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The S-pseudo radical, the S-prime radical and new classes of ideals

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Abstract. This paper explores the S-pseudo radical (respectively, the S-prime radical) of an ideal of noncommutative rings (respectively, commutative rings), where S is an m-system (respectively, a multiplicatively closed subset) of R. We highlight the distinction between these two radical concepts in the commutative and noncommutative settings. Our study includes properties, examples, and connections between S-pseudo radicals (respectively, the S-prime radical), and related structures. We also introduce the notions of S-nilary and right S-primary (respectively, S_S-primary) ideals, study their properties, and characterize their behavior in various related ring constructions.

Key Words: S-prime ideal; S-primary ideal; primary ideal; prime radical; S-prime radical; commutative rings; noncommutative rings.

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1 Introduction

Consider a noncommutative ring R. The *pseudo radical* of an ideal I in R is symbolized as $\sqrt{I}$, and was defined in [10] as the sum of all ideals A in R such that $A^n \subseteq I$ for some positive integer n; i.e.,

$$\sqrt{I} = \sum \{A \lhd R : A^n \subseteq I \text{ for some } n \in \mathbb{N}\}.$$

This definition generalizes the concept of the radical of an ideal as known in the commutative setting. Specifically, in a commutative ring, the radical of an ideal I coincides with the set

$$\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\},$$

which is the intersection of all prime ideals of R that contain I. However, in noncommutative rings, the two concepts diverge: the pseudo radical is in general strictly contained within the prime radical. Nevertheless, it is noted in Lemma 1.2 of [10] that they coincide when $\sqrt{I}$ is finitely generated. Therefore throughout the paper, when we say "pseudo radical", we mean that the ring R is noncommutative. Referring to [10], many properties and corollaries of the pseudo radical have been shown. In particular see in Section 3 of [10] the results related to nilary and right primary ideals, which are generalizations of quasi-primary and primary ideals in commutative rings, respectively. It is well known that an ideal K of a commutative ring R is said to be primary (quasi-primary) if $ab \in K$ implies $a \in K$ ($a \in \sqrt{K}$) or $b \in \sqrt{K}$, for $a, b \in R$.

Let S be a multiplicatively closed set of a commutative ring R; i.e., $1 \in S$, $0 \notin S$, and $s_1 s_2 \in S$ for all $s_1, s_2 \in S$. The concept of S-Noetherian rings was first introduced in [8] in commutative rings, then, many papers have generalized this new concept into noncommutative rings, notably in [24], we see an S-analogue of Cohen's Theorem (where S is a multiplicatively closed set) as a generalization of the prior work in [23].

The concept of S-prime ideals was introduced in commutative rings by Hamed and Malek in [15]. An ideal $K \subseteq R$ that does not intersect S is called S-prime if there exists $s \in S$ such that for all $\alpha, \beta \in R$, $\alpha\beta \in K$ implies $s\alpha \in K$ or $s\beta \in K$. It was shown therein that R is S-Noetherian if and only if every S-prime ideal is S-finite. This framework has been explored in many directions—see, for example, [3, 4, 7, 11, 12, 18, 20, 21]—including studies of S-primary ideals. According to [18, 21], an ideal K disjoint from S is S-primary if there exists $s \in S$ such that for all $\alpha, \beta \in R$, $\alpha\beta \in K$ implies $s\alpha \in K$ or $s\beta \in \sqrt{K}$. With this, in refer to [22], the notion $\sqrt[S]{I}$ of the S-prime radical of an ideal I disjoint from S, was introduced as an S-analogue of the prime radical of I as

$$\sqrt[S]{I} = \{r \in R : r^n s \in I,\ \exists n \in \mathbb{N}, \exists s \in S\}.$$

The notion of S-prime radical of an ideal I was used in [14], to give a correspondence between S-idempotents of R and clopen sets of S-Zariski topology, and to demonstrate some relations between pure ideals and S-pure ideals. Recently, several studies in noncommutative rings generalized the concept of S-Noetherian rings such as [2, 9, 17]. In addition, the concept of S-prime ideals was also generalized to its noncommutative analogue; e.g. see in [1], where S was an m-system, with the motivation that the concept of m-system is closely tied to the concept of prime ideals in noncommutative rings. In addition, a definition of the S-pseudo radical, was given in [5] as the following. Let $S \subseteq R$ be an m-system of a ring R, and let I be in ideal of R disjoint from S. Then, the sum of ideals A with $A^n \subseteq (I : \langle s \rangle)$ for some $n \in \mathbb{N}$ and some $s \in S$ is called the S-pseudo radical $\sqrt[S]{I}$ of I; i.e.,

$$\sqrt[S]{I} = \sum \{A \lhd R : A^n \subseteq (I : \langle s \rangle) \text{ for some } n \in \mathbb{N} \text{ and } s \in S\}.$$

The objective. Main results of the paper are grouped in two main parts: Section 2 and Section 3 consist of results in noncommutative rings, while Section 4 and Section 5 are devoted to commutative rings. In the current article, we give a more general definition of S-pseudo radical, and we give several examples and properties, specifically, in case that S is contained in the center $C(R)$ of the ring R. In the third section, we introduce the concept of S-nilary (principally S-nilary) ideals and right S-primary ideals as generalizations of the nilary and right primary ideals, respectively, and we show how these ideals are closely tied to the S-pseudo radical. In particular, Theorem 3.19 states that if a ring R has ACC (the ascending chain condition) on the S-nilpotent ideals, where S is an m-system of R such that $S \subseteq C(R)$, and if K is right S-primary of R, then $\sqrt[S]{K}$ is prime. In the fourth section, we study the structure of the S-prime radical of an ideal in commutative rings. We give several examples, properties, and show that the S-prime radical of an ideal K is the intersection of all prime ideals that contain K and disjoint from S, which naturally implies $\sqrt{K} \subseteq \sqrt[S]{K}$. We investigate when $\sqrt[S]{K} = K$, referring to such ideals as S-radical, and prove that the S-radicalization operation is idempotent. Finally, Section 5 introduces the concept of S_S-primary ideals, and provides several characterizations and examples. One key result is that for an ideal K, if $K \cap S = \emptyset$, then $(K : s)$ is primary if and only if it is S_S-primary for some $s \in S$. In addition, we characterize the S_S-primary ideals of related rings to the ring R such as the S_S-primary ideal $K(+)M$ of the trivial ring extension $R(+)M$, amalgamated duplication $R \bowtie I$, and amalgamated structures $R \bowtie^{\varphi} J$ with respect to ring homomorphisms.

Why use an m-system? In commutative ring theory, given a prime ideal K, the complement $R \setminus K$ forms a multiplicatively closed set. This underpins the definitions of S-prime and S-primary ideals. However, in the noncommutative setting, $R \setminus K$ need not be multiplicatively closed, though it still forms an m-system. Thus, one cannot define S-primeness in the same way unless S is taken to be an m-system.

Recognizing this, the first author in [1] introduced the concept of right S-prime ideals for noncommutative rings. Here, an ideal K disjoint from an m-system S is called right S-prime if for all ideals

$\mathfrak{a}, \mathfrak{b} \lhd R$ with $\mathfrak{a}\mathfrak{b} \subseteq K$, it follows that $\mathfrak{a}\langle s\rangle \subseteq K$ or $\mathfrak{b}\langle s\rangle \subseteq K$ for some $s \in S$. Under this definition, any prime ideal K becomes $(R\backslash K)$-prime, aligning this noncommutative framework with its commutative counterpart when S is generalized to an m-system rather than a multiplicative set.

2 S-pseudo radical of an ideal in noncommutative rings

All the rings in this section are noncommutative and with identity. S is assumed to be an m-system of a ring R, even if not explicitly mentioned.

Definition 2.1. Let I be an ideal of R disjoint from S. Then, we call the ideal

$$\sqrt[S]{I} = \sum \{A_i \lhd R : A_i^{n_i} \subseteq (I : \langle s_i\rangle), \exists n_i \in \mathbb{N}, \exists s_i \in S, i \in \mathbb{A} \text{ for some index set } \mathbb{A}\}$$

the S-pseudo radical of I. Note that if $a \in \sqrt[S]{I}$, then $a = a_{i_1} + \cdots + a_{i_k}$ for some $a_{i_l} \in A_{i_l}$, $1 \leq l \leq k < \infty$.

Remark 2.2. The following are some quick investigations.
• If $S \subseteq U(R)$, then the concepts of S-pseudo radical and pseudo radical of I coincide.
• Let J be an ideal of R such that $J^n \subseteq (I : \langle s\rangle)$ for some $s \in S$ and $n \in \mathbb{N}$. Then, $J \subseteq \sqrt[S]{I}$. In particular, we have $I \subseteq \sqrt[S]{I}$, since $I \subseteq (I : \langle s\rangle)$.
• Note that if $I = R$, then $\sqrt[S]{I} = R$. However, it is worthy to emphasize that if $I \cap S = \phi$, then $\sqrt[S]{I} \neq R$. Because, if $\sqrt[S]{I} = R$, then $R\langle s\rangle \subseteq I$, which implies $s \in I$, which is a contradiction. Also, it is clear that $\sqrt[S]{I} \cap S = \phi$, because if $s_1 \in \sqrt[S]{I} \cap S$, then, $s_1 \in W$ and $W^m \subseteq (I : \langle s\rangle)$ for some ideal W of R and some $m \in \mathbb{N}$, hence, $\langle s_1\rangle^m \langle s\rangle \subseteq W^m \langle s\rangle \subseteq I$. By the m-system property, there exists $s' \in S$ such that $s' \in \langle s_1\rangle^m \langle s\rangle \subseteq I$, that contradicts with $I \cap S = \phi$.

Remark 2.3. In this remark, we give a simpler equivalent definition of the S-pseudo radical of I in R. Let J_1, J_2 be any ideals with $J_i^{n_i} \subseteq (I : \langle s_i\rangle)$ for some $n_i \in \mathbb{N}$, and $s_i \in S$ where $i \in \{1, 2\}$. Because S is an m-system, there exists $s' \in S$ such that $s' = s_1 r s_2$ for some $r \in R$. Now $J_1^{n_1}\langle s_1\rangle \subseteq I$, and $J_2^{n_2}\langle s_2\rangle \subseteq I$ implying that $J_1^{n_1}\langle s'\rangle \subseteq J_1^{n_1} R s_1 R^3 \subseteq I$ and $J_2^{n_2}\langle s'\rangle \subseteq J_2^{n_1} R^3 s_2 R \subseteq I$, respectively. Thus, $J_i^{n_i} \subseteq (I : \langle s'\rangle)$ for $i = 1, 2$. Hence, the Definition 2.1 can be set as the following (Definition 2.28 of [5]):

$$\sqrt[S]{I} = \sum \{A_i \lhd R : A_i^{n_i} \subseteq (I : \langle s\rangle), \exists n_i \in \mathbb{N}, \exists s \in S, i \in \mathbb{A} \text{ for some index set } \mathbb{A}\},$$

and we call s the maximal element with respect to $\sqrt[S]{I}$. The next proposition is a consequence of the renewed definition.

Proposition 2.4. *Let I be an ideal of a ring R such that $I \cap S = \phi$. Then*
(*i*) $I \subseteq \sqrt{I} \subseteq \sqrt[S]{I}$.
(*ii*) $\sqrt[S]{I} = \{r \in R : \exists n \in \mathbb{N}, \exists s \in S, \exists W_r \lhd R$ *such that* $r \in W_r$ *and* $W_r^n \subseteq (I : \langle s\rangle)\}$.

Proof. (*i*) The inclusion $I \subseteq \sqrt{I}$ is by definition. Now assume that $r \in \sqrt{I}$, where $r = r_{i_1} + \cdots + r_{i_k} \in A_{i_1} + \cdots + A_{i_k} = W$ for some A_{i_j}, such that $A_{i_j}^{m_{i_j}} \subseteq I \subseteq (I : \langle s\rangle)$ for some $m_{i_j} \in \mathbb{N}$ and $s \in S$. Put $m = \sum m_{i_j}$. It is straightforward to see that $W^m \subseteq (I : \langle s\rangle)$. Thus, $r \in W \subseteq \sqrt[S]{I}$.
(*ii*) Let r be in the set right side of the equality. Then assuming $W = W_r$, similar arguments in the first part of the proof yield that $r \in W_r \subseteq \sqrt[S]{I}$ with desired conditions. Conversely, let $r \in W_r$ with $W_r^m \subseteq (I : \langle s\rangle)$, $\exists m \in \mathbb{N}$, for some ideal W_r of R. Then, $r \in W_r \subseteq \sqrt[S]{I}$. □

Lemma 2.5. *Let A, B, I be ideals of R disjoint from S. Then the following hold.*

(1) *If* $A \subseteq B$, *then* $\sqrt[S]{A} \subseteq \sqrt[S]{B}$.
(2) *If* $A \subseteq \sqrt[S]{I}$ *and A is finitely generated, then* $A^m \subseteq (I : \langle s\rangle)$, $\exists m \in \mathbb{N}$, $\exists s \in S$.

(3) *If* $\sqrt[S]{I}$ *is finitely generated, then* $(\sqrt[S]{I})^m \subseteq (I:\langle s\rangle)$, $\exists m\in\mathbb{N}$, $\exists s\in S$. *Hence,* $\exists m\in\mathbb{N}$ *such that* $A^m\subseteq (I:\langle s\rangle)$ *for any ideal* $A\subseteq \sqrt[S]{I}$.
(4) $\sqrt[S]{A\cap B}=\sqrt[S]{A}\cap\sqrt[S]{B}$.
(5) $\sqrt[S]{(I:\langle s\rangle)}=\sqrt[S]{I}$ *for some* $s\in S$.

Proof. (1) Let $x\in\sqrt[S]{A}$. Then, there exists an ideal $W\triangleleft R$ such that $x\in W$ and $W^n\subseteq (A:\langle s\rangle)$ for some $m\in\mathbb{N}$ and $s\in S$. Thus, $W^n\subseteq (B:\langle s\rangle)$ and $x\in\sqrt[S]{B}$.

(2) Suppose $A\subseteq\sqrt[S]{I}$. Because A is finitely generated, $A\subseteq J_1+\cdots+J_k$ for some ideals $J_1,\ldots,J_k$, where $J_i^{m_i}\subseteq (I:\langle s\rangle)$, $\exists m_i\in\mathbb{N}$, $\exists s\in S$. Put $m=m_1+\cdots+m_k$. Then, $A^m\subseteq (J_1+J_2+\cdots+J_k)^m\subseteq (I:\langle s\rangle)$.

(3) Clear by (2).

(4) By (1), $\sqrt[S]{A\cap B}\subseteq\sqrt[S]{A}\cap\sqrt[S]{B}$. Let $x\in\sqrt[S]{A}\cap\sqrt[S]{B}$. Then, there exist ideals $U,W\triangleleft R$ such that $x\in U\cap W$ and $U^n\subseteq (A:\langle s_1\rangle)$, $W^m\subseteq (B:\langle s_2\rangle)$, for some $n,m\in\mathbb{N}$ and $s_1,s_2\in S$. Let $s=s_1rs_2\in S$ for some $r\in R$. Then,

$$(U\cap W)^{n+m}\subseteq U^n\cap W^m\subseteq (A:\langle s_1\rangle)\cap(B:\langle s_2\rangle)\subseteq (A:\langle s\rangle)\cap(B:\langle s\rangle).$$

Thus, $(U\cap W)^{n+m}\subseteq (A\cap B:\langle s\rangle)$, and hence, $x\in\sqrt[S]{A\cap B}$.

