Moroccan Journal of Algebra
and Geometry with Applications
Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco

Volume 4, Issue 2 (2025), pp 268-274

ISSN: 2820-7114

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Vol. 4(2) (2025), 268–274

Supported by Sidi Mohamed Ben Abdellah University, Fez, Morocco **ISSN: 2820-7114**

On the lattice of basic z°-ideals

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Communicated by Najib Mahdou
(Received 02 February 2025, Revised 19 May 2025, Accepted 21 May 2025)

Abstract. It has been shown that for a reduced f-ring R with bounded inversion property, the lattice of basic z°-ideals is a complemented lattice if and only if the space of minimal prime ideals, Min(R), is compact and R satisfies the annihilator condition. Examples are provided to illustrate and delineate these results.

Key Words: Commutative ring, z-ideal, ring of continuous functions, reduced ring.
2020 MSC: 13A15, 16G30, 54C40

1 Introduction

In this paper, R is assumed to be an f-ring with *bounded inversion property*. In [12], it was prove that for such a ring, the set

$$BZ(R) = \{M_f : f \in R\},$$

partially ordred by inclusion forms a distributive lattice with operations:

$$M_a \vee M_b = M_{a^2+b^2} \quad \text{and} \quad M_a \wedge M_b = M_{ab}.$$

Moreover, it was also shown that the set

$$BZ^\circ(R) = \{P_f : f \in R\},$$

partially ordred by inclusion, forms a distributive lattice with the operations:

$$P_a \vee P_b = P_{a^2+b^2} \quad \text{and} \quad P_a \wedge P_b = P_{ab}.$$

In [12, Theorem 2], it was proved that for an f-ring R with bounded inversion property, the lattice $BZ(R)$ is a complemented lattice and R is a semiprimitive ring if and only if the lattice $BZ^\circ(R)$ is a complemented lattice and R is a reduced ring if and only if the base elements for closed sets in the space of Max(R) are open and R is semiprimitive if and only if the base elements for closed sets in the space Min(R) are open and R is reduced. We observe that this theorem holds for basic z-ideals. Therefore, Theorem 3.5 is essentially a restatement (or translation) of the original proof from [12], adapted to the language of the basic z-ideals. However, we will demonstrate that [12, Theorem 2] does not hold when extended to the context of basic z°-ideals. This highlights a fundamental distinction between the behavior of these two classes of ideals. It is proved that for a reduced f-ring R with bounded inversion property, the lattice of basic z°-ideals is a complemented lattice if and only if the space of minimal prime ideals, Min(R), is compact and R satisfies the annihilator condition (Theorem 3.6). Finally, we prove that for a reduced f-ring R with bounded inversion property, two lattices $\{\mathrm{Ann}(f) : f \in R\}$ and $BZ^\circ(R)$ are isomorphic (Theorem 3.8).

2 Preliminaries

An f-ring is a lattice ordered ring which satisfies the condition that, for any $a, b \in R$ and any $c \geq 0$ in R, $(a \wedge b)c = (ac) \wedge (bc)$. By a positive element of an f-ring R we mean an element $a \geq 0$, and we set $R^+ = \{a \in R : a \geq 0\}$. An f-ring has bounded inversion if every element $a \geq 1$ is invertible. We write $J(R)$ for the *Jacobson radical* of R, which is the intersection of all maximal ideals of R and we say R is *semiprimitive* if $J(R) = 0$. A ring R is *semiprime* (*reduced*) if it has no nonzero nilpotent element.
As usual, we write $\mathrm{Max}(R)$ for the set of maximal ideals of R. We shall view $\mathrm{Max}(R)$ as a topological space with the Zariski topology. To recall, for $a \in R$ and $A \subseteq R$, we set

$$\mathcal{M}(a) = \{M \in \mathrm{Max}(R) \colon a \in M\} \quad \text{and} \quad \mathcal{M}(A) = \{M \in \mathrm{Max}(R) \colon A \subseteq M\},$$

and also

$$\mathcal{D}(a) = \{M \in \mathrm{Max}(R) \colon a \notin M\} \quad \text{and} \quad \mathcal{D}(A) = \{M \in \mathrm{Max}(R) \colon A \nsubseteq M\},$$

so that

$$\mathcal{D}(a) = \mathrm{Max}(R) \setminus \mathcal{M}(a) \quad \text{and} \quad \mathcal{D}(A) = \mathrm{Max}(R) \setminus \mathcal{M}(A).$$

The family $\{\mathcal{M}(a) \colon a \in R\}$ is a base for the closed sets in Zariski topology on $\mathrm{Max}(R)$.

