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On 2-nil-clean commutative rings

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Abstract. A ring R is said to be 2-nil-clean if any element in R can be written as the sum of two idempotents and a nilpotent [7]. In this paper, we exhibit the connections between this ring and other related classes of rings. Our specific aim is to illustrate how the 2-nil clean condition behaves with respect to several types of ring extensions such as polynomial ring, power series ring, amalgamated algebra and pullback. Our results produce new and original classes of 2-nil-clean rings subject to various ring theoretic properties.

Key Words: Amalgamated algebra, trivial ring extension, nil-clean ring, 2-nil-clean ring, weakly-nil-clean ring, pullback.
2020 MSC: 13F05, 13A15, 13E05, 13F20, 13C10, 13C11, 13F30, 13D05.

Dedicated to our Professor David E. Dobbs for his 80$^{\text{th}}$ Birthday.

1 Introduction

Throughout this paper all rings are assumed to be commutative with unity. We denote respectively by $U(R)$, $Idem(R)$ and $Nilp(R)$ the multiplicative group of units of R, the set of all idempotent elements of R, and the set of all nilpotent elements of the ring R. The determinant of a square matrix B is denoted by $det(B)$. Let A be a ring and M be an A-module. The trivial ring extension of A by M (also called the idealization of M over A) is the ring $R = A \ltimes M$ whose underlying group is $A \times M$ with multiplication given by $(a,m)(b,n) = (ab, an + bm)$ for each $a, b \in A$ and $m, n \in M$. This notion has been the subject of several works in the area, and it is a fertile field of research in the theory of rings (see [16, 18] and the reference therein). Trivial ring extensions have been studied extensively; and considerable work, part of is summarized in Glaz's book [16] and Huckaba's book [18], has been concerned with these extensions. Mainly, Trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory.

In 2006, M. D'Anna and M. Fontana [13] introduced a new construction, called amalgamated duplication of a ring A along an $A-$submodule E of $Q(A)$ (the total ring of fractions of A) such that $E^2 \subseteq E$. When $E^2 = \{0\}$, this construction coincides with the trivial ring extension of A by E. Motivations and more applications of the amalgamated duplication $A \bowtie E$ of A along an $A-$submodule E of $Q(A)$ are discussed in more details, especially in the case E is an ideal of A, see for instance [13, 11]. In 2010, D'Anna, Finocchiaro and Fontana [11] extended the notion of amalgamated duplication construction $A \bowtie I$ of a ring A along an ideal I of A to the general context of ring homomorphism extensions as follows:

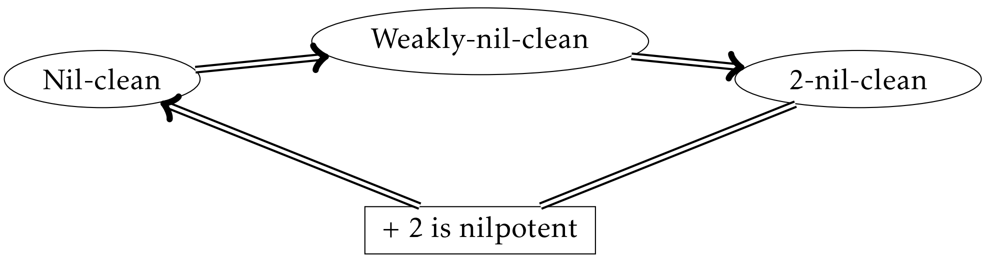
Let A and B be two rings with identity elements, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$; $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$ called the amalgamation of A and B along J with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [13, 11]). The usefulness of amalgamation, as mentioned in [11], comes from the fact that many of the classical constructions (such as $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as a special case of amalgamation, and amalgamation itself can be studied using pullbacks.

