his/her helpful suggestions and comments for the improvement of the paper.

*Corresponding author.

such that the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\gamma} & M \\ \downarrow f & & \downarrow g \\ Y & \xrightarrow{\gamma} & M. \end{array}$$

From [9], the category Ξ is an abelian category and the $R[x]$ -Mod is isomorphic to category Ξ . Note that a sequence $0 \rightarrow (Y_1, f_1) \rightarrow (Y_2, f_2) \rightarrow (Y_3, f_3) \rightarrow 0$ in Ξ is exact if and only if the sequence $0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow Y_3 \rightarrow 0$ in R -Mod is exact.

There are some functors as follows.

The functor $\mathbf{T} : R\text{-Mod} \rightarrow \Xi$ is given, for every object $M \in R\text{-Mod}$, by $\mathbf{T}(M) = (M^{(\mathbb{N})}, \mu)$ with $\mu(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ and for morphisms by $\mathbf{T}(\alpha) = \alpha^{(\mathbb{N})}$.

The functor $\mathbf{U} : \Xi \rightarrow R\text{-Mod}$ is given, for every object $(Y, f) \in \Xi$, by $\mathbf{U}(Y, f) = Y$ and for morphisms by $\mathbf{U}(\alpha) = \alpha$.

The functor $\mathbf{Z} : R\text{-Mod} \rightarrow \Xi$ is given, for every object $M \in R\text{-Mod}$, by $\mathbf{Z}(M) = (M, 0)$ and for morphisms by $\mathbf{Z}(\alpha) = \alpha$.

The functor $\mathbf{C} : \Xi \rightarrow R\text{-Mod}$ is given, for every object $(Y, f) \in \Xi$, by $\mathbf{C}(Y, f) = \text{coker}(f)$ and for morphisms by $\mathbf{C}(\alpha) =$ the induced morphism.

The functor $\mathbf{H} : R\text{-Mod} \rightarrow \Xi$ is given, for every object $M \in R\text{-Mod}$, by $\mathbf{H}(M) = (M^{\mathbb{N}}, \nu)$ with $\nu(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$ and for morphisms by $\mathbf{H}(\alpha) = \alpha^{\mathbb{N}}$.

The functor $\mathbf{K} : \Xi \rightarrow R\text{-Mod}$ is given, for every object $(Y, f) \in \Xi$, by $\mathbf{K}(Y, f) = \ker(f)$ and for morphisms by $\mathbf{K}(\alpha) =$ the induced morphism.

From [9], functors \mathbf{U} , \mathbf{Z} , \mathbf{T} and \mathbf{H} are exact, \mathbf{C} is right exact and \mathbf{K} is left exact. (\mathbf{T}, \mathbf{U}) , (\mathbf{U}, \mathbf{H}) , (\mathbf{C}, \mathbf{Z}) and (\mathbf{Z}, \mathbf{K}) are adjoint pairs with $\mathbf{CT} = id_{R\text{-Mod}}$, $\mathbf{UZ} = id_{R\text{-Mod}}$ and $\mathbf{KH} = id_{R\text{-Mod}}$.

Let M be a left R -module and x an indeterminate. Recall that $M[x] = \{ \sum_{i=0}^n a_i x^i : a_i \in M, n \in \mathbb{N} \} \cong R[x] \otimes_R M$ is a left $R[x]$ -module. It is clear that the left $R[x]$ -module $M[x]$ corresponds to $\mathbf{T}(M)$ in Ξ . Note that $R[x] \cong R^{(\mathbb{N})}$ as an R -module. In particular, the regular left $R[x]$ -module $R[x]$ corresponds to $\mathbf{T}(R)$. Similarly, the left $R[x]$ -module $\text{Hom}_R(R[x], M)$ corresponds to $\mathbf{H}(M)$ in Ξ .

Replacing the objects in Ξ with right R -modules, we obtain the category Υ . We have that the category Υ is abelian and $\text{Mod-}R[x]$ is isomorphic to the category Υ .

The main goal of this paper is to study duality pairs over polynomial rings using an equivalent categories of the module categories over the polynomial rings. In Section 2, we construct some duality pairs over $R[x]$ using duality pairs over R . Symmetric duality pairs and complete duality pairs over polynomial rings are obtained. In Section 3, we construct Gorenstein projective modules relative to complete duality pairs over polynomial rings.

We will identify $R[x]$ -Mod with Ξ and $\text{Mod-}R[x]$ with Υ in the rest of the paper.

2 Duality pairs over Polynomial Rings

Definition 2.1. Let \mathcal{L} be a class of left R -modules and \mathcal{A} be a class of right R -modules. Recall that a pair $(\mathcal{L}, \mathcal{A})$ is a *duality pair* [6] if the following two conditions hold:

- (1) For a left R -module L , one has $L \in \mathcal{L}$ if and only if $L^+ \in \mathcal{A}$.
- (2) \mathcal{A} is closed under direct summands and finite direct sums.

