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A note on S-AB-(m,n)-absorbing ideals

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Abstract. Let R be a commutative ring with identity, S a multiplicative subset of R, and I an ideal of R such that $I \cap S = \emptyset$. In this paper, we present further results on S-AB-(m,n)-absorbing ideals, a class that extends the concept of n-absorbing ideals. We also explore the behavior of these ideals under quotient rings, trivial ring extensions, and amalgamations of rings.

Key Words: 2-absorbing ideal, S-prime ideal, homomorphic image, amalgamation of rings, trivial ring extension.

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1 Introduction

Throughout this paper, we let R denote a commutative ring with identity, and S a multiplicative subset of R (not necessarily saturated). In [33], Querré demonstrated that if D is an integrally closed domain and B is an integral ideal of $D[X]$ with $B \cap D \neq 0$, then the divisorial closure of B coincides with the divisorial closure of $A_B[X]$, where A_B is the ideal in D generated by the coefficients of the elements in B. Later, Anderson, Kwak, and Zafrullah [3] introduced the notions of *almost finitely generated ideals* and *agreeable domains* as a framework to further study Querré's characterization of divisorial ideals.

Building on this, Anderson and Dumitrescu [4] extended these concepts to arbitrary ideals in a commutative ring with identity. They introduced the ideas of S-*finite ideals* and S-*Noetherian rings*. Specifically, for a given multiplicative subset $S \subseteq R$ and an ideal I of R, the ideal I is said to be S-*finite* if there exists an element $s \in S$ and a finitely generated ideal $J \subseteq R$ such that $sI \subseteq J \subseteq I$. A ring R is called S-*Noetherian* if every ideal of R is S-finite. Moreover, Anderson and Dumitrescu generalized several classical results from Noetherian ring theory, including the Eakin–Nagata Theorem and Hilbert's Basis Theorem [4, Corollary 7, Proposition 10], to the S-Noetherian setting.

These ideas sparked a series of papers focusing on S-Noetherianity within specific classes of rings and extending classical notions in multiplicative ideal theory to their S-analogs. For instance, Liu [30] studied conditions under which generalized power series rings become S-Noetherian. Lim and Oh [28, 29] examined the behavior of the S-Noetherian property in amalgamated algebras along ideals and in composite ring extensions. Kim, Kim, and Lim [27] introduced S-*strong Mori domains* as an extension of strong Mori domains. Likewise, Hamed and Hizem [22] generalized GCD-domains to S-*GCD-domains*, a notion that was further studied by Anderson, Hamed, and Zafrullah [6].

In 2020, Hamed and Malek [21, 23] introduced the concept of S-*prime ideals* as a natural generalization of prime ideals. Given a multiplicative subset $S \subseteq R$ and an ideal I of R such that $I \cap S = \emptyset$, the ideal I is called an S-*prime ideal* if for all $a, b \in R$, whenever $ab \in I$, there exists $s \in S$ such that either $sa \in I$ or $sb \in I$.

Among the various generalizations of prime ideals, *n-absorbing ideals*, introduced by Anderson and Badawi [5], have received significant attention. For a natural number n and an ideal I in a commutative ring R, I is said to be an *n-absorbing ideal* if for any elements $a_1,\ldots,a_{n+1} \in R$ satisfying $a_1\cdots a_{n+1} \in I$, there exists an index $j \in \{1,\ldots,n+1\}$ such that the product of the remaining n elements lies in I, i.e., $\prod_{\substack{1\leq i\leq n+1 \\ i\neq j}} a_i \in I$. The theory of n-absorbing ideals has since developed into a vibrant area of research (see the survey [10] and its references). Furthermore, I is called an (m,n)-absorbing ideal of R as in [1] if whenever $a_1\cdots a_m \in I$ for some nonunits $a_1,\ldots,a_m \in R$, then there are n of the a_i 's whose product is in I. In [11], Badawi, El Khalfi, and Mahdou defined (m,n)-absorbing prime ideals: a proper ideal I of R is called (m,n)-absorbing prime if, whenever nonunit elements $a_1,\ldots,a_m \in R$ satisfy $a_1\cdots a_m \in I$, then either $a_1\cdots a_n \in I$ or $a_{n+1}\cdots a_m \in I$. They also introduced the notion of AB-(m,n)-absorbing ideals, where a proper ideal I of R is called AB-(m,n)-absorbing if, whenever $a_1\cdots a_m \in I$ for some $a_1,\ldots,a_m \in R$, there exist n of the a_i 's whose product belongs to I. Recently, Assalami, Kharbouch and Kim introduced the notions of S-(m,n)-absorbing ideals and S-(m,n)-absorbing prime ideals [8]. An ideal I of a rings R is called an S-(m,n)-absorbing ideal of R if there exists $s \in S$ such that, whenever nonunit elements $a_1,\ldots,a_m \in R$ satisfy $a_1\cdots a_m \in I$, there exist n distinct indices $i_1,\ldots,i_n \in \{1,\ldots,m\}$ such that $sa_{i_1}\cdots a_{i_n} \in I$. Also I is said to be an S-(m,n)-absorbing prime ideal of R if there exists $s \in S$ such that, whenever nonunit elements $a_1,\ldots,a_m \in R$ satisfy $a_1\cdots a_m \in I$, then either $sa_1\cdots a_n \in I$ or $sa_{n+1}\cdots a_m \in I$. Moreover, I is defined as an S-AB-(m,n)-absorbing ideal of R if there exists $s \in S$ such that, whenever elements $a_1,\ldots,a_m \in R$ satisfy $a_1\cdots a_m \in I$, there exist n distinct indices $i_1,\ldots,i_n \in \{1,\ldots,m\}$ such that $sa_{i_1}\cdots a_{i_n} \in I$.