In [10], *A. J. Diesl* introduced a new class of rings and called it nil-clean ring. Recall that a ring R is called nil-clean if each of its elements can be written as the sum of an idempotent and a nilpotent. It was proved in [10] that every nil-clean ring is clean. Various authors have studied nil-clean rings and related notions in [9, 10, 19]. Examples of nil clean elements include all idempotents, nilpotents and unipotents (that is an element which can be written in the form $1 + n$ for some nilpotent n), also, any Boolean ring is a nil-clean ring. It is worthwhile noting that any homomorphic image of a nil-clean ring is also nil-clean.

In [9], *P. V. Danchev* and *W. Wm. McGovern* introduced a generalization of the nil-clean rings, and they called it the weakly-nil-clean rings, recall that a ring is said to be weakly-nil-clean if each of its elements can be written as a sum or difference of a nilpotent and an idempotent. They proved that the class of weakly-nil-clean rings is closed under homomorphic images. They proved, again, that if R is a ring and I is an ideal of R then if R is weakly-nil-clean then R/I so is, the converse holds when I is nil ideal of R, ie., there is a positive integer $n \geq 2$ such that $I^n = 0$.

In [7], *H. Chen* and *M. Sheibani* introduced a new generalization of the nil-clean rings, and called it 2-nil-clean ring. Recall that a ring R is called 2-nil-clean if each of its elements can be written as the sum of two idempotents and a nilpotent. It is clear that any homomorphic image of a 2-nil-clean ring is, again, 2-nil-clean. It was proved that a ring R is strongly 2-nil-clean if and only if for all $a \in R$, $a - a^3 \in R$ is nilpotent, if and only if for all $a \in R$, $a^2 \in R$ is nil-clean, if and only if every element in R is the sum of a tripotent and a nilpotent, recall that an element $e \in R$ is tripotent if $e^3 = e$. It is proved, again, that a ring R is nil-clean if and only if $2 \in R$ is nilpotent and R is 2-nil-clean.

The following diagram of implications summarizes the relations between the main three notions involved in this paper:



In this paper, we exhibit the connections between this ring and other related classes of rings. We examine how the 2-nil-clean property behave with respect to several ring-theoretic constructions such as polynomial ring, power series ring, amalgamated algebra and pullback. Our results produce new and original classes of 2-nil-clean rings subject to various ring theoretic properties.

2 Main Results

Our first proposition of this section establishes the relationship between the three notions of nil-clean, weakly-nil-clean and 2-nil-clean.

Proposition 2.1. 1. *Every nil-clean ring is weakly-nil-clean.*

2. *Every weakly-nil-clean ring is 2-nil-clean.*

Proof.
(1) Trivial.
(2) This follows from [7, Corollary 4.6]. □

Remark 2.2. Note that a 2-nil-clean need not be weakly-nil-clean. For instance, take $A := \mathbb{Z}_3$, the ring of integers modulo 3. One can easily check that $A \times A$ is 2-nil-clean which is not weakly-nil-clean.

In the next proposition, we show that a 2-nil-clean ring R with unit stable range 1 is a 4-good ring. Recall that a ring R is 4-good if each of its element can be written as a sum of 4 units.

Proposition 2.3. *Let R be a ring with unit stable range* 1*. If R is* 2*-nil-clean, then R is a* 4*-good ring.*

Proof. Assume that R is 2-nil-clean with unit stable range 1. Let $x \in R$. Then $x = e_1 + e_2 + w$ for some idempotent elements e_1, e_2 and a nilpotent element w of R. Using the fact that R has unit stable range 1, there exists $u \in U(R)$ such that $e_1 - 1.u \in U(R)$. Likewise, $e_2 - 1.u' \in U(R)$ for some $u' \in U(R)$. Therefore, $e_1 = u + v$ and $e_2 = u' + v'$ for some $v, v' \in U(R)$. Hence, $x = u + v + u' + v' + w$, making x, a 4-good element. Thus, R is a 4-good ring.
□

It is well known that if $f(X) = \sum_{i=0}^{\infty} a_i X^i$ is nilpotent then for all i, a_i is nilpotent. The next proposition shows that the polynomial and power series rings are never 2-nil-clean.

