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Evolution algebras satisfying a train identity of degree 2 and exponent $m > 3$

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Abstract. The present paper is devoted to the study of evolution algebras that satisfy the identity $x^m x^m = \omega(x)^m x^m$, where the integer $m > 3$. This polynomial identity is known as the train identity of degree 2 and exponent m. We determine the Peirce decomposition, the set of nonzero idempotents and characterise the derivations and automorphisms of the algebras of this class. In passing, we show that they strictly contain the class of finite-dimensional baric evolution algebras that are power-associative. Finally, we provide the classification of evolution algebras of dimension at most 5 that satisfy strictly the train identity of degree 2 and exponent m, for some integer $m > 3$.

Key Words: Bernstein algebra, Evolution algebra, Train identity of degree 2 and exponent m, Peirce decomposition, Derivation, Idempotent.
2010 MSC: Primary 17A30, 17D92 ; Secondary 17A60, 17D99.

1 Introduction

In 1939, Etherington established the first link between the concept of algebra and that of population genetics. Bernstein algebras [6] are an efficient algebraic tool for studying populations reaching a steady state after the first generation. However, the notion of Bernstein algebra does not appear explicitly in the work of S. Bernstein, nor that of Lyubich [11, 12]. They were introduced by Ph. Holgate in 1975 [9]. Since then, work on this class of non-associative algebras has expanded considerably. In [1], Abraham gives the mathematical formulation of a genetic population that reaches its equilibrum state at the $(k+1)$ th generation: Bernstein algebras of order k thus make their appearance in the literature.

Given a commutative field F, a commutative F-algebra A is a:

- *Jordan algebra* if $(xy)x^2 = x(yx^2)$, $\forall x \in A$;
- *baric algebra* if there is a non-zero morphism of algebras $\omega : A \longrightarrow F$.

Let (A, ω) be a baric F-algebra. The algebra A is a:

- *Bernstein algebra* of order $k \geq 1$, if $x^{[k+2]} = \omega(x)^{2^k} x^{[k+1]}$ for any $x \in A$;
- *train algebra* of rank $n \geq 2$ if there are scalars $\gamma_1, \ldots, \gamma_{n-1}$ such that $x^n + \gamma_1 \omega(x) x^{n-1} + \cdots + \gamma_{n-1} \omega(x)^{n-1} x = 0$, for all $x \in A$.

The plenary power $x^{[k]}$ and *principal power* x^k of $x \in A$, are defined by:

$$x^{[1]} = x^1 = x \text{ and } x^{[r+1]} = x^{[r]}x^{[r]};\ x^{r+1} = x^r x \text{ with } r \in \mathbb{N}^*,$$

and those of algebra A are given by:

$$A^{[1]} = A^1 = A \text{ and } A^{[r+1]} = (A^{[r]})^2;\ A^{r+1} = A^r A \text{ with } r \in \mathbb{N}^*.$$

Bernstein algebras appear as Bernstein algebras of order 1. In [18], the author shows that in Bernstein algebra, the identity

$$2x^i x^j = \omega(x)^i x^j + \omega(x)^j x^i \tag{1}$$

is hold for all integers $i, j \geq 2$ and the structure of this identity has been studied for $2 \leq i, j \leq 3$; for $i = 2, j = 4$ and for $i = j = 4$ in [19, 3, 5, 10, 18]. In [7, 16], the authors study respectively evolution algebras that are Bernstein algebras and evolution algebras satisfying the train identity of degree 2 and exponent 3, i.e. the identity

$$x^3 x^3 = \omega(x)^3 x^3. \tag{2}$$

They give the general form of the idempotents and a classification in dimension ≤ 4, of algebras of this class. They also show that evolution algebras satisfying the identity (2) are subclass of Bernstein algebras of order 2. Thus, in the present paper, we study evolution algebras that satisfy the identity

$$x^m x^m = \omega(x)^m x^m, \tag{3}$$

for some integer $m > 3$. This identity is known as *the train identity of degree* 2 *and exponent* m. An evolution algebra is not defined by an identity like the Bernstein algebras of order k. A F-algebra of dimension n is said to be an evolution algebra if it has a basis $B = \{e_1, \ldots, e_n\}$ such that

$$e_i e_j = 0 \text{ for } 1 \leq i \neq j \leq n \text{ and } e_i^2 = \sum_{k=1}^{n} a_{ik} e_k \tag{4}$$

Such a basis is called a natural basis of the evolution algebra A and the matrix $M = (a_{ik})_{1 \leq i,k \leq n}$ is the matrix of the structural constants of A relative to the natural basis B. Evolution algebras are commutative but they are not associative in general [17]. In Section 2, we give the Peirce decomposition of baric evolution algebra satisfying a train identity of degree 2 and exponent $m > 3$. The general form of nonzero idempotents, a characterisation of automorphisms groups and the connection with Bernstein algebras are given. We also show that the class of finite-dimensional baric evolution algebras that are power-associative is strictly included in the class of evolution algebras satisfying the identity (3) for some $m > 3$. Section 3 is devoted to the study of the classification in dimension ≤ 5, of the algebras strictly satisfying the train identity of degree 2 and exponent $m > 3$. In Section 4, we study the Lie algebra of derivations of finite-dimensional evolution algebras satisfying our identity and we show that this Lie algebra is abelian for an evolution algebra of dimension 5 strictly satisfying the identity (3) for some $m > 3$.

2 Structure of evolution algebras satisfying the identity (3)

In this paper, assume that F is a commutative field of $Char(F) \neq 2$. In [13, Proof of Corollary 3.4], the authors show that any baric evolution algebra of dimension n has a natural basis $B = \{e_1, \ldots, e_n\}$ whose multiplication table is given by:

$$e_1^2 = e_1 + \sum_{k=2}^{n} a_{1k} e_k \text{ and } e_j^2 = \sum_{k=2}^{n} a_{jk} e_k \text{ with } \omega(e_1) = 1 \text{ and } \omega(e_j) = 0 \text{ for all } 2 \leq j \leq n.$$

In the following, a baric evolution algebra of dimension n will be provided with such a natural basis.

2.1 About Peirce Decomposition

We give here the Peirce decomposition of evolution algebra satisfying the identity (3) for some $m > 3$.

Lemma 2.1. *Let (A,ω) be a baric evolution F-algebra of dimension n satisfying the identity (3) for some $m > 3$. For all $2 \leq j,k \leq n$, the following assertions are satisfied:*
(i) $e_1^2 e_1^2 = e_1^2$;
(ii) $2e_1^2(e_1^2 e_j) = e_1^2 e_j$;
(iii) $2e_1^2 e_j^3 = e_j^3$;
(iv) $(e_1^2 e_j)(e_1^2 e_k) = 0$;
(v) $2e_1^2((e_1^2 e_j)e_k) = (e_1^2 e_j)e_k$.

Proof. The equality $e_1^3 = e_1^2$ leads to $e_1^m = e_1^2$ for $m > 3$ and the identity (3) tells us that $e_1^2 = e_1^m \omega(e_1)^m = e_1^m e_1^m = e_1^2 e_1^2$: we obtain the assertion (i). Let $x = \alpha e_1 + y$ with $y \in \ker\omega$ and $\alpha \in F$. We have $x^2 = \alpha^2 e_1^2 + y^2$ and we show by induction that for any integer $t \geq 3$, $x^t = \alpha^t e_1^2 + \sum_{k=1}^{t-2} \alpha^{t-k} R_y^k(e_1^2) + y^t$ where R_y is the right multiplication of A determined by y. It follows that

$$x^m = \alpha^m e_1^2 + \sum_{k=1}^{m-2} \alpha^{m-k} R_y^k(e_1^2) + y^m$$

and

$$\begin{aligned} x^m x^m \quad = \quad & \alpha^{2m} e_1^2 e_1^2 + 2\alpha^m e_1^2 y^m + \sum_{k,j=1}^{m-2} \alpha^{2m-(k+j)} R_y^k(e_1^2) R_y^j(e_1^2) \\ & + 2\sum_{k=1}^{m-2} \alpha^{2m-k} e_1^2 R_y^k(e_1^2) + 2\sum_{k=1}^{m-2} \alpha^{m-k} y^m R_y^k(e_1^2) \end{aligned}$$

By identifying the coefficients of α^{2m-1}, α^{2m-2} and α^m in the equality $x^m x^m = \omega(x)^m x^m$, we obtain respectively identities (5), (6) and (7) below:

$$R_y(e_1^2) = 2e_1^2 R_y(e_1^2) \tag{5}$$

$$R_y^2(e_1^2) = (R_y(e_1^2))^2 + 2e_1^2 R_y^2(e_1^2) \tag{6}$$

$$y^m = 2e_1^2 y^m + \sum_{k=2}^{m-2} R_y^{m-k}(e_1^2) R_y^k(e_1^2) \tag{7}$$

For $2 \leq j \leq n$, replacing y by e_j in the identity (5), we obtain the assertion (ii) and replacing y by e_j in the identity (6), we get

$$(e_1^2 e_j)^2 = e_j(e_1^2 e_j) - 2e_1^2(e_j(e_1^2 e_j)) = a_{1j} a_{jj}(e_j^2 - 2e_1^2 e_j^2).$$