(5) Since $I\subseteq (I:\langle s\rangle)$, then by (1) we have that $\sqrt[S]{I}\subseteq\sqrt[S]{(I:\langle s\rangle)}$. Conversely, let $x\in\sqrt[S]{(I:\langle s\rangle)}$. Then, $\exists W\triangleleft R$ such that $x\in W$ and $W^n\subseteq ((I:\langle s\rangle):\langle s\rangle)$ for some $n\in\mathbb{N}$ and $s\in S$. Thus, $W^n\subseteq (I:\langle s_1\rangle)$ for some $s_1\in S$, and hence, $x\in W\subseteq\sqrt[S]{I}$. □

Lemma 2.6. *Let* A,B,I *be ideals of* R *disjoint from* S. *If* $S\subseteq C(R)$, *then the following statements hold.*
(1) $\sqrt[S]{AB}=\sqrt[S]{A\cap B}=\sqrt[S]{A}\cap\sqrt[S]{B}$.
(2) *If the ideal* $\sqrt[S]{I}$ *is finitely generated, then* $\sqrt[S]{\sqrt[S]{I}}=\sqrt[S]{I}$.
(3) $\sqrt[S]{I^n}=\sqrt[S]{I}$ *for all* $n\geq 1$.
(4) *If* $A^n\subseteq (I:\langle s\rangle)\subseteq A$, $\exists m\in\mathbb{N}$, $\exists s\in S$, *then* $\sqrt[S]{A}=\sqrt[S]{I}$.

Proof. (1) Since $AB\subseteq A\cap B$, then by (1) of Lemma 2.5, $\sqrt[S]{AB}\subseteq\sqrt[S]{A\cap B}$. Let $x\in\sqrt[S]{A\cap B}$. Then, there exists an ideal W such that $x\in W$ and

$$W^n\subseteq (A\cap B:\langle s\rangle)=(A:\langle s\rangle)\cap(B:\langle s\rangle).$$

Hence, $W^{2n}\subseteq (A:\langle s\rangle)(B:\langle s\rangle)$ for some $n\in\mathbb{N}$ and some $s\in S$. Now for each $y\in (A:\langle s\rangle)(B:\langle s\rangle)$, there exist $a\in (A:\langle s\rangle)$ and $b\in (B:\langle s\rangle)$ such that $y=ab$, $a\langle s\rangle\subseteq A$ and $b\langle s\rangle\subseteq B$. Consequently, $a\langle s\rangle b\langle s\rangle\subseteq AB$. Since $S\subseteq C(R)$, then, $ab\langle s\rangle\langle s\rangle\subseteq AB$. Hence, $ab\langle s_1\rangle\subseteq ab\langle s\rangle\langle s\rangle\subseteq AB$ for $s_1=srs$ and $r\in R$. Thus, $y=ab\in (AB:\langle s_1\rangle)$, and hence, $W^{2n}\subseteq (AB:\langle s_1\rangle)$. Thus, $x\in W\subseteq\sqrt[S]{AB}$, which implies that $\sqrt[S]{A\cap B}\subseteq\sqrt[S]{AB}$. The second equality follows from Lemma 2.5.

(2) Clearly $\sqrt[S]{I}\subseteq\sqrt[S]{\sqrt[S]{I}}$. Let $x\in\sqrt[S]{\sqrt[S]{I}}$. Then, there exists an ideal W such that $x\in W$ and $W^n\subseteq (\sqrt[S]{I}:\langle s\rangle)$ for some $n\in\mathbb{N}$, which yields that $W^n\langle s\rangle\subseteq\sqrt[S]{I}$. Since $\sqrt[S]{I}$ is finitely generated, then by (3) of Lemma 2.5, there exists $m\in\mathbb{Z}^+$ such that $(\sqrt[S]{I})^m\subseteq (I:\langle s\rangle)$. Consequently, $(W^n\langle s\rangle)^m\subseteq (I:\langle s\rangle)$. Because $S\subseteq C(R)$, we get $W^{nm}\langle s\rangle^m\subseteq (I:\langle s\rangle)$. Hence, there exists $s'\in S$ with $W^{nm}\subseteq (I:\langle s'\rangle)$. Thus, $x\in W\subseteq\sqrt[S]{I}$, and hence $\sqrt[S]{\sqrt[S]{I}}\subseteq\sqrt[S]{I}$.

(3) Clear by (1).

(4) Suppose that $A^n\subseteq (I:\langle s\rangle)\subseteq A$. Then, we have by (1) of Lemma 2.5 that $\sqrt[S]{A^n}\subseteq\sqrt[S]{(I:\langle s\rangle)}\subseteq\sqrt[S]{A}$. Hence, by (5) of Lemma 2.5, and by (3) of current lemma, we obtain $\sqrt[S]{A}\subseteq\sqrt[S]{I}\subseteq\sqrt[S]{A}$. □

We put next result that is Lemma 2.1 of [10], and its proof follows from Lemma 2.5 and Lemma 2.6, by taking $S=\{1\}$.

Lemma 2.7. *[10] Let A,B,I be ideals of R. Then following statements hold.*

(1) *If $A \subseteq B$, then $\sqrt{A} \subseteq \sqrt{B}$.*
(2) *If $A \subseteq \sqrt{I}$ and A is finitely generated, then $A^n \subseteq I$.*
(3) *$\sqrt{AB} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.*
(4) *If $\sqrt{I}$ is finitely generated, then $\sqrt{\sqrt{I}} = \sqrt{I}$.*

Proper generalization. In the following example we show that the definition of S-pseudo radical is a proper generalization of the pseudo radical of an ideal I.

Example 2.8. Let $R = M_2(\mathbb{Z})$, and $S = \{s^{2^k} : k \geq 0\} = \{s, s^2, s^4, s^8, \ldots\}$, where $s = 9(e_{11} + e_{22})$. Then, S is an m-system of R. Consider the ideal $I = M_2(\langle 12 \rangle)$. Since $\mathbb{Z}$ is Noetherian, then $\sqrt{I} = M_2(J)$ is finitely generated, for some ideal J of $\mathbb{Z}$. Hence, by Lemma 1.2 of [10], $\sqrt{I} = \mathrm{rad}(I) = M_2(\langle 6 \rangle)$, and $I \cap S = \varnothing$. In addition, $[M_2(\langle 2 \rangle)]^4 \langle s \rangle \subseteq I$, and hence, $M_2(\langle 2 \rangle) \subseteq \sqrt[S]{I} \neq R$. But $M_2(\langle 2 \rangle)$ is a maximal ideal which implies that $M_2(\langle 2 \rangle) = \sqrt[S]{I}$.

Definition 2.9. An ideal I of a ring R disjoint from S is called S-radical if $\sqrt[S]{I} = I$.

In the next remark, we state some investigations.

Remark 2.10. (i) Every maximal (prime) ideal disjoint from S is an S-radical ideal. If P is a prime ideal and disjoint from S, then, for all $x \in \sqrt[S]{P}$, there exists an ideal W_x such that $x \in W_x$ and $W_x^n \langle s \rangle \subseteq P$. Because $\langle s \rangle \nsubseteq P$, we have $W_x \subseteq P$. Hence, $\sqrt[S]{P} = P$.
(ii) If I is S-radical, then by Remark 2.2, I is radical, and the converse is not true in general. In the following example we show an ideal which is radical, but not S-radical.

Example 2.11. Let R and S be the same as in Example 2.8. Assume that $I = M_2(\langle 6 \rangle)$. As in Example 2.8, one may show that $\sqrt{I} = I$. However, since $I \cap S = \varnothing$ and $M_2(\langle 2 \rangle) \subseteq (I : \langle s \rangle)$, then $M_2(\langle 2 \rangle) = \sqrt[S]{I}$.

Example 2.12. Let $R = M_2(\mathbb{Z}_6)$ and $I = 0$. Clearly, $\sqrt{I} = I$. Assume that $S = \{s^{2^k} : k \geq 0\} = \{s, s^2, s^4, s^8, \ldots\}$, where $s = 3(e_{11} + e_{22})$. Then $I \cap S = \varnothing$ and $\sqrt[S]{I} = M_2(\langle 2 \rangle)$. Since $\sqrt[S]{I}$ is finitely generated and $S \subseteq C(M_2(\mathbb{Z}_6))$, then by item (2) of Lemma 2.6, we obtain $\sqrt[S]{\sqrt[S]{I}} = \sqrt[S]{I}$. Hence $\sqrt[S]{M_2(\langle 2 \rangle)} = M_2(\langle 2 \rangle)$.

We close this section by the theorem below.

Theorem 2.13. Let $\varphi : R_1 \to R_2$ be a ring epimorphism. Consider an m-system $S \subseteq R_1$, and an ideal K of R_1 disjoint from S which contains $\ker(\varphi)$. Then

$$\varphi(\sqrt[S]{K}) = \sqrt[\varphi(S)]{\varphi(K)}.$$

Proof. Notice that if $\varphi(S) \cap \varphi(K) \neq \phi$, then given $a \in \varphi(S) \cap \varphi(K)$, there exist $s_1 \in S$ and $b \in K$ such that $a = \varphi(s_1) = \varphi(b)$. Hence, $s_1 - b \in \ker(\varphi) \subseteq K$, and thus $s_1 \in K$, contradiction. That is, $\varphi(S) \cap \varphi(K) = \phi$.

Now, given $\beta \in \varphi(\sqrt[S]{K})$, there exists $\alpha \in \sqrt[S]{K}$ such that $\beta = \varphi(\alpha)$, and hence, there exists an ideal W of R_1 such that $\alpha \in W$ and $W^n \langle s \rangle \subseteq K$ for some $n \in \mathbb{N}$ and $s \in S$. Hence, $\beta = \varphi(\alpha) \in \varphi(W)$ and $\varphi(W)^n \langle \varphi(s) \rangle \subseteq \varphi(K)$. Consequently, $\beta \in \sqrt[\varphi(S)]{\varphi(K)}$, and thus, $\varphi(\sqrt[S]{K}) \subseteq \sqrt[\varphi(S)]{\varphi(K)}$.

Conversely, for $\beta \in \sqrt[\varphi(S)]{\varphi(K)}$, there exists an ideal U of R_2 such that $\beta \in U$ and $U^n \langle \varphi(s) \rangle \subseteq \varphi(K)$ for some $n \in \mathbb{N}$ and $\varphi(s) \in \varphi(S)$. Notice that $\ker(\varphi) \subseteq \varphi^{-1}(U) = W$ for some ideal W of R_1. Hence, $\varphi(W) = U$ because φ is an epimorphism. Then, we have that $U^n = [\varphi(W)]^n = \varphi(W^n) \subseteq \varphi(K)$. Thus,

$$W^n \subseteq \varphi^{-1}(\varphi(W^n)) \subseteq \varphi^{-1}(\varphi(K)) = K \subseteq (K : \langle s \rangle).$$

Hence, $\beta \in \varphi(W) \subseteq \varphi(\sqrt[S]{K})$, and consequently, $\sqrt[\varphi(S)]{\varphi(K)} \subseteq \varphi(\sqrt[S]{K})$. □

3 Right S-primary and S-nilary ideals in noncommutative rings

In this section all the rings are noncommutative and with identity, unless explicitly mentioned otherwise. S is assumed to be an m-system of a ring R, even if not explicitly mentioned.

Remark 3.1. The concept of nilary (right primary) ideals were introduced in [13]. In this context, an ideal P of a ring R is called nilary (right primary), if whenever A and B are ideals of R with $AB \subseteq P$, then either $A^m \subseteq P$ ($A \subseteq P$) or $B^n \subseteq P$ for some positive integers n, m. While the concept of principally right primary ideals was introduced in [16], by considering the ideals A and B in the previous definition of right primary ideals, as principal ideals. Later the concept of principally nilary ideals was introduced in [10], by considering the ideals A and B, in the previous definition of nilary ideals, as principal ideals.

In the following, we present the concepts of S-nilary (S-principally-nilary) and right S-primary (right S-principally-primary).

Definition 3.2. Let K be an ideal of a ring R such that $K \cap S = \emptyset$.

(i) The ideal K is said to be *right S-primary* (or *right S-p-primary*) if for any pair of (principal) ideals $\mathfrak{a}, \mathfrak{b}$ of R with $\mathfrak{a}\mathfrak{b} \subseteq K$, it follows that either $\mathfrak{a}\langle s\rangle \subseteq K$ or $\mathfrak{b}^n\langle s\rangle \subseteq K$ for some $n \in \mathbb{N}$ and an (fixed) $s \in S$.

(ii) The ideal K is called *S-nilary* (or *S-p-nilary*) if for any pair of (principal) ideals $\mathfrak{a}, \mathfrak{b}$ of R with $\mathfrak{a}\mathfrak{b} \subseteq K$, we have either $\mathfrak{a}^m\langle s\rangle \subseteq K$ or $\mathfrak{b}^n\langle s\rangle \subseteq K$ for some $m, n \in \mathbb{N}$ and an (fixed) $s \in S$.

Remark 3.3. (1) Every right S-prime (right S-primary) ideal is an S-nilary ideal.
(2) Every right primary ideal disjoint from S, is a right S-primary ideal.
(3) Every nilary ideal that is disjoint from S, is an S-nilary ideal. However, the converse does not always hold, as illustrated in the following example.

Example 3.4. Let $R = M_2(\mathbb{Z}[x])$, and $K = M_2(\langle 9x\rangle)$. It is shown in Example 2.12 of [1], that K is a right S-prime ideal which is not prime, where

$$S = \{s^{2^k} : k \geq 0\} = \{s, s^2, s^4, s^8, \ldots\},$$

and $s = 3(e_{11} + e_{22})$. Hence, K is S-nilary. In addition, assuming that $I = M_2(\langle 9\rangle)$ and $J = M_2(\langle x\rangle)$, we obtain $IJ = M_2(\langle 9\rangle)M_2(\langle x\rangle) \subseteq K$, and $I^n, J^n \not\subseteq K$ for all $n \in \mathbb{N}$. Hence K is not nilary, i.e., K is not right primary.