Proposition 2.4. *Let A be a commutative ring A. Then:*

1. *The ring of formal power series $A[[X]]$ is never 2-nil-clean.*

2. *The polynomial ring $A[X]$ is never 2-nil-clean.*

Proof. Assume by the way of contradiction that the element of $A[[X]]$(resp., A[X]) X is 2-nil-clean. Then $X = e + f + b$ (resp., $X = e' + f' + b'$) where $e, f \in Idem(A[[X]])$ (resp., $e', f' \in Idem(A[X])$) and $b \in Nilp(A[[X]])$ (resp., $b' \in Nilp(A[X])$). Note that $Idem(A[[X]]) = Idem(A[X]) = Idem(A)$ since A is commutative ring, so $-e - f + X \in Nilp(A[[X]])$ (resp.,$-e' - f' + X \in Nilp(A[X])$), which is a contradiction. Hence, $A[[X]]$(resp., $A[X]$) is never 2-nil-clean. □

The next proposition exhibits the relationship between the 2-nil-clean and 2-nil-good properties. Recall that a ring R is said to be 2-nil-good if each of its element is sum of two units and a nilpotent element of R.

Proposition 2.5. *Let R be a ring in which 2 is invertible. If R is 2-nil-clean, then R is 2-nil-good. The converse holds if R is UU(every unit is unipotent).*

Proof. Assume that R is 2-nil-clean. Let $a \in R$. Then $\frac{a+2}{2} = e_1 + e_2 + w$ for some $e_1, e_2 \in Idem(R)$ and $w \in Nilp(R)$. So, $a + 2 = 2e_1 + 2e_2 + 2w$ and therefore, $a = (2e_1 - 1) + (2e_2 - 1) + 2w$, with $(2e_1 - 1)^2 = 1$ and $(2e_2 - 1)^2 = 1$, making a, a 2-nil-good element. Hence, R is 2-nil-good. Conversely, assume that R is a UU 2-nil-good ring. Let $x \in R$. Then $x = u_1 + u_2 + w$ for some $u_1, u_2 \in U(R)$ and $w \in Nilp(R)$. Using the fact that R is UU, it follows that $x = 1 + b_1 + 1 + b_2 + w = 1 + 1 + W$, where $W = b_1 + b_2 + w \in Nilp(R)$. Hence, x is a 2-nil-good element, making R, a 2-nil-good ring.
□

Recall that a ring A is tripotent if for any element a of A, $a^3 = a$. Also, in view of [7, Theorem 3.6] we have a ring A is 2-nil-clean if and only if $A/Nilp(A)$ is tripotent. Next, we examine the 2-nil-clean property in reduced ring.

Proposition 2.6. *For a commutative ring A, the following assertions are equivalent:*

1. *A is tripotent.*
2. *A is 2-nil-clean and reduced ring.*

Proof.
(1) ⇒ (2) It is clear that any potent ring is 2-nil-clean (for all $x \in A$, we have $x^3 = x$ and we can write $x = x + 0$). Furthermore, every potent ring (that is, a ring R is potent if for any $r \in R$, there exist an integer $n > 1$ such that $r^n = r$) is reduced (see [15, Lemma 2.19 and Proposition 3.1]). In particular, for $n = 3$, any tripotent ring is reduced.
(2) ⇒ (1) It is known by [7, Theorem 3.6] that a ring A is 2-nil-clean if and only if $A/Nilp(A)$ is tripotent. Since A is reduced, A is tripotent.

□

3 2-nil-clean property in amalgamated algebra

The following result is to study the transfer of tripotent property to amalgamated algebras, recall that a ring R is tripotent if for every element $r \in R$, $r^3 = r$.