If $a_{1j} = 0$, then $(e_1^2 e_j)^2 = 0$. Otherwise, the assertion (ii) tells us that $2e_1^2 e_j^2 = e_j^2$ and it follows that $(e_1^2 e_j)^2 = 0$. We deduce that

$$(e_1^2 e_j)^2 = 0, \text{ for all } 2 \leq j \leq n. \tag{8}$$

Replacing y by e_j with $2 \leq j \leq n$ in the identity (7), we obtain

$$e_j^m = 2e_1^2 e_j^m + \sum_{k=2}^{m-2} R_{e_j}^{m-k}(e_1^2) R_{e_j}^{k}(e_1^2) = 2e_1^2 e_j^m + a_{jj}^{m-2} \sum_{k=2}^{m-2} (e_1^2 e_j)^2 = 2e_1^2 e_j^m.$$

Since $e_j^m = a_{jj}^{m-2} e_j^2$ and $m > 3$, the relation $e_j^m = 2e_1^2 e_j^m$ gives $a_{jj} e_j^2 = 2a_{jj} e_1^2 e_j^2$, i.e. $e_j^3 = 2e_1^2 e_j^3$. So, we have the assertion (iii).
For $2 \leq j \neq k \leq n$, by setting in the identity (6), $y = e_j + e_k$, we obtain

$$(e_1^2 e_j)e_k + (e_1^2 e_k)e_j = 2(e_1^2 e_k)(e_1^2 e_j) + 2e_1^2((e_1^2 e_j)e_k + (e_1^2 e_k)e_j) \tag{9}$$

It follows that

$$2(e_1^2 e_j)(e_1^2 e_k) = a_{1k} a_{kj} (e_j^2 - 2e_1^2 e_j^2) + a_{1j} a_{jk} (e_k^2 - 2e_1^2 e_k^2).$$

If $a_{1j} a_{1k} = 0$, then $(e_1^2 e_j)(e_1^2 e_k) = 0$. Otherwise, the assertion (ii) tells us that $0 = e_j^2 - 2e_1^2 e_j^2 = e_k^2 - 2e_1^2 e_k^2$ and it follows that

$$(e_1^2 e_j)(e_1^2 e_k) = 0, \text{ for all } 2 \leq j \neq k \leq n. \tag{10}$$

The assertion (iv) results from the identities (8) and (10). Since $(e_1^2 e_j)(e_1^2 e_k) = 0$, for $2 \leq j \neq k \leq n$, the identity (9), leads to

$$(e_1^2 e_j)e_k + (e_1^2 e_k)e_j = 2e_1^2((e_1^2 e_j)e_k + (e_1^2 e_k)e_j). \tag{11}$$

If $a_{1k} = 0$ then the identity (11) becomes $2e_1^2((e_1^2 e_j)e_k) = (e_1^2 e_j)e_k$. Otherwise, the assertion (ii) tells us that $2e_1^2 e_k^2 = e_k^2$ and we have $(e_1^2 e_j)e_k = a_{1j} a_{jk} e_k^2 = 2a_{1j} a_{jk} e_1^2 e_k^2 = 2e_1^2(a_{1j} a_{jk} e_k^2) = 2e_1^2((e_1^2 e_j)e_k)$. This gives us the assertion (v). □

An evolution F-algebra satisfying the five assertions of Lemma 2.1 does not necessarily satisfy a train identity of degree 2 and exponent 3. This is illustrated by the following example.

Example 2.2. Let (A, ω) be a five-dimensional evolution algebra with the natural basis $B = \{e_1, \ldots, e_5\}$ whose multiplication table is given by:

$$e_1^2 = e_1 + e_2 + e_3, \ e_2^2 = \frac{1}{4}(e_2 - e_3), \ e_3^2 = -e_2^2, e_4^2 = e_2 + e_3, e_5^2 = e_4.$$

Since $e_5^2 e_5^2 \neq 0$, this algebra is not a Bernstein algebra. Moreover, it does not satisfy the identity (2). Indeed, for $y = e_1 + e_4 + e_5$, we have $y^2 = e_1^2 + e_4^2 + e_5^3, y^3 = e_1^2 + e_4^2 = e_1 + 2(e_2 + e_3)$ and $y^3 y^3 = e_1^2 \neq \omega(y)^3 y^3$. It satisfies the assertions of Lemma 2.1 because it satisfies the train identity of degree 2 and exponent 4.

Definition 2.3. Let A be an evolution algebra of dimension n in the natural basis $B = \{e_1, \ldots, e_n\}$ and $x = \sum_{i=1}^n x_i e_i$ an element of A. The support of x relative to B, denoted $\text{Support}(x)$ is defined as the set $\text{Support}(x) := \{i \in \{1, \ldots, n\} \mid x_i \neq 0\}$.

Notation. Let (A, ω) be a finite-dimensional baric evolution algebra. Let $\Lambda_B := \text{Support}(e_1^2) \backslash \{1\} = \{j \in \{2, \ldots, n\} \mid a_{1j} \neq 0\}$ and denote its cardinal by $\text{Card}(\Lambda_B)$.

Theorem 2.4. Let (A, ω) be a baric evolution F-algebra of dimension n satisfying the identity (3) for some $m > 3$. Then, the algebra A admits the following Peirce decomposition:

$$A = Fe_1^2 \oplus A_0(e_1^2) \oplus A_{\frac{1}{2}}(e_1^2) \text{ where } A_\lambda(e_1^2) = \{x \in \ker(\omega) \mid e_1^2 x = \lambda x\} \text{ with } \lambda \in \left\{0, \frac{1}{2}\right\}.$$

Furthermore,

$$A_{\frac{1}{2}}(e_1^2) = \langle e_j^2 \mid j \in \Lambda_B \rangle \text{ and } A_0(e_1^2) = \langle e_j - 2e_1^2 e_j \mid 2 \leq j \leq n \rangle.$$

Proof. A partial linearisation of identity (3) gives us

$$2x^m\left(\sum_{k=0}^{m-3} R_x^k(yx^{m-1-k}) + 2R_x^{m-1}(y)\right) = \omega(x)^m\left(\sum_{k=0}^{m-3} R_x^k(yx^{m-1-k}) + 2R_x^{m-1}(y)\right) \tag{12}$$

for all $x \in A$ and $y \in \ker(\omega)$. By relpacing x by e_1 in the identity (12) and taking into account the equalities $e_1 \ker(\omega) = 0$ and $e_1^2 \ker(\omega) \subset \ker(\omega)$, we obtain

$$0 = 2e_1^2(e_1^2 y) - e_1^2 y = R_{e_1^2}(2R_{e_1^2} - Id_{\ker(\omega)})(y).$$

It follows that $P = X(2X-1)$ is the annihilating polynomial of $R_{e_1^2}|\ker(\omega) : \ker(\omega) \to \ker(\omega), x \mapsto e_1^2 x$. From kernel lemma we deduce that $\ker(\omega) = A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ with $A_{\frac{1}{2}}(e_1^2) = \ker(2R_{e_1^2} - Id_{\ker(\omega)}) = \{y \in \ker(\omega) \mid 2e_1^2 y = y\}$ and $A_0(e_1^2) = \ker(R_{e_1^2}) = \{y \in \ker(\omega) \mid e_1^2 y = 0\}$. Thus, we obtain the Peirce decomposition of the algebra A given by $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$. Let $x = x_2e_2 + \cdots + x_ne_n \in \ker(\omega)$; we have $e_1^2 x = x_2 e_1^2 e_2 + \cdots + x_n e_1^2 e_n = \sum_{k\in\Lambda_B} x_k e_1^2 e_k$.

- Assume that $x \in A_{\frac{1}{2}}(e_1^2)$. Then $x = 2e_1^2 x = \sum_{k\in\Lambda_B} 2x_k e_1^2 e_k \in \langle e_1^2 e_j \mid j \in \Lambda_B\rangle$. Reciprocally, the assertion (ii) of Lemma 2.1 tells us that $e_1^2 e_j \in A_{\frac{1}{2}}(e_1^2)$ for all $j \in \Lambda_B$. It follows that $A_{\frac{1}{2}}(e_1^2) = \langle e_1^2 e_j \mid a_{1j} \neq 0 \text{ for all } 2 \leq j \leq n\rangle = \langle e_j^2 \mid j \in \Lambda_B\rangle$.
- It is assumed that $x \in A_0(e_1^2)$. Then, $0 = e_1^2 x = x_2 e_1^2 e_2 + \cdots + x_n e_1^2 e_n$ and $x = x_2(e_2 - 2e_1^2 e_2) + \cdots + x_n(e_n - 2e_1^2 e_n) \in \langle e_j - 2e_1^2 e_j \mid 2 \leq j \leq n\rangle$. Conversely, according to the assertion (ii) of Lemma 2.1, we have $e_1^2(e_j - 2e_1^2 e_j) = 0$ for all $2 \leq j \leq n$, i.e. $e_j - 2e_1^2 e_j \in A_0(e_1^2)$ for all $2 \leq j \leq n$. We deduce that $A_0(e_1^2) = \langle e_j - 2e_1^2 e_j \mid 2 \leq j \leq n\rangle$.