In this paper, we present further results on S-AB-(m,n)-absorbing ideals. We also explore the behavior of these ideals under quotient rings, trivial ring extensions, and amalgamation of rings.

2 Main Results

We start this section by the following proposition.

Proposition 2.1. *Let R be a ring and let S be a multiplicative subset of R. Then the following assertions hold.*

(1) Let I be an ideal of R disjoint from S and let J be an ideal of R such that $J\cap S \neq \emptyset$. If I is an S-AB-(m,n)-absorbing ideal of R, then IJ is an S-AB-(m,n)-absorbing ideal of R.

(2) Let T be a commutative ring with identity containing R and let J be an ideal of T disjoint from S. If J is an S-AB-(m,n)-absorbing ideal of T, then $J\cap R$ is an S-AB-(m,n)-absorbing ideal of R.

(3) Let I be an ideal of R disjoint from S. If I is an S-AB-(m,n)-absorbing ideal of R, then $\sqrt{I}$ is an S-AB-(m,n)-absorbing ideal of R, and for each $r \in \sqrt{I}$, there exists $z \in S$ such that $zr^n \in I$.

Proof. (1) Suppose that I is an S-AB-(m,n)-absorbing ideal of R for some $s \in S$. As $J\cap S \neq \emptyset$, we can pick an element $b \in J\cap S$. Let $a_1,\ldots,a_m \in R$ with $a_1\cdots a_m \in IJ$. Then $a_1\cdots a_m \in I$, so we may assume that $sa_1\cdots a_n \in I$. Then $sba_1\cdots a_n \in IJ$. Thus IJ is an S-AB-(m,n)-absorbing ideal of R.

(2) Suppose that J is an S-AB-(m,n)-absorbing ideal of T. Let $a_1,\ldots,a_m \in R$ with $a_1\cdots a_m \in J\cap R$. Then we may assume that $sa_1\cdots a_n \in J$ for some $s \in S$. Thus $sa_1\cdots a_n \in J\cap R$, which means that $J\cap R$ is an S-AB-(m,n)-absorbing ideal of R.

(3) Suppose that I is an S-AB-(m,n)-absorbing ideal of R. Let $a_1,\ldots,a_m \in R$ with $a_1\cdots a_m \in \sqrt{I}$. Then $a_1^p\cdots a_m^p \in I$ for some $p \in \mathbb{N}$. Hence we may assume that $sa_1^p\cdots a_n^p \in I$ for some $s \in S$, so $(sa_1\cdots a_n)^p \in I$. Hence $sa_1\cdots a_n \in \sqrt{I}$, and thus $\sqrt{I}$ is an S-AB-(m,n)-absorbing ideal of R. Now, let $a \in \sqrt{I}$. Then $a^p \in I$ for some $p \in \mathbb{N}$. If $p \leq n$, then we are done. Now, suppose that $p > n$. Then there exists $w \in \mathbb{N}$ such that $s^w r^n \in I$. Since $s^w \in S$, the proof is done. □

Theorem 2.2. Let I be a radical ideal of a ring R. Then I is an S-AB-(m,n)-absorbing ideal of R if and only if I is an S-n-absorbing ideal of R.

Proof. Let I be an S-AB-(m,n)-absorbing ideal of R. It suffices to show that I is an S-AB-$(m-1,n)$-absorbing ideal of R.

Let I be a radical ideal of R and $a_1\cdots a_{m-1}\in I$ for $a_1,\ldots,a_{m-1}\in R$. Since I is an S-AB-(m,n)-absorbing ideal of R and $a_1^2a_2\cdots a_{m-1}\in I$, either there are n of a_i's whose product with some $s\in S$ is in I or there are $n-2$ of $a_2,\ldots,a_{m-1}$ whose product with a_1^2 and some $s\in S$ is in I. If there are n of a_i's whose product with s is in I, then we are done. In the other case, there are $n-2$ of $a_2,\ldots,a_{m-1}$ whose product with a_1^2 and s is in I, we may assume that $sa_1^2a_2\cdots a_{n-2}\in I$, then $sa_1a_2\cdots a_{n-2}\in\sqrt{I}=I$. Hence I is an S-AB-$(m-1,n)$-absorbing ideal of R. □

Let R and T be rings, and let S be a multiplicative subset of R. If $f:R\to T$ is a ring homomorphism, then the image $f(S)$ is a multiplicative subset of $f(R)$.