Proposition 3.1. *Let (A,B) be a pair of commutative rings, J be an ideal of B and $f : A \to B$ be a ring homomorphism. Then the following are equivalent.*
(1) $A \bowtie^f J$ is tripotent.
(2) A and $f(A)+J$ are tripotent.

Proof.
(1) ⇒ (2) In virtue of [11, Proposition 5.1], A and $f(A)+J$ are homomorphic images of $A \bowtie^f J$, the result follows, since any homomorphic image of a tripotent element is, again, tripotent.
(2) ⇒ (1) Assume that A and $f(A)+J$ are tripotent. Let $(x, f(x)+j) \in A \bowtie^f J$ such that $x \in A$, $j \in J$. Then x and $f(x)+j$ are tripotent. Then $x^3 = x$ and $(f(x)+j)^3 = f(x)+j$. Thus $(x, f(x)+j) = (x, f(x)+j)^3$ is tripotent. □

In particular, we have the following result:

Corollary 3.2. *Let A be a ring and I be an ideal of A. Then:*
$A \bowtie I$ is tripotent if and only if so is A.

Proof. In this case we have $f = id_A$ and $f(A)+I = A$. In view of Proposition 3.1, we have the desired result. □

Now we investigate the transfer of the 2-nil-clean property to the amalgamated algebras.

Theorem 3.3. Let (A,B) be a pair of rings, J be an ideal of B and $f : A \to B$ be a ring homomorphism. Then the following assertions hold:
(a) $A \bowtie^f J$ is 2-nil-clean if and only if A and $f(A)+J$ are 2-nil-clean.
(b) Assume that $J \subseteq Nilp(B)$. Then $A \bowtie^f J$ is 2-nil-clean if and only if so is A.
(c) If $J \cap Idem(B) = 0$, then the following assertions are equivalent:

1. $A \bowtie^f J$ is 2-nil-clean.

2. A is 2-nil-clean and $J \subseteq Nilp(B)$.

Proof.
$(a) \Rightarrow$) By [11, Proposition 5.1], A and $f(A)+J$ are homomorphic images of $A \bowtie^f J$.
$\Leftarrow$) Assume that A and $f(A)+J$ are 2-nil-clean. Let $(x, f(x)+j) \in A \bowtie^f J$ such that $x \in A$, $j \in J$. Then x and $f(x)+j$ are 2-nil-clean. From [7, Theorem 2.3], $x-x^3 \in Nilp(A)$ and $(f(x)+j)-(f(x)+j)^3 \in Nilp(B)$. Thus $(x, f(x)+j)-(x, f(x)+j)^3 = (x-x^3, f(x)+j-(f(x)+j)^3) \in A \bowtie^f J$ is nilpotent. Hence, $(x, f(x)+j)$ is 2-nil-clean by [7, Theorem 2.3].
(b) Assume that $J \subseteq Nilp(B)$. If $A \bowtie^f J$ is 2-nil-clean, then by assertion (a) above, so is A. Conversely, assume that A is 2-nil-clean. Let $(a, f(a)+j) \in A \bowtie^f J$ with $a \in A$ and $j \in J$. Then, there exist two idempotents $e_1, e_2 \in A$ and a nilpotent $w \in A$ such that $a = e_1 + e_2 + w$. So,

$$(a, f(a)+j) = (e_1, f(e_1)) + (e_2, f(e_2)) + (w, f(w)+j)$$

It is clear that $(e_1, f(e_1))$ and $(e_2, f(e_2))$ are idempotents in $A \bowtie^f J$. Furthermore, since $J \subseteq Nilp(B)$, one can easily check that $(w, f(w)+j)$ is a nilpotent element of $A \bowtie^f J$. Hence, $(a, f(a)+j)$ is 2-nil-clean.