Proposition 3.5. *Let K be an ideal of a ring R such that $K \cap S = \emptyset$. Then the following conditions are equivalent:*

(1) *K is an S-p-nilary ideal.*
(2) *For any ideals $\mathfrak{a}, \mathfrak{b}$ of R satisfying $\mathfrak{a}\mathfrak{b} \subseteq K$, one of the inclusions $\mathfrak{a} \subseteq \sqrt[S]{K}$ or $\mathfrak{b} \subseteq \sqrt[S]{K}$ holds.*
(3) *If $\mathfrak{a}$ and $\mathfrak{b}$ are finitely generated ideals with $\mathfrak{a}\mathfrak{b} \subseteq K$, then for some $s \in S$, either $\mathfrak{a}^m \subseteq (K : \langle s\rangle)$ or $\mathfrak{b}^n \subseteq (K : \langle s\rangle)$, for some positive integers m, n.*

Proof. (1) $\Rightarrow$ (2) Assume that K is an S-p-nilary ideal, and let $\mathfrak{a}\mathfrak{b} \subseteq K$ for ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$. If $\mathfrak{a} \not\subseteq \sqrt[S]{K}$, then take $\alpha \in \mathfrak{a} \setminus \sqrt[S]{K}$, so $\langle\alpha\rangle \not\subseteq \sqrt[S]{K}$. For any $\beta \in \mathfrak{b}$, we have $\langle\alpha\rangle\langle\beta\rangle \subseteq K$. By the definition of S-p-nilary, either $\langle\alpha\rangle^m\langle s\rangle \subseteq K$ or $\langle\beta\rangle^n\langle s\rangle \subseteq K$ for some $m, n \in \mathbb{Z}^+$, $s \in S$. The first case contradicts our assumption. Hence, each $\langle\beta\rangle \subseteq \sqrt[S]{K}$, and so $\mathfrak{b} \subseteq \sqrt[S]{K}$.

(2) $\Rightarrow$ (3) Assume $\mathfrak{a}, \mathfrak{b}$ are finitely generated ideals such that $\mathfrak{a}\mathfrak{b} \subseteq K$. Then by (2), either $\mathfrak{a} \subseteq \sqrt[S]{K}$ or $\mathfrak{b} \subseteq \sqrt[S]{K}$. By Lemma 2.5 part (2), we obtain either $\mathfrak{a}^m \subseteq (K : \langle s\rangle)$ or $\mathfrak{b}^n \subseteq (K : \langle s\rangle)$ for some $s \in S$, $m, n \in \mathbb{Z}^+$.

(3) $\Rightarrow$ (1) This implication directly follows from the definition of S-p-nilary ideals. □

One may set a proof for the following proposition similar to Proposition 3.5.

Proposition 3.6. *Let K be an ideal of R disjoint from S. Then the following statements are equivalent.*

(1) *K is an S-p-primary ideal.*
(2) *If $\mathfrak{a},\mathfrak{b}$ are ideals of R with $\mathfrak{a}\mathfrak{b} \subseteq K$, then either $\mathfrak{a}\langle s\rangle \subseteq K$ or $\mathfrak{b} \subseteq \sqrt[S]{K}$.*
(3) *If $\mathfrak{a},\mathfrak{b}$ are finitely generated ideals of R with $\mathfrak{a}\mathfrak{b} \subseteq K$, then either $\mathfrak{a} \subseteq (K : \langle s\rangle)$ or $\mathfrak{b}^m \subseteq (K : \langle s\rangle)$ for some positive integers n, m and some $s \in S$.*

Remark 3.7. In referring to [11], an ideal K of a *commutative* ring R disjoint from a multiplicatively closed subset $S \subseteq R$ is called a quasi-S-primary ideal, if there exists an element $s \in S$ such that for all $a, b \in R$ if $ab \in K$, then $sa \in \sqrt{K}$ or $sb \in \sqrt{K}$. We give the next result in accordance with this definition. Recall that every multiplicatively closed subset is an m-system.

Corollary 3.8. *Let S be a multiplicatively closed subset of a commutative ring R, and let k be an ideal of R such that $k \cap S = \emptyset$. Then the following statements are equivalent:*
(1) *k is an S-p-nilary ideal if and only if it is a quasi-S-primary ideal of R.*
(2) *k is an S-p-primary ideal if and only if it is an S-primary ideal of R.*

Proof. (1) Suppose k is an S-p-nilary ideal, and let $a, b \in R$ such that $ab \in k$. Then $\langle a\rangle\langle b\rangle \subseteq k$, so by the definition, either $\langle a\rangle^m\langle s\rangle \subseteq k$ or $\langle b\rangle^n\langle s\rangle \subseteq k$ for some $m, n \in \mathbb{Z}^+$, $s \in S$. This implies either $a^m s \in P$ or $b^n s \in k$, i.e., $(as)^m \in k$ or $(bs)^n \in k$, so $as \in \sqrt{k}$ or $bs \in \sqrt{k}$. Thus, k is quasi-S-primary.

Conversely, assume K is a quasi-S-primary ideal, and suppose $\langle a\rangle\langle b\rangle \subseteq K$ for some $a, b \in R$. Then $ab \in K$, and since K is quasi-S-primary, we have either $as \in \sqrt{K}$ or $bs \in \sqrt{K}$ for some $s \in S$. Hence, there exist integers $m, n \geq 1$ such that $(as)^m \in K$ or $(bs)^n \in K$, i.e., $a^m s^m \in K$ or $b^n s^n \in K$. Let $s'' = s^m s^n \in S$, then either $\langle a\rangle^m\langle s''\rangle \subseteq K$ or $\langle b\rangle^n\langle s''\rangle \subseteq K$, and hence K is S-p-nilary.

(2) The argument follows the same pattern as in (1). □

Theorem 3.9. Let S be an m-system of a ring R such that $S \subseteq C(R)$, and let $P_1, \ldots, P_n$ be S-nilary ideals of R such that

$$(P_j^m : \langle s\rangle) \subseteq \left(\bigcap_{i=1}^{n} P_i : \langle s\rangle\right),$$

for a fixed $j \in \{1, \ldots, n\}$, $\exists m \in \mathbb{N}$, and $\forall s \in S$. Then, $\prod_{i=1}^n P_i = P_1 \cdots P_n$ is an S-nilary ideal of R.

Proof. Suppose $AB \subseteq P_1 \cdots P_n$ for some ideals A, B of R. Then $AB \subseteq P_j$, and hence, either $A^{m_1}\langle s\rangle \subseteq P_j$ or $B^{m_2}\langle s\rangle \subseteq P_j$ for some positive integers m_1, m_2 and some $s \in S$. If $A^{m_1}\langle s\rangle \subseteq P_j$, then $A^{m_1} \subseteq (P_j : \langle s\rangle)$, and hence, $A^{m_1 m} \subseteq (P_j : \langle s\rangle)^m$. Since $S \subseteq C(R)$, it follows, by an argument similar to that in part (1) of Lemma 2.6, that the following holds:

$$(P_j : \langle s\rangle)^m \subseteq (P_j^m : \langle s'\rangle),$$

where $s' = sr_1sr_2 \cdots r_{m-1}s \in S$ for some $r_1, \ldots r_{m-1} \in R$. Hence, by the assumption,

$$A^{m_1 m} \subseteq (P_j : \langle s\rangle)^m \subseteq \left(\bigcap_{i=1}^{n} P_i : \langle s'\rangle\right) = (P_1 : \langle s'\rangle) \cap \cdots \cap (P_n : \langle s'\rangle),$$

and hence,

$$(A^{m_1 m})^n \subseteq (P_1 : \langle s'\rangle) \cdots (P_n : \langle s'\rangle).$$

Again, because $S \subseteq C(R)$, we obtain:

$$A^{m_1 mn} \subseteq (P_1 \cdots P_n : \langle s''\rangle),$$

for some $s'' = s't_1s't_2 \cdots t_{n-1}s' \in S$, and some $t_1, \ldots t_{n-1} \in R$. If $B^{m_2}\langle s\rangle \subseteq P_j$, then, similarly one may see that $B^{m_2 mn} \subseteq (P_1 \cdots P_n : \langle s''\rangle)$ for some $s'' \in S$. Thus, $P_1 \cdots P_n$ is an S-nilary ideal of R. □

The proof of the following theorem is similar to the proof of Theorem 3.9.

Theorem 3.10. Suppose that $S \subseteq C(R)$, and let $K_1, K_2, \ldots, K_n$ be S-p-nilary ideals of R such that: $(K_j^m : \langle s\rangle) \subseteq (\cap_{i=1}^n K_i : \langle s\rangle)$, for a fixed $1 \leq j \leq n$, $\exists m \in \mathbb{N}$, $\forall s \in S$. Then, $K_1 \cdots K_n$ is an S-p-nilary ideal of R.

The proof of the next corollary follows from Theorem 3.9 (Theorem 3.10).

Corollary 3.11. *Let $S \subseteq C(R)$. If K is S-nilary (S-p-nilary) ideal of R, then so is K^n for all $n \geq 1$.*

Theorem 3.12. Let $S \subseteq C(R)$, and let K be an ideal of R such that $K \cap S = \emptyset$. Then, K is an S-nilary ideal if and only if $(K : \langle s\rangle)$ is a nilary ideal of R for some $s \in S$.

Proof. Assume that $(K : \langle s\rangle)$ is a nilary ideal of R. Let $\mathfrak{a}\mathfrak{b} \subseteq K$ for some ideals $\mathfrak{a}, \mathfrak{b}$ of R. Then $\mathfrak{a}\mathfrak{b} \subseteq (K : \langle s\rangle)$. Therefore, K is an S-nilary ideal of R.

Conversely, suppose that K is an S-nilary ideal of R, and let $\mathfrak{a}\mathfrak{b} \subseteq (K : \langle s\rangle)$ for some ideals $\mathfrak{a}, \mathfrak{b}$ of R. Then $\mathfrak{a}\mathfrak{b}\langle s\rangle \subseteq K$. By assumption, either $\mathfrak{a}^n\langle s\rangle \subseteq K$ or $(\mathfrak{b}\langle s\rangle)^m\langle s\rangle \subseteq K$ for some $m, n \in \mathbb{N}$. Assume $\mathfrak{a}^n \not\subseteq (K : \langle s\rangle)$. Since $S \subseteq C(R)$, it follows that $\mathfrak{b}^m\langle s\rangle^{m+1} \subseteq K$. Hence, either $\mathfrak{b}^{mk}\langle s\rangle \subseteq K$ or $\langle s\rangle^l \subseteq K$ for some $k, l \in \mathbb{N}$. The latter contradicts $K \cap S = \emptyset$, so we must have $\mathfrak{b}^{mk}\langle s\rangle \subseteq K$, and therefore $\mathfrak{b}^{mk} \subseteq (K : \langle s\rangle)$. □

One may prove the following theorem similarly.

Theorem 3.13. Let $S \subseteq C(R)$, and K be an ideal of R disjoint from S. Then K is a right S-primary ideal if and only if $(K : \langle s\rangle)$ is a right primary ideal of R for some $s \in S$.

Proposition 3.14. *Let I, J, K be ideals of R disjoint from S. Then the following hold.*

(1) *If $\sqrt[S]{K}$ is prime, then K is an S-p-nilary ideal of R.*
(2) *If $\sqrt[S]{I}$ and $\sqrt[S]{J}$ are prime, then $\sqrt[S]{I \cap J}$ is prime if and only if $\sqrt[S]{I} \subseteq \sqrt[S]{J}$ or $\sqrt[S]{J} \subseteq \sqrt[S]{I}$.*
(3) *If $\sqrt[S]{I}$ and $\sqrt[S]{J}$ are prime, and $S \subseteq C(R)$, then $\sqrt[S]{IJ}$ is prime if and only if $\sqrt[S]{I} \subseteq \sqrt[S]{J}$ or $\sqrt[S]{J} \subseteq \sqrt[S]{I}$.*

Proof. (1) Let $\langle a\rangle\langle b\rangle \subseteq K$ for some $a, b \in R$. Then, $\langle a\rangle\langle b\rangle \subseteq \sqrt[S]{K}$. Hence, either $\langle a\rangle \subseteq \sqrt[S]{K}$ or $\langle b\rangle \subseteq \sqrt[S]{K}$. Thus, by (2) of Lemma 2.5, either $\langle a\rangle^n \subseteq (K : \langle s\rangle)$ or $\langle b\rangle^m \subseteq (K : \langle s\rangle)$ for some $n, m \in \mathbb{N}$.

(2) Suppose $\sqrt[S]{I}$ and $\sqrt[S]{J}$ are prime ideals. If $\sqrt[S]{I \cap J}$ is prime, then

$$\sqrt[S]{I}\,\sqrt[S]{J} \subseteq \sqrt[S]{I} \cap \sqrt[S]{J} = \sqrt[S]{I \cap J},$$

by (4) of Lemma 2.5. Hence, either $\sqrt[S]{I} \subseteq \sqrt[S]{I \cap J} \subseteq \sqrt[S]{J}$ or $\sqrt[S]{J} \subseteq \sqrt[S]{I \cap J} \subseteq \sqrt[S]{I}$. Conversely, without loss of generality, suppose $\sqrt[S]{I} \subseteq \sqrt[S]{J}$. Then,

$$\sqrt[S]{I \cap J} = \sqrt[S]{I} \cap \sqrt[S]{J} = \sqrt[S]{I}.$$

(3) The proof follows from (2) and from the item (1) of Lemma 2.6. □

Corollary 3.15. *Let Q, P be ideals of R disjoint from S. Then the following hold.*

(1) *If Q is right S-prime, $S \subseteq C(R)$, and $Q^n \subseteq P \subseteq Q$, for some $n \in \mathbb{N}$, then P is S-nilary.*
(2) *If Q is prime and $Q^n \subseteq (P : \langle s\rangle) \subseteq Q$, for some $n \in \mathbb{N}$ and $s \in S$, then P is S-nilary.*

Proof. (1) Let $\mathfrak{a}\mathfrak{b} \subseteq P$ for some ideals $\mathfrak{a}, \mathfrak{b}$ of R. Then, $\mathfrak{a}\mathfrak{b} \subseteq Q$. Hence, either $\mathfrak{a}\langle s\rangle \subseteq Q$ or $\mathfrak{b}\langle s\rangle \subseteq Q$ for some $s \in S$. Therefore, either $(\mathfrak{a}\langle s\rangle)^n \subseteq Q^n \subseteq P$ or $(\mathfrak{b}\langle s\rangle)^n \subseteq Q^n \subseteq P$. Since $S \subseteq C(R)$, then either $\mathfrak{a}^n\langle s_1\rangle \subseteq P$ or $\mathfrak{b}^n\langle s_1\rangle \subseteq P$. Thus, P is S-nilary.