□

Corollary 2.5. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of a finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$. Then,*

$$A_{\frac{1}{2}}(e_1^2)^2 = 0 \textit{ and } A_{\frac{1}{2}}(e_1^2)\ker(\omega) \subset A_{\frac{1}{2}}(e_1^2).$$

In particular $A_{\frac{1}{2}}(e_1^2)A_0(e_1^2) \subset A_{\frac{1}{2}}(e_1^2)$.

Proof. This corollary is a consequence of the assertions (iv) and (v) of Lemma 2.1 and Theorem 2.4. □

Proposition 2.6. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of a finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$. Then, the Peirce subspace $A_0(e_1^2)$ is nonzero.*

Proof. It is assumed that $A_0(e_1^2) = 0$ and let $x = \alpha e_1^2 + x_{\frac{1}{2}} \in A$. We have $x^2 = \alpha^2 e_1^2 + \alpha x_{\frac{1}{2}} = \alpha(\alpha e_1^2 + x_{\frac{1}{2}}) = \omega(x)x$. It follows that A is train algebra of rank 2. This is absurd by [13, Proposition 3.7] and the result follows. □

Lemma 2.7. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of a finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$. Then*

$$y^m \in A_{\frac{1}{2}}(e_1^2) \textit{ for all } y \in \ker(\omega).$$

Proof. Since $A_{\frac{1}{2}}(e_1^2)\ker(\omega) \subset A_{\frac{1}{2}}(e_1^2)$, the lemma follows from the identity (7). Indeed, for all $2 \leq k \leq m-2$ and $y \in \ker(\omega)$, we have $R_y^{m-k}(e_1^2), R_y^k(e_1^2) \in A_{\frac{1}{2}}(e_1^2)$ which leads to $R_y^{m-k}(e_1^2)R_y^k(e_1^2) = 0$, hence the desired result. □

Proposition 2.8. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of a finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$. If the Peirce subspace $A_{\frac{1}{2}}(e_1^2)$ is zero, then algebra A is a train algebra of rank at most $m+1$.*

Proof. Assume that $A_{\frac{1}{2}}(e_1^2) = 0$ and let $y \in \ker\omega = A_0(e_1^2)$. Lemma 2.7, tells us that $y^m = 0$. It follows that $\ker\omega$ is a nilalgebra of nilindex at most m. The result follows by [13, Theorem 3.8]. □

Theorem 2.9. [Characterisation] Let (A, ω) be a baric evolution F-algebra of dimension n. Then, the algebra A satisfies the identity (3) if, and only if, the following assertions hold:

(i) $e_1^2 e_1^2 = e_1^2$;
(ii) $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$;
(iii) $(A_{\frac{1}{2}}(e_1^2))^2 = 0$; $A_{\frac{1}{2}}(e_1^2)\ker(\omega) \subseteq A_{\frac{1}{2}}(e_1^2)$;
(iv) $y^m \in A_{\frac{1}{2}}(e_1^2)$ for all $y \in \ker(\omega)$.

Proof. It is assumed that the algebra A satisfies the identity (3). Then assertions (i) to (iv) follow from Lemma 2.1, Theorem 2.4, Corollary 2.5 and Lemma 2.7. Reciprocally, we assume that the assertions (i) to (iv) are satisfied. Let $x = \alpha e_1^2 + x_{\frac{1}{2}} + x_0 \in A$ with $x_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$ and $x_0 \in A_0(e_1^2)$. We have $x^2 = \alpha^2 e_1^2 + \alpha x_{\frac{1}{2}} + 2x_{\frac{1}{2}} x_0 + x_0^2$ and we show that for any integer $k \geq 3$, $x^k = \alpha^k e_1^2 + z_k + x_0^k$ where (z_j) is a sequence of elements in $A_{\frac{1}{2}}(e_1^2)$. It follows that $x^m = \alpha^m e_1^2 + z_m + x_0^m$ with $z_m \in A_{\frac{1}{2}}(e_1^2)$. Since $A_{\frac{1}{2}}(e_1^2)^2 = 0$ and $y^m \in A_{\frac{1}{2}}(e_1^2)$ for all $y \in \ker(\omega)$, we have $x^m x^m = \alpha^{2m} e_1^2 + \alpha^m(z_m + x_0^m) = \alpha^m(\alpha^m e_1^2 + z_m + x_0^m) = \omega(x)^m x^m$. Thus algebra A satisfies the identity (3) and the proposition is thus established. □

Corollary 2.10. *Let (A, ω) be a finite-dimensional baric evolution algebra satisfying the identity (3) for an integer $m_0 > 3$. Then, the algebra A satisfies the identity (3) for all integer $m_1 > m_0$.*

Proof. It is assumed that the algebra A satisfies the identity (3) for $m = m_0$ and let $y \in \ker\omega$. The relations $A_{\frac{1}{2}}(e_1^2)\ker(\omega) \subset A_{\frac{1}{2}}(e_1^2)$ and $y^{m_0} \in A_{\frac{1}{2}}(e_1^2)$ tell us that for all integer $m_1 > m_0$, $y^{m_1} \in A_{\frac{1}{2}}(e_1^2)$. We deduce from Theorem 2.9 that the algebra A satisfies the identity (3) for $m = m_1$. □

Proposition 2.11. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of finite-dimension baric evolution algebra of dimension n satisfying the identity (3) for some $m > 3$. If $\ker(\omega)^m \subset A_{\frac{1}{2}}(e_1^2)$, then the algebra A is a Bernstein algebra of order $m-1$.*

Proof. Let $x = \alpha e_1^2 + x_0 + x_{\frac{1}{2}} \in A$ with $\alpha \in F$, $x_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$ and $x_0 \in A_0(e_1^2)$. We show that for any integer $k \geq 2$, $x^{[k]} = \alpha^{2^{k-1}} e_1^2 + z_k + x_0^{[k]}$ where (z_j) is a sequence of elements in $A_{\frac{1}{2}}(e_1^2)$. We deduce that $x^{[m]} = \alpha^{2^{m-1}} e_1^2 + z_m + x_0^{[m]}$ with $x_0^{[m]} \in \ker(\omega)^{[m]} \subset \ker(\omega)^m \subset A_{\frac{1}{2}}(e_1^2)$. It follows that $x^{[m+1]} = (x^{[m]})^2 = \alpha^{2^m} e_1^2 + \alpha^{2^{m-1}}(z_m + x_0^{[m]}) = \alpha^{2^{m-1}}(\alpha^{2^{m-1}} e_1^2 + z_m + x_0^{[m]}) = \omega(x)^{2^{m-1}} x^{[m]}$. We deduce that the algebra A is a Bernstein of order $m-1$. □

2.2 About the idempotents

Proposition 2.12. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of a finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$. Then, the set $\mathcal{I}(A)$ of nonzero idempotents of A is given by:*

$$\mathcal{I}(A) = \{e_1^2 + u \mid u \in A_{\frac{1}{2}}(e_1^2)\}.$$

Furthermore, for any idempotent $f = e_1^2 + u$ with $u \in A_{\frac{1}{2}}(e_1^2)$, we have

$$A_{\frac{1}{2}}(f) = A_{\frac{1}{2}}(e_1^2) \text{ and } A_0(f) = \{x_0 - 2ux_0 \mid x_0 \in A_0(e_1^2)\}.$$

Proof. Let $f = e_1^2 + f_{\frac{1}{2}} + f_0$ with $f_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$ and $f_0 \in A_0(e_1^2)$, a nonzero idempotent of A. The equality $f = f^2 = e_1^2 + f_{\frac{1}{2}} + f_0^2 + 2f_{\frac{1}{2}} f_0$ gives

$$f_0 = f_0^2 + 2f_{\frac{1}{2}} f_0.$$

Multiplying this last equality by f_0, we get $f_0^2 = f_0^3 + 2R_{f_0}^2(f_{\frac{1}{2}})$ and we deduce that

$$f_0 = f_0^3 + 2\sum_{k=1}^{2} R_{f_0}^k(f_{\frac{1}{2}}).$$

We show by induction that for any $j \geq 2$, $f_0 = f_0^j + 2\sum_{k=1}^{j-1} R_{f_0}^k(f_{\frac{1}{2}})$. Thus,

$$f_0 = f_0^m + 2\sum_{k=1}^{m-1} R_{f_0}^k(f_{\frac{1}{2}}).$$

Since $1 \leq k \leq m-1$ and $f_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$, we have $R_{f_0}^k(f_{\frac{1}{2}}) \in A_{\frac{1}{2}}(e_1^2)$. Also $f_0^m \in A_{\frac{1}{2}}(e_1^2)$ according to Lemma 2.7. It follows that $f_0 \in A_{\frac{1}{2}}(e_1^2) \cap A_0(e_1^2) = \{0\}$ and $f = e_1^2 + f_{\frac{1}{2}} \in \mathcal{I}(A)$. Conversely, for any $f_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$, the vector $e_1^2 + f_{\frac{1}{2}}$ is a nonzero idempotent of the algebra A.