Proposition 2.3. *Let R and T be rings, S a multiplicative subset of R and $f:R\to T$ a homomorphism.*

(1) Let I be an ideal of R disjoint from S with $\ker(f)\subseteq I$. Then I is an S-AB-(m,n)-absorbing ideal of R if and only if $f(I)$ is an $f(S)$-AB-(m,n)-absorbing ideal of $\mathrm{Im}(f)$.

(2) Let J be an ideal of $\mathrm{Im}(f)$ disjoint from $f(S)$. Then J is an $f(S)$-AB-(m,n)-absorbing ideal of $\mathrm{Im}(f)$ if and only if $f^{-1}(J)$ is an S-AB-(m,n)-absorbing ideal of R.

Proof. (1) Suppose that I is an S-AB-(m,n)-absorbing ideal of R. First, we claim that $f(I)\cap f(S)=\emptyset$. Suppose to the contrary that there exists an element $i\in I$ such that $f(i)\in f(S)$. Hence $i-s\in\ker(f)\subseteq I$ for some $s\in S$, so $s\in I$. This is a contradiction since $I\cap S=\emptyset$. Therefore $f(I)$ is disjoint from $f(S)$. Let $f(a_1),\ldots,f(a_m)\in\mathrm{Im}(f)$ such that $f(a_1)\cdots f(a_m)\in f(I)$. Then there exists an element $i\in I$ such that $a_1\cdots a_m-i\in\ker(f)\subseteq I$. Hence $a_1\cdots a_m\in I$. Since I is an S-AB-(m,n)-absorbing ideal of R, we may assume that there exists an element $s\in S$ such that $sa_1\cdots a_n\in I$. Hence $f(s)f(a_1)\cdots f(a_n)\in f(I)$, which means that $f(I)$ is an $f(S)$-AB-(m,n)-absorbing ideal of $\mathrm{Im}(f)$. For the converse, suppose that $f(I)$ is an $f(S)$-AB-(m,n)-absorbing ideal of $\mathrm{Im}(f)$. It is clear that $I\cap S=\emptyset$. Let $a_1,\ldots,a_m\in R$ with $a_1\cdots a_m\in I$. Then $f(a_1)\cdots f(a_m)\in f(I)$, so we may assume that there exists an element $s\in S$ such that $f(sa_1\cdots a_n)=f(s)f(a_1)\cdots f(a_n)\in f(I)$. It follows that there exists an element $i\in I$ such that $sa_1\cdots a_n-i\in\ker(f)\subseteq I$. Hence $sa_1\cdots a_n\in I$. Thus I is an S-AB-(m,n)-absorbing ideal of R.

(2) It is clear that $J\cap f(S)=\emptyset$ if and only if $f^{-1}(J)\cap S=\emptyset$. Suppose that J is an $f(S)$-AB-(m,n)-absorbing ideal of $\mathrm{Im}(f)$. Let $a_1,\ldots,a_m\in R$ with $a_1\cdots a_m\in f^{-1}(J)$. Then $f(a_1)\cdots f(a_m)=f(a_1\cdots a_m)\in J$, so we may assume that $f(sa_1\cdots a_n)=f(s)f(a_1)\cdots f(a_n)\in J$ for some $s\in S$. Thus $sa_1\cdots a_n\in f^{-1}(J)$. Consequently, $f^{-1}(J)$ is an S-AB-(m,n)-absorbing ideal of R. The converse follows directly from (1). □

By Proposition 2.3, we obtain:

Corollary 2.4. *Let R be a ring, S a multiplicative subset of R and $f:R\to T$ a homomorphism with $\ker(f)\cap S=\emptyset$. Then there is a one-to-one order-preserving correspondence between the S-AB-(m,n)-absorbing ideals of R containing $\ker(f)$ and the $f(S)$-AB-(m,n)-absorbing ideals of $f(R)$.*

Let R be a ring, I an ideal of R and S a multiplicative subset of R. Then $S/I:=\{s+I\,|\,s\in S\}$ is a multiplicative subset of R/I.

Corollary 2.5. *Let R be a ring, S a multiplicative subset of R, and I,J ideal of R such that $I\subseteq J$ and $J\cap S=\emptyset$. Then J is an S-AB-(m,n)-absorbing ideal of R if and only if J/I is an S/I-AB-(m,n)-absorbing ideal of R/I.*

Let R be a ring, S a multiplicative subset of R and I an ideal of R. Recall that $(I:s)=\{r\in R|sr\in I\}$ is an ideal of R. In the following, we examine the behavior of the ideal $(I:s)$ under the assumption that I is an S-AB-(m,n)-absorbing ideal. The next result can play a key role in studying S-AB-(m,n)-absorbing ideals by making use of the properties of (m,n)-absorbing ideals.