(c) In virtue of Theorem 3.3(b), we have (2) implies (1). Furthermore, if $A \bowtie^f J$ is 2-nil-clean then so is A, as homomorphic image of $A \bowtie^f J$. It remains to show, only, $J \subseteq Nilp(B)$. Let $j \in J$. By [8, Lemma 2.10], we obtain:

$$(0, j) = (e_1, f(e_1)) + (e_2, f(e_2)) + (w, f(w)+k).$$

with $e_1, e_2 \in Idem(A)$, $w \in Nilp(A)$ and $k \in Nilp(B)$. It follows that $j \in Nilp(B)$. □

The following corollary is an immediate consequence of Theorem 3.3 on the transfer of the 2-nil-clean property to duplications.

Corollary 3.4. *Let A be a ring and I be an ideal of A. Then $A \bowtie I$ is 2-nil-clean if and only if so is A.*

Proof. In this case, we have $f = id_A$ and $f(A)+I = A$. In view of Theorem 3.3(a), the result follows. □

Example 3.5.
Let T be be a ring, I be an ideal of T and let D be a subring of T such that $I \cap D = (0)$. Then the ring $D+I$ is 2-nil-clean if and only if D so is.

Proof.
By [11, Example 2.6], $D+I$ is isomorphic to $D \bowtie^i I$ where $i : D \hookrightarrow T$ is the natural embedding. Thus, by Theorem 3.3(a), $D+I$ is 2-nil-clean if and only if D is 2-nil-clean. □

The next result is to study the transfer of 2-nil-clean property to the trivial extension.

Corollary 3.6. *Let A be a ring, E be an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then R is 2-nil-clean if and only if A is 2-nil-clean.*

Proof.
In this case $J = 0 \propto E \subseteq Nilp(R)$. Thus, in view of Theorem 3.3(b), we obtain the desired result. □

The following example shows the failure of assertion (b) of Theorem 3.3, beyond the context "$J \subseteq Nilp(B)$".

Example 3.7. Let A be a 2-nil-clean ring (for instance, $A := \mathbb{Z}_6$, the ring of integers modulo 6), $B = A[X]$, $J := XA[X]$ and $f : A \hookrightarrow B$ the natural injection. Then:
(1) $J \not\subseteq Nilp(B)$.
(2) A is 2-nil-clean.
(3) $A \bowtie^f J$ is not 2-nil-clean.

Proof.
(1) This follows from assumption.
(2) Trivial.
(3) Indeed, $f(A)+J = A+XA[X] = A[X]$ is not 2-nil-clean since $f(1)+X = 1+X \in f(A)+J$ is not 2-nil-clean. Therefore, in virtue of Theorem 3.3(1), $A \bowtie^f J$ is not 2-nil-clean. □

Theorem 3.3 allows us to construct new original class of 2-nil-clean which is not nil-clean.

Example 3.8. Let B be a 2-nil-clean ring (for instance take $B := \mathbb{Z}_3[X]/ < X^m(X-1)^m(X-2)^m >$), $A := B \propto E$ be the trivial ring extension of B by a B-module E, $f : A \to B$ be a surjective ring homomorphism and $J :=< \overline{X} >$ be an ideal of B. Then:

1. $A \bowtie^f J$ is 2-nil-clean.

2. $A \bowtie^f J$ is not nil-clean.

Proof. (1) From [20, Remark 2], it follows that B is 2-nil-clean. By Corollary 3.6, A is 2-nil-clean. Using assertion (a) of Theorem 3.3, $A \bowtie^f J$ is 2-nil-clean.
(2) One can easily check that $((\overline{X},0),\overline{X})^2 - ((\overline{X},0),\overline{X}) = ((\overline{X}^2 - \overline{X},0),\overline{X}^2 - \overline{X}) \notin Nilp(A \bowtie^f J)$ since $(\overline{X}^2 - \overline{X},0) \notin Nilp(A)$. Therefore, $A \bowtie^f J$ is not nil-clean.

□

Theorem 3.3 enriches the current literature with a new original class of 2-nil-clean rings which are not weakly nil-clean.