Let $f = e_1^2 + f_{\frac{1}{2}}$ be a nonzero idempotent of A and $x = x_{\frac{1}{2}} + x_0$ an element of $\ker(\omega)$ with $f_{\frac{1}{2}}, x_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$ and $x_0 \in A_0(e_1^2)$. We have $fx = \frac{1}{2}(x_{\frac{1}{2}} + 2f_{\frac{1}{2}} x_0)$

- Assume that $x \in A_{\frac{1}{2}}(f)$. The equality $fx = \frac{1}{2}x$ leads to $x_0 = 2f_{\frac{1}{2}} x_0 \in A_{\frac{1}{2}}(e_1^2)$. It follows that $x_0 = 0$ and $x = x_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$. Reciprocally, for any $y \in A_{\frac{1}{2}}(e_1^2)$, we have $fy = (e_1^2 + f_{\frac{1}{2}})y = \frac{1}{2}y$ and, consequently, $y \in A_{\frac{1}{2}}(f)$. Thus, $A_{\frac{1}{2}}(f) = A_{\frac{1}{2}}(e_1^2)$.
- Assume that $x \in A_0(f)$. The equality $fx = 0$ gives $x_{\frac{1}{2}} = -2f_{\frac{1}{2}} x_0$. So $x = -2f_{\frac{1}{2}} x_0 + x_0 \in \{y_0 - 2f_{\frac{1}{2}} y_0 \mid y_0 \in A_0(e_1^2).\}$ Conversely, for all $x_0 \in A_0(e_1^2)$, we have $f(-2f_{\frac{1}{2}} x_0 + x_0) = (e_1^2 + f_{\frac{1}{2}})(-2f_{\frac{1}{2}} x_0 + x_0) = -f_{\frac{1}{2}} x_0 + f_{\frac{1}{2}} x_0 = 0$. We deduce that $A_0(f) = \{y_0 - 2f_{\frac{1}{2}} y_0 \mid y_0 \in A_0(e_1^2)\}$.

□

In [16], the authors give a characterisation of the automorphisms of an evolution algebra satisfying the identity (2). However, in the relation (6) of their Theorem 5.6, the term "$- 2u_\sigma g_\sigma((x_0 y_0)_0)$" is too much, as shown by the proposition below:

Proposition 2.13. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of a finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$. A bijective linear map σ is an algebra automorphism of the algebra A if, and only if, there exists an element $u_\sigma \in A_{\frac{1}{2}}(e_1^2)$ and two morphisms $f_\sigma \in \mathrm{GL}_F(A_{\frac{1}{2}}(e_1^2))$ and $g_\sigma \in \mathrm{GL}_F(A_0(e_1^2))$ such that for all $x_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$ and $x_0 \in A_0(e_1^2)$ the following conditions hold:*

(i) $\sigma(e_1^2) = e_1^2 + u_\sigma$;
(ii) $\sigma(x_{\frac{1}{2}}) = f_\sigma(x_{\frac{1}{2}})$;
(iii) $\sigma(x_0) = g_\sigma(x_0) - 2u_\sigma g_\sigma(x_0)$;
(iv) $f_\sigma(x_{\frac{1}{2}} x_0) = f_\sigma(x_{\frac{1}{2}}) g_\sigma(x_0)$;
(v) $g_\sigma((x_0 y_0)_0) = (g_\sigma(x_0) g_\sigma(y_0))_0$;
(vi) $f_\sigma((x_0 y_0)_{\frac{1}{2}}) = (g_\sigma(x_0) g_\sigma(y_0))_{\frac{1}{2}} - 2g_\sigma(x_0)(u_\sigma g_\sigma(y_0)) - 2g_\sigma(y_0)(u_\sigma g_\sigma(x_0))$.

Proof. Assume that σ is an algebras automorphism. The assertions (i), (ii) and (iii) follow from Proposition 2.12. The relation $f_\sigma(x_0 x_{\frac{1}{2}}) = \sigma(x_0 x_{\frac{1}{2}}) = \sigma(x_0)\sigma(x_{\frac{1}{2}}) = f_\sigma(x_0) g_\sigma(x_{\frac{1}{2}})$ gives the assertion (iv). The assertions (v) and (vi) follow from the relation

$$\begin{aligned}
f_\sigma((x_0y_0)_{\frac{1}{2}}) + g_\sigma((x_0y_0)_0) &= \sigma(x_0y_0)\\
&= \sigma(x_0)\sigma(y_0)\\
&= (g_\sigma(x_0) - 2u_\sigma g_\sigma(x_0))(g_\sigma(y_0) - 2u_\sigma g_\sigma(y_0))\\
&= (g_\sigma(x_0)g_\sigma(y_0))_{\frac{1}{2}} + (g_\sigma(x_0)g_\sigma(y_0))_0 - 2(g_\sigma(x_0)(u_\sigma g_\sigma(y_0)) -\\
&\quad 2(g_\sigma(y_0)(u_\sigma g_\sigma(x_0)).
\end{aligned}$$

By identifying traces on $A_{\frac{1}{2}}(e_1^2)$ and on $A_0(e_1^2)$, we obtain the desired statements. Conversely, It is assumed that the assertions (i) to (vi) are satisfied and let $x = \alpha e_1^2 + x_{\frac{1}{2}} + x_0$, $y = \beta e_1^2 + y_{\frac{1}{2}} + y_0 \in A$ with $\alpha, \beta \in F$, $x_{\frac{1}{2}}, y_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2), x_0, y_0 \in A_0(e_1^2)$. We have

$$\begin{aligned}
xy &= \alpha\beta e_1^2 + \frac{1}{2}(\alpha y_{\frac{1}{2}} + \beta x_{\frac{1}{2}}) + x_{\frac{1}{2}}y_0 + x_0y_{\frac{1}{2}} + x_0y_0\\
\sigma(xy) &= \alpha\beta(e_1^2 + u_\sigma) + \frac{1}{2}(\alpha f_\sigma(y_{\frac{1}{2}}) + \beta f_\sigma(x_{\frac{1}{2}})) + f_\sigma(x_{\frac{1}{2}}y_0) + f_\sigma(x_0y_{\frac{1}{2}})\\
&\quad + f_\sigma((x_0y_0)_{\frac{1}{2}}) + g_\sigma((x_0y_0)_0)\\
&= \alpha\beta(e_1^2 + u_\sigma) + \frac{1}{2}(\alpha f_\sigma(y_{\frac{1}{2}}) + \beta f_\sigma(x_{\frac{1}{2}})) + f_\sigma(x_{\frac{1}{2}})g_\sigma(y_0) + f_\sigma(y_{\frac{1}{2}})g_\sigma(x_0)\\
&\quad + g_\sigma(x_0)g_\sigma(y_0) - 2g_\sigma(x_0)(u_\sigma g_\sigma(y_0)) - 2g_\sigma(y_0)(u_\sigma g_\sigma(x_0))\\
&= (\alpha(e_1^2 + u_\sigma) + f_\sigma(x_{\frac{1}{2}}) + g_\sigma(x_0) - 2u_\sigma g_\sigma(x_0))\\
&\quad (\beta(e_1^2 + u_\sigma) + f_\sigma(y_{\frac{1}{2}}) + g_\sigma(y_0) - 2u_\sigma g_\sigma(y_0)) = \sigma(x)\sigma(y).
\end{aligned}$$

We get the proposition. ☐

Proposition 2.12 and Proposition 2.13 show that the form of the idempotents and the characterisation of an algebras automorphisms of the algebra satisfying the identity (3) for some $m > 3$ do not depend on the integer m.

2.3 Link with Bernstein algebras

Let (A, ω) be a Bernstein algebra over a commutative field F of $Char(F) \neq 2$. In [18], the author shows that the algebra A admits nonzero idempotents and for any nonzero idempotent e, the algebra A admits the following Peirce decomposition $A = Ke \oplus U_e \oplus V_e$ where $U_e = \{x \in A \mid ex = \frac{1}{2}x\}$ and $V_e = \{x \in A \mid ex = 0\}$. Furthermore, the Peirce subspaces U_e and V_e satisfy the following relations:

$$U_eV_e \subseteq U_e,\ V_e^2 \subseteq U_e,\ U_e^2 \subseteq V_e \text{ and } U_eV_e^2 = 0.$$

Theorem 2.14. Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$. Then A is a Bernstein algebra if, and only if, $(A_0(e_1^2))^2 \subset A_{\frac{1}{2}}(e_1^2)$.

Proof. It is assumed that A is a Bernstein algebra. Then $(A_0(e_1^2))^2 \subset A_{\frac{1}{2}}(e_1^2)$. Reciprocally, if $(A_0(e_1^2))^2 \subset A_{\frac{1}{2}}(e_1^2)$ then, for any $x = \alpha e_1^2 + x_0 + x_{\frac{1}{2}} \in A$, we have $x^2 = \alpha^2 e_1^2 + \alpha x_{\frac{1}{2}} + x_0^2 + 2x_0x_{\frac{1}{2}}$ and $x^2x^2 = \alpha^4 e_1^2 + \alpha^3 x_{\frac{1}{2}} + \alpha^2(x_0^2 + 2x_0x_{\frac{1}{2}}) = \alpha^2 x^2$. We deduce that A is a Bernstein algebra. ☐

Proposition 2.15. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$ such that $(\ker\omega)^2 = A_{\frac{1}{2}}(e_1^2)$. Then, A is a Bernstein algebra.*

Proof. This proposition follows from Proposition 2.11. It is sufficient to take $m = 2$ in Proposition 2.11. ☐

The following results given for a finite-dimensional baric evolution algebra satisfying a train identity of degree 2 and exponent 3, remain true for a finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$. We give proofs based on Proposition 2.15.