Let \mathcal{L} (resp. \mathcal{A}) be a class of left (resp. right) R -modules. We write

$$\mathbf{T}(\mathcal{L}) = \{ \mathbf{T}(L) \in R[x]\text{-Mod} : L \in \mathcal{L} \},$$

$$\mathbf{H}(\mathcal{L}) = \{ \mathbf{H}(L) \in R[x]\text{-Mod} : L \in \mathcal{L} \},$$

$$\mathbf{Z}(\mathcal{L}) = \{ \mathbf{Z}(L) \in R[x]\text{-Mod} : L \in \mathcal{L} \},$$

$$\begin{aligned}\mathcal{U}^{\mathcal{L}} &= \{(X, \alpha) \in R[x]\text{-Mod} : X \in \mathcal{L}\}, \\ \mathcal{C}^{\mathcal{L}} &= \{(X, \alpha) \in R[x]\text{-Mod} : \text{coker}\alpha \in \mathcal{L}\}, \\ \mathcal{K}^{\mathcal{L}} &= \{(X, \alpha) \in R[x]\text{-Mod} : \text{ker}\alpha \in \mathcal{L}\}, \\ \mathcal{U}_{\mathcal{A}} &= \{(X, \alpha) \in \text{Mod-}R[x] : X \in \mathcal{A}\}, \\ \mathcal{C}_{\mathcal{A}} &= \{(X, \alpha) \in \text{Mod-}R[x] : \text{coker}\alpha \in \mathcal{A}\}, \\ \mathcal{K}_{\mathcal{A}} &= \{(X, \alpha) \in \text{Mod-}R[x] : \text{ker}\alpha \in \mathcal{A}\}.\end{aligned}$$

Clearly, $\mathbf{T}(\mathcal{L}) \subseteq \mathcal{C}^{\mathcal{L}}$ and $\mathbf{H}(\mathcal{L}) \subseteq \mathcal{K}^{\mathcal{L}}$.

Lemma 2.2. *Let (X, α) be a left $R[x]$ -module and Y be a left R -module. Then*

$$(1) (X, \alpha)^+ \cong (X^+, \alpha^+), \mathbf{T}(Y)^+ \cong \mathbf{H}(Y^+).$$

(2) *If $\{(X_i, \mu_i)\}_{i \in I}$ is a family of left $R[x]$ -modules, and $\lambda_i : X_i \rightarrow \bigoplus_{i \in I} X_i$ is the i -th injection, then $\bigoplus_{i \in I} (X_i, \mu_i) \cong (\bigoplus_{i \in I} X_i, \eta)$, where $\eta : \bigoplus_{i \in I} X_i \rightarrow \bigoplus_{i \in I} X_i$ is induced by the composition $X_i \xrightarrow{\mu_i} X_i \xrightarrow{\lambda_i} \bigoplus_{i \in I} X_i$.*

Proof. (1) This is obvious.

(2) By the universal property of the direct sums in $R\text{-Mod}$, there is a unique $\eta : \bigoplus_{i \in I} X_i \rightarrow \bigoplus_{i \in I} X_i$ such that $\lambda_i \mu_i = \eta \lambda_i$ in $R\text{-Mod}$ for any $i \in I$. Thus, $\lambda_i \in \text{Hom}_{R[x]}((X_i, \mu_i), (\bigoplus_{i \in I} X_i, \eta))$.

Let $(Y, \theta) \in R[x]\text{-Mod}$ and $\gamma_i \in \text{Hom}_{R[x]}((X_i, \mu_i), (Y, \theta))$ for any $i \in I$. By the universal property of the direct sums in $R\text{-Mod}$, there is a unique $\xi : \bigoplus_{i \in I} X_i \rightarrow Y$ such that $\xi \lambda_i = \gamma_i$ in $R\text{-Mod}$ for any $i \in I$. Therefore, we have $\xi \eta \lambda_i = \xi \lambda_i \mu_i = \gamma_i \mu_i = \theta \gamma_i = \theta \xi \lambda_i$. So $\xi \eta = \theta \xi$. Hence, $\xi \in \text{Hom}_{R[x]}((\bigoplus_{i \in I} X_i, \eta), (Y, \theta))$. By the universal property of the direct sums in $R[x]\text{-Mod}$, we conclude that $\bigoplus_{i \in I} (X_i, \mu_i) \cong (\bigoplus_{i \in I} X_i, \eta)$. \square

Lemma 2.3. *Let \mathcal{L} (resp. \mathcal{A}) be a class of left (resp. right) R -modules and X be a left R -module. Then*

- (1) $X \in \mathcal{L}$ if and only if $\mathbf{T}(X) \in \mathcal{C}^{\mathcal{L}}$.
- (2) $X \in \mathcal{L}$ if and only if $\mathbf{H}(X) \in \mathcal{K}^{\mathcal{L}}$.
- (3) $X \in \mathcal{L}$ if and only if $\mathbf{Z}(X) \in \mathcal{U}^{\mathcal{L}}$.
- (4) $X^+ \in \mathcal{A}$ if and only if $\mathbf{T}(X)^+ \in \mathcal{K}_{\mathcal{A}}$.
- (5) $X^+ \in \mathcal{A}$ if and only if $\mathbf{Z}(X)^+ \in \mathcal{U}_{\mathcal{A}}$.