Proposition 2.6. *Let R be a ring, S a multiplicative subset of R and I an ideal of R disjoint from S. Then I is an S-AB-(m,n)-absorbing ideal of R if and only if $(I:s)$ is an (m,n)-absorbing ideal of R for some $s\in S$.*

Proof. Suppose that I is an S-AB-(m,n)-absorbing ideal of R for some $r\in S$. We claim that $(I:r^m)$ is an (m,n)-absorbing ideal of R. Let $a_1,\ldots,a_m\in R$ with $a_1\cdots a_m\in(I:r^m)$. Then $(ra_1)(ra_2)\cdots(ra_m)\in I$, which means that $r(ra_{i_1})\cdots(ra_{i_n})\in I$ for some distinct indices $i_1,\ldots,i_n\in\{1,\ldots,m\}$. This implies that $a_{i_1}\cdots a_{i_n}\in(I:r^m)$. Thus $(I:s)$ is an (m,n)-absorbing ideal of R, where $s=r^m\in S$.
Conversely, suppose that $(I:s)$ is an (m,n)-absorbing ideal of R for some $s\in S$ and let $a_1,\ldots,a_m\in R$ such that $a_1\cdots a_m\in I$. Since $a_1\cdots a_m\in(I:s)$, $a_{i_1}\cdots a_{i_n}\in(I:s)$ for some distinct indices $i_1,\ldots,i_n\in\{1,\ldots,m\}$. In other words, $sa_{i_1}\cdots a_{i_n}\in I$ for some distinct indices $i_1,\ldots,i_n\in\{1,\ldots,m\}$. Thus I is an S-AB-(m,n)-absorbing ideal of R. □

Now, we examine the quotient extension of S-AB-(m,n)-absorbing ideals.

Proposition 2.7. *Let R be a ring, S a multiplicative subset of R and I an ideal of R disjoint from S. If I is an S-AB-(m,n)-absorbing ideal of R, then IR_S is an (m,n)-absorbing ideal of R_S.*

Proof. Suppose that I is an S-AB-(m,n)-absorbing ideal of R. Let $a_1,\ldots,a_m\in R_S$ such that $a_1\cdots a_m\in IR_S$. Since for each $1\le i\le m$, $a_i=\frac{r_i}{s_i}$ for some $r_i\in R$ and $s_i\in S$, we obtain that $\frac{r_1\cdots r_m}{s_1\cdots s_m}\in IR_S$. Hence there exist elements $s\in S$ such that $sr_1\cdots r_m\in I$. As I is an S-AB-(m,n)-absorbing ideal of R, there exist $t\in S$ and some distinct indices $i_1,\ldots,i_n\in\{1,\ldots,m\}$ such that $tr_{i_1}\cdots r_{i_n}\in I$. so, $\frac{r_{i_1}}{s_{i_1}}\cdots\frac{r_{i_n}}{s_{i_n}}\in IR_S$. Thus IR_S is an (m,n)-absorbing ideal of R_S. □

Let R be a ring, m a positive integer, S a multiplicative subset of R and I an ideal of R disjoint from S. Hence, if I is an S-AB-(m,k)-absorbing ideal for some $k\in\mathbb{N}$, then there exists the minimal positive integer n such that I is an S-AB-(m,n)-absorbing ideal. Such minimal integer n is denoted by $\omega_{R,S,m}(I)$ and we set $\omega_{R,S,m}(I)=\infty$ if I is not an S-AB-(m,n)-absorbing ideal of R for any $n\in\mathbb{N}$.

Proposition 2.8. *Let R_1 and R_2 be rings, and let $S_1\subseteq R_1$ and $S_2\subseteq R_2$ be multiplicative subsets. For $j=1,2$, let I_j be an ideal of R_j such that $I_j\cap S_j=\emptyset$. Define $R=R_1\times R_2$, $S=S_1\times S_2$, and $I=I_1\times I_2$. Then*

$$\omega_{R,S,m+m'}(I)=\omega_{R_1,S_1,m}(I_1)+\omega_{R_2,S_2,m'}(I_2).$$

Proof. Assume that $\omega_{R_1,S_1}(I_1)=n$ and $\omega_{R_2,S_2}(I_2)=r$. Then there exist elements $a_1,\ldots,a_n\in R_1$ and $b_1,\ldots,b_r\in R_2$ such that $a_1\cdots a_n\in I_1$ and $b_1\cdots b_r\in I_2$, and for all $1\le i\le m$, $1\le j\le n$, $s\in S_1$, and $t\in S_2$, we have

$$s\prod_{\substack{1\le\ell\le n\\ \ell\ne i}}a_\ell\notin I_1\quad\text{and}\quad t\prod_{\substack{1\le\ell\le r\\ \ell\ne j}}b_\ell\notin I_2.$$