Corollary 2.16. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of a finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$ such that $A_{\frac{1}{2}}(e_1^2) \neq 0$ and $\dim(\ker\omega)^2 = 1$. Then, A is a Bernstein algebra.*

Proof. We have $0 \neq A_{\frac{1}{2}}(e_1^2) \subset (\ker\omega)^2$. Since $\dim(\ker\omega)^2 = 1$, it follows that $A_{\frac{1}{2}}(e_1^2) = (\ker\omega)^2$. Proposition 2.15 tells us that A is a Bernstein algebra. ☐

Corollary 2.17. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of a finite-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$ such that $\mathrm{Card}(\mathrm{Support}(e_1^2)) = \dim(A)$. Then, A is a Bernstein algebra.*

Proof. The equality $\mathrm{Card}(\mathrm{Support}(e_1^2)) = \dim(A)$ leads to $(\ker\omega)^2 = A_{\frac{1}{2}}(e_1^2)$ because the annihilator of A is $\langle e_i \in B \mid e_i^2 = 0\rangle$ [8, Lemma 2.7]. Thus, we obtain the corollary. ☐

2.4 Link with power-associative algebras

Definition 2.18. A commutative F-algebra A is power-associative if for any $x \in A$, $x^i x^j = x^{i+j}$ for all $i, j \geq 1$ integers.

Theorem 2.19. [2] Let F be a commutative field of $Char(F) \neq 2, 3, 5$. A commutative F-algebra A is power-associative if, and only if, $x^2x^2 = x^4$ for all $x \in A$.

The proposition below shows that the class of power-associative baric evolution algebra is included in the class of the algebras satisfying the identity (3) for some $m > 3$.

Proposition 2.20. *Let (A, ω) be a finite-dimensional power-associative baric evolution algebra. Then, the algebra A satisfies the identity (3) with $m = 4$.*

Proof. It is assumed that A is a power-associative algebra. [13, Remark 3.20] tells us that the Peirce decomposition of the algebra A is given by the direct sum of algebras $A = Fe_1^2 \oplus A_0(e_1^2)$ where e_1^2 is the unique nonzero idempotent of the algebra A and $\ker(\omega) = A_0(e_1^2)$. Since $A_{\frac{1}{2}}(e_1^2) = 0$, Lemma 2.7 tells us that $A_0(e_1^2)$ is a nilalgebra. Furthermore, following [14, Theorem 5 and Corollary 2] we deduce that $A_0(e_1^2)$ is a nil power-associative evolution algebra of nilindex at most 4. It follows that for all $y \in \ker(\omega)$ we have $y^4 = 0 \in A_{\frac{1}{2}}(e_1^2)$. Theorem 2.9 shows that the algebra A satisfies the identity (3) for $m = 4$. ☐

The class of power-associative baric evolution algebras is strictly included in that of algebras satisfying the identity (3) for some $m > 3$. Indeed, the algebra defined in Example 2.2 is not a power-associative algebra because $e_5^2 e_5^2 \neq 0 = e_5^4$.

Corollary 2.21. *Let (A, ω) be a finite-dimensional power-associative baric evolution algebra. Then, the algebra A is Jordan algebra and train algebra with train equation $x^4 - \omega(x)x^3 = 0$.*

Proof. This result follows from Proposition 2.20 and from [4, Corollary 4.5 and Proposition 5.2]. ☐

3 Classification

In [16, 7], the authors give the classification of evolution algebra satisfying strictly the identities $x^2x^2 = \omega(x)^2x^2$ and $x^3x^3 = \omega(x)^3x^3$ in dimension ≤ 4. We continue this classification in dimension ≤ 5 for the evolution algebras verifying the identity (3) with $m > 3$.

Definition 3.1. We say that a baric evolution algebra (A, ω) satisfies *strictly the identity* (3) *for some* $m > 3$ if (3) holds for m but it does not hold for any integer strictly less than m.

Proposition 3.2. *There is no two-dimensional baric evolution algebra satisfying the identity* (3) *for some* $m > 3$.

Proof. Assume that A is 2-dimensional baric evolution algebra satisfying the identity (3) for some $m > 3$. Since $A_0(e_1^2) \neq 0$ and $\dim(\ker\omega) = 1$, we have $A_0(e_1^2) = \ker\omega = Ke_2$ and $A_{\frac{1}{2}}(e_1^2) = 0$. Let $e_2^2 = \alpha e_2$ with $\alpha \in F$. According to the assertion (iii) of Lemma 2.1, we have $e_2^3 = 2e_1^2e_2^3$, this leads to $0 = e_2^3 = \alpha^2 e_2$, i.e. $\alpha = 0$ and $e_2^2 = 0$. Let $x = ae_1^2 + x_2e_2 \in A$, we have $x^2 = a^2e_1^2$ and $x^2x^2 = \omega(x)^2x^2$. It follows that A is a Bernstein algebra and since $A_{\frac{1}{2}}(e_1^2) = 0$, it is also a train algebra. [18, Lemma 9.11] tells us that the algebra A is a train evolution algebra of rank 2: this is impossible. Thus, there is no evolution algebra of 2-dimensional satisfying the identity (3) for some $m > 3$. □

Lemma 3.3. *Let* (A, ω) *be a baric evolution algebra of dimension* $n \geq 3$ *satisfying the identity* (3) *for some* $m > 3$. *If* $\operatorname{Card}(\operatorname{Support}(e_1^2)) = 1$ *or* 2, *then* A *is a train algebra of rank at most* $m+1$.

Proof.
- If $\operatorname{Card}(\operatorname{Support}(e_1^2)) = 1$, then $e_1^2 = e_1$ and $A_{\frac{1}{2}}(e_1^2) = 0$. Proposition 2.8 tells us that (A, ω) is a train algebra of rank at most $m+1$.
- If $\operatorname{Card}(\operatorname{Support}(e_1^2)) = 2$, then without loss of generality, set $a_{12} \neq 0$ and $a_{1i} = 0$ for $3 \leq i \leq n$. Since $a_{12} \neq 0$, the equality $e_1^2 = e_1^2e_1^2 = e_1^2 + a_{12}^2e_2^2$ leads to $a_{12}^2e_2^2 = 0$, i.e. $e_2^2 = 0$ and we have $e_1^2e_2 = 0$. Furthermore, $e_1^2e_j = 0$ for $3 \leq j \leq n$. It follows that $A_{\frac{1}{2}}(e_1^2) = 0$ and (A, ω) is a train algebra of rank at most $m+1$.

□

Lemma 3.4. *Let* (A, ω) *be a baric evolution F-algebra of dimension* $n \geq 3$ *and satisfying the identity* (3) *for some* $m > 3$. *If* $\operatorname{Card}(\operatorname{Support}(e_1^2)) = n-1$, *then the algebra* A *satisfies the identity* (2).

Proof. Without loss of generality, we set $a_{1n} = 0$, $a_{1j} \neq 0$ for $2 \leq j \leq n-1$ and $x = x_1e_1 + \sum_{j=2}^{n-1} x_je_j + x_ne_n \in A$. We have $x^2 = x_1^2e_1^2 + \sum_{j=2}^{n-1} x_j^2e_j^2 + x_n^2e_n^2$ and $x^3 = x_1^3e_1^2 + z$ with $z = x_n^3e_n^3 + \sum_{j=2}^{n-1}(x_1^2x_je_1^2e_j + x_n^2x_je_n^2e_j + x_j^2x_ne_j^2e_n + \sum_{k=2}^{n-1} x_j^2x_ke_j^2e_k) \in A_{\frac{1}{2}}(e_1^2)$ because $e_n^3, e_1^2e_j, e_n^2e_j, e_j^2e_n, e_j^2e_k \in A_{\frac{1}{2}}(e_1^2)$ for $2 \leq j, k \leq n-1$. It follows that $x^3x^3 = x_1^6e_1^2 + x_1^3z = x_1^3(x_1^3e_1^2 + z) = \omega(x)^3x^3$. Thus, we obtain the lemma. □

Proposition 3.5. *There is no baric evolution algebra of dimension* 3 *or* 4 *satisfying strictly the identity* (3) *for* $m > 3$.

Proof.
- Suppose that $\dim(A) = 3$, then $\operatorname{Card}(\operatorname{Support}(e_1^2)) = 1, 2$ or 3. Thus, the algebra A is either a Bernstein algebra or a train algebra. In the particular case where $\operatorname{Card}(\operatorname{Support}(e_1^2)) = 2$, it verifies the identity (2) and it is a train algebra of rank at most 4.
- It is assumed that $\dim(A) = 4$, $\operatorname{Card}(\operatorname{Support}(e_1^2)) = 1, 2, 4-1$ or 4, then A is a Bernstein algebra, a train algebra or satisfies the identity (2).

□

Lemma 3.6. *Let* $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ *be a Peirce decomposition of baric evolution algebra of dimension* $n \geq 3$ *and satisfying the identity* (3) *for some* $m > 3$. *If there is a unique* $i_0 \in \{2, \ldots, n\}$ *such that* $e_{i_0}^2 \notin A_{\frac{1}{2}}(e_1^2)$, *then the algebra* A *verifies the identity* (2).