Proof. (1) The “if” part is obvious, and the “only if” part follows from $\mathbf{T}(\mathcal{L}) \subseteq \mathcal{C}^{\mathcal{L}}$.

(2) The “if” part is obvious, and the “only if” part follows from $\mathbf{H}(\mathcal{L}) \subseteq \mathcal{K}^{\mathcal{L}}$.

(3) This is obvious.

(4) By Lemma 2.2(1), $\mathbf{T}(X)^+ \cong \mathbf{H}(X^+)$. The assertion follows from the dual statement of (2).

(5) This is obvious. \square

Theorem 2.4. *Let \mathcal{L} be a class of left R -modules and \mathcal{A} be a class of right R -modules. Then the following statements are equivalent.*

- (1) $(\mathcal{L}, \mathcal{A})$ is a duality pair over R .
- (2) $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ is a duality pair over $R[x]$.
- (3) $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ is a duality pair over $R[x]$.

Proof. (1) \Rightarrow (2). $(X, \alpha) \in \mathcal{C}^{\mathcal{L}}$ if and only if $\text{coker}(\alpha) \in \mathcal{L}$, which is equivalent to $(\text{coker}(\alpha))^+ \cong \text{ker}(\alpha^+) \in \mathcal{A}$ because $(\mathcal{L}, \mathcal{A})$ is a duality pair. This, in turn, is equivalent to $(X, \alpha)^+ \cong (X^+, \alpha^+) \in \mathcal{K}_{\mathcal{A}}$ by Lemma 2.2(1).

Next, we show that $\mathcal{K}_{\mathcal{A}}$ is closed under direct summands and finite direct sums.

Let $(X_1, \alpha_1) \oplus (X_2, \alpha_2) \in \mathcal{K}_{\mathcal{A}}$. There is split exact sequence in $\text{Mod-}R[x] : 0 \rightarrow (X_1, \alpha_1) \xrightarrow{\varepsilon_1} (X_1, \alpha_1) \oplus (X_2, \alpha_2) \xrightarrow{\pi_2} (X_2, \alpha_2) \rightarrow 0$ where ε_1 is an embedding, and π_2 is a projection. Note that $(X_1, \alpha_1) \oplus (X_2, \alpha_2) \cong (X_1 \oplus X_2, \alpha)$ with $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$. Let $i = \begin{pmatrix} f \\ g \end{pmatrix} : \text{ker}(\alpha) \rightarrow X_1 \oplus X_2$, $i_1 : \text{ker}(\alpha_1) \rightarrow X_1$, and $i_2 : \text{ker}(\alpha_2) \rightarrow X_2$.

By Snake lemma, there is an exact sequence in $\text{Mod-}R : 0 \rightarrow \text{ker}(\alpha_1) \xrightarrow{\tau_1} \text{ker}(\alpha) \xrightarrow{p_2} \text{ker}(\alpha_2), f\tau_1 = i_1,$

and $g = i_2 p_2$. We have the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \ker(\alpha_1) & \xrightarrow{\tau_1} & \ker(\alpha) & \xrightarrow{p_2} & \ker(\alpha_2) \\
 & & \downarrow i_1 & & \downarrow i & & \downarrow i_2 \\
 0 & \longrightarrow & X_1 & \longrightarrow & X_1 \oplus X_2 & \longrightarrow & X_2 \longrightarrow 0 \\
 & & \downarrow \alpha_1 & & \downarrow \alpha & & \downarrow \alpha_2 \\
 0 & \longrightarrow & X_1 & \longrightarrow & X_1 \oplus X_2 & \longrightarrow & X_2 \longrightarrow 0.
 \end{array}$$

Similarly, there is a split exact sequence in $\text{Mod-}R[x] : 0 \rightarrow (X_2, \alpha_2) \xrightarrow{\varepsilon_2} (X_1, \alpha_1) \oplus (X_2, \alpha_2) \xrightarrow{\pi_1} (X_1, \alpha_1) \rightarrow 0$ where ε_2 is an embedding, and π_1 is a projection. By Snake lemma, there is an exact sequence in $\text{Mod-}R : 0 \rightarrow \ker(\alpha_2) \xrightarrow{\tau_2} \ker(\alpha) \xrightarrow{p_1} \ker(\alpha_1)$, $f = i_1 p_1$ and $i_2 = g \tau_2$. We have the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \ker(\alpha_2) & \xrightarrow{\tau_2} & \ker(\alpha) & \xrightarrow{p_1} & \ker(\alpha_1) \\
 & & \downarrow i_2 & & \downarrow i & & \downarrow i_1 \\
 0 & \longrightarrow & X_2 & \longrightarrow & X_1 \oplus X_2 & \longrightarrow & X_1 \longrightarrow 0 \\
 & & \downarrow \alpha_2 & & \downarrow \alpha & & \downarrow \alpha_1 \\
 0 & \longrightarrow & X_2 & \longrightarrow & X_1 \oplus X_2 & \longrightarrow & X_1 \longrightarrow 0.
 \end{array}$$

We have $i_1 p_1 \tau_1 = f \tau_1 = i_1$ and $i_2 p_2 \tau_2 = g \tau_2 = i_2$. Thus, $p_1 \tau_1 = id_{\ker(\alpha_1)}$ and $p_2 \tau_2 = id_{\ker(\alpha_2)}$. It follows that $\ker(\alpha_1) \oplus \ker(\alpha_2) \cong \ker(\alpha) \in \mathcal{A}$. Since \mathcal{A} is closed under direct summands, $\ker(\alpha_1) \in \mathcal{A}$ and $\ker(\alpha_2) \in \mathcal{A}$. Therefore, $\mathcal{K}_{\mathcal{A}}$ is closed under direct summands.