(2) Let $\mathfrak{a}\mathfrak{b} \subseteq P$ for some ideals $\mathfrak{a}, \mathfrak{b}$ of R. Then, $\mathfrak{a}\mathfrak{b} \subseteq (P : \langle s\rangle) \subseteq Q$. Hence, either $\mathfrak{a} \subseteq Q$ or $\mathfrak{b} \subseteq Q$. Consequently, either $\mathfrak{a}^n \subseteq Q^n \subseteq (P : \langle s\rangle)$ or $\mathfrak{b}^n \subseteq Q^n \subseteq (P : \langle s\rangle)$. Thus, P is S-nilary. □

Proposition 3.16. *Let K be an ideal of R disjoint from S, and $\sqrt[S]{K}$ be finitely generated. Then the following statements hold.*

(1) *If $\sqrt[S]{K}$ is S-prime, and $S \subseteq C(R)$, then K is S-nilary.*
(2) *If $\sqrt[S]{K}$ is prime, then K is S-nilary.*
(3) *If $\sqrt[S]{K}$ is S-nilary (S-p-nilary) and $S \subseteq C(R)$, then K is S-nilary.*

Proof. (1) Suppose $\sqrt[S]{K}$ is S-prime, and $S \subseteq C(R)$. Since $\sqrt[S]{K}$ is finitely generated, then, by (3) of Lemma 2.5, we get $(\sqrt[S]{K})^m \subseteq (K : \langle s\rangle)$ for some $m \in \mathbb{N}$ and $s \in S$. Let $\mathfrak{a}\mathfrak{b} \subseteq K \subseteq \sqrt[S]{K}$ for some ideals $\mathfrak{a}, \mathfrak{b}$ of R. Hence, either $\mathfrak{a}\langle s\rangle \subseteq \sqrt[S]{K}$ or $\mathfrak{b}\langle s\rangle \subseteq \sqrt[S]{K}$ for some $s \in S$. Because $S \subseteq C(R)$, we obtain, either $\mathfrak{a}^m\langle s_1\rangle \subseteq (K : \langle s\rangle)$ or $\mathfrak{b}^m\langle s_1\rangle \subseteq (K : \langle s\rangle)$ for some $s_1 \in S$. Thus, either $\mathfrak{a}^m\langle s_2\rangle \subseteq K$ or $\mathfrak{b}^m\langle s_2\rangle \subseteq K$ for some $s_2 \in S$.

(2) Since $\sqrt[S]{K}$ is finitely generated, then, by (3) of Lemma 2.5, we get $(\sqrt[S]{K})^m \subseteq (K : \langle s\rangle) \subseteq \sqrt[S]{K}$ for some $m \in \mathbb{N}$ and $s \in S$. By (2) of Corollary 3.15, we get that K is S-nilary.

(3) The proof can be obtained similarly to (1). □

Corollary 3.17. *Let R be a ring in which for any two ideals A, B, and any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $A^m B^m \subseteq (AB)^n$. Asuume that $S \subseteq C(R)$, and let K be an ideal of R disjoint from S. If $\sqrt[S]{K}$ is finitely generated, then $\sqrt[S]{K}$ is S-prime if and only if K is S-nilary.*

Proof. Suppose K is S-nilary. Since $\sqrt[S]{K}$ is finitely generated, then by (3) of Lemma 2.5, we get $(\sqrt[S]{K})^n \subseteq (K : \langle s\rangle)$ for some $n \in \mathbb{N}$ and $s \in S$. Let $AB \subseteq \sqrt[S]{K}$ for some ideals A, B of R. Then, there exists $m \in \mathbb{N}$ such that

$$A^m B^m \subseteq (AB)^n \subseteq (\sqrt[S]{K})^n \subseteq (K : \langle s\rangle).$$

But, by Theorem 3.12, $(K : \langle s\rangle)$ is a nilary ideal. Hence, either $A^m\langle s\rangle \subseteq K \subseteq \sqrt[S]{K}$ or $B^m\langle s\rangle \subseteq K \subseteq \sqrt[S]{K}$ for some $s \in S$. The converse follows from (1) of Proposition 3.16. □

Definition 3.18. An ideal K of a ring R, is called S-nilpotent, if $K^n\langle s\rangle = 0$ for some $n \in \mathbb{N}$ and some $s \in S$.

The next theorem examines the case when the R has the ascending chain condition (ACC) on the S-nilpotent ideals, where S is an m-system of R.

Theorem 3.19. Let R be a ring having ACC on the S-nilpotent ideals, and $S \subseteq C(R)$. If K is a right S-primary ideal, then $\sqrt[S]{K}$ is a prime ideal.

Proof. Let A_j be one of the decomposed ideals of $\sqrt[S]{K}$. Then, $A_j^n\langle s\rangle \subseteq K$ for some $n \in \mathbb{N}$ and $s \in S$. Hence,

$$\left[\frac{A_j^n + K}{K}\right]\left[\frac{\langle s + K\rangle + K}{K}\right] \subseteq \frac{A_j^n\langle s\rangle + K}{K} \subseteq \frac{K}{K}.$$

Thus, A_j is $\bar{S}$-nilpotent in R/K. The assumption of the theorem implies that R/K has ACC on the $\bar{S}$-nilpotent ideals. Hence, the ascending sequence

$$A_{j_1} \subseteq A_{j_1} + A_{j_2} \subseteq \cdots$$

has a largest element, after which the chain remains constant, for each decomposed ideal A_{j_i} of $\sqrt[S]{K}$. Hence, $\sqrt[S]{K}$ is a finite sum of ideals, which implies by Remark 2.2, that $\sqrt[S]{K} = W$, where $W^n \subseteq (K : \langle s\rangle)$ for some $n \in \mathbb{N}$. Thus, $(\sqrt[S]{K})^n \subseteq (K : \langle s\rangle)$.

Now let $\mathfrak{a}\mathfrak{b} \subseteq \sqrt[S]{K}$ for some ideals $\mathfrak{a}, \mathfrak{b}$ of R. Then, $(\mathfrak{a}\mathfrak{b})^n \subseteq (\sqrt[S]{K})^n \subseteq (K : \langle s\rangle)$. Assume that n is the minimal positive integer that satisfies the inclusion. Assume $\mathfrak{a} \not\subseteq \sqrt[S]{K}$. Then, $\mathfrak{a}^m\langle s\rangle \not\subseteq K$ for all $m \in \mathbb{N}$ and $s \in S$. If $n = 1$, then $\mathfrak{a}\mathfrak{b} \subseteq (K : \langle s\rangle)$. Hence, by Theorem 3.13, $(K : \langle s\rangle)$ is a right primary ideal, and thus, $\mathfrak{b}^k\langle s\rangle \subseteq K$ for some $k \in \mathbb{N}$. If $n \geq 2$, then

$$\left[(\mathfrak{a}\mathfrak{b})^{n-1}\mathfrak{a}\right]\mathfrak{b} \subseteq (K : \langle s\rangle).$$

Consequently, either $[(\mathfrak{a}\mathfrak{b})^{n-1}\mathfrak{a}] \subseteq (K : \langle s\rangle)$ or $\mathfrak{b}^{n_2}\langle s\rangle \subseteq K$ for some $n_1, n_2 \in \mathbb{N}$. If $(\mathfrak{a}\mathfrak{b})^{n-1}\mathfrak{a} \subseteq (K : \langle s\rangle)$, then either $(\mathfrak{a}\mathfrak{b})^{n-1} \subseteq (K : \langle s\rangle)$, which contradicts the minimality of n, or $\mathfrak{a}^l\langle s\rangle \subseteq K$ for some $l \in \mathbb{N}$, which is a contradiction. Thus, $\mathfrak{b} \subseteq \sqrt[S]{K}$. □

Theorem 3.20. Let R be a ring has ACC on the S-nilpotent ideals. If $K_1, \ldots, K_n$ are S-nilary ideals of R with the same S-radical, then $K = \bigcap_{i=1}^{n} K_i$ is an S-nilary ideal of R.

Proof. As in the proof of Theorem 3.19, one may see that $(\sqrt[S]{K_i})^{n_i} \subseteq (K_i : \langle s_i\rangle)$ for some $n_i \in \mathbb{N}$ and some $s_i \in S$ depending on K_i. For $n = 1$ the proof is trivial. Let $n = 2$, and $AB \subseteq K = K_1 \cap K_2$ for some ideals A, B of R. Assume that $A^m\langle s\rangle \not\subseteq K$ for all $m \in \mathbb{N}$ and $s \in S$. Without loss of generality, we may assume that $A^m\langle s\rangle \not\subseteq K_1$. Since $AB \subseteq K \subseteq K_1$, then $B^{k_1}\langle s\rangle \subseteq K_1$, and hence, $B \subseteq \sqrt[S]{K_1} = \sqrt[S]{K_2}$. Thus by (3) of Lemma 2.5, $B^{k_2}\langle s\rangle \subseteq K_2$. Hence, $B^{max\{k_1,k_2\}}\langle s\rangle \subseteq K$. Consequently, K is an S-nilary ideal of R. By induction one can complete the proof easily. □

In what follows, we consider a ring epimorphism of noncommutative rings and its action on corresponding ideals.

Theorem 3.21. Let $\varphi : R_1 \to R_2$ be a ring epimorphism, $S \subseteq R_1$ be an m-system, and K be an ideal of R_1 such that $\ker(\varphi) \subseteq K$. Then, K is an S-nilary ideal of R_1 associated with $s \in S$ if and only if $\varphi(K)$ is an $\varphi(S)$-nilary ideal of R_2 associated with $\varphi(s)$.

Proof. Initially, one may easily check that $K \cap S = \varnothing$ if and only if $\varphi(K) \cap \varphi(S) = \varnothing$. Suppose that K is an S-nilary ideal, and that $B_1B_2 \subseteq \varphi(K)$ for some ideals B_1, B_2 of R_2. Then, $\exists A_1, A_2 \lhd R_1$ such that $A_1 = \varphi^{-1}(B_1)$ and $A_2 = \varphi^{-1}(B_2)$. Since φ is an epimorphism, then $\varphi(A_1) = B_1$ and $\varphi(A_2) = B_2$, which implies that $\varphi(A_1A_2) = B_1B_2 \subseteq \varphi(K)$. Thus, $A_1A_2 \subseteq \varphi^{-1}(\varphi(A_1A_2)) \subseteq \varphi^{-1}(\varphi(K)) = K$. Hence, either $A_1^n\langle s\rangle \subseteq K$ or $A_2^m\langle s\rangle \subseteq K$, for some $n, m \in \mathbb{N}$, and hence, either $B_1^n\langle\varphi(s)\rangle \subseteq K$ or $B_2^m\langle\varphi(s)\rangle \subseteq K$. Finally, it is easy to check that $\varphi(S)$ is an m-system of R_2, and $\varphi(S) \cap \varphi(K) = \varnothing$. Thus, $\varphi(S)$ is disjoint from $\varphi(K)$, and hence $\varphi(K)$ is an $\varphi(S)$-nilary ideal of R_2.

Conversely, assume that $\varphi(K)$ is an $\varphi(S)$-nilary ideal of R_2, and that A_1, A_2 are ideals of R_1 with $A_1A_2 \subseteq K$. Then $\varphi(A_1)\varphi(A_2) \subseteq \varphi(K)$. Thus, for some $\varphi(s) \in \varphi(S)$ and $n, m \in \mathbb{N}$, either $\varphi(A_1)^n\langle\varphi(s)\rangle \subseteq \varphi(K)$ or $\varphi(A_2)^m\langle\varphi(s)\rangle \subseteq \varphi(K)$. Hence, either

$$A_1^n\langle s\rangle \subseteq \varphi^{-1}(\varphi(A_1^n\langle s\rangle)) \subseteq \varphi^{-1}(\varphi(K)) = K$$

or $A_2^m\langle s\rangle \subseteq \varphi^{-1}(\varphi(A_2^m\langle s\rangle)) \subseteq \varphi^{-1}(\varphi(K)) = K$. Consequently, K is an S-nilary ideal of R_1. □

Corollary 3.22. *Let $\varphi : R_1 \to R_2$ be a ring epimorphism, $S \subseteq R_1$ be an m-system, and B be an ideal of R_2 such that $\varphi^{-1}(B)$ is an S-nilary ideal of R_1. Then, B is an $\varphi(S)$-nilary ideal of R_2.*

Proof. Since the inverse image of any ideal of R_2 is an ideal of R_1 that contains $\ker(\varphi)$, the proof follows from Theorem 3.21. □

Theorem 3.23. Let I, K be ideals of R such that $I \subseteq K$ and $K \cap S = \varnothing$. Then the following hold.

(1) If K is an S-nilary ideal of R, then $\frac{K}{I}$ is an $\bar{S}$-nilary ideal of $\frac{R}{I}$.
(2) If $\frac{K}{I}$ is an $\bar{S}$-nilary ideal of $\frac{R}{I}$, and I is an S-nilary ideal of R, then K is an S-nilary ideal of R.

Proof. Assume that $\varphi\colon R \to \frac{R}{I}$ is the natural epimorphism.
(1) We have $\ker\varphi = I \subseteq K$. Since K is an S-nilary ideal of R, then by Theorem 3.21, $\varphi(K) = \frac{K}{I}$ is an $\varphi(S)$-nilary ideal of $\frac{R}{I}$.

(2) Suppose that $AB \subseteq K$ for some ideals A, B of R. If $AB \subseteq I$, then $A^n\langle s\rangle \subseteq I \subseteq K$ or $B^m\langle s\rangle \subseteq I \subseteq K$, for some $n, m \in \mathbb{N}$ and $s \in S$. If $AB \nsubseteq I$, then $\frac{A+I}{I} \cdot \frac{B+I}{I} \subseteq \frac{K}{I}$. Thus, either $\left(\frac{A+I}{I}\right)^n \langle \bar{s}\rangle \subseteq \frac{K}{I}$ or $\left(\frac{B+I}{I}\right)^m \langle \bar{s}\rangle \subseteq \frac{K}{I}$, for some $\bar{s} = s + I \in \bar{S}$ and $n, m \in \mathbb{N}$. Hence, $A^n\langle s\rangle \subseteq K$ or $B^m\langle s\rangle \subseteq K$. □

4 S-prime radical of an ideal in commutative rings

All rings considered in this section are assumed to be commutative with identity. Moreover, S denotes a multiplicatively closed subset, even if not stated explicitly.

4.1 The S-prime radical of an ideal

The following definition is Definition 1 of [22].

Definition 4.1. Let I be an ideal of a ring R such that $I \cap S = \phi$. Then,

$$\sqrt[S]{I} = \{r \in R : r^n s \in I \text{ for some } n \in \mathbb{N} \text{ and some } s \in S\}.$$

Remark 4.2. In what follows, we give some investigations.

• It is easy to check that our definition of the S-pseudo radical (Definition 2.1) of an ideal in non-commutative rings is equivalent to the definition of the S-prime radical of an ideal (Definition 1 of [22]) in the case where the ring R is commutative, when considering S as a multiplicatively closed subset of R. Notice that every multiplicatively closed set is an m-system, and the converse is not true even in the commutative case.

• One may easily prove that $\sqrt[S]{I}$ is an ideal of R containing I, and that

$$\sqrt[S]{I} = \{r \in R : \exists J \lhd R \text{ such that } r \in J \text{ and } J^n s \subseteq I \text{ for some } n \in \mathbb{N} \text{ and } s \in S\}.$$

• For each $x \in \sqrt{I}$, there exists $n_x \in \mathbb{N}$, such that $x^{n_x} \in I$, and hence, $x^{n_x} s \in Is \subseteq I$ for all $s \in S$. Therefore, $x \in \sqrt[S]{I}$ that yields $I \subseteq \sqrt{I} \subseteq \sqrt[S]{I}$.

• If $S \subseteq U(R)$, then the concepts of the radical and the S-radical of an ideal coincide.

• If $I = R$, then $\sqrt[S]{I} = R$. However, it is worthy to emphasize that if $I \cap S = \phi$, then $\sqrt[S]{I} \neq R$. Because, if $\sqrt[S]{I} = R$, then $1 \in \sqrt[S]{I}$, which implies $s \in I$ for some $s \in S$, that is a contradiction. Also, we assume that $\sqrt[S]{I} \cap S = \phi$, because if $s_1 \in \sqrt[S]{I} \cap S$, then, $s_1^n s \in I \cap S$, which contradicts with $I \cap S = \phi$.