Proof. Without loss the generality, we set $i_0 = n$ and let $x = x_1e_1 + \sum_{j=2}^{n-1} x_je_j + x_ne_n \in A$. We have $x^2 = x_1^2e_1^2 + \sum_{j=2}^{n-1} x_j^2e_j^2 + x_n^2e_n^2$, $x^3 = x_1^3e_1^2 + z$ with $z = x_1^2x_ne_1^2e_n + x_n^3e_n^3 + \sum_{j=2}^{n-1}(x_1^2x_je_1^2e_j + x_n^2x_je_n^2e_j + x_j^2x_ne_j^2e_n) + \sum_{j,k=2}^{n-1} x_j^2x_ke_j^2e_k \in A_{\frac{1}{2}}(e_1^2)$ because for all $2 \leq j,k \leq n-1$, we have
$e_1^2e_j, e_1^2e_n, e_n^2e_j, e_j^2e_n, e_j^2e_k, e_n^3 \in A_{\frac{1}{2}}(e_1^2)$. It follows that $x^3x^3 = x_1^6e_1^2 + x_1^3z = \omega(x)^3x^3$ and the algebra A verifies the identity (2). □

Lemma 3.7. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of baric evolution algebra of dimension $n \geq 3$ satisfying the identity (3) for some $m > 3$. For all $i,j \in \{2,\ldots,n\}$ verifying $a_{ii} = a_{jj} = 0$, there is an integer $k_0 \geq 1$ such that $(e_i + e_j)^{2k_0} = (a_{ij}a_{ji})^{k_0-1}(e_i^2 + e_j^2) \in A_{\frac{1}{2}}(e_1^2)$. Moreover, if $a_{ij}a_{ji} \neq 0$, then $e_i^2, e_j^2 \in A_{\frac{1}{2}}(e_1^2)$.*

Proof. Let $i,j \in \{2,\ldots,n\}$, we have $(e_i + e_j)^2 = e_i^2 + e_j^2 = (a_{ij}a_{ji})^{1-1}(e_i^2 + e_j^2)$ and suppose that $(e_i + e_j)^{2k} = (a_{ij}a_{ji})^{k-1}(e_i^2 + e_j^2)$. Then, $(e_i + e_j)^{2(k+1)} = (a_{ij}a_{ji})^{k-1}((e_i^2 + e_j^2)(e_i + e_j))(e_i + e_j) = (a_{ij}a_{ji})^{k-1}((a_{ij}e_j^2 + a_{ji}e_i^2))(e_i + e_j) = (a_{ij}a_{ji})^k(e_i^2 + e_j^2)$. We denote by k_0, the smallest integer such that $2k_0 \geq m$. Since $(e_i + e_j)^m \in A_{\frac{1}{2}}(e_1^2)$, we have $(e_i + e_j)^{2k_0} = (a_{ij}a_{ji})^{k_0-1}(e_i^2 + e_j^2) \in A_{\frac{1}{2}}(e_1^2)$. Furthermore, for $a_{ij}a_{ji} \neq 0$, we will get $e_j^2 + e_i^2 \in A_{\frac{1}{2}}(e_1^2)$ and by multiplying the vector $e_j^2 + e_i^2$ by e_j and e_i respectively, we obtain $e_i^2e_j, e_j^2e_i \in A_{\frac{1}{2}}(e_1^2)$, i.e. $e_i^2, e_j^2 \in A_{\frac{1}{2}}(e_1^2)$. □

Proposition 3.8. *Let (A,ω) be a baric evolution algebra of dimension 5 satisfying strictly the identity (3) with $m > 3$. It is assumed that A is neither a Bernstein algebra nor a power-associative algebra. Then A is isomorphic to the algebra $A(\gamma,\gamma_1,\gamma_2,\gamma_3)$: $e^2 = e, eu = \frac{1}{2}u, uv_1 = \gamma u, v_2^2 = \gamma_1 u + v_1, v_3^2 = \gamma_2 u + \gamma_3 v_1 + v_2$ with $\gamma_1,\gamma_2,\gamma_3 \in K$ and $\gamma \in K^*$. The products that are not mentioned are void.*

Proof. It is assumed that A is an evolution algebra with the natural basis $B = \{e_1,\ldots,e_5\}$. If Card(Support(e_1^2)) $= 1, 2, 5-1, 5$ then A is a Bernstein algebra, a train algebra or verifies the identity (2). Thus, Card(Support(e_1^2)) $= 3$ and without loss of generality, we assume that $a_{12}a_{13} \neq 0$ and $a_{14} = a_{15} = 0$. We deduce that $e_2^2, e_3^2 \in A_{\frac{1}{2}}(e_1^2)$ and $e_4, e_5 \in A_0(e_1^2)$.

- The equality $e_1^2 = e_1^2e_1^2 = e_1^2 + a_{12}^2e_2^2 + a_{13}^2e_3^2$ leads to
$$e_3^2 = -a_{13}^{-2}a_{12}^2e_2^2.$$
We deduce that $A_{\frac{1}{2}}(e_1^2) = Fe_2^2$.
- The equality $e_2^2 = 2e_1^2e_2^2 = 2(a_{12}a_{22}e_2^2 + a_{13}a_{23}e_3^2) = 2a_{12}(a_{22} - a_{13}^{-1}a_{12}a_{23})e_2^2$ gives
$$a_{22} = \frac{1}{2}a_{12}^{-1} + a_{13}^{-1}a_{12}a_{23}.$$
- Lemma 3.6 and Proposition 2.15, tell us that $a_{24} = a_{25} = a_{44} = a_{55} = 0$, otherwise e_4^2 or e_5^2 would belong to $A_{\frac{1}{2}}(e_1^2)$ and the algebra A would be a Bernstein algebra or would satisfy the identity (2).
- The relation $0 = e_2^2e_2^2 = a_{22}^2e_2^2 + a_{23}^2e_3^2 = (\frac{1}{4}a_{12}^{-2} + a_{13}^{-1}a_{23})e_2^2$ leads to
$$a_{23} = -\frac{1}{4}a_{12}^{-2}a_{13} \text{ and } a_{22} = \frac{1}{4}a_{12}^{-1}.$$
- Lemma 3.7 tells us that $a_{45}a_{54} = 0$, otherwise A would be a Bernstein algebra. Thus, we distinguish three cases:
  (*i*) $a_{45} = a_{54} = 0$, then $e_4^2 = a_{42}e_2 + a_{43}e_3$, $e_5^2 = a_{52}e_2 + a_{53}e_3$ and we set $x = x_1e_1 + \sum_{j=2}^{5} x_je_j$. We have $x^2 = x_1^2e_1^2 + \sum_{j=2}^{5} x_j^2e_j^2$ and $x^3 = x_1^3e_1^2 + z$ with $z = \sum_{j=2}^{5}(x_1^2x_je_1^2e_j + \sum_{k=2}^{5} x_j^2x_ke_j^2e_k) \in A_{\frac{1}{2}}(e_1^2)$ because for all $2 \leq j,k \leq 5$, $e_1^2e_j, e_j^2e_k \in A_{\frac{1}{2}}(e_1^2)$. We deduce that $(x^3)^2 = x_1^6e_1^2 + x_1^3z = \omega(x)^3x^3$. It follows that the algebra A verifies the identity (2).

(*ii*) $a_{45} = 0$ and $a_{54} \neq 0$, then $e_4^2 = a_{42}e_2 + a_{43}e_3$ and $e_5^2 = a_{52}e_2 + a_{53}e_3 + a_{54}e_4$. Since the family (e_2^2, e_5^2) is linearly independent, Lemma 3.6 indicate that the family (e_2^2, e_4^2) is linearly independent otherwise the algebra A would satisfy the identity (2). It follows that $(a_{42} + a_{12}a_{13}^{-1}a_{43}) \neq 0$.

(*iii*) $a_{45} \neq 0$ and $a_{54} = 0$, then if we swap the vectors e_4 and e_5 of the natural basis, we return to the previous case.

From above, we deduce that the multiplication table of (A, ω) in the natural basis $B = (e_1, e_2, e_3, e_4', e_5)$, where $e_4' = a_{54}e_4$, is the following:

$$\begin{cases} e_1^2 &= e_1 + a_{12}e_2 + a_{13}e_3, \\ e_2^2 &= \frac{1}{4}a_{12}^{-2}(a_{12}e_2 - a_{13}e_3), \\ e_3^2 &= -a_{13}^{-2}a_{12}^2e_2^2, \\ e_4'^2 &= a_{42}'e_2 + a_{43}'e_3, \\ e_5^2 &= a_{52}e_2 + a_{53}e_3 + e_4', \end{cases} \qquad \text{with } a_{42}' + a_{12}a_{13}^{-1}a_{43}' \neq 0.$$