Similarly, we write *Min(R)* for the set of minimal prime ideals of R. We shall also endow $\mathrm{Min}(R)$ with the Zariski topology, and for $a \in R$ and $A \subseteq R$, we set

$$\mathcal{P}(a) = \{P \in \mathrm{Min}(R) \colon a \in P\} \quad \text{and} \quad \mathcal{P}(A) = \{P \in \mathrm{Min}(R) \colon A \subseteq P\},$$

The family $\{\mathcal{P}(a) \colon a \in R\}$ is a base for the closed sets in Zariski topology on $\mathrm{Min}(R)$. There will be no confusion since at no stage are we going to consider $\mathrm{Max}(R)$ and $\mathrm{Min}(R)$ simultaneously.
An ideal I of a ring R is a z-ideal (resp., z^0-ideal) if for each $a, b \in R$ and $\mathcal{M}(a) = \mathcal{M}(b)$ (resp., $\mathcal{P}(a) = \mathcal{P}(b)$) , $a \in I$, imply $b \in I$, or equivalently, $M_a \subseteq I$ (resp., $P_a \subseteq I$) for each $a \in I$, where M_a (resp., P_a) is the intersection of all maximal ideals (resp., all minimal prime ideals) of R containing a. Trivially, we have $M_a \subseteq M_b$ (resp., $P_a \subseteq P_b$) if and only if $\mathcal{M}(b) \subseteq \mathcal{M}(a)$ (resp., $\mathcal{P}(b) \subseteq \mathcal{P}(a)$). See [9], for a detailed study of z-ideals in rings.

Recall that an ideal I of an f-ring R is an ℓ-ideal, if for any $a, b \in R$, $|a| \leq |b|$ and $b \in I$ imply $a \in I$. As showed in [10], if R is an f-ring with bounded inversion, then every z-ideal in R is an ℓ-ideal. In fact, an f-ring R has bounded inversion precisely when every z-ideal in R is an ℓ-ideal. Thus in this case of R every maximal ideal is an ℓ-ideal which is known to be equivalent to having the bounded inversion property. Whenever *J(R)* = 0, then every minimal prime ideal of R is a z-ideal and hence is an ℓ-ideal.

In this paper, $C(X)$ ($C^*(X)$) is the ring of all (bounded) real-valued continuous functions on a completely regular Hausdorff space X. In fact, for every topological space X there exists a completely regular Hausdorff space Y such that $C(X)$ and $C(Y)$ are isomorphic as two rings. So, whenever we speak about $C(X)$, X is a completely regular and Hasdorff space.
In studying relations between topological properties of a space X and algebraic properties of $C(X)$, it is natural to look at the subsets of X of the form $f^{-1}\{0\}$, for each $f \in C(X)$. The set $f^{-1}\{0\}$ is called the zero-set of f and denoted by $Z(f)$. Any set that is a zero-set of some function in $C(X)$ is called a zero-set in X. Thus, Z is a mapping from the ring $C(X)$ onto the set of all zero-sets in X. A $\mathrm{coz} f$ is the set $X \setminus Z(f)$ which is called the cozero-set of f. The set of all zero-sets in X is denoted by $Z[X]$ and for each ideal I in $C(X)$, $Z[I]$ is the set of all zero-sets of the form $Z(f)$, where $f \in I$.

Lemma 2.1. *Let R be an f-ring with bounded inversion.*

1. *$BZ(R) = \{M_a : a \in R\}$ is a distributive bounded lattice with inclusion order.*
2. *If R is reduced ring, then $BZ^o(R) = \{P_a : a \in R\}$ is a distributive bounded lattice with inclusion order.*

Proof. (1) It suffices to show that for each $a,b \in R$, $M_a \vee M_b$ and $M_a \wedge M_b$ exist in $BZ(R)$. We first claim that $M_a \vee M_b = M_{a^2+b^2}$. To see this, note that since $0 \leq a^2 \leq a^2+b^2$, and since every maximal ideal is assumed to be an ℓ-ideal (by the comment preceding the lemma), we have $M_a = M_{a^2} \subseteq M_{a^2+b^2}$ and similarly $M_b = M_{b^2} \subseteq M_{a^2+b^2}$. Thus, $M_{a^2+b^2}$ is an upper bounded for $\{M_a, M_b\}$. Now suppose $M_a \subseteq M_c$ and $M_b \subseteq M_c$. Then we have $\mathcal{M}(c) \subseteq \mathcal{M}(a)$ and $\mathcal{M}(c) \subseteq \mathcal{M}(b)$, which implies $\mathcal{M}(c) \subseteq \mathcal{M}(a) \cap \mathcal{M}(b)$. By the hypothesis that every maximal ideal is an ℓ-ideal, it follows that $\mathcal{M}(a) \cap \mathcal{M}(b) = \mathcal{M}(a^2+b^2)$, and hence $\mathcal{M}(c) \subseteq \mathcal{M}(a^2+b^2)$, which implies $M_{a^2+b^2} \subseteq M_c$. Therefore, $M_{a^2+b^2}$ is the least upper bound (join) of M_a and M_b. Next, it is straightforward to show that:

$$M_a \wedge M_b = M_a \cap M_b = M_{ab},$$

as also noted in Lemma 3.1, Part (1) of [4]. To verify the distributivity of the lattice $BZ(R)$, consider $a,b,c \in R$. Then we have

$$M_a \vee (M_b \wedge M_c) = M_a \vee M_{bc} = M_{a^2+(bc)^2},$$

and

$$(M_a \vee M_b) \wedge (M_a \vee M_c) = M_{a^2+b^2} \wedge M_{a^2+c^2} = M_{(a^2+b^2)(a^2+c^2)}.$$

On the other hand, it is easy to see that both expressions coincide. So we are done. Finally, the least and greatest elements of $BZ(R)$ are $M_0 = J(R)$ and $M_1 = R$, respectively.

(2) Similarly, by hypothesis and the earlier comments preceding the lemma, we see that for each $a,b \in R$, $P_a \vee P_b = P_{a^2+b^2}$ and $P_a \wedge P_b = P_{ab}$. Analogous to the proof of Part (1) and using the hypothesis, for each $a,b,c \in R$ we have,

$$P_a \vee (P_b \wedge P_c) = (P_a \vee P_b) \wedge (P_a \vee P_c).$$

This shows $BZ^o(R)$ is a distributive lattice. Moreover, the elements $P_0 = N(R)$ and $P_1 = R$ serve as the least and greatest elements of $BZ^o(R)$. □

3 Some properties of the lattice of basic z°-ideals

In this section, we begin by presenting some preliminary results, followed by the main result of the paper.

Lemma 3.1. *For a ring R the following statements are equivalent:*

1. *For each $a \in R$, $P_a = \mathrm{Ann}(\mathrm{Ann}(a))$;*
2. *The ring R is reduced;*
3. *Every annihilator ideal of R is a z°-ideal.*

Proof. (1)⇒(2) We have $P_\circ = \mathrm{Ann}(\mathrm{Ann}(0)) = 0$. This says that the intersection of all minimal prime ideals of R is zero, i.e., R is a reduced ring.

(2)⇒(3) Let I be an annihilator ideal in R, $a \in I$ and $x \in P_a$. Then $\mathrm{Ann}(I) \subseteq \mathrm{Ann}(a) \subseteq \mathrm{Ann}(x)$, by [1, Proposition 1.5]. Thus $x \in I$, i.e., $P_a \subseteq I$. That is I is a z°-ideal.

(3)⇒(1) $\mathrm{Ann}(\mathrm{Ann}(a))$ is an annihilator ideal and hence it is a z°-ideal. Thus, $a \in \mathrm{Ann}(\mathrm{Ann}(a))$, implies $P_a \subseteq \mathrm{Ann}(\mathrm{Ann}(a))$. On the other hand, $b \in \mathrm{Ann}(\mathrm{Ann}(a))$ implies $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(b)$. By [1, Proposition 1.5], $b \in P_a$. So we are done. □

Similar to the above result, we can prove the following result.

Lemma 3.2. *For a ring R the following statements are equivalent:*

1. *For each $a \in R$, $M_a \subseteq \operatorname{Ann}(\operatorname{Ann}(a))$;*
2. *The ring R is semiprimitive;*
3. *Every annihilator ideal of R is a z-ideal.*

Corollary 3.3. *Let R be a reduced ring. A principal ideal Ra is an annihilator ideal if and only if it is a z°-ideal.*

Theorem 3.4. Let R be an f-ring with bounded inversion property. The following statements are equivalent:

1. $BZ(R)$ is a complemented lattice and R is a semiprimitive ring;
2. R is a regular ring;
3. The base elements for closed sets in $\operatorname{Max}(R)$ are open and R is semiprime.

Proof. (1)⇒(2) Let $a \in R$. By hypothesis, there exists $b \in R$ such that

$$M_a \vee M_b = M_1 = R \quad \text{and} \quad M_a \wedge M_b = M_{ab} = M_{\circ} = 0.$$

Thus $M_{a^2+b^2} = R$ and $M_{ab} = 0$. This shows $ab = 0$ and hence $a^2 + b^2$ is unit in R. It follows there exists $c \in R$ such that;

$$c(a^2 + b^2) = 1 \Rightarrow a = a^3c + ab^2c = a^2(ac).$$

This shows that R is a regular ring.