Proper generalization. In the following example we show that the definition of S-radical is a proper generalization of the prime radical of an ideal I.

Example 4.3. Initially, it is known that if $(n) = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}\mathbb{Z}$ for the primes p_i and positive integers n_i, $1 \le i \le k$, then $\sqrt{(n)} = p_1 p_2 \cdots p_k\mathbb{Z}$. Now let $R = \mathbb{Z}$, and $I = (12)$. Hence, $\sqrt{I} = \sqrt{(12)} = (6)$.

• Let $S_1 = \{9^k : k \geq 0\} = \{1, 9, 81, \ldots\}$. Clearly S_1 is multiplicatively closed. Then, $(12) \cap S_1 = \emptyset$, and $12^2 = 2^4.9 \in (12)$. Hence, $2 \in \sqrt[S_1]{(12)}$. Thus, $(2) \subseteq \sqrt[S_1]{(12)} \neq R$, but since the ideal (2) is a maximal ideal, then $\sqrt[S_1]{(12)} = (2)$.

• Let $S_2 = \{4^k : k \geq 0\} = \{1, 4, 16, \ldots\}$. Then $(12) \cap S_2 = \emptyset$ and $12 = 3^1.4 \in (12)$, and hence $3 \in \sqrt[S_2]{(12)}$. Thus, $\sqrt[S_2]{(12)} = (3)$.

Theorem 4.4. For an ideal I of R with $I \cap S = \emptyset$:

$$\sqrt[S]{I} = \bigcap_{I \subseteq P_i} P_i = \bigcap_{I \subseteq (Q_i : s_i)} (Q_i : s_i),$$

where, for each i, P_i is a prime ideal containing I and disjoint from S, and Q_i is an S-prime ideal with $I \subseteq (Q_i : s_i)$, $s_i \in S$.

Proof. Let $x \in \sqrt[S]{I}$, then there exist $n \in \mathbb{N}$ and $s \in S$ such that $x^n s \in I$. Let P be a prime ideal containing I and disjoint from S. Then the inclusion implies $x^n s \in P$, and hence $x \in P$ due to the fact that $s \notin P$.

Assuming that P_i, $i \in \mathbb{A}$, are all prime ideals containing I and disjoint from S for some index set $\mathbb{A}$, we conclude that $\sqrt[S]{I} \subseteq \bigcap_{i \in \mathbb{A}} P_i$.

Conversely, let $x \notin \sqrt[S]{I}$, with $x^n s \notin I$ for all $s \in S$ and all $n \in \mathbb{N}$. Let us define

$$\mathcal{A} = \{J : J \text{ is an ideal of } R,\ I \subseteq J, \text{ and } x^n s \notin J\}.$$

We have that $I \in \mathcal{A}$ by definition. Now Zorn's lemma yields that $\mathcal{A}$ has a maximal element, denoted by P. Clearly, $P + (a)$ and $P + (b)$ are not in $\mathcal{A}$ for $a, b \notin P$. Hence, there exist $n_1, n_2 \in \mathbb{N}$ and $s_1, s_2 \in S$ such that $x^{n_1} s_1 \in P + (a)$ and $x^{n_2} s_2 \in P + (b)$. Consequently,

$$x^{n_1+n_2} s_1 s_2 \in (P + (a))(P + (b)) \subseteq P + (ab),$$

since $s_1 s_2 \in S$. Then $P + (ab) \notin \mathcal{A}$, and hence $ab \notin P$. Therefore, P is prime.

Clearly, $S \cap P = \emptyset$ and $x \notin P$, which yields that $\bigcap_{i \in \mathbb{A}} P_i \subseteq \sqrt[S]{I}$. Thus, $\bigcap_{i \in \mathbb{A}} P_i = \sqrt[S]{I}$.

Finally, we have

$$\sqrt[S]{I} = \bigcap_{I \subseteq (Q_i : s_i)} (Q_i : s_i)$$

by Proposition 5 of [22], which completes the proof. □

Referring to [19], Definition 2.1. states that an ideal P of a commutative ring R, disjoint from S, is called an S-semiprime ideal of R, if there exists an $s \in S$ such that for all $a \in R$ with $a^2 \in P$, we have $sa \in P$.

Corollary 4.5. *Let I be an ideal of R such that $I \cap S = \phi$. Then, $\sqrt[S]{I}$ is an S-semiprime ideal which contains I.*

Proof. Let $a^2 \in \sqrt[S]{I} = \bigcap_{i \in \mathbb{A}} P_i$, where P_i is a prime ideal which contains I and is disjoint from S. Hence, $a \in \sqrt[S]{I}$, and therefore, $\sqrt[S]{I}$ is a semiprime ideal disjoint form S by Remark 4.2. Consequently, $\sqrt[S]{I}$ is an S-semiprime ideal. □

Lemma 4.6. *Let A, B, and I be ideals of R disjoint from S. Then the following hold.*

(i) If $A \subseteq B$ then $\sqrt[S]{A} \subseteq \sqrt[S]{B}$.
(ii) $\sqrt[S]{\sqrt[S]{I}} = \sqrt[S]{I}$; i.e., the S-radicalization is idempotent.
(iii) $\sqrt[S]{AB} = \sqrt[S]{A \cap B} = \sqrt[S]{A} \cap \sqrt[S]{B}$.
(iv) $\sqrt[S]{I^n} = \sqrt[S]{I}$. In particular, if I is prime, then, $\sqrt[S]{I^n} = I$.

Proof. (i) The proof is routine.

(ii) Clearly, $\sqrt[S]{I} \subseteq \sqrt[S]{\sqrt[S]{I}}$, since $I \subseteq \sqrt[S]{I}$, by (i). Conversely, let $x \in \sqrt[S]{\sqrt[S]{I}}$. Then, $x^n s \in \sqrt[S]{I}$ for some $n \in \mathbb{N}$ and $s \in S$. This yields that $(x^n s)^m s_1 \in I$ for some $m \in \mathbb{N}$ and $s_1 \in S$. Thus, $x^{nm} s^m s_1 \in I$. Since $s^m s_1 \in S$, $x \in \sqrt[S]{I}$, and hence, $\sqrt[S]{\sqrt[S]{I}} = \sqrt[S]{I}$.

(iii) By (i), $\sqrt[S]{AB} \subseteq \sqrt[S]{A \cap B}$, since $AB \subset A \cap B$ for ideals A and B. Conversely, let $x \in \sqrt[S]{A \cap B}$. Then, $x^n s \in A \cap B$ for some $n \in \mathbb{N}$ and $s \in S$, and hence, $x^{2n} s^2 \in AB$. This implies that $x \in \sqrt[S]{AB}$, since $s^2 \in S$. The second equality can be proved similarly.

(iv) Straightforward by (iii). □

Corollary 4.7. *Let I be an ideal of R such that $I \cap S = \phi$. Then the following hold.*

(1) *If I is an S-prime (S-primary) ideal, then $\sqrt[S]{I}$ is S-prime.*
(2) $\sqrt[S]{I} = \sqrt[S]{(I : s)}$ *for some $s \in S$.*

Proof. (1) We set the proof for the S-primary case, and the proof of S-prime case is similar. Assume that I is an S-primary ideal, and that $ab \in \sqrt[S]{I}$ for some $a, b \in R$. Then, $(ab)^n s \in I$, for some $n \in \mathbb{N}$ and $s \in S$. Hence, $a^n (b^n s) \in I$, and thus, either $a^n s \in I$ which implies $a \in \sqrt[S]{I}$, or $b^n s^2 \in \sqrt{I} \subseteq \sqrt[S]{I}$ which implies $b \in \sqrt[S]{I}$. As a consequence, $\sqrt[S]{I}$ is prime and hence an S-prime ideal.

(2) It is easy to check that $\sqrt[S]{(I : s)} \subseteq \sqrt[S]{I}$. Now since $I \subseteq (I : s)$, then we have $\sqrt[S]{I} \subseteq \sqrt[S]{(I : s)}$ by (i) of Lemma 4.6. Thus $\sqrt[S]{I} = \sqrt[S]{(I : s)}$. □

Remark 4.8. For some ring epimorphism $f\colon R \to f(R)$, $f(S)$ is a multiplicatively closed subset of $f(R)$. In additon, the set

$$\bar{S} = \{\bar{s} = s + I : s \in S\}$$

is a multiplicatively closed subset of R/I for any ideal I of R. Besides, if A is a commutative ring with identity, and S_1 is a multiplicatively closed subset of A, then, it is easy to check that $S \times S_1$ is a multiplicatively closed subset of the decomposable ring $R \times A$. Moreover, for a unitary R-module M, the idealization of M in R is the commutative ring $R(+)M = \{(r, m) : r \in R, m \in M\}$, with the usual addition and the multiplication defined as

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$$

for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. Therefore, if S is a multiplicatively closed subset of R, then $S_M = S(+)M$ is a multiplicatively closed subset of $R(+)M$.

Theorem 4.9. Let $\varphi : R_1 \to R_2$ be a ring epimorphism. Let $S \subseteq R_1$ be a multiplicatively closed set, and let K be an ideal of R_1 that is disjoint from S and contains $\ker(\varphi)$. Then,

$$\varphi(\sqrt[S]{K}) = \sqrt[\varphi(S)]{\varphi(K)}.$$

Proof. We first verify that $\varphi(S) \cap \varphi(K) = \varnothing$. Suppose there exists $a \in \varphi(S) \cap \varphi(K)$, then $a = \varphi(s_1) = \varphi(k_1)$ for some $s_1 \in S$, $k_1 \in K$. This implies $s_1 - k_1 \in \ker(\varphi) \subseteq K$, hence $s_1 \in K$, contradicting $S \cap K = \varnothing$. Thus, $\varphi(S) \cap \varphi(K) = \varnothing$.

Let $y \in \varphi(\sqrt[S]{K})$. Then there exists $x \in \sqrt[S]{K}$ such that $y = \varphi(x)$, and by definition, $x^n s \in K$ for some $n \in \mathbb{N}$ and $s \in S$. Applying φ, we obtain $y^n \varphi(s) = \varphi(x^n s) \in \varphi(K)$, which implies $y \in \sqrt[\varphi(S)]{\varphi(K)}$. Therefore, $\varphi(\sqrt[S]{K}) \subseteq \sqrt[\varphi(S)]{\varphi(K)}$.

Conversely, let $y \in \sqrt[\varphi(S)]{\varphi(K)}$. Then $y^n \varphi(s) \in \varphi(K)$ for some $s \in S$ and $n \in \mathbb{N}$. Let $x \in R_1$ satisfy $\varphi(x) = y$. Then

$$\varphi(x^n s) = y^n \varphi(s) \in \varphi(K),$$

which implies $x^n s \in \varphi^{-1}(\varphi(K)) = K$, since $\ker(\varphi) \subseteq K$. Therefore, $x \in \sqrt[S]{K}$, so $y = \varphi(x) \in \varphi(\sqrt[S]{K})$. Hence, equality holds. □

Theorem 4.10. Let $\varphi : R_1 \to R_2$ be a ring epimorphism, $S \subseteq R_1$ a multiplicatively closed subset, and $K \subseteq R_2$ an ideal with $K \cap \varphi(S) = \emptyset$. Then,

$$\varphi^{-1}(\sqrt[\varphi(S)]{K}) = \sqrt[S]{\varphi^{-1}(K)}.$$

Proof. First, we check that $\varphi^{-1}(K) \cap S = \emptyset$. If there exists $s \in \varphi^{-1}(K) \cap S$, then $\varphi(s) \in K \cap \varphi(S)$, contradicting the assumption that $K \cap \varphi(S) = \emptyset$. Therefore, $\varphi^{-1}(K) \cap S = \emptyset$.

Let $x \in \varphi^{-1}(\sqrt[\varphi(S)]{K})$. Then $\varphi(x) \in \sqrt[\varphi(S)]{K}$, so there exist $n \in \mathbb{N}$ and $s \in S$ such that

$$\varphi(x)^n \varphi(s) = \varphi(x^n s) \in K,$$

which implies $x^n s \in \varphi^{-1}(K)$, so $x \in \sqrt[S]{\varphi^{-1}(K)}$.

Conversely, suppose $x \in \sqrt[S]{\varphi^{-1}(K)}$. Then $x^n s \in \varphi^{-1}(K)$ for some $s \in S$, $n \in \mathbb{N}$, which implies

$$\varphi(x)^n \varphi(s) = \varphi(x^n s) \in \varphi(\varphi^{-1}(K)) \subseteq K.$$

Thus, $\varphi(x) \in \sqrt[\varphi(S)]{K}$, and so $x \in \varphi^{-1}(\sqrt[\varphi(S)]{K})$. This completes the proof. □

The following is Definition 2.8 of [14].

Definition 4.11. In a ring R:
(1) An element r of R is called S-nilpotent if $r^n s = 0$, for some $n \in \mathbb{N}$ and $s \in S$.
(2) The set of all S-nilpotent elements of R forms an ideal. It is called the S-Nilradical of R, and denoted by $N_S(R)$.

It is clear that the Nilradical $N(R)$ of R is contained in the S-Nilradical $N_S(R)$ of R.

Corollary 4.12. *Let I be an ideal of R disjoint from S. Then the following hold.*
(1) $N(R/\sqrt[S]{I}) = N_{\bar{S}}(R/\sqrt[S]{I}) = \bar{0}$.
(2) $\sqrt[S]{I}/I = N_S(R/I)$.

Proof. (1) Let $x + \sqrt[S]{I} \in R/\sqrt[S]{I}$ be an $\bar{S}$-nilpotent element. Then, there exist $n \in \mathbb{N}$ and $\bar{s} \in \bar{S}$ such that $x^n s + \sqrt[S]{I} = (x + \sqrt[S]{I})^n (s + \sqrt[S]{I}) \in \sqrt[S]{I}$. Thus, $x \in \sqrt[S]{\sqrt[S]{I}} = \sqrt[S]{I}$ and that $N_{\bar{S}}(R/\sqrt[S]{I}) = \bar{0}$. Similarly we can show $N(R/\sqrt[S]{I}) = \bar{0}$.

(2) The proof is routine. □

Corollary 4.13. *Let K be an ideal of R disjoint from S_1. Then the following hold.*

(1) *If I is an ideal of R such that $I \subseteq K$, then, $\sqrt[S_1]{K}/I = \sqrt[\bar{S}_1]{K/I}$.*
(2) *If A is a commutative ring with identity, and S_2 is a multiplicatively closed subset of A, then, $\sqrt[S]{K \times A} = \sqrt[S_1]{K} \times A$, where $S = S_1 \times S_2$.*
(3) *If M is a unitary R-module, then $\sqrt[S_M]{K(+)M} = \sqrt[S_1]{K}(+)M$, $S_M = S_1(+)M$.*

Proof. We set the proof of (iii) only, since the proof of other options are straightforward and similar. An element $(a, m) \in \sqrt[S_M]{K(+)M}$, if and only if,

$$(a, m)^n (s, m_1) \in K(+)M$$

for some $n \in \mathbb{N}$ and $(s, m_1) \in S_M$, if and only if, $a^n s \in K$, if and only if, $a \in \sqrt[S]{K}$, if and only if, $(a, m) \in \sqrt[S]{K}(+)M$. □

4.2 S-radical ideals

Definition 4.14. An ideal K of a ring R disjoint from S, is called an S-radical ideal if $\sqrt[S]{K} = K$.