We have $(\ker\omega)^2 = \langle e_2^2, e_4'^2, e_5^2\rangle$, $(\ker\omega)^3 = \langle e_2^2, e_4'^2\rangle$ and $(\ker\omega)^4 = \langle e_2^2\rangle = A_{\frac{1}{2}}(e_1^2)$. Set $e = e_1^2$, $u = 2a_{12}e_2^2$, $v_1 = e_2 - 2e_1^2e_2$, $v_2 = e_4'$ and $v_3 = e_5$. Solving the system

$$\begin{cases} u = \frac{1}{2}e_2 - \frac{1}{2}a_{12}^{-1}a_{13}e_3, \\ v_1 = \frac{1}{2}e_2 + \frac{1}{2}a_{12}^{-1}a_{13}e_3, \end{cases}$$

we get $e_2 = u + v_1$ and $e_3 = -a_{12}a_{13}^{-1}(u - v_1)$. Thus, the multiplication table of the algebra A in the basis $\{e, u, v_1, v_2, v_3\}$ is given by: $e^2 = e$, $eu = \frac{1}{2}u, uv_1 = \frac{1}{4}a_{12}^{-1}u, v_2^2 = (a_{42}' - a_{12}a_{13}^{-1}a_{43}')u + (a_{42}' + a_{13}^{-1}a_{12}a_{43}')v_1 = \delta_1 u + \delta_2 v_1$ with $\delta_2 \neq 0, v_3^2 = (a_{52} - a_{12}a_{13}^{-1}a_{53})u + (a_{52} + a_{13}^{-1}a_{12}a_{53})v_1 + v_2 = \delta_3 u + \delta_4' v_1 + v_2$. Furthermore, the equality $(\ker\omega)^4 = A_{\frac{1}{2}}(e_1^2)$ and Theorem 2.9 tell us that the algebra A verifies the identity (3) with $m = 4$. By setting $v_1' = \delta_2 v_1$, we get the desired algebra. □

Proposition 3.9. *Assume that F is an algebraically closed field and* (A, ω) *is the algebra defined in Proposition 3.8. Then, the automorphism group of the algebra A is given by Table 1 below:*

Table 1

Conditions on structural constants	$Aut_K(A)$	*Conditions on the matrix coefficients*
$\gamma_1 = \gamma_2 = \gamma_3 = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_{33}^2 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix}$	$\varepsilon_{33}^4 = 1$ *and* $\beta \neq 0$
$\gamma_1 = \gamma_2 = 0$ *and* $\gamma_3 \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix}$	$\varepsilon_{33}^2 = 1$ *and* $\beta \neq 0$
$\gamma_1 = \gamma_3 = 0$ *and* $\gamma_2 \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{33}^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_{33}^2 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix}$	$\varepsilon_{33}^4 = 1$
$\gamma_1 \neq 0$ *and* $\gamma_2 = \gamma_3 = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_{33}^2 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix}$	$\varepsilon_{33}^4 = 1$

Table 1

Conditions on structural constants	$Aut_K(A)$	*Conditions on the matrix coefficients*
$(\gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3) \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix}$	$\varepsilon_{33}^2 = 1$

Proof. Let $\sigma \in Aut_K(A)$. The assertions (i) to (iii) of Proposition 2.13 tell us that $\sigma(e_1^2) = e_1^2 + \alpha u$, $f_\sigma(u) = \beta u$ and we set $g_\sigma(v_i) = \varepsilon_{i1}v_1 + \varepsilon_{i2}v_2 + \varepsilon_{i3}v_3$ with $1 \leq i \leq 3$.

(1) The assertion (iv) of Proposition 2.13 leads to:
  - $\gamma f_\sigma(u) = f_\sigma(uv_1) = f_\sigma(u)g_\sigma(v_1) = \beta\varepsilon_{11}uv_1$, i.e. $\gamma\beta(\varepsilon_{11} - 1)u = 0$. Since $\gamma\beta \neq 0$, we have $\varepsilon_{11} = 1$.
  - For $i = 2$ or 3, we have $0 = f_\sigma(uv_i) = f_\sigma(u)g_\sigma(v_i) = \gamma\beta\varepsilon_{i1}u$, i.e. $\varepsilon_{21} = \varepsilon_{31} = 0$.

(2) The assertion (v) of Proposition 2.13 gives:
  - $0 = g_\sigma(v_1^2) = ((v_1 + \varepsilon_{12}v_2 + \varepsilon_{13}v_3)^2)_0 = (\varepsilon_{12}^2 + \varepsilon_{13}^2\gamma_3)v_1 + \varepsilon_{13}^2 v_2$. It follows that $\varepsilon_{12} = \varepsilon_{13} = 0$ and $g_\sigma(v_1) = v_1$.
  - $v_1 = g_\sigma(v_1) = g_\sigma((v_2^2)_0) = ((g_\sigma(v_2))^2)_0 = (\varepsilon_{22}^2 + \varepsilon_{23}^2\gamma_3)v_1 + \varepsilon_{23}^2 v_2$. We deduce that $\varepsilon_{23} = 0$, $\varepsilon_{22}^2 = 1$ and $g_\sigma(v_2) = \varepsilon_{22}v_2$.
  - $0 = g_\sigma(v_2v_3) = (g_\sigma(v_2)g_\sigma(v_3))_0 = \varepsilon_{32}\varepsilon_{22}v_1$, i.e. $\varepsilon_{32} = 0$. So $g_\sigma(v_3) = \varepsilon_{33}v_3$.
  - $\gamma_3 v_1 + \varepsilon_{22}v_2 = g_\sigma((v_3^2)_0) = ((g_\sigma(v_3))^2)_0 = \varepsilon_{33}^2\gamma_3 v_1 + \varepsilon_{33}^2 v_2$, i.e. $\varepsilon_{33}^2 = \varepsilon_{22}$ and $\gamma_3(\varepsilon_{33}^2 - 1) = 0$.

(3) The assertion (vi) of Proposition 2.13 gives:
  - $0 = f_\sigma(v_1^2) = -4g_\sigma(v_1)(\alpha u g_\sigma(v_1)) = -4v_1(\alpha u v_1) = -4\alpha\gamma^2 u$, i.e. $\alpha = 0$ and $\sigma(e_1^2) = e_1^2$.
  - $\gamma_1\beta u = f_\sigma((v_2^2)_{\frac{1}{2}}) = ((g_\sigma(v_2))^2)_{\frac{1}{2}} = \varepsilon_{22}^2\gamma_1 u$, i.e. $\gamma_1(\beta - 1) = 0$.
  - $\gamma_2\beta u = f_\sigma((v_3^2)_{\frac{1}{2}}) = ((g_\sigma(v_3))^2)_{\frac{1}{2}} = \varepsilon_{33}^2\gamma_2 u$ i.e. $\gamma_2(\beta - \varepsilon_{33}^2) = 0$.

From the above, we deduce that the matrix of σ in the basis $(e_1^2, u, v_1, v_2, v_3)$ is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_{33}^2 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix}$$

with

$$\beta \neq 0,\ \varepsilon_{33}^4 = 1,\ \gamma_1(\beta - 1) = 0,\ \gamma_2(\beta - \varepsilon_{33}^2) = 0 \text{ and } \gamma_3(\varepsilon_{33}^2 - 1) = 0. \tag{13}$$

Taking into account the identities in the relation (13), we obtain automorphism group of the algebra A given by Table 1. ☐

From Proposition 3.8 and Proposition 3.9, we deduce the corollary below:

Corollary 3.10. *Let F be an algebraically closed field and (A, ω) be a five-dimensional baric evolution F-algebra satisfying strictly the identity (3). It is assumed that A is neither a Bernstein algebra nor a power-associative algebra. Then, the algebra A is isomorphic to one of the following pairwise non-isomorphic algebras (where $\gamma \neq 0$ and the products that are not mentioned are void):*

- $A_1(\gamma)$: $e^2 = e$; $eu = \frac{1}{2}u$; $uv_1 = \gamma u$; $v_2^2 = v_1$; $v_3^2 = v_2$.
- $A_2(\gamma, \gamma_3)$: $e^2 = e$; $eu = \frac{1}{2}u$; $uv_1 = \gamma u$; $v_2^2 = v_1$; $v_3^2 = \gamma_3 v_1 + v_2$ *with* $\gamma_3 \neq 0$.
- $A_3(\gamma, \gamma_2)$: $e^2 = e$; $eu = \frac{1}{2}u$; $uv_1 = \gamma u$; $v_2^2 = v_1$; $v_3^2 = \gamma_2 u + v_2$ *with* $\gamma_2 \neq 0$.
- $A_4(\gamma, \gamma_1)$: $e^2 = e$; $eu = \frac{1}{2}u$; $uv_1 = \gamma u$; $v_2^2 = \gamma_1 u + v_1$; $v_3^2 = v_2$ *with* $\gamma_1 \neq 0$.
- $A_5(\gamma, \gamma_1, \gamma_2, \gamma_3)$: $e^2 = e$; $eu = \frac{1}{2}u$; $uv_1 = \gamma u$; $v_2^2 = \gamma_1 u + v_1$; $v_3^2 = \gamma_2 u + \gamma_3 v_1 + v_2$ *with* $(\gamma_1\gamma_2, \gamma_1\gamma_3, \gamma_2\gamma_3) \neq 0$.