Conversely, suppose that $(X_1, \alpha_1), (X_2, \alpha_2) \in \mathcal{K}_{\mathcal{A}}$. There is an exact sequence in $\text{Mod-}R : 0 \rightarrow \ker(\alpha_1) \oplus \ker(\alpha_2) \rightarrow X_1 \oplus X_2 \xrightarrow{\alpha} X_1 \oplus X_2$ with $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$. Since \mathcal{A} is closed under finite direct sums, $\ker(\alpha) \cong \ker(\alpha_1) \oplus \ker(\alpha_2) \in \mathcal{A}$. Thus, $(X_1, \alpha_1) \oplus (X_2, \alpha_2) \cong (X_1 \oplus X_2, \alpha) \in \mathcal{K}_{\mathcal{A}}$. Therefore, $\mathcal{K}_{\mathcal{A}}$ is closed under finite direct sums.

(2) \Rightarrow (1). $X \in \mathcal{L}$ if and only if $\mathbf{T}(X) \in \mathcal{C}^{\mathcal{L}}$ by Lemma 2.3(1), if and only if $\mathbf{T}(X)^+ \in \mathcal{K}_{\mathcal{A}}$ since $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ is a duality pair, if and only if $X^+ \in \mathcal{A}$ by Lemma 2.3(4).

Next, we prove that \mathcal{A} is closed under direct summands and finite direct sums.

Note that \mathbf{H} is an exact functor, $X_1 \oplus X_2 \in \mathcal{A}$ if and only if $\mathbf{H}(X_1) \oplus \mathbf{H}(X_2) \cong \mathbf{H}(X_1 \oplus X_2) \in \mathcal{K}_{\mathcal{A}}$ by Lemma 2.3(2), if and only if $\mathbf{H}(X_1), \mathbf{H}(X_2) \in \mathcal{K}_{\mathcal{A}}$ since $\mathcal{K}_{\mathcal{A}}$ is closed under direct summands and finite direct sums, if and only if $X_1, X_2 \in \mathcal{A}$ by Lemma 2.3(2).

(1) \Rightarrow (3). $(X, \alpha) \in \mathcal{U}^{\mathcal{L}}$ if and only if $X \in \mathcal{L}$, if and only if $X^+ \in \mathcal{A}$ by $(\mathcal{L}, \mathcal{A})$ being a duality pair, if and only if $(X, \alpha)^+ \cong (X^+, \alpha^+) \in \mathcal{U}_{\mathcal{A}}$ by Lemma 2.2(1).

Next, we prove that $\mathcal{U}^{\mathcal{A}}$ is closed under direct summands and finite direct sums.

$(X_1, \alpha_1) \oplus (X_2, \alpha_2) \in \mathcal{U}_{\mathcal{A}}$ if and only if $X_1 \oplus X_2 \in \mathcal{A}$, if and only if $X_1, X_2 \in \mathcal{A}$ since $(\mathcal{L}, \mathcal{A})$ is a duality pair, if and only if $(X_1, \alpha_1), (X_2, \alpha_2) \in \mathcal{U}_{\mathcal{A}}$.

(3) \Rightarrow (1). $X \in \mathcal{L}$ if and only if $\mathbf{Z}(X) \in \mathcal{U}^{\mathcal{L}}$ by Lemma 2.3(3), if and only if $\mathbf{Z}(X)^+ \in \mathcal{U}_{\mathcal{A}}$ since $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ is a duality pair, and if and only if $X^+ \in \mathcal{A}$ by Lemma 2.3(5).

Next, we prove that \mathcal{A} is closed under direct summands and finite direct sums.

Note that \mathbf{Z} is an exact functor, $X_1 \oplus X_2 \in \mathcal{A}$ if and only if $\mathbf{Z}(X_1 \oplus X_2) \cong \mathbf{Z}(X_1) \oplus \mathbf{Z}(X_2) \in \mathcal{U}_{\mathcal{A}}$ by Lemma 2.3(3), and if and only if $\mathbf{Z}(X_1), \mathbf{Z}(X_2) \in \mathcal{U}_{\mathcal{A}}$ since $\mathcal{U}_{\mathcal{A}}$ is closed under direct summands and finite direct sums, and if and only if $X_1, X_2 \in \mathcal{A}$ by Lemma 2.3(3). \square

A duality pair $(\mathcal{L}, \mathcal{A})$ over R is said to be *symmetric* [6] if both $(\mathcal{L}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{L})$ are duality pairs.