Note that if S_1 and S_2, with $S_1 \subseteq S_2$, are multiplicatively closed subsets of R, then every S_1-radical ideal is an S_2-radical ideal. In addition, every maximal (prime) ideal disjoint from S is an S-radical ideal, by Remark 2.2.

Example 4.15. Let $R = \mathbb{Z}$, and $S = \{3^k : k \geq 0\} = \{1, 3, 9, \ldots\}$. Then clearly $(2) \cap S = \phi$. Given $x \in \sqrt[S]{(2)}$, we have $x^n s \in (2)$ for some $s \in S$ and $n \in \mathbb{N}$. Hence, x is even, and so $\sqrt[S]{(2)} \subseteq (2) \subseteq \sqrt[S]{(2)}$. Thus, (2) is an S-radical ideal.

Corollary 4.16. *Let K be an ideal of R such that $K \cap S = \phi$. If K is a prime ideal, then, $\sqrt[S]{K} = K = \sqrt{K}$.*

Proof. Let $x \in \sqrt[S]{K}$. Then there exist $n \in \mathbb{N}$ and $s \in S$ such that $x^n s \in K$, and $s \notin K$. Hence, $x \in K$. By Remark 4.2, we obtain that $\sqrt[S]{K} = K = \sqrt{K}$. □

Corollary 4.17. *Let S be a multiplicatively closed subset of a commutative ring R, and I be an ideal of R such that $I \cap S = \phi$. Then the following hold.*

(i) If I is S-radical, then I is radical.
(ii) If I is S-radical, then $(I : s) = I$ for some $s \in S$.

Proof. (i) Suppose $\sqrt[S]{I} = I$. Then, by Remark 4.2, $I = \sqrt{I}$.

(ii) We have $I \subseteq (I : s) \subseteq \sqrt[S]{I} = I$. □

The converse of (i) of Corollary 4.17 is not true in general; i.e., a radical ideal is not necessarily an S-radical, as we show in the following example.

Example 4.18. It is a well-known fact that the ideal (n) of $\mathbb{Z}$ coincides with its radical if and only if n is a product of distinct primes to the first power. Now let $R = \mathbb{Z}$, and $I = (6)$. Then, $\sqrt{(6)} = (6)$. Assuming that $S = \{9^k : k \geq 0\} = \{1, 9, 81, \cdots\}$, we have that $(6) \cap S = \phi$ and $2^1.9 \in (6)$. Hence, $2 \in \sqrt[S]{(6)}$ and that $(2) \subseteq \sqrt[S]{(6)} \neq \mathbb{Z}$. Consequently, $(2) = \sqrt[S]{(6)}$ because (2) is maximal.

Example 4.19. Let $R = 6\mathbb{Z}$, and $I = 0$. It is well-known that the ideal (0) of $n\mathbb{Z}$ is radical if and only if n is a product of distinct primes to the first power. Hence, $\sqrt{(0)} = (0)$. Now let $S = \{1, 3\}$. Then, $(0) \cap S = \phi$, $\sqrt[S]{(0)} = (2)$.

In the following, we give another example of an S-radical ideal that is not radical.

Example 4.20. Consider the ring $R = 6\mathbb{Z}$ and the multiplicatively closed set $S = \{1, 3\}$. As shown in Example 4.19, we have $\sqrt[S]{(0)} = (2)$. Then, by part (ii) of Lemma 4.6, it follows that

$$\sqrt[S]{\sqrt[S]{(0)}} = \sqrt[S]{(2)}.$$

Since $\sqrt[S]{(2)} = (2)$, we conclude that the ideal (2) is S-radical.

Proposition 4.21. *Let I be an ideal of R such that $I \cap S = \phi$. Then, the following statement are equivalent.*

(1) *$\sqrt[S]{I} = (I : s)$ for some $s \in S$.*
(2) *$(I : s)$ is an intersection of prime ideals.*
(3) *$(I : s)$ is a semiprime ideal.*

Proof. (1) ⇒ (2): Clear by Theorem 4.4.

(2) ⇒ (3): Clear by Corollary 4.5.

(3) ⇒ (1): Obviously, $(I : s) \subseteq \sqrt[S]{I}$. Given $x \in \sqrt[S]{I}$, there exist $n \in \mathbb{N}$ and $s \in S$ such that $x^n s \in I$. Then, $x^n \in (I : s)$. Since $(I : s)$ is semiprime, we get $x \in (I : s)$ and hence, $\sqrt[S]{I} \subseteq (I : s)$. □

Corollary 4.22. *Let I be an ideal of a ring R. Then the following are equivalent.*

(1) $\sqrt{I} = I$.
(2) *I is an intersection of prime ideals.*
(3) *I is a semiprime ideal.*

Proof. The proof follows from Proposition 4.21, by taking S=$\{1\}$. □

5 S_S-Primary ideals in commutative rings

All rings considered in this section are assumed to be commutative with identity. Moreover, S denotes a multiplicatively closed subset, even if not stated explicitly. In the following, we introduce the notion of S_S-primary ideals.

Definition 5.1. Let K be an ideal of R disjoint from S. We call K an S_S-primary ideal if for all $a, b \in R$ with $ab \in K$ either $a \in K$ or $b \in \sqrt[S]{K}$.

Due to of the symmetry between a and b, we conclude that if K is an S_S-primary ideal with $a, b \notin K$, then $a, b \in \sqrt[S]{K}$.

Remark 5.2. In a ring R:

(1) Every primary ideal of R disjoint from S is an S_S-primary ideal, because $\sqrt{K} \subseteq \sqrt[S]{K}$ for any ideal K disjoint from S, from Remark 4.2.

(2) In the following lines we show that every S_S-primary ideal is S-primary: Let K be an S_S-primary ideal of R. We have to show that K is an S-primary ideal. Let $ab \in K$ for some $a, b \in R$, then either $a \in K$ which implies $as \in K$, or $b \in \sqrt[S]{K}$ which implies $b^n s \in K$ for some $n \in \mathbb{N}$ and $s \in S$. Therefore, $(bs)^n \in K$, and thus $bs \in \sqrt{K}$.

That is, the concept of S-primary ideals is generalization of S_S-primary ideals. However, the converse is not true in general as we show further in Example 5.5 and Example 5.6.

(3) *A new definition of S-primary ideals.* It is easy to check that $b \in \sqrt[S]{K}$ is equivalent to $bs \in \sqrt{K}$ for some $s \in S$. Hence, we give an equivalent definition of S-primary ideal using the S-prime radical as follows:

"An ideal K of R disjoint from S is called S-primary ideal if for all $a, b \in R$ with $ab \in K$, either $as \in K$ or $b \in \sqrt[S]{K}$ for some $s \in S$. Throughout this section we will use this definition when referring to S-primary ideals."

(4) Let K be an ideal disjoint from S. Then we have the following.

$$S_S\text{-primary} \Longrightarrow S\text{-primary} \Longrightarrow \text{quasi-}S\text{-primary}.$$

(5) Every prime ideal of R disjoint from S, is an S_S-primary ideal.

(6) If $S \subseteq U(R)$, then the notions of primary, S-primary, and S_S-primary ideals coincide.

Corollary 5.3. *Let K be an ideal of R disjoint from S. Then following hold.*

(1) *If K is S-radical, then K is primary if and only if K is S_S-primary.*
(2) *K is S_S-primary and S-radical if and only if K is prime.*

Proof. Straightforward. □

Corollary 5.4. *Let K be an ideal of a ring R disjoint from S. If K is S_S-primary, then $\sqrt[S]{K}$ is prime.*

Proof. Suppose that K is S_S-primary, and that $ab \in \sqrt[S]{K}$ for some $a, b \in R$. Then, there exists $s \in S$ and $n \in \mathbb{N}$ such that $a^n b^n s = (ab)^n s \in K$. Hence, either $a^n s \in K$ or $b^n \in \sqrt[S]{K}$. If $a^n s \in K$, then, $a \in \sqrt[S]{K}$. If $b^n \in \sqrt[S]{K}$, then, $b^{nk} s \in K$ for some $k \in \mathbb{N}$, hence, either $b^{nk} \in K$ which implies $b \in \sqrt{K} \subseteq \sqrt[S]{K}$ or $s \in \sqrt[S]{K}$, contradiction. Thus $\sqrt[S]{K}$ is prime. □

Proper generalization. We showed in (2) of Remark 5.2 that every S_S-primary ideal is an S-primary ideal. The next two examples show that the converse is not true in general.

Example 5.5. In the ring $R = \mathbb{Z}_6$, the ideal $K = (0)$ is not primary since $2 \cdot 3 \in K$, but $2, 3 \notin K = \sqrt{K}$. In addition, K is not S_S-primary, when $S = \{1, 3\}$. Because $2 \cdot 3 \in K$; $2, 3 \notin K$, and $3 \notin \sqrt[S]{K} = (2)$ by Example 4.19. However, K is S-primary, since $ab \in K$ for $a, b \in R$ implies that $3a \in K$ or $3b \in K$ or $a, b \in \sqrt[S]{K}$.

Example 5.6. Let $R = \mathbb{Z}_{24}$, and define $S = \{1, 3, 9\}$, a multiplicatively closed subset of R. Consider the ideal $K = \{0, 12\}$. The ideal K is neither primary nor quasi-primary, as we observe that $3 \cdot 4 \in K$, yet $3 \notin K$, $4 \notin K$, and also $3, 4 \notin \sqrt{K} = (6)$. Moreover, K fails to be S_S-primary since both $3, 4 \notin K$, and $3 \notin \sqrt[S]{K} = (2)$. Nevertheless, K qualifies as an S-primary ideal with respect to the element $s = 3$. That is, for any $a, b \in R$ satisfying $ab \in K$, one of the following must hold: $3a \in K$, $3b \in K$, or $a, b \in \sqrt[S]{K}$.

Proposition 5.7. *Let K be an ideal of R. Then the following statements hold.*

(1) *For a given $s \in S$, the ideal K is S-primary if and only if the colon ideal $(K : s)$ is S_S-primary.*
(2) *If K is S_S-primary for some $s \in S$, then $(K : s)$ is an S-primary ideal.*

Proof. (1) Assume K is an S-primary ideal and let $ab \in (K : s)$ for some $a, b \in R$. Then $abs \in K$. By the assumption, either $as^2 \in K$ or $b \in \sqrt[S]{K}$. If $b \notin \sqrt[S]{K}$, then $as^2 \in K$, and again by the definition of S-primary, either $s^2 \in \sqrt[S]{K}$ (which leads to a contradiction) or $a \in (K : s)$. Therefore, $(K : s)$ must be S_S-primary.

Conversely, suppose $(K : s)$ is S_S-primary, and let $ab \in K$ for some $a, b \in R$. Then $ab \in (K : s)$ as $s \in S$, so by assumption, either $a \in (K : s)$ or $b \in \sqrt[S]{(K : s)} = \sqrt[S]{K}$, using part (2) of Corollary 4.7. This shows that K is S-primary.

(2) Now suppose K is S_S-primary, and take any $a, b \in R$ with $ab \in (K : s)$. Then $asb \in K$. Since K is S_S-primary, it follows that either $as \in K$ or $b \in \sqrt[S]{K}$. This implies either $a \in (K : s)$ or there exists $s_1 \in S$ such that $bs_1 \in \sqrt{K} \subseteq \sqrt{(K : s)}$. Set $s_2 = ss_1 \in S$. Then either $as_2 \in (K : s)$ or $bs_2 \in \sqrt{(K : s)}$, proving that $(K : s)$ is S-primary. □

Let K be an ideal of a ring R. Proposition 2.3 of [18] and Proposition 2.4 of [21] state that $(K : s)$ is a primary ideal, if and only if, K is S-primary. Hence, by Proposition 5.7, we conclude that $(K : s)$ is primary if and only if $(K : s)$ is S_S-primary.

Proposition 5.8. *Let K be an ideal of a ring R. Then K is S_S-primary if and only if, for all ideals I, J of R with $IJ \subseteq K$, either $I \subseteq K$ or $J \subseteq \sqrt[S]{K}$.*

Proof. Suppose K is S_S-primary, and let $IJ \subseteq K$ for some ideals I, J of R. If $I \nsubseteq K$ and $J \nsubseteq \sqrt[S]{K}$, then there exist $a \in I$ and $b \in J$ such that $a \notin K$ and $b \notin \sqrt[S]{K}$, which contradicts with $ab \in IJ \subseteq K$. Conversely, let $ab \in K$ for some $a, b \in R$. Then, $(a)(b) \subseteq K$. Thus, either $a \in (a) \subseteq K$ or $b \in (b) \subseteq \sqrt[S]{K}$, which implies that K is S_S-primary. □

Corollary 5.9. *Let K be an ideal of a ring R disjoint from S. Then following statements are equivalent.*

(1) *K is an S_S-primary ideal.*
(2) *Let $\prod_{i=1}^{n} I_i = I_1 \cdots I_n \subseteq K$ for some ideals $I_1, \ldots, I_n$ of R. Then, there exist $i, j \in \{1, 2, \ldots, n\}$ such that either $I_i \subseteq K$ or $I_j \subseteq \sqrt[S]{P}$.*

Proof. (1) ⇒ (2) Suppose that for some ideals $I_1, \ldots, I_n$ of R, we have $I_1 \cdots I_n \subseteq K$. Then either $I_n \subseteq \sqrt[S]{K}$ or $(I_1 \cdots I_{n-1}) \subseteq K$. If $(I_1 \cdots I_{n-1}) \subseteq K$, then either $I_{n-1} \subseteq \sqrt[S]{K}$ or $(I_1 \cdots I_{n-2}) \subseteq K$. If $I_1 \cdots I_{n-2} \subseteq K$, then by repeating the process, we obtain $I_1 I_2 \subseteq K$, hence, either $I_2 \subseteq \sqrt[S]{K}$, or $I_1 \subseteq K$.

(2) ⇒ (1) Immediate by Proposition 5.8. □

Proposition 5.10. *Let K be an ideal of a ring R such that $K \cap S = \varnothing$. Then the following hold.*

(1) *The ideal K is S_S-primary if and only if every zero divisor in the quotient ring R/K is $\bar{S}$-nilpotent.*
(2) *If the S-radical $\sqrt[S]{K}$ is a maximal ideal of R, then K is S-primary.*
(3) *Let M be a maximal ideal disjoint from S, and suppose $M^n s \subseteq K \subseteq M$ for some $s \in S$ and positive integer n. Then K is S-primary.*

Proof. (1) Assume that K is S_S-primary, and let $a + K \in R/K$ be a zero divisor. Then there exists some $b \in R \setminus K$ such that $ab \in K$, i.e., $(a+K)(b+K) = K$. Since $b \notin K$, the S_S-primary property implies that $a \in \sqrt[S]{K}$, so there exist $n \in \mathbb{N}$ and $s \in S$ such that $a^n s \in K$. This means $(a+K)^n(s+K) = K$, where $s + K \neq K$ because $s \notin K$. Therefore, $a + K$ is $\bar{S}$-nilpotent.