(2)⇒(1) Trivially, the regularity of R implies $J(R) = 0$. Now, let $a \in R$. By hypothesi, there exists $b \in R$ such that $a = a^2b$. Hence, $a(1 - ab) = 0$. This implies that

$$M_a \wedge M_{1-ab} = M_{a(1-ab)} = M_0 = 0.^{(1)}$$

Since R has the bounded inversion property, every maximal ideal is an l-ideal. Therefore, no maximal ideal contains $a^2+(1-ab)^2$. In fact, a maximal ideal contains $a^2+(1-ab)^2$ if and only if it is containing a and $(1 - ab)$. Hence we have,

$$M_a \vee M_{1-ab} = M_{a^2+(1-ab)^2} = M_1.^{(2)}$$

The equalities (1) and (2) show that for each $a \in R$, M_a has complement. Thus, $BZ(R)$ is a complemented lattice.

(2)⇒(3) Since R is regular, it follows that $J(R) = 0$ and hence R is a reduced ring. By hypothesis, for each $a \in R$, there exists a $b \in R$ such that $a = a^2b$. This implies that $a(1 - ab) = 0$. Therefore, every maximal ideal must contain either a or $1 - ab$, which yields

$$M(a) \cup M(1 - ab) = \operatorname{Max}(R) \quad \text{and} \quad M(a) \cap M(1 - ab) = \emptyset.$$

Thus $M(a) = \operatorname{Max}(R) \setminus M(1 - ab)$ is open in the Zariski topology.

(3)⇒(2) Let $a \in R$ such that $M(a)$ is clopen in $\operatorname{Max}(R)$. Since $J(R) = 0$, there exists an idempotent $e \in R$ such that $M(a) = \operatorname{Max}(R) \setminus M(e)$. Thus we have,

$$M(ae) = M(a) \cup M(e) = \operatorname{Max}(R) \Rightarrow ae \in J(R) = 0.$$

On the other hand, we compute:

$$M(a^2 + e^2) = M(a) \cap M(e) = D(e) \cap M(e) = \emptyset.$$

Which implies that $a^2+e^2=a^2+e$ is not contained in any maximal ideal, and therefore is a unit in R. So there exists $c\in R$ such that

$$c(a^2+e)=1\Rightarrow ca^2+ce=1.$$

Multiplying both sides by a, we get $a=ca^3+cea$. But since $ae=0$, we have $a=a^2(ca)$. Thus, R is a regular ring. □

It is well-known that $C(X)$ is a regular ring if and only if X is a P-space (i.e., every zero-set, see [5, 4J]. This fact together with the above result imply the following.

Corollary 3.5. *$BZ(C(X))$ is a complemented lattice if and only if X is a P-space.*

A space X is *cozero complemented* if, given any cozero set U, there is a disjoint cozero set V such that $U\cup V$ is dense in X. There are many equivalent algebraic or topological conditions for cozero complemented spaces, see [6] and [8].

Remark 3.6. *Let X be a cozero complemented space. Then we claim that $BZ^\circ(C(X))$ is a complemented lattice and hence this shows that the results of Theorem 2 in [12] for basic z°-ideals is not true. In fact, it is enough to consider the space $\mathbb{R}$ with standard topology. Then $BZ^\circ(C(\mathbb{R}))$ is a complemented lattice, while $C(\mathbb{R})$ is not a regular ring. To see our claim, let $P_f\in BZ^\circ(C(X))$. Since X is a cozero complemented space, there exists $g\in C(X)$ such that*

$$\mathrm{cl}_X(X\setminus Z(f))\cup(\mathrm{cl}_X(X\setminus Z(g))=X \quad \textit{and} \quad (X\setminus Z(f))\cap(X\setminus Z(g))=\emptyset.$$

$$\textit{Thus}\quad \mathrm{int}_X Z(f^2+g^2)=\mathrm{int}_X Z(f)\cap\mathrm{int}_X Z(g)=\emptyset,$$

$$\textit{and}\quad \mathrm{int}_X Z(fg)=\mathrm{int}_X(Z(f)\cup Z(g))=X.$$

This implies

$$P_f\wedge P_g=P_{fg}=P_{\mathrm{int}_X Z(fg)}=P_X=0,\quad \textit{and}$$

$$P_f\vee P_g=P_{f^2+g^2}=P_{\mathrm{int}_X Z(f^2+g^2)}=P_\emptyset=R.$$

See also [11, Proposition 3.8].