Example 3.9. Consider the homomorphism $h : \mathbb{Z}_6 \to \mathbb{Z}_3$ by $h(0) = h(3) = 0$, $h(1) = h(4) = 1$ and $h(2) = h(5) = 2$. Now, let $A := \mathbb{Z}_6 \times \mathbb{Z}_3$ be 2-nil-clean ring, $B := \mathbb{Z}_3 \times \mathbb{Z}_3$ be a ring and $f : A \to B$ be a ring homomorphism defined by $f(a,b) = (h(a),b)$ and $J := 0 \times \mathbb{Z}_3$ be an ideal of B. Then:
(1) $A \bowtie^f J$ is 2-nil-clean.
(2) $A \bowtie^f J$ is not weakly nil clean.

Proof. (1) Since A and $f(A)+J = B$ are 2-nil-clean, then by assertion (a) of Theorem 3.3, it follows that $A \bowtie^f J$ is 2-nil-clean.
(2) We claim that $A \bowtie^f J$ is not weakly nil-clean. Indeed, observe that $f(A)+J = \mathbb{Z}_3 \times \mathbb{Z}_3$ is not weakly nil clean. Since $f(A)+J$ is a homomorphic image of $A \bowtie^f J$ which is not weakly nil clean, it follows that $A \bowtie^f J$ is not weakly nil clean, as the class of weakly nil clean rings is closed under homomorphic images [9, Proposition 1.9]. □

4 Pullbacks

In this section, we examine the 2-nil-clean (resp. nil-clean) property into pullback constructions. It is worth to mention that many of our proofs of this section are straightforward. We are very grateful to [7] for their results. Recall that a pullback can be defined as follows:
Let T be a ring, I be a nonzero ideal of T, $\phi : T \to T/I$ be the natural surjection and D be a subring of T/I. Let R be the pullback of the following diagram:

$$\begin{array}{ccc} R = \phi^{-1}(D) & \xrightarrow{\phi_{/R}} & D = R/I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ T & \xrightarrow{\phi} & T/I \end{array} \qquad (\triangle)$$

where i and j are the natural injections. We assume that $R \subset T$ and refer to this diagram as a diagram of type ($\triangle$). If $I = M$ is a maximal ideal of T, we refer to this diagram as diagram of type ($\square$).

Proposition 4.1. *For the diagram of type* ($\triangle$)*:*
(1) *If R is 2-nil-clean (resp., nil-clean), then so is D.*
(2) *If T is 2-nil-clean (resp., nil-clean), then so is R.*
(3) *Assume that I is nil. Then R is 2-nil-clean (resp., nil-clean) if and only if so is D.*

Proof.
(1) Assume that R is 2-nil-clean. Then D is 2-nil-clean as a homomorphic image of R.
(2) If T is 2-nil-clean then so is R by [7, Corollary 2.4].
(3) Assume that I is nil. Then, in virtue of [7, Lemma 3.1], R is 2-nil-clean if and only if so is D.
The proof for nil-clean rings is similar. □

Remark 4.2. It is worthwhile noting that the converse of the assertion (1) of Proposition 4.1 does not hold in general. Indeed, let R be a non-2-nil-clean ring (For instance, take $R := A[[X]]$ where A is 2-nil-clean). It is easy to see that A is 2-nil-clean. However, $R := A[[X]]$ is never 2-nil-clean by Proposition 2.4. Also, note that in the assertion (3) of Proposition 4.1, the assumption "I is nil" is necessary for the equivalence. Indeed, Let $I = (X)$ be an ideal of $R := A[[X]]$, where A is 2-nil-clean. Observe that I is not nil. Consider the canonical surjection $\theta : A[[X]] \to A[[X]]/(X)$. Clearly, R is not 2-nil-clean. However, $\theta(R) = R/(X) = A[[X]]/(X) \simeq A$ is 2-nil-clean.

The following example illustrates Proposition 4.1 by providing a new original class of 2-nil-clean rings which are not nil-clean.