Conversely, suppose every zero divisor in R/K is $\bar{S}$-nilpotent. Let $a, b \in R$ satisfy $ab \in K$. If $a \notin K$, then $a + K$ is a zero divisor in R/K, and hence $b + K$ must be $\bar{S}$-nilpotent. Thus, $b^n s \in K$ for some $n \in \mathbb{N}$ and $s \in S$, which implies $b \in \sqrt[S]{K}$. Hence, K is S_S-primary.

(2) Let $a, b \in R$ with $ab \in K$, and suppose $b \notin \sqrt[S]{K}$. Since $\sqrt[S]{K}$ is maximal, the ideal $\sqrt[S]{K} + (b) = R$. Thus, there exist $x \in \sqrt[S]{K}$ and $r \in R$ such that $x + br = 1$. Then for some $n \in \mathbb{N}$ and $s \in S$, we have $x^n s \in K$. Using the binomial expansion, $(x + br)^n = 1$, so multiplying both sides by as, we get $as = x^n sa + abr_1 s \in K$, for some $r_1 \in R$. Therefore, K is S-primary.

(3) Since $K \subseteq M$, it follows from part (i) of Lemma 4.6 that $\sqrt[S]{K} \subseteq \sqrt[S]{M} = M$. For any $x \in M$, the assumption $M^n s \subseteq K$ implies $x^n s \in K$, so $x \in \sqrt[S]{K}$, and thus $M \subseteq \sqrt[S]{K}$. Therefore, $\sqrt[S]{K} = M$, and by part (2), we conclude that K is S-primary. □

Corollary 5.11. *Let K be an ideal of a ring R. Then following hold.*
(1) *K is a primary ideal if and only if every zero divisor in R/K is nilpotent.*
(2) *If $\sqrt{K}$ is a maximal ideal, then K is primary.*
(3) *If M is a maximal ideal such that $M^n \subseteq K \subseteq M$, then K is primary.*

Proof. The proof follows from Proposition 5.10, by taking $S = \{1\}$. □

Proposition 5.12. *Let K be an ideal of a von Neumann regular ring R, and disjoint from S. If K is an S_S-primary ideal, then K is an S-prime ideal.*

Proof. Suppose that K is an S_S-primary ideal, and that $ab \in K$ for some $a, b \in R$. Since R is von Neumann regular, then $rR = r^2R = r^nR$ for all $n \geq 3$ and $r \in R$. Let $x \in \sqrt[S]{K}$. Then, $x^m s \in K$ for some $m \in \mathbb{N}$. Consequently,

$$x \in Rx = Rx^m \subseteq R(K:s) = (K:s).$$

Hence, $\sqrt[S]{K} \subseteq (K:s)$, which implies that $\sqrt[S]{K} = (K:s)$. If $b \in \sqrt[S]{K}$, then $bs \in K$. If $b \notin \sqrt[S]{K}$, then $a \in K$, and hence, $as \in K$, because K is S_S-primary. Thus, K is S-prime. □

In refer to [6], Definition 2.1 states that an ascending sequence of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is said to be S-stationary if there exist $s \in S$ and $n \in \mathbb{N}$ such that $sI_j \subseteq I_n$ for all $j \geq n$.

Theorem 5.13. Let K be an ideal of a ring R such that $K \cap S = \emptyset$. Then the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) hold, where:
(1) The ideal $(K:s)$ is an S_S-primary ideal for some $s \in S$.
(2) The ascending chain of ideals

$$(K:\alpha s) \subseteq (K:\alpha^2 s) \subseteq (K:\alpha^3 s) \subseteq \cdots$$

stabilizes for some $\alpha \in R$ and $s \in S$.
(3) The chain

$$(K:\alpha) \subseteq (K:\alpha^2) \subseteq (K:\alpha^3) \subseteq \cdots$$

is S-stationary for some $\alpha \in R$.

Proof. (1) $\Rightarrow$ (2): Suppose $(K:s)$ is an S_S-primary ideal for some $s \in S$, and let $\alpha \in R$. If $\alpha \in \sqrt[S]{(K:s)}$, then by part (2) of Corollary 4.7, we have $\alpha \in \sqrt[S]{K}$. Therefore, there exists $n \in \mathbb{N}$ such that $\alpha^n s \in K$, which implies that $(K:\alpha^m s) = R$ for all $m \geq n$.

If $\alpha \notin \sqrt[S]{(K:s)}$, then $\alpha \notin \sqrt[S]{K}$, and so $\alpha^n s \notin K$ for any $n \in \mathbb{N}$. Let $x \in (K:\alpha^n s)$. Then $x\alpha^n s \in K$, hence $x\alpha^n \in (K:s)$. Since $(K:s)$ is primary and $\alpha^n \notin \sqrt{(K:s)}$, it follows that $x \in (K:s)$.

Conversely, if $x \in (K:s)$, then $\alpha^n xs \in K$, so $x \in (K:\alpha^n s)$. Thus, $(K:\alpha^n s) = (K:s)$ for all $n \in \mathbb{N}$, and the chain stabilizes.

(2) $\Rightarrow$ (3): This follows from Proposition 2.8 in [21]. □

Proposition 5.14. *Let K be an ideal of a ring R such that $K \cap S = \emptyset$, and define $\bar{S} = \{s + K \mid s \in S\}$. If $Z(R/K) \cap \bar{S} = \emptyset$, then K is S_S-primary if and only if K is S-primary.*

Proof. The forward direction is immediate, since any S_S-primary ideal is, by definition, also S-primary. Conversely, suppose that K is S-primary. Then clearly $K \subseteq (K:s)$ for any $s \in S$. Now let $\alpha \in (K:s)$, so that $s\alpha \in K$. This implies that in the quotient ring R/K, we have:

$$(s+K)(\alpha+K) = 0.$$

Since $s + K \in \bar{S}$ and by assumption $\bar{S} \cap Z(R/K) = \emptyset$, it follows that $s + K$ is not a zero-divisor in R/K. Thus, the product above being zero implies $\alpha + K = 0$, i.e., $\alpha \in K$. Therefore, $(K:s) \subseteq K$, and so $K = (K:s)$ for all $s \in S$. By part (1) of Proposition 5.7, we conclude that K is S_S-primary. □

Theorem 5.15. Let $\varphi : R_1 \to R_2$ be a ring epimorphism, $S \subseteq R_1$ a multiplicatively closed set, and K an ideal of R_1 disjoint from S such that $\ker(\varphi) \subseteq K$. If K is an S_S-primary ideal of R_1 associated with $s \in S$, then $\varphi(K)$ is an $\varphi(S)_{\varphi(S)}$-primary ideal of R_2 associated with $\varphi(s)$.

Proof. Suppose that K is an S_S-primary ideal of R_1, and that $y_1y_2 \in \varphi(K)$ for some $y_1, y_2 \in R_2$. Then there exist $x_1, x_2 \in R_1$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Hence, $x_1x_2 \in \varphi^{-1}(\varphi(x_1x_2)) \subseteq \varphi^{-1}(\varphi(K)) = K$, and thus either $x_1 \in K$ or $x_2 \in \sqrt[S]{K}$. Consequently, either $\varphi(x_1) \in \varphi(K)$ or $\varphi(x_2) \in \varphi(\sqrt[S]{K}) = \sqrt[\varphi(S)]{\varphi(K)}$ by Theorem 4.9. Since $\varphi(S) \cap \varphi(K) = \emptyset$ by Theorem 4.9, it follows that $\varphi(K)$ is an $\varphi(S)_{\varphi(S)}$-primary ideal of R_2. □

Theorem 5.16. Let $\varphi : R_1 \to R_2$ be a ring epimorphism, $S \subseteq R_1$ a multiplicatively closed set, and let K be an ideal of R_2 disjoint from $\varphi(S)$ such that $\ker(\varphi) \subseteq K$. If K is an $\varphi(S)_{\varphi(S)}$-primary ideal of R_2 associated with $\varphi(s)$ for some $s \in S$, then the preimage $\varphi^{-1}(K)$ is an S_S-primary ideal of R_1 associated with s.

Proof. Assume that K is an $\varphi(S)_{\varphi(S)}$-primary ideal of R_2, and consider elements $x_1, x_2 \in R_1$ such that $x_1x_2 \in \varphi^{-1}(K)$. Then, applying the homomorphism, we have:

$$\varphi(x_1)\varphi(x_2) = \varphi(x_1x_2) \in \varphi(\varphi^{-1}(K)) \subseteq K.$$

Since K is $\varphi(S)_{\varphi(S)}$-primary, either $\varphi(x_1) \in K$ or $\varphi(x_2) \in \sqrt[\varphi(S)]{K}$. In the first case, $x_1 \in \varphi^{-1}(K)$; in the second case, Theorem 4.9 implies that:

$$\varphi(x_2) \in \sqrt[\varphi(S)]{K} = \varphi(\sqrt[S]{\varphi^{-1}(K)}),$$

so $x_2 \in \sqrt[S]{\varphi^{-1}(K)}$. By Theorem 4.10, $\varphi^{-1}(K) \cap S = \emptyset$. Therefore, $\varphi^{-1}(K)$ is an S_S-primary ideal of R_1 associated with s. □

Theorem 5.17. Let I, K be ideals of a ring R such that $I \subseteq K$. Then K is an S_S-primary ideal of R if and only if K/I is a $\bar{S}_{\bar{S}}$-primary ideal of R/I.

Proof. Suppose first that K is an S_S-primary ideal. Then clearly $K \cap S = \emptyset$, which implies $K/I \cap \bar{S} = \emptyset$. Now let $(\alpha + I)(\beta + I) \in K/I$ for some $\alpha, \beta \in R$. Then $\alpha\beta \in K$, so either $\alpha \in K$ or $\beta \in \sqrt[S]{K}$. It follows that either $\alpha + I \in K/I$ or $\beta + I \in \sqrt[S]{K}/I = \sqrt[\bar{S}]{K/I}$ by Corollary 4.12. Hence, K/I is a $\bar{S}_{\bar{S}}$-primary ideal of R/I.

Conversely, assume that K/I is a $\bar{S}_{\bar{S}}$-primary ideal of R/I. Then $K/I \cap \bar{S} = \emptyset$, which implies $K \cap S = \emptyset$. Let $\alpha, \beta \in R$ such that $\alpha\beta \in K$. Then $(\alpha + I)(\beta + I) \in K/I$, so by the $\bar{S}_{\bar{S}}$-primarity of K/I, we have either $\alpha + I \in K/I$ or $\beta + I \in \sqrt[\bar{S}]{K/I} = \sqrt[S]{K}/I$ by part (1) of Corollary 4.13. Thus, either $\alpha \in K$ or $\beta \in \sqrt[S]{K}$. Therefore, K is an S_S-primary ideal of R. □

Theorem 5.18. Let $R = R_1 \times R_2$ for some commutative rings R_1, R_2 with identites, and let $S = S_1 \times S_2$ for some multiplicatively closed subsets $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$. If P_1 and P_2 are ideals of R_1 and R_2, respectively, then the following hold.

(1) P_1 is an ${S_1}_{S_1}$-primary ideal of R_1 if and only if $P_1 \times R_2$ is an S_S-primary ideal of R.
(2) P_2 is an ${S_2}_{S_2}$-primary ideal of R_2 if and only if $R_1 \times P_2$ is an S_S-primary ideal of R.

Proof. (1) Suppose P_1 is an ${S_1}_{S_1}$-primary ideal of R_1. Then, $P_1 \cap S_1 = \phi$, which implies $(P_1 \times R_2) \cap S = \phi$. Now let $(a_1, b_1)(a_2, b_2) \in (P_1 \times R_2)$, for some $a_1, a_2 \in R_1$ and $b_1, b_2 \in R_2$. Then, $a_1a_2 \in P_1$, and hence, either $a_1 \in P_1$ or $a_2 \in \sqrt[S_1]{P_1}$. Consequently, $(a_1, b_1) \in P_1 \times R_2$ or $(a_2, b_2) \in \sqrt[S_1]{P_1} \times R_2 = \sqrt[S]{P_1 \times R_2}$, by Corollary 4.13. Thus, $P_1 \times R_2$ is an S_S-primary ideal of R.
Conversely, suppose $P_1 \times R_2$ is an S_S-primary ideal of R. Then, $(P_1 \times R_2) \cap S = \phi$, which implies $P_1 \cap S_1 = \phi$. Let $a_1a_2 \in P_1$ for some $a_1, a_2 \in R_1$. Then, $(a_1, b_1)(a_1, b_2) \in P_1 \times R_2$ for any $b_1, b_2 \in R_2$. Thus, either $(a_1, b_1) \in P_1 \times R_2$ or $(a_2, b_2) \in \sqrt[S]{P_1 \times R_2} = \sqrt[S_1]{P_1} \times R_2$. Consequently, either $a_1 \in P_1$ or $a_2 \in \sqrt[S_1]{P_1}$. Hence, P_1 is an ${S_1}_{S_1}$-primary ideal of R_1.

(2) Similar to the proof of (1). □

Theorem 5.19. Let $R = R_1 \times R_2$ for some commutative rings R_1, R_2, and let $S = S_1 \times S_2$ for some multiplicatively closed subsets $S_1 \subseteq R_1$ and $S_2 \subseteq R_2$. If $P = P_1 \times P_2$ is an ideal of R disjoint from S, for some ideals $P_1 \lhd R_1$ and $P_2 \lhd R_2$, then following are equivalent.

(1) P is S_S-primary of R.
(2) Either [P_1 is an $(S_1)_{S_1}$-primary ideal of R_1 and $P_2 \cap S_2 \neq \phi$] or [P_2 is an $(S_2)_{S_2}$-primary ideal and $P_1 \cap S_1 \neq \phi$].

Proof. (1) ⇒ (2) Assume P is an S_S-primary ideal of R. Since $(a,0)\cdot(0,b) \in P$ for all $a \in R_1$ and all $b \in R_2$, then either $(a,0) \in P$ or $(0,b) \in \sqrt[S]{P}$. Hence, either $a \in P_1$ or $(0,b^n)\cdot(s,s') \in P$ for some $(s,s') \in S$ and $n \in \mathbb{N}$. Taking $a = 1_{R_1}$ and $b = 1_{R_2}$, we obtain that either $R_1 = P_1$ or $s' \in P_2$, contradiction in both cases. Thus, either $P_1 \cap S_1 \neq \phi$ or $P_2 \cap S_2 \neq \phi$.
Without any loss of generality, we may assume that $P_2 \cap S_2 \neq \phi$. Then we have to show that P_1 is an $(S_1)_{S_1}$-primary ideal of R_1. Let $a_1a_2 \in P_1$ for some $a_1, a_2 \in R_1$. Then, $(a_1,0)\cdot(a_2,0) \in P_1 \times P_2$, and hence, either $(a_1,0) \in P_1 \times P_2$ or $(a_2,0) \in \sqrt[S]{P_1 \times P_2}$. Consequently, either $a_1 \in P_1$ or $(a_2^n,0)\cdot(s,s') \in P$ for some $(s,s') \in S$ and $n \in \mathbb{N}$. Thus, either $a_1 \in P_1$ or $a_2 \in \sqrt[S_1]{P_1}$. Since P is disjoint from S and $P_2 \cap S_2 \neq \phi$, we obtain $P_1 \cap S_1 = \phi$. Thus, P_1 is an $(S_1)_{S_1}$-primary ideal of R_1.
If $P_1 \cap S_1 \neq \phi$, then as in the preceding case, P_2 is an $(S_2)_{S_2}$-primary ideal of R_2.