Therefore,

$$\left(\prod_{1\le\ell\le n}(a_\ell,1)\right)\left(\prod_{1\le\ell\le r}(1,b_\ell)\right)(1,1)\cdots(1,1)=(a_1\cdots a_n,b_1\cdots b_r)\in I_1\times I_2.$$

However, for all $1\le i\le n$ and $1\le j\le r$, we have:

$$(s,t)\left(\prod_{\substack{1\le\ell\le n\\ \ell\ne i}}(a_\ell,1)\right)\left(\prod_{1\le\ell\le r}(1,b_\ell)\right)\notin I_1\times I_2\ \ \textit{and}\ \ (s,t)\left(\prod_{1\le\ell\le n}(a_\ell,1)\right)\left(\prod_{\substack{1\le\ell\le r\\ \ell\ne j}}(1,b_\ell)\right)\notin I_1\times I_2.$$

This implies that $n+r \leq \omega_{R,S,m+m'}(I)$.

For the reverse inequality, let $k = m+m'$, and suppose that $(a_1,b_1),\ldots,(a_k,b_k) \in R_1 \times R_2$ with

$$\prod_{1\leq \ell\leq k}(a_\ell,b_\ell) \in I_1 \times I_2.$$

Then there exist subsets $\{i_1,\ldots,i_n\}$ and $\{j_1,\ldots,j_r\}$ of $\{1,\ldots,k\}$ such that for some $(s_1,s_2) \in S_1 \times S_2$, we have

$$s_1 a_{i_1}\cdots a_{i_n} \in I_1 \quad \text{and} \quad s_2 b_{j_1}\cdots b_{j_r} \in I_2.$$

Let $K = \{i_1,\ldots,i_n\} \cup \{j_1,\ldots,j_r\}$. Then

$$(s_1,s_2)\prod_{i\in K}(a_i,b_i) \in I_1 \times I_2.$$

Since $|K| \leq n+r$, we conclude that $\omega_{R,S,m+m'}(I) \leq n+r$.

Combining both directions, we have

$$\omega_{R,S,m+m'}(I) = \omega_{R_1,S_1,m}(I_1) + \omega_{R_2,S_2,m'}(I_2).$$

Finally, it is easy to adapt this argument to show that $\omega_{R,S,m+m'}(I) = \infty$ if and only if either $n = \infty$ or $r = \infty$, completing the proof. □

Corollary 2.9. *Let R_1,R_2 be rings and let S_1,S_2 be multiplicative subsets of R_1,R_2, respectively. For $j=1,2$, let I_j be an ideal of R_j disjoint from S_j. If I_1 is an S_1-AB-(m,n)-absorbing ideal of R_1 and I_2 is an S_2-AB-(m',r)-absorbing ideal of R_2, then $I_1\times I_2$ is an $(S_1\times S_2)$-AB-$(m+m',n+r)$-absorbing ideal of $R_1\times R_2$.*

Let A be a ring and E an A-module. Then $A \ltimes E$, the *trivial (ring) extension of A by E*, is the ring whose additive structure is that of the external direct sum $A\oplus E$ and whose multiplication is defined by $(a,e)(b,f) := (ab, af+be)$ for all $a,b \in A$ and all $e,f \in E$. (This construction is also known by other terminology and other notation, such as the *idealization* $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [20], [19]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [2, 7, 11, 16, 17, 18, 24, 25]). In addition, for an ideal I of A and a submodule F of E, $I\ltimes F$ is an ideal of $A\ltimes E$ if and only if $IE \subseteq F$.

Theorem 2.10. Let A be a ring and $R = A\ltimes A$, S a multiplicative subset of A. Let m,n be two positive integers such that $n \leq m-2$ and I be an S-AB-(m,n)-absorbing ideal of A. Let $S_\ltimes = S\ltimes 0$. Then, $I\ltimes I$ is an $S_\ltimes$-AB-$(m+n-1,2n)$-absorbing ideal of R.

Proof. Let $c_1 = (a_1,b_1),\ldots,c_{m+n-1} = (a_{m+n-1},b_{m+n-1}) \in R$ such that $c_1\cdots c_{m+n-1} \in I\ltimes I$. Then $a_1\cdots a_{m+n-1} \in I$ and $\Sigma_{i=1}^{m+n-1}\hat{a}_i b_i \in I$, where $\hat{a}_i = a_1\cdots a_{i-1}a_{i+1}\cdots a_{m+n-1}$ for $1\leq i\leq m+n-1$. Thus there are n of a_i 's whose product with some $s\in S$ is in I, say $sa_1\cdots a_n \in I$. Therefore $s\hat{a}_i \in I$ for $n+1 \leq i \leq m+n-1$ and hence

$$s(b_1a_2\cdots a_n+\cdots+a_1\cdots a_{j-1}b_ja_{j+1}\cdots a_n+\cdots+a_1\cdots a_{n-1}b_n)a_{n+1}\cdots a_{m+n-1} \in I$$