We recall that for a,b in a reduced f-ring R, $\mathrm{Ann}(a^2+b^2)=\mathrm{Ann}(a)\cap\mathrm{Ann}(b)$. For, we have $\mathrm{Ann}(a)\cap\mathrm{Ann}(b)\subseteq\mathrm{Ann}(a^2+b^2)$. Now, let $r\in\mathrm{Ann}(a^2+b^2)$. Then $(ra)^2+(rb)^2=0$. As $0\le(ra)^2\le(ra)^2+(rb)^2=0$. Thus $(ra)^2=0$ and hence $ra=0$ and similarly $rb=0$. These imply $r\in\mathrm{Ann}(a)\cap\mathrm{Ann}(b)$.
In the next result we give a correct form of Theorem 2 in [12] for basic z°-ideals.

Theorem 3.7. For a reduced f-ring R with bounded inversion property, the following statements are equivalent:

1. $\mathrm{Min}(R)$ is compact, and R satisfies the annihilator condition;
2. For each $x\in R$ there exists $y\in R$ such that $\mathrm{Ann}(\mathrm{Ann}(y))=\mathrm{Ann}(x)$;
3. For each $x\in R$ there exists $y\in R$ such that $\mathrm{Ann}(x)=P_y$;
4. $BZ^\circ(R)$ is a complemented lattice.

Proof. (1)⇔(2) See Theorem 3.4 in [7].
(2)⇔(3) This follows from Lemma 3.1.
(3)⇒(4) It is enough to show that every element of $BZ^0(R)$ has a complement. Consider $P_r\in BZ^0(R)$. By Lemma 3.1, $P_r=\mathrm{Ann}(\mathrm{Ann}(r))$. Since R is a reduced ring, $\mathrm{Ann}(\mathrm{Ann}(r))\cap\mathrm{Ann}(r)=0=P_0$. By hypothesis, there exists $y\in R$ such that $\mathrm{Ann}(r)=P_y$. Hence $P_r\wedge P_y=P_r\cap P_y=0$. On the other hand,

$$P_r\vee P_y=P_{r^2+y^2}=\mathrm{Ann}(\mathrm{Ann}(r^2+y^2))=\mathrm{Ann}(\mathrm{Ann}(r)\cap\mathrm{Ann}(y))$$

$$= \mathrm{Ann}(P_y \cap \mathrm{Ann}(y)) = \mathrm{Ann}(\mathrm{Ann}(\mathrm{Ann}(y)) \cap \mathrm{Ann}(y)) = \mathrm{Ann}(0) = R.$$

Thus P_y is a complement of P_r.

(4)⇒(3) Let $x \in R$. Then there exists $y \in R$ such that $P_y \cap P_x = 0$ and $\mathrm{Ann}(\mathrm{Ann}(x^2 + y^2)) = P_{x^2+y^2} = P_x \vee P_y = R$. The first equality shows that $\mathrm{Ann}(\mathrm{Ann}(x)) \cap P_y = 0$ and hence $P_y \subseteq \mathrm{Ann}(\mathrm{Ann}(\mathrm{Ann}(x))) = \mathrm{Ann}(x)$. The second equality shows that $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) = \mathrm{Ann}(x^2 + y^2) = 0$ and hence $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(\mathrm{Ann}(y)) = P_y$. Thus $\mathrm{Ann}(x) = P_y$. □

It is also easy to see that for a reduced f-ring R, the set $\{\mathrm{Ann}(f) : f \in R\}$, partially ordered with inclusion, forms a lattice with the following operations;

$$\mathrm{Ann}(f) \vee \mathrm{Ann}(g) = \mathrm{Ann}(fg), \quad \mathrm{Ann}(f) \wedge \mathrm{Ann}(g) = \mathrm{Ann}(f^2 + g^2).$$

We need the following well-known result in the sequel, see [2, Lemma 4.2] and [1, Proposition 1.5]. We note that for a subset A of $\mathrm{Min}(R)$, $\mathrm{int}\, A = \mathrm{int}_{\mathrm{Min}(R)} A$.

Lemma 3.8. *For a reduced ring R the following statements hold.*

1. *If $a \in R$, then $P_a = \{x \in R : \mathrm{int}\, V(a) \subseteq \mathrm{int}\, V(x)\}$.*
2. *For ideals I, J of R, $\mathrm{Ann}(I) \subseteq \mathrm{Ann}(J)$ if and only if $\mathrm{int}\, V(I) \subseteq \mathrm{int}\, V(J)$.*

Theorem 3.9. Let R be a reduced f-ring with bounded inversion property. Then two lattices $(\{\mathrm{Ann}(f) : f \in R\}, \subseteq)$