(2) ⇒ (1) Without loss of generality, suppose that P_1 is an $(S_1)_{S_1}$-primary ideal of R_1 and $P_2 \cap S_2 \neq \phi$. Let $(a_1,b_1)\cdot(a_2,b_2) \in P$ for some $a_1,a_2 \in R_1$ and some $b_1,b_2 \in R_2$. Then, $(a_1a_2, b_1b_2) \in P_1 \times P_2$, and hence, $a_1a_2 \in P_1$. Consequently, either $a_1 \in P_1$ or $a_2 \in \sqrt[S_1]{P_1}$. Hence, either $a_1 \in P_1$ or $a_2^n s_1 \in P_1$ for some $s_1 \in S_1$ and $n \in \mathbb{N}$. Now for $s_2 \in P_2 \cap S_2$, we obtain

$$\text{either}(a_1,b_1) \in P_1 \times P_2 \text{ or } (a_2^n s_1, b_2^n s_2) \in P_1 \times P_2.$$

Hence,

$$\text{either}(a_1,b_1) \in P_1 \times P_2 \text{ or } (a_2,b_2)^n \cdot (s_1,s_2) \in P_1 \times P_2.$$

Finally,

$$\text{either}(a_1,b_1) \in P_1 \times P_2 \text{ or } (a_2,b_2) \in \sqrt[S]{P_1 \times P_2}.$$

Thus, P is an S_S-primary ideal of R.
If P_2 is an $(S_2)_{S_2}$-primary and $P_1 \cap S_1 \neq \phi$, then as in the preceding case, P is an S_S-primary ideal of R. □

Theorem 5.20. Let P be an ideal of a ring R disjoint from S, and M be an R-module. Then, $P(+)M$ is an $(S_M)_{S_M}$-primary ideal of $R(+)M$ if and only if P is an S_S-primary ideal of R.

Proof. Suppose P is an S_S-primary ideal of R. Notice that $S_M \cap (P(+)M) = \phi$. Let $(a_1,m_1)(a_2,m_2) \in P(+)M$ for some $a_1,a_2 \in R$ and $m_1,m_2 \in M$. Then, $a_1a_2 \in P$, and hence, either $a_1 \in P$ or $a_2 \in \sqrt[S]{P}$. If $a_2 \in \sqrt[S]{P}$, then by (3) of Corollary 4.13, $(a_2,m_2) \in \sqrt[S]{P}(+)M = \sqrt[S_M]{P(+)M}$. If $a_1 \in P$, then $(a_1,m_1)(1,m) \in P(+)M$, where $(1,m) \in S_M$ for any $m \in M$. Thus, $P(+)M$ is an $(S_M)_{S_M}$-primary ideal of $R(+)M$.
Conversely, suppose $P(+)M$ is an $(S_M)_{S_M}$-primary ideal of $R(+)M$. Let $ab \in P$, for some $a,b \in R$. Then, $(a,m_1)(b,m_2) \in P(+)M$ for some $m_1,m_2 \in M$, and hence, either $(a_1,m_1) \in P(+)M$, or $(b,m_2) \in \sqrt[S_M]{P(+)M} = \sqrt[S]{P}(+)M$. Consequently, either $a_1 \in P$ or $b \in \sqrt[S]{P}$. Thus, P is an S_S-primary ideal of R. □

5.1 Amalgamation of S_S-primary ideals

Let R and A be commutative rings and let $\varphi : R \to A$ be a ring homomorphism. Suppose P is an ideal of A. The structure called the *amalgamation* of R with A along P, relative to the map φ, is denoted by $R \bowtie^{\varphi} P$, and it is defined as the following subring of the direct product $R \times A$:

$$R \bowtie^{\varphi} P = \{(r, \varphi(r) + p) \mid r \in R,\ p \in P\}.$$

This construction extends the notion of *amalgamated duplication* of a ring along one of its ideals. To recall, if P is an ideal of a ring R, then the amalgamated duplication of R along P, denoted by $R \bowtie P$, is the subring of $R \times R$ given by

$$R \bowtie P = \{(r, r+p) \mid r \in R,\ p \in P\}.$$

Furthermore, if K is an ideal of R, then the set

$$K \bowtie^{\varphi} P = \{(k, \varphi(k)+p) \mid k \in K,\ p \in P\}$$

forms an ideal of the ring $R \bowtie^{\varphi} P$. In addition, when $S \subseteq R$ is a multiplicatively closed set, the subset

$$S^{\bowtie} = \{(s, \varphi(s)) \mid s \in S\}$$

constitutes a multiplicatively closed subset of $R \bowtie^{\varphi} P$.

Proposition 5.21. *Let R and A be commutative rings, $\varphi : R \to A$ a ring homomorphism, P an ideal of the ring A, and $R \bowtie^{\varphi} P$ the amalgamation of R with A along P with respect to φ. If K is an ideal of R disjoint from S, then*

$$\sqrt[S^{\bowtie}]{K \bowtie^{\varphi} P} = \sqrt[S]{K} \bowtie^{\varphi} P.$$

Proof. Initially, since $S \cap K = \varnothing$, then $S^{\bowtie} \cap (K \bowtie^{\varphi} P) = \varnothing$. Now let $(x, \varphi(x)+p) \in \sqrt[S^{\bowtie}]{K \bowtie^{\varphi} P}$, for some $x \in R$ and $p \in P$. Then, there exists $(s, \varphi(s)) \in S^{\bowtie}$ such that

$$(x, \varphi(x)+p)^n (s, \varphi(s)) \in K \bowtie^{\varphi} P$$

for some $s \in S$ and $n \in \mathbb{N}$. Hence, $x^n s \in K$, which implies $x \in \sqrt[S]{K}$. Thus, $(x, \varphi(x)+p) \in \sqrt[S]{K} \bowtie^{\varphi} P$.

Conversely, let $(x, \varphi(x)+p) \in \sqrt[S]{K} \bowtie^{\varphi} P$, for some $x \in R$ and $p \in P$. Then $x \in \sqrt[S]{K}$, and hence, there exist $s \in S$ and $n \in \mathbb{N}$ such that $x^n s \in K$. Additionally, define

$$p_1 = \sum_{k=0}^{n-1} \binom{n}{k} \varphi(s)\varphi(x)^k p^{n-k} \in P.$$

Then,

$$(\varphi(x)+p)^n \varphi(s) = \sum_{k=0}^{n} \binom{n}{k} \varphi(s)\varphi(x)^k p^{n-k} = \varphi(x^n s) + p_1 \in \varphi(K) + P.$$

Consequently, $(x, \varphi(x)+p)^n (s, \varphi(s)) = (x^n s, \varphi(x^n s) + p_1) \in K \bowtie^{\varphi} P$, and hence $(x, \varphi(x)+p) \in \sqrt[S^{\bowtie}]{K \bowtie^{\varphi} P}$. □

Corollary 5.22. *Let R be a ring, P an ideal of R, and $R \bowtie P$ the amalgamated duplication of R along P. If K is an ideal of R disjoint from S, then*

$$\sqrt[S^{\bowtie}]{K \bowtie P} = \sqrt[S]{K} \bowtie P.$$

Proof. The proof follows from Proposition 5.21. □

Theorem 5.23. Let $\varphi : R \to A$ be a ring homomorphism, P an ideal of the ring A, and $R \bowtie^{\varphi} P$ the amalgamation of R with A along P with respect to φ. If K is an ideal of R disjoint from S, then $K \bowtie^{\varphi} P$ is an $S^{\bowtie}_{S^{\bowtie}}$-primary ideal of $R \bowtie^{\varphi} P$ if and only if K is an S_S-primary ideal of R.

Proof. Since $S \cap K = \emptyset$, then $S^{\bowtie} \cap (K \bowtie^{\varphi} P) = \emptyset$. Suppose that $K \bowtie^{\varphi} P$ is an $S^{\bowtie}_{S^{\bowtie}}$-primary ideal of $R \bowtie^{\varphi} P$, and that $ab \in K$ for some $a, b \in R$. Then, $(a, \varphi(a))(b, \varphi(b)) \in K \bowtie^{\varphi} P$, and hence, by assumption, either $(a, \varphi(a)) \in K \bowtie^{\varphi} P$ or $(b, \varphi(b)) \in \sqrt[S^{\bowtie}]{K \bowtie^{\varphi} P}$. If $(a, \varphi(a)) \in K \bowtie^{\varphi} P$, then $a \in K$. If $(b, \varphi(b)) \in \sqrt[S^{\bowtie}]{K \bowtie^{\varphi} P}$, then by Proposition 5.21, we have $b \in \sqrt[S]{K}$. Thus, K is an S_S-primary ideal of R.

Conversely, suppose that K is an S_S-primary ideal of R, and that

$$(a, \varphi(a) + p_1)(b, \varphi(b) + p_2) \in K \bowtie^{\varphi} P$$

for some $a, b \in R$ and $p_1, p_2 \in P$. Then $ab \in K$, and hence, by assumption, either $a \in K$ or $b \in \sqrt[S]{K}$. If $a \in K$, then $(a, \varphi(a)+p_1) \in K \bowtie^{\varphi} P$. If $b \in \sqrt[S]{K}$, then $(b, \varphi(b)+p_2) \in \sqrt[S]{K} \bowtie^{\varphi} P = \sqrt[S^{\bowtie}]{K \bowtie^{\varphi} P}$ by Proposition 5.21. Thus, $K \bowtie^{\varphi} P$ is an $S^{\bowtie}_{S^{\bowtie}}$-primary ideal of $R \bowtie^{\varphi} P$. □

Corollary 5.24. *Let R be a ring, P an ideal of R, and $R \bowtie P$ the amalgamated duplication of R along P. If K is an ideal of R disjoint from S, then $K \bowtie P$ is an $S^{\bowtie}_{S^{\bowtie}}$-primary ideal of $R \bowtie P$ if and only if K is an S_S-primary ideal of R.*

Proof. The proof follows from Theorem 5.23 and Corollary 5.22. □

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Conflict of interest

The authors declare no conflicts of interest.

References

[1] A. Abouhalaka, S-prime ideals, S-Noetherian noncommutative rings, and the S-Cohen's theorem, Mediterr. J. Math. (2024), 21(43), doi: 10.1007/s00009-023-02578-w

[2] A. Abouhalaka, Non-$\mathcal{P}$-Right Noetherian and Non-$\mathcal{P}$-Right-S-Noetherian Rings, Moroccan J. Algebra Geom. Appl., (2025), accepted for publication.

[3] S. Yavuz, B. A. Ersoy, Ü. Tekir and E. Y. Çelikel, Weakly S-2-prime ideals of commutative rings, Moroccan J. Algebra Geom. Appl. 4(1) (2024), 1-8.

[4] A. Hamed, Generalized S-prime ideals of commutative rings, Moroccan J. Algebra Geom. Appl. 3(2) (2024), 1-9.

[5] A. Abouhalaka and N. Groenewald, The structure of S-semiprime ideals and S-semiprime submodules, Commun. Algebra 53(12) (2025), 5315–5327, https://doi.org/10.1080/00927872.2025.2509830

[6] H. Ahmed and H. Sana, Modules satisfying the S-Noetherian property and S-ACCR, Commun. Algebra 44(5) (2016), 1941–1951.

[7] F. A. Almahdi, E. M. Bouba and M. Tamekkante, On weakly S-prime ideals of commutative rings, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. 29(2) (2021), 173–186.

[8] D. D. Anderson and T. Dumitrescu, S-Noetherian rings, Commun. Algebra 30(9) (2002), 4407–4416. https://doi.org/10.1081/AGB-120013328

[9] J. Baeck, On S-principal right ideal rings, AIMS Math. 7 (2022), 12106–12122.

[10] G. F. Birkenmeier, J. Y. Kim and J. K. Park, Right primary and nilary rings and ideals, J. Algebra 378 (2013), 133–152.

[11] E. Y. Çelikel and A. Hamed, Quasi-S-primary ideals of commutative rings, Commun. Algebra 51(10) (2023), 4285–4298. doi: 10.1080/00927872.2023.2204970

[12] E. Y. Çelikel and H. Khashan, On weakly S-primary ideals of commutative rings, J. Algebra Appl. (2024). doi: 10.1142/S021949882450155X

[13] R. P. Dilworth, Noncommutative residuated lattices, Trans. Amer. Math. Soc. 46 (1939), 426–444.

[14] B. Ersoy, Ü. Tekir and E. Yıldız, S-versions and S-generalizations of idempotents, pure ideals and Stone type theorems, Bull. Korean Math. Soc. 61 (2024), 83–92. doi: 10.4134/BKMS.b230023

[15] A. Hamed and A. Malek, S-prime ideals of a commutative ring, Beitr. Algebra Geom. 61 (2020), 533–542. https://doi.org/10.1007/s13366-019-00476-5

[16] C. Gorton, Generalized primary rings, Ph.D. thesis, University of Louisiana at Lafayette, 2006.

[17] G. Lee, J. Baeck and J. Lim, Eakin–Nagata–Eisenbud theorem for right S-Noetherian rings, Taiwanese J. Math. 27 (2023), 237–257.

[18] E. Massaoud, S-primary ideals of a commutative ring, Commun. Algebra 50(3) (2022), 988–997. https://doi.org/10.1080/00927872.2021.1977939

[19] K. K. Pathak and J. Goswami, S-semiprime ideals and weakly S-semiprime ideals of rings, Palestine J. Math. 12(4) (2023), 115–124.

[20] Ü. Tekir, S. Koç, K. H. Oral and K. Shum, On 2-absorbing quasi-primary ideals in commutative rings, Commun. Math. Stat. 4 (2016), 55–62. doi: 10.1007/s40304-015-0075-9

[21] S. Visweswaran, Some results on S-primary ideals of a commutative ring, Beitr. Algebra Geom. 63 (2022), 247–266. doi: 10.1007/s13366-021-00580-5

[22] E. Yıldız, B. A. Ersoy, Ü. Tekir and S. Koç, On S-Zariski topology, Commun. Algebra 49(3) (2021), 1212–1224.

[23] M. L. Reyes, Noncommutative generalizations of theorems of Cohen and Kaplansky, Algebr. Represent. Theory 15 (2012), 933–975.

[24] Z. Bilgin, M. L. Reyes and Ü. Tekir, On right S-Noetherian rings and S-Noetherian modules, Commun. Algebra 46(2) (2018), 863–869. https://doi.org/10.1080/00927872.2017.1332199