Since I is an S-AB-(m,n)-absorbing ideal of A, either the product of

$$s(b_1a_2\cdots a_n+\ldots+a_1\cdots a_{j-1}b_ja_{j+1}\cdots a_n+\ldots a_1\cdots a_{n-1}b_n)$$

with $n-1$ of the a_i 's $(n+1 \leq i \leq m+n-1)$ and s is in I or the product of n of the a_i 's $(n+1\leq i\leq m+n-1)$ with s is in I. In both cases, the product of $c_1\cdots c_n$ with n of the c_i 's $(n+1\leq i\leq m+n-1)$ and $(s^2,0)$ is in $I\ltimes I$. Hence $I\ltimes I$ is an $S_\ltimes$-AB-$(m+n-1,2n)$-absorbing ideal of R. □

Theorem 2.11. Let A be a ring, I be an ideal of A, S a multiplicative subset of A and E be an A-module. Then I is an S-(m,n)-absorbing ideal of A if and only if $I \ltimes E$ is an $(S \ltimes 0)$-AB-(m,n)-absorbing ideal of $A \ltimes E$.

Proof. Let $R = A \ltimes E$ and $(a_1,b_1),\cdots,(a_m,b_m) \in R$ such that $(a_1,b_1)\dots(a_m,b_m) \in I \ltimes E$. Thus there are n of the a_i 's whose product with some $s \in S$ is in I. Therefore there are n of the (a_i,b_i) 's whose product with $(s,0)$ is in $I \ltimes E$. Therefore, $I \ltimes E$ is an $(S \ltimes 0)$-AB-(m,n)-absorbing ideal of R. The converse is clear. □

Theorem 2.12. Let D be an integral domain, $R = D \ltimes D$, S a multiplicative subset of D. Let m,n be two positive integers such that $n \leq m-2$ and I be an S-AB-(m,n)-absorbing ideal of D that is not an S-AB-$(m,n-1)$-absorbing ideal of D. Then, $0 \ltimes I$ is an $S_\ltimes$-AB-$(m+1,n+1)$-absorbing ideal of R that is not an $S_\ltimes$-AB-$(m+1,n)$-absorbing ideal of R with $S_\ltimes = S \ltimes 0$.

Proof. Since I is an S-AB-(m,n)-absorbing ideal of D that is not an S-AB-$(m,n-1)$-absorbing ideal, there exist elements $a_1,\dots,a_m \in D$ such that $a_1\cdots a_m \in I$, but the product of any $n-1$ of the a_i's with any $s \in S$ does not belong to I. Define $b_1 = (a_1,0),\dots,b_m = (a_m,0)$, and $b_{m+1} = (0,1)$. Then

$$b_1\cdots b_{m+1} = (0,a_1\cdots a_m) \in 0 \ltimes I,$$

and it is evident that no product of n of the b_i's with any $t \in S_\ltimes$ belongs to $0 \ltimes I$. Hence, $0 \ltimes I$ is not an $S_\ltimes$-AB-$(m+1,n)$-absorbing ideal of R.

We now show that $0 \ltimes I$ is an $S\ltimes$-$(m+1,n+1)$-absorbing ideal of R. Let

$$c_1 = (a_1,e_1),\dots,c_{m+1} = (a_{m+1},e_{m+1}) \in R$$

such that $c_1\cdots c_{m+1} \in 0 \ltimes I$. Then $a_1\cdots a_{m+1} = 0$, which implies that $a_j = 0$ for some $1 \leq j \leq m+1$, without loss of generality, let $a_1 = 0$. Thus,

$$c_1\cdots c_{m+1} = (0,e_1a_2\cdots a_{m+1}) \in 0 \ltimes I,$$

then there is $s \in S$ such that either the product of e_1 with s and $n-1$ of the a_i's is in I, or the product of n of the a_i's with s is in I. Therefore, the product of $(s,0)$ and c_1 with either $n-1$ of the c_i's (where $i \neq 1$), or with n of the c_i's (where $i \neq 1$), lies in $0 \ltimes I$.

Thus, $0 \ltimes I$ is an $S_\ltimes$-AB-$(m+1,n+1)$-absorbing ideal of R. □

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a,f(a)+j) \in A \times B | a \in A, j \in J\}$$

is called the *amalgamation of A and B along J with respect to f*. This construction is a generalization of the amalgamated duplication of a ring along an ideal denoted $A \bowtie I$ (introduced and studied by D'Anna and Fontana in [15]). In [13, 14], D'Anna, Finocchiaro and Fontana introduced the more general context of amalgamations. They have studied these constructions in the frame of pullbacks which allowed them to establish numerous results on the transfer of various ideal and ring-theoretic properties from A and $f(A)+J$ to $A \bowtie^f J$. The concept of amalgamation is an important and an interesting concept that received a considerable attention by well-known established algebraists. The interest of amalgamations resides in their ability to cover basic constructions in commutative algebra, including classical pullbacks and trivial ring extensions. Moreover, other classical constructions (such as $A+XB[X]$, $A+XB[[X]]$ and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([13, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [12]) are strictly related to it ([13, Example 2.7 and Remark

2.8]). In [13], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation.