Recall that an exact sequence of left R -modules $\epsilon : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is *pure exact* [3], if sequence $M \otimes \epsilon$ is exact for any right R -module M . In this case, A is said to be pure submodule of B .

Recall that a left R -module E is *pure injective* [3], if sequence $\text{Hom}_R(\epsilon, E)$ is exact for any pure exact sequence of left R -modules $\epsilon : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

Corollary 2.5. *Let \mathcal{L} be a class of left R -modules and \mathcal{A} be a class of right R -modules. Then the following statements are equivalent.*

(1) $(\mathcal{L}, \mathcal{A})$ is a symmetric duality pair over R .

(2) $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ is a symmetric duality pair over $R[x]$.

Moreover, if \mathcal{A} is closed under pure submodules and \mathcal{L} contains all pure injective modules, then the above statements are equivalent to

(3) $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ is a symmetric duality pair over $R[x]$.

Proof. (1) \Leftrightarrow (2). This is immediately from Theorem 2.4 and its dual statement.

(1) \Rightarrow (3). By Theorem 2.4, $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ and $(\mathcal{C}_{\mathcal{A}}, \mathcal{K}^{\mathcal{L}})$ are duality pairs over $R[x]$.

$(X, \alpha) \in \mathcal{K}_{\mathcal{A}}$ if and only if $\ker(\alpha) \in \mathcal{A}$, which is equivalent to $\text{coker}(\alpha^+) \cong (\ker \alpha)^+ \in \mathcal{L}$ because $(\mathcal{A}, \mathcal{L})$ is a duality pair. This, in turn, is equivalent to $(X, \alpha)^+ \cong (X^+, \alpha^+) \in \mathcal{C}^{\mathcal{L}}$ by Lemma 2.2(1).

Next, we show that $\mathcal{C}^{\mathcal{L}}$ is closed under direct summands and finite direct sums.

Let $(X_1, \alpha_1), (X_2, \alpha_2)$ be left $R[x]$ -modules. Note that $(X_1, \alpha_1) \oplus (X_2, \alpha_2) \cong (X_1 \oplus X_2, \alpha)$ with $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$. By Lemma 2.1, $(X_1 \oplus X_2, \alpha) \in \mathcal{C}^{\mathcal{L}}$ if and only if $(X_1 \oplus X_2, \alpha)^+ \cong (X_1^+ \oplus X_2^+, \alpha^+) \cong (X_1^+, \alpha_1^+) \oplus (X_2^+, \alpha_2^+) \in \mathcal{K}^{\mathcal{A}}$ because $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}^{\mathcal{A}})$ is a duality pairs over $R[x]$, which is equivalent to $(X_1^+, \alpha_1^+), (X_2^+, \alpha_2^+) \in \mathcal{K}_{\mathcal{A}}$ because $\mathcal{K}_{\mathcal{A}}$ is closed under direct summands and finite direct sums. This, in turn, is equivalent to $(X_1, \alpha_1), (X_2, \alpha_2) \in \mathcal{C}^{\mathcal{L}}$ because $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ is a duality pairs over $R[x]$.

Thus, $(\mathcal{K}_{\mathcal{A}}, \mathcal{C}^{\mathcal{L}})$ is a duality pairs over $R[x]$. Therefore, $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ is a symmetric duality pair over $R[x]$.

(3) \Rightarrow (1). By Theorem 2.4, $(\mathcal{L}, \mathcal{A})$ is a duality pair over R .

Let $X \in \mathcal{A}$. X^+ is a pure injective left R -module by [3, Proposition 5.3.7]. Thus, $X^+ \in \mathcal{L}$.

Conversely, let $X^+ \in \mathcal{L}$. $X^{++} \in \mathcal{A}$ because $(\mathcal{L}, \mathcal{A})$ is a duality pair over R . Thus, $X \in \mathcal{A}$ because X is a pure submodules of X^{++} by [3, Proposition 5.3.9].

Next, we prove that \mathcal{L} is closed under direct summands and finite direct sums.

Let X_1, X_2 be left R -modules. Note that $X_1 \oplus X_2 \in \mathcal{L}$ if and only if $\mathbf{T}(X_1 \oplus X_2) \in \mathcal{C}^{\mathcal{L}}$ by Lemma 2.3(1), if and only if $\mathbf{T}(X_1) \oplus \mathbf{T}(X_2) \in \mathcal{C}^{\mathcal{L}}$, and if and only if $\mathbf{T}(X_1), \mathbf{T}(X_2) \in \mathcal{C}^{\mathcal{L}}$ since $\mathcal{C}^{\mathcal{L}}$ is closed under direct summands and finite direct sums, and if and only if $X_1, X_2 \in \mathcal{L}$ by Lemma 2.3(1).

Thus, $(\mathcal{A}, \mathcal{L})$ is a duality pair over R . Therefore, $(\mathcal{L}, \mathcal{A})$ is a symmetric duality pair over R . \square

Lemma 2.6. *Let \mathcal{L} be a class of left R -modules.*

(1) \mathcal{L} is closed under direct sums if and only if $\mathcal{U}^{\mathcal{L}}$ is closed under direct sums.