Proposition 3.11. *Suppose that $F = \mathbb{R}$ and (A, ω) is the algebra defined in Proposition 3.8. Then, the automorphism group of the algebra A is given by Table 2 below:*

Table 2

Conditions on structural constants	$Aut_K(A)$	*Conditions on the matrix coefficients*
$\gamma_1 = \gamma_2 = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix}$	$\varepsilon_{33}^2 = 1$ *and* $\beta \neq 0$
$(\gamma_1, \gamma_2) \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_{33} \end{pmatrix}$	$\varepsilon_{33}^2 = 1$

Proof. This proposition is an immediate consequence of the previous proposition because the equation $\varepsilon_{33}^4 = 1$ is equivalent to $\varepsilon_{33}^2 = 1$ on $\mathbb{R}$. □

From Proposition 3.8 and Proposition 3.11, we deduce the corollary below:

Corollary 3.12. *Let (A, ω) be a five-dimensional baric evolution $\mathbb{R}$-algebra satisfying strictly the identity (3). It is assumed that A is neither a Bernstein algebra nor a power-associative algebra. Then, the algebra A is isomorphic to one of the following pairwise non-isomorphic algebras (where $\gamma \neq 0$; $\gamma_3 \in K$ and the products that are not mentioned are void):*

- $A_1(\gamma, \gamma_3)$: $e^2 = e$; $eu = \frac{1}{2}u$; $uv_1 = \gamma u$; $v_2^2 = v_1$; $v_3^2 = \gamma_3 v_1 + v_2$.
- $A_2(\gamma, \gamma_1, \gamma_2, \gamma_3)$: $e^2 = e$; $eu = \frac{1}{2}u$; $uv_1 = \gamma u$; $v_2^2 = \gamma_1 u + v_1$; $v_3^2 = \gamma_2 u + \gamma_3 v_1 + v_2$ *with* $(\gamma_1, \gamma_2) \neq 0$.

4 Derivations

Let F be a commutative field and A be an F-algebra. A *derivation* d of A is a linear operator on A satisfying the following condition:

$$d(xy) = d(x)y + xd(y) \text{ for all } x, y \in A.$$

Since, the commutator $[d, d'] = d \circ d' - d' \circ d$ of two derivations is still a derivation, the set $Der_K(A)$ of all derivations of A is a subalgebra of $(End_K(A), [\ ,\ ])$. Thus, $Der_K(A)$ is a Lie algebra, called the *Lie algebra of derivations of A* [15].

In [16, Proposition 5.2], the authors give a characterisation of the derivation of a baric evolution algebra satisfying the identity (2). The following proposition shows that this characterisation does not depend on the integer m.

Proposition 4.1. *Let $A = Fe_1^2 \oplus A_{\frac{1}{2}}(e_1^2) \oplus A_0(e_1^2)$ be a Peirce decomposition of evolution algebra of dimension n satisfying the identity (3) for some $m > 3$. The linear operator $d \in End_K(A)$ is a derivation of the algebra A if, and only if, the following assertions are satisfied:*

(i) $d(e_1^2) = 0$;

(ii) *the subspaces $A_{\frac{1}{2}}(e_1^2)$ and $A_0(e_1^2)$ are invariants with respect to d. Let $d_{\frac{1}{2}}$ and d_0 be the endomorphisms induced by d on $A_{\frac{1}{2}}(e_1^2)$ and $A_0(e_1^2)$ respectively;*

(iii) $d_{\frac{1}{2}}(x_{\frac{1}{2}} x_0) = d_{\frac{1}{2}}(x_{\frac{1}{2}})x_0 + x_{\frac{1}{2}} d_0(x_0)$ *for all* $x_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$ *and* $x_0 \in A_0(e_1^2)$;

(iv) $d_{\frac{1}{2}}((x_0 y_0)_{\frac{1}{2}}) = (d_0(x_0)y_0 + x_0 d_0(y_0))_{\frac{1}{2}}$ *for all* $x_0, y_0 \in A_0(e_1^2)$;

(*v*) $d_0((x_0y_0)_0) = (d_0(x_0)y_0 + x_0d_0(y_0))_0$ *for all* $x_0, y_0 \in A_0(e_1^2)$.
Furthermore, if d *is a derivation of* A*, then* $d_{\frac{1}{2}}$ *is a derivation of* $A_{\frac{1}{2}}(e_1^2)$.

Proof. Assume that d is a derivation of the algebra A and let $d(e_i) = \sum_{k=1}^{n} d_{ik}e_k$. For $e_j \in ann(A)$, we have $e_j^2 = 0$ and the relation $0 = d(e_1e_j) = d(e_1)e_j + e_1d(e_j) = d_{j1}e_1^2$, leads to $d_{j1} = 0$. For $e_j \notin ann(A)$ with $2 \le j \le n$, the family (e_1^2, e_j^2) is linearly independent and we have $d_{j1} = d_{1j} = 0$. We deduce that $d(e_1) = d_{11}e_1 + z$ with $z \in ann(A)$ and $d(e_j) \in \ker(\omega)$ for all $2 \le j \le n$.

- We have $d(e_1^2) = 2e_1d(e_1) = 2d_{11}e_1^2$ and $d(e_1^2e_1^2) = 2e_1^2(d(e_1^2)) = 4d_{11}e_1^2e_1^2 = 4d_{11}e_1^2$. The equality $d(e_1^2) = d(e_1^2e_1^2)$ gives $d_{11} = 0$ and it follows that $d(e_1^2) = 0$. We obtain the assertion (i).
- For $x_{\frac{1}{2}} \in A_{\frac{1}{2}}(e_1^2)$ and $x_0 \in A_0(e_1^2)$, the equalities $\frac{1}{2}d(x_{\frac{1}{2}}) = d(e_1^2x_{\frac{1}{2}}) = e_1^2d(x_{\frac{1}{2}})$ and $0 = d(e_1^2x_0) = e_1^2d(x_0)$ tell us that the subspaces $A_{\frac{1}{2}}(e_1^2)$ and $A_0(e_1^2)$ are invariants with respect to d. Thus, we set $d_{\frac{1}{2}}$ and d_0 the induced endomorphisms by d on $A_{\frac{1}{2}}(e_1^2)$ and on $A_0(e_1^2)$ respectively. By the same reasoning as in the proof of [16, Corollary 5.4], we deduce that the endomorphism $d_{\frac{1}{2}}$ is a derivation of the algebra A. Thus, we get the assertion (ii).
- The assertions (iii) to (v) follow from a direct calculation of $d(x_{\frac{1}{2}}x_0)$ and $d(x_0y_0)$ and identifying the traces on $A_{\frac{1}{2}}(e_1^2)$ and $A_0(e_1^2)$.

The proof of the reciprocal is identical to [16, Proof of Proposition 5.3]. Indeed, the Peirce decomposition and the product between the Peirce subspaces of the evolution algebra satisfying the identity (3) for some $m > 3$ and one verifying the identity (2) are identical. ☐

Proposition 4.2. *Let* (A, ω) *be the algebra defined in Proposition 3.8. Then, the Lie algebra of derivations of* A *is the abelian algebra.*

Proof. Let d be a derivation of A. Set $d_{\frac{1}{2}}(u) = \alpha u$ and $d_0(v_i) = \sum_{k=1}^{3} d_{ik}v_k$ for $1 \le i \le 3$.

- We have $\gamma\alpha u = d_{\frac{1}{2}}(uv_1) = d_{\frac{1}{2}}(u)v_1 + ud_0(v_1) = \gamma\alpha u + d_{11}\gamma u$, i.e. $d_{11} = 0$. Also, for $j = 2$ or 3, we have $0 = d_{\frac{1}{2}}(uv_j) = d_{\frac{1}{2}}(u)v_j + ud_0(v_j) = d_{j1}\gamma u$, i.e. $d_{j1} = 0$.
- For $j = 2$ or 3, we have $0 = d_0((v_1v_j)_0) = (d_0(v_1)v_j + v_1d_0(v_j))_0 = (d_{1j}v_j^2)_0$. Thus, for $j = 2$, we have $0 = d_{12}v_1$, i.e. $d_{12} = 0$ and for $j = 3$, we have $0 = d_{13}(\gamma_3v_1 + v_2)$, i.e. $d_{13} = 0$. We deduce that $d_0(v_1) = 0$.
- We have $0 = d_0(v_1) = d_0((v_2^2)_0) = 2(v_2d_0(v_2))_0 = 2d_{22}v_1$, i.e. $d_{22} = 0$ and $d_{23}v_3 = d_0(v_2) = d_0((v_3^2)_0) = 2(v_3d_0(v_3))_0 = 2d_{33}(\gamma_3v_1 + v_2)$, i.e. $d_{23} = d_{33} = 0$. It follows that $d_0(v_2) = 0$ and $d_0(v_3) = d_{32}v_2$. Also, we have $0 = d_0((v_2v_3)_0) = (v_2d_0(v_3))_0 = d_{32}v_2^2$, i.e. $d_{32} = 0$ and $d_0(v_3) = 0$.
- We have $\gamma_1\alpha u = d_{\frac{1}{2}}((v_2^2)_{\frac{1}{2}}) = 2(v_2d_0(v_2)) = 0$, i.e. $\gamma_1\alpha = 0$ and $\gamma_2\alpha u = d_{\frac{1}{2}}((v_3^2)_{\frac{1}{2}}) = 2(v_3d_0(v_3)) = 0$, i.e. $\gamma_2\alpha = 0$.

From above, we deduce that the matrix of the derivation d in the basis $\{e_1^2, u, v_1, v_2, v_3\}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Particulary, when $\gamma_1 \ne 0$ or $\gamma_2 \ne 0$ the only derivation is the zero one. Moreover, for $d, d' \in Der_K(A)$, we have $[d, d'] = 0$ and we get the proposition. ☐

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