Let I be an ideal of A, S a multiplicative subset of A and K an ideal of $f(A)+J$. Throughout this section, we set

$$S^{\bowtie^f} := \{(s, f(s)) \mid s \in S\},$$
$$I \bowtie^f J := \{(i, f(i)+j) \mid i \in I, j \in J\},$$
$$\overline{K}^f := \{(a, f(a)+j) \mid a \in A, j \in J, f(a)+j \in K\}$$

and

$$\overline{I \times K}^f := \{(i, f(i)+j) \mid i \in I, j \in J, f(i)+j \in K\}.$$

Then clearly, $S^{\bowtie^f}$ is a multiplicative subset of $A \bowtie^f J$; and $I \bowtie^f J, \overline{K}^f$ and $\overline{I \times K}^f$ are ideals of $A \bowtie^f J$.

Theorem 2.13. Let A, B be rings, S a multiplicative subset of A and $f: A \to B$ a homomorphism. Let I be an ideal of A disjoint from S, J an ideal of B and K an ideal of $f(A)+J$ disjoint from $f(S)$. Then the following assertions hold.

(1) I is an S-AB-(m,n)-absorbing ideal if and only if $I \bowtie^f J$ is an $S^{\bowtie^f}$-AB-(m,n)-absorbing ideal of $A \bowtie^f J$.

(2) K is an $f(S)$-AB-(m,n)-absorbing ideal of $f(A)+J$ if and only if $\overline{K}^f$ is an $S^{\bowtie^f}$-AB-(m,n)-absorbing ideal of $A \bowtie^f J$.

(3) If I is an S-AB-(m,n)-absorbing ideal of A and K is an $f(S)$-AB-(m',n')-absorbing ideal of $f(A)+J$, then $\overline{I \times K}^f$ is an $S^{\bowtie^f}$-AB-$(m+m', n+n')$-absorbing ideal of $A \bowtie^f J$.

Proof. (1) We consider $\varphi: A \to (A \bowtie^f J)/(\{0\} \bowtie^f J)$ defined by $\varphi(a) = (a, f(a)) + (\{0\} \bowtie^f J)$. It is clear that $\varphi(S) = S^{\bowtie^f}/(\{0\} \bowtie^f J)$. Hence by Proposition 2.3(1), I is an S-AB-(m,n)-absorbing ideal of A if and only if $(I \bowtie^f J)/(\{0\} \bowtie^f J)$ is a $\varphi(S)$-AB-(m,n)-absorbing ideal of $(A \bowtie^f J)/(\{0\} \bowtie^f J)$. Also, by Corollary 2.5, $(I \bowtie^f J)/(\{0\} \bowtie^f J)$ is a $\varphi(S)$-AB-(m,n)-absorbing ideal of $(A \bowtie^f J)/(\{0\} \bowtie^f J)$ if and only if $I \bowtie^f J$ is an $S^{\bowtie^f}$-AB-(m,n)-absorbing ideal of $A \bowtie^f J$. So, the result holds.

(2) Let $\varphi: A \bowtie^f J \to f(A)+J$ such that $\varphi(a, f(a)+j) = f(a)+j$. Then φ is an epimorphism with $\ker(\varphi) = f^{-1}(J) \times \{0\}$, $\varphi(\overline{K}^f) = K$ and $\varphi(S^{\bowtie^f}) = f(S)$. Since $f^{-1}(J) \times \{0\} \subseteq \overline{K}^f$, we have that K is an $f(S)$-AB-(m,n)-absorbing ideal of $f(A)+J$ if and only if $\overline{K}^f$ is an $S^{\bowtie^f}$-AB-(m,n)-absorbing ideal of $A \bowtie^f J$ by Proposition 2.3(1).

(3) $I \times K$ is an $(S \times f(S))$-AB-$(m+n)$-absorbing ideal of $A \times (f(A)+J)$ by Corollary 2.9, Let $a_1, \ldots, a_{m+m'} \in A \times (f(A)+J)$ with $a_1 \cdots a_{m+m'} \in I \times K$. Then we may assume that there exist elements $s_1, s_2 \in S$ such that $(s_1, f(s_2))a_1 \cdots a_{n+n'} \in I \times K$. Hence $(s_1 s_2, f(s_1 s_2))a_1 \cdots a_{n+n'} \in I \times K$. Therefore $I \times K$ is an $S^{\bowtie^f}$-AB-$(m+m', n+n')$-absorbing ideal of $A \times (f(A)+J)$. As $A \times (f(A)+J)$ is a ring extension of $A \bowtie^f J$ and $\overline{I \times K}^f = (I \times K) \cap (A \bowtie^f J)$, $\overline{I \times K}^f$ is an $S^{\bowtie^f}$-AB-$(m+m', n+n')$-absorbing ideal of $A \bowtie^f J$ by Proposition 2.1(2). □