(2) \mathcal{L} is closed under extensions if and only if $\mathcal{U}^{\mathcal{L}}$ is closed under extensions.

(3) \mathcal{L} is closed under direct sums if and only if $\mathcal{C}^{\mathcal{L}}$ is closed under direct sums.

(4) If $\mathcal{C}^{\mathcal{L}}$ is closed under extensions, then \mathcal{L} is closed under extensions.

Proof. (1) Let $\{(X_i, \alpha_i)\}_{i \in I}$ be a family of left $R[x]$ -modules with $(X_i, \alpha_i) \in \mathcal{U}^{\mathcal{L}}$ for any $i \in I$. By the lemma 2.2(2), $\oplus_{i \in I} (X_i, \alpha_i) \cong (\oplus_{i \in I} X_i, \oplus_{i \in I} \alpha_i)$. Since $\oplus_{i \in I} X_i \in \mathcal{L}$, $\oplus_{i \in I} (X_i, \alpha_i) \in \mathcal{U}^{\mathcal{L}}$.

Conversely, let $\{X_i\}_{i \in I}$ be a family of left R -modules with $X_i \in \mathcal{L}$ for any $i \in I$. $\mathbf{Z}(X_i) \in \mathcal{U}^{\mathcal{L}}$ by Lemma

2.3(3). Since $\mathcal{U}^{\mathcal{L}}$ is closed under direct sums, $\mathbf{Z}(\oplus_{i \in I} X_i) \in \mathcal{U}^{\mathcal{L}}$. Thus $\oplus_{i \in I} X_i \in \mathcal{L}$.

(2) Let $0 \rightarrow (A, \alpha) \rightarrow (B, \beta) \rightarrow (C, \gamma) \rightarrow 0$ be an exact sequence in $R[x]\text{-Mod}$ with $(A, \alpha), (C, \gamma) \in \mathcal{U}^{\mathcal{L}}$. This induces the exact sequence in $R\text{-Mod}$: $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $A, C \in \mathcal{L}$. Since \mathcal{L} is closed under extensions, $B \in \mathcal{L}$. It follows that $(B, \beta) \in \mathcal{U}^{\mathcal{L}}$.

Conversely, let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence in $R\text{-Mod}$ with $A, C \in \mathcal{L}$, which induces the exact sequence in $R[x]\text{-Mod}$: $0 \rightarrow \mathbf{Z}(A) \rightarrow \mathbf{Z}(B) \rightarrow \mathbf{Z}(C) \rightarrow 0$ with $\mathbf{Z}(A), \mathbf{Z}(C) \in \mathcal{U}^{\mathcal{L}}$ by Lemma 2.3(3). Since $\mathcal{U}^{\mathcal{L}}$ is closed under extensions, $\mathbf{Z}(B) \in \mathcal{U}^{\mathcal{L}}$. Thus $B \in \mathcal{L}$ by Lemma 2.3(3).

(3) The proof is similar to that of (1).

(4) The proof is similar to that of (2). □

A duality pair $(\mathcal{L}, \mathcal{A})$ over R is said to be *perfect* [6] if \mathcal{L} contains the regular module ${}_R R$, and is closed under direct sums and extensions.

Theorem 2.7. *Let \mathcal{L} be a class of left R -modules and \mathcal{A} be a class of right R -modules. Then the following statements are equivalent.*

(1) $(\mathcal{L}, \mathcal{A})$ is a perfect duality pair over R .

(2) $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ is a perfect duality pair over $R[x]$.

Moreover, if $\text{Tor}_1^{R[x]}(\mathbf{Z}({}_R R), \mathcal{C}^{\mathcal{L}}) = 0$, then the above statements are equivalent to

(3) $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ is a perfect duality pair over $R[x]$.

Proof. (1) \Leftrightarrow (2). For left $R[x]$ -module $R[x] = \mathbf{T}({}_R R) \in \mathcal{U}^{\mathcal{L}}$ if and only if ${}_R R \in \mathcal{L}$ because $(\mathcal{L}, \mathcal{A})$ is a duality pair over R . The assertion follows from Lemma 2.6(1) and (2), and Theorem 2.4.

(1) \Leftrightarrow (3). For left $R[x]$ -module $R[x] = \mathbf{T}({}_R R) \in \mathcal{C}^{\mathcal{L}}$ if and only if ${}_R R \in \mathcal{L}$ since $\mathbf{T}(\mathcal{L}) \subseteq \mathcal{C}^{\mathcal{L}}$.