Example 4.3. Let $R = \mathbb{Z}_6$ be the ring of integers modulo 6. Consider the ring $T := \mathbb{Z}_6 \times \mathbb{Z}_3$. Consider the following pullback:

$$\begin{array}{ccc} R = \pi^{-1}(D) = \mathbb{Z}_6 & \xrightarrow{\pi/R} & D = \mathbb{Z}_3 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ T = \mathbb{Z}_6 \times \mathbb{Z}_3 & \xrightarrow{\pi} & \mathbb{Z}_3 \end{array}$$

One can easily check that T is 2-nil-clean which is not nil-clean. Hence, by Proposition 4.1(2), it follows that R is 2-nil-clean. Moreover, by assertion (1) of Proposition 4.1(1), R is not nil-clean, as D is not nil-clean.

Next, we examine the 2-nil-clean (resp. nil-clean) property in pullback when T is a ring, M is a maximal ideal of T, $\phi : T \to T/M$ is the natural surjection and D is a subring of T/M. Let R be the pullback of the following diagram:

$$\begin{array}{ccc} R = \phi^{-1}(D) & \xrightarrow{\phi_{/R}} & D = R/M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ T & \xrightarrow{\phi} & T/M \end{array} \qquad (\square)$$

where i and j are the natural injections. We assume that $R \subset T$ and we refer to this diagram as diagram of type ($\square$).

Now, we examine the 2-nil-clean (resp. nil-clean) property in pullback, where M is a maximal ideal of T and Q is a maximal ideal of R.

Theorem 4.4. For the diagram of type (□), with M (resp., Q) a maximal ideal of T (resp., R). Then the following assertions are equivalent:
(1) R is 2-nil-clean (resp., nil-clean) ring.
(2) $R \setminus Q$ is 2-nil-clean (resp., nil-clean) multiplicative set.
(3) $R \setminus M$ is 2-nil-clean (resp., nil-clean) multiplicative set.
Recall that $R \setminus Q$ (resp., $R \setminus M$) is 2-nil-clean if any element of $R \setminus Q$ (resp., $R \setminus M$) is 2-nil-clean.

Proof.
(1) $\Longrightarrow$ (2) By [7, Corollary 2.4], we have the desired result.
(2) $\Longrightarrow$ (3) Let $x \in R \setminus M$. If $x \in R \setminus Q$, then we are done. If $x \in Q$, then $1-x \notin Q$. Then $1-x = e+f+n$ for two idempotents e, f and a nilpotent n in $R \setminus Q$. Then $x = 1-e-f-n$ is 2-nil-clean by [7, Lemma 2.2] .
(3) $\Longrightarrow$ (1) Let $x \in R$. If $x \in R \setminus M$, then we are done. If $x \in M$, then $1-x \notin M$. Hence, with a similar argument as previously, it follows that x is 2-nil-clean.
The proof for nil-clean rings is similar. □

Recall that a ring R is UU if $U(R) = 1+Nilp(R)$. In the case R is a local ring, we obtain the following result:

Corollary 4.5. *Let R be a commutative local ring. Then the following assertions hold:*

1. *R is 2-nil-clean (resp., nil-clean) if and only if every unit of R is 2-nil-clean (resp., nil-clean).*
2. *R is nil-clean if and only if R is UU.*

Proof.
(1) This follows from Theorem 4.4, with Q is the unique maximal ideal of R.
(2) Assume that R is nil-clean and let $u \in U(R)$. Then $u = e+n$ where e (resp., n) is idempotent (resp., nilpotent) of R. Then $e = u-n \in Idem(R) \cap U(R) = \{1\}$. Hence $u = 1+n$. Conversely, assume that R is UU. Then in virtue of statement (1), it suffices to show that every unit is nil-clean. Indeed, since R is UU, then every unit $u \in R$ can be written as $u = 1+b$ for some nilpotent b of R which completes the proof of the corollary. □

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