The following corollaries are immediate applications of Theorem 2.13,

Corollary 2.14. *Let A, B be rings such that $A \subseteq B$ and let S be a multiplicative subset of A. Then, I is an S-AB-(m,n)-absorbing ideal of A if and only if $I + XB[X]$ is an S-AB-(m,n)-absorbing ideal of $A + XB[X]$, if and only if $I + XB[[X]]$ is an S-AB-(m,n)-absorbing ideal of $A + XB[[X]]$.*

Proof. Set $J_1 = XB[X]$ and $J_2 = XB[[X]]$. Consider the natural embedding $\iota_1 : A \to B[X]$. Then it is easy to check that $A \bowtie^{\iota_1} J_1$ is isomorphic to $A + XB[X]$. Hence by Theorem 2.13(1), I is an S-AB-(m,n)-absorbing ideal of A if and only if $I + XB[X]$ is an S-AB-(m,n)-absorbing ideal of $A + XB[X]$.
Now, consider the natural embedding $\iota_2 : A \to B[[X]]$. Then it is easy to check that $A \bowtie^{\iota_2} J_2$ is isomorphic to $A + XB[[X]]$. Hence by Theorem 2.13(1), I is an S-AB-(m,n)-absorbing ideal of A if and only if $I + XB[[X]]$ is an S-AB-(m,n)-absorbing ideal of $A + XB[[X]]$. □

Corollary 2.15. *Let A, B be rings, S a multiplicative subset of A and $f : A \to B$ a homomorphism. Let I be an ideal of A disjoint from S, J an ideal of B and K an ideal of $f(A) + J$ disjoint from $f(S)$. Then the following assertions hold.*

(1) *Every $S^{\bowtie^f}$-AB-(m,n)-absorbing ideal of $A \bowtie^f J$ containing $\{0\} \times J$ is of the form $I \bowtie^f J$, where I is an S-AB-(m,n)-absorbing ideal of A.*

(2) *Every $S^{\bowtie^f}$-AB-(m,n)-absorbing ideal of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$ is of the form $\overline{K}^f$, where K is an $f(S)$-AB-(m,n)-absorbing ideal of $f(A) + J$.*

Proof. Let L be an ideal of $A \bowtie^f J$ disjoint from $S^{\bowtie^f}$. Define $\pi_1 : A \bowtie^f J \to A$ by $\pi_1(a, f(a)+j) = a$ and define $\pi_2 : A \bowtie^f J \to f(A) + J$ by $\pi_2(a, f(a)+j) = f(a) + j$. Then π_1 and π_2 are epimorphisms.

(1) By Theorem 2.13(1), $I \bowtie^f J$ is an $S^{\bowtie^f}$-(m,n)-absorbing ideal of $A \bowtie^f J$ for any S-AB-(m,n)-absorbing ideal I of A. Suppose that L is an $S^{\bowtie^f}$-AB-(m,n)-absorbing ideal of $A \bowtie^f J$ containing $\{0\} \times J$. It is easy to show that $L = \pi_1(L) \bowtie^f J$. Since $\ker(\pi_1) = \{0\} \times J$, we obtain that $\pi_1(L)$ is an S-AB-(m,n)-absorbing ideal of A by Proposition 2.3(1). Thus the result holds.

(2) By Theorem 2.13(2), $\overline{K}^f$ is an $S^{\bowtie^f}$-AB-(m,n)-absorbing ideal of $A \bowtie^f J$ for any $f(S)$-(m,n)-absorbing ideal K of $f(A)+J$. Suppose that L is an $S^{\bowtie^f}$-(m,n)-AB-absorbing ideal of $A \bowtie^f J$ containing $f^{-1}(J) \times \{0\}$. It is easy to show that $L = \overline{\pi_2(L)}^f$. Since $\ker(\pi_2) = f^{-1}(J) \times \{0\}$, we obtain that $\pi_2(L)$ is an $f(S)$-AB-(m,n)-absorbing ideal of $f(A) + J$ by Proposition 2.3(1). Thus the result holds. □

Corollary 2.16. *Let D and T be integral domains with $D \subseteq T$, let $\{M_\alpha \mid \alpha \in \Lambda\}$ a subset of* Max(T) *and $J = \bigcap_{\alpha \in \Lambda} M_\alpha$ with $J \cap D = (0)$,* Max(T) *is the set of maximal ideals of T. Let S be a multiplicative subset of D, I an ideal of D disjoint from S. Then I is an S-AB-(m,n)-absorbing ideal of D if and only if $I + J$ is an $S^{\bowtie^\iota}$-AB-(m,n)-absorbing ideal of $D + J$ with $\iota : D \to T$ is the natural embedding.*

Proof. It is easy to show that $D + J$ is canonically isomorphic to $D \bowtie^\iota J$. So, we obtain the result directly from Theorem 2.13(1). □

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