Let $0 \rightarrow (A, \alpha) \rightarrow (B, \beta) \rightarrow (C, \gamma) \rightarrow 0$ be an exact sequence in $R[x]\text{-Mod}$ with $(A, \alpha), (C, \gamma) \in \mathcal{C}^{\mathcal{L}}$. This induces the exact sequence in $R\text{-Mod}$: $0 \rightarrow \text{coker}(\alpha) \rightarrow \text{coker}(\beta) \rightarrow \text{coker}(\gamma) \rightarrow 0$ with $\text{coker}(\alpha), \text{coker}(\gamma) \in \mathcal{L}$ by [8, Lemma 1.1(2)] and $\text{Tor}_1^{R[x]}(\mathbf{Z}({}_R R), \mathcal{C}^{\mathcal{L}}) = 0$. Since \mathcal{L} is closed under extensions, $\text{coker}(\beta) \in \mathcal{L}$. Thus, $(B, \beta) \in \mathcal{C}^{\mathcal{L}}$. Therefore, the assertion follows from Lemma 2.6(3) and (4), and Theorem 2.4. □

A duality pair $(\mathcal{L}, \mathcal{A})$ is said to be *complete* [6] if $(\mathcal{L}, \mathcal{A})$ is symmetric and perfect. The following result can be obtained immediately from Corollary 2.5 and Theorem 2.7.

Corollary 2.8. *Let \mathcal{L} be a class of left R -modules and \mathcal{A} be a class of right R -modules. Then the following statements are equivalent.*

(1) $(\mathcal{L}, \mathcal{A})$ is a complete duality pair over R .

(2) $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ is a complete duality pair over $R[x]$.

Moreover, if \mathcal{A} is closed under pure submodules and \mathcal{L} contains all pure injective modules, and $\text{Tor}_1^{R[x]}(\mathbf{Z}({}_R R), \mathcal{C}^{\mathcal{L}}) = 0$, then the above statements are equivalent to

(3) $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ is a complete duality pair over $R[x]$.

3 Application

Definition 3.1. Let \mathcal{L} be a class of left R -modules and \mathcal{A} be a class of right R -modules such that $(\mathcal{L}, \mathcal{A})$ is a complete duality pair over R . A left R -module X is *Gorenstein $(\mathcal{L}, \mathcal{A})$ -projective* [4] if there exists an exact sequence of projective left R -modules

$$\mathbb{P} : \dots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \rightarrow \dots$$

such that $X \cong \ker(d^0)$, $\text{Hom}_R(\mathbb{P}, L)$ is exact for any $L \in \mathcal{L}$.

Proposition 3.2. *Let \mathcal{L} be a class of left R -modules and \mathcal{A} be a class of right R -modules.*

(1) *Let \mathcal{A} is closed under pure submodules and \mathcal{L} contains all pure injective modules. If $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ is a complete duality pair over $R[x]$, and (X, α) is a Gorenstein $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ -projective left R -module, then α is a monomorphism and $\text{coker}(\alpha)$ is a Gorenstein $(\mathcal{L}, \mathcal{A})$ -projective left R -module.*

(2) *Let $\text{Tor}_1^{R[x]}(\mathbf{Z}(R), \mathcal{C}^{\mathcal{L}}) = 0$. If $(\mathcal{L}, \mathcal{A})$ is a complete duality pair over R , and (X, α) is a Gorenstein $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ -projective left R -module, then α is a monomorphism and $\text{coker}(\alpha)$ is a Gorenstein $(\mathcal{L}, \mathcal{A})$ -projective left R -module.*

(3) *If $(\mathcal{L}, \mathcal{A})$ is a complete duality pair over R , and (X, α) is a Gorenstein $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ -projective left R -module, then α is a monomorphism and $\text{coker}(\alpha)$ is a Gorenstein $(\mathcal{L}, \mathcal{A})$ -projective left R -module.*

Proof. (1) By Corollary 2.8, $(\mathcal{L}, \mathcal{A})$ is a complete duality pair over R . There is an exact sequence in $R[x]$ -Mod :

$$\mathbb{P} : \cdots \rightarrow (P^{-1}, \beta^{-1}) \xrightarrow{g^{-1}} (P^0, \beta^0) \xrightarrow{g^0} (P^1, \beta^1) \rightarrow \cdots$$

with (P^i, β^i) projective left $R[x]$ -modules and $(X, \alpha) \cong \ker(g^0)$.

Consider the following commutative diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \longrightarrow & P^0 \\ & & \downarrow \alpha & & \downarrow \beta^0 \\ 0 & \longrightarrow & X & \longrightarrow & P^0. \end{array}$$

Since β^0 is a monomorphism by [9, Proposition 2.6(1)], α is a monomorphism.

Note that $pd_{R[x]}(\mathbf{Z}(R)) \leq 1$ by [8, Corollary 2.7(1) and 9, Lemma 2.8]. Thus $\mathbf{Z}(R) \otimes_{R[x]} \mathbb{P}$ is exact by [2, Lemma 2.3]. Since $\mathbf{Z}(R) \otimes_{R[x]} (P^i, \beta^i) \cong \text{coker}(\beta^i)$ by [8, Lemma 1.1(2)], we obtain an exact sequence of projective left R -modules :

$$\mathbf{C}(\mathbb{P}) : \cdots \rightarrow \text{coker}(\beta^{-1}) \xrightarrow{\mathbf{C}(g^{-1})} \text{coker}(\beta^0) \xrightarrow{\mathbf{C}(g^0)} \text{coker}(\beta^1) \rightarrow \cdots$$

with $\text{coker}(\alpha) \cong \ker(\mathbf{C}(g^0))$ by [9, Proposition 2.6(1)].

Let $Y \in \mathcal{L}$. $\mathbf{Z}(Y) \in \mathcal{C}^{\mathcal{L}}$. Then $\text{Hom}_R(\mathbf{C}(\mathbb{P}), Y) \cong \text{Hom}_{R[x]}(\mathbb{P}, \mathbf{Z}(Y))$ is exact because $\mathbf{Z}(Y) \in \mathcal{C}^{\mathcal{L}}$ and (\mathbf{C}, \mathbf{Z}) is an adjoint pair. Thus, $\text{coker}(\alpha)$ is a Gorenstein $(\mathcal{L}, \mathcal{A})$ -projective left R -module.

(2) $(\mathcal{C}^{\mathcal{L}}, \mathcal{K}_{\mathcal{A}})$ is a complete duality pair over $R[x]$ by Corollary 2.8. The assertion is similar to (1).

(3) $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ is a complete duality pair over $R[x]$ by Corollary 2.8. The assertion is similar to (1). \square

Corollary 3.3. *Let \mathcal{L} be a class of left R -modules and \mathcal{A} be a class of right R -modules. If $(\mathcal{L}, \mathcal{A})$ is a complete duality pair over R , then X is a Gorenstein $(\mathcal{L}, \mathcal{A})$ -projective left R -module if and only if $\mathbf{T}(X)$ is a Gorenstein $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ -projective left $R[x]$ -module.*

Proof. Assume that X is a Gorenstein $(\mathcal{L}, \mathcal{A})$ -projective left R -module. There exists an exact sequence of projective left R -modules

$$\Lambda : \cdots \rightarrow P_{-1} \rightarrow P_0 \xrightarrow{f_0} P_1 \rightarrow P_2 \rightarrow \cdots$$

with $X \cong \ker(f_0)$. This induces an exact sequence of projective left $R[x]$ -modules

$$\mathbf{T}(\Lambda) : \cdots \rightarrow \mathbf{T}(P_{-1}) \rightarrow \mathbf{T}(P_0) \xrightarrow{\mathbf{T}(f_0)} \mathbf{T}(P_1) \rightarrow \mathbf{T}(P_2) \rightarrow \cdots$$

with $\mathbf{T}(X) \cong \ker(\mathbf{T}(f_0))$ by [9, Corollary 2.7(1)].

For any $(Y, \alpha) \in \mathcal{U}^{\mathcal{L}}$, since (\mathbf{T}, \mathbf{U}) is an adjoint pair by [9, Theorem 2.1], we have $\text{Hom}_{R[x]}(\mathbf{T}(\Lambda), (Y, \alpha)) \cong \text{Hom}_R(\Lambda, Y)$, which is exact. Thus, $\mathbf{T}(X)$ is a Gorenstein $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ -projective left $R[x]$ -module.

Conversely, if $\mathbf{T}(X)$ is a Gorenstein $(\mathcal{U}^{\mathcal{L}}, \mathcal{U}_{\mathcal{A}})$ -projective left $R[x]$ -module, then X is a Gorenstein $(\mathcal{L}, \mathcal{A})$ -projective left R -module by Proposition 3.2(2). \square

References

- [1] G. W. Chang and P. T. Toan, *Polynomial and power series ring extensions from sequences*, J. Algebra Appl. **21**(3) (2022), 2250048.
- [2] E. E. Enochs, M. C. Izurdiaga and B. Torrecillas, *Gorenstein conditions over triangular matrix rings*, J. Pure Appl. Algebra **218** (2014), 1544-1554.
- [3] E. E. Enochs, O. M. G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, Berlin-New York, 2000.
- [4] J. Gillespie, *Duality pairs and stable module categories*, J. Pure Appl. Algebra **223**(8) (2019), 3425-3435.
- [5] J. Gillespie and A. Iacob, *Duality pairs, generalized Gorenstein modules, and Ding injective envelopes*, Comptes Rendus Math. **360**, (2022) 381-398.
- [6] H. Holm and P. Jørgensen, *Cotorsion pairs induced by duality pairs*, J. Commut. Algebra **1**(4) (2009), 621-633.
- [7] Y. J. Ma, J. F. Lü, H. H. Li and J. S. Hu, *How to construct Gorenstein projective modules relative to complete duality pairs over Morita rings*, J. Algebra Appl. **23**(10) (2024), 2450153.
- [8] L. X. Mao, *Gorenstein projective, injective, flat modules and dimensions over polynomial rings*, J. Algebra Appl. **25** (2026), 2650056.
- [9] L. X. Mao, *A structure result on modules over polynomial rings*, J. Algebra Appl. **24**(10) (2025), 2550230.
- [10] L. X. Mao, *Pure projective and pure injective modules over polynomial rings*, Bull. Korean Math. Soc. **62**(2) (2025), 527-540.
- [11] L. X. Mao, *Duality pairs and FP-injective modules over formal triangular matrix rings*, Comm. Algebra **48** (2020), 5296-5310.
- [12] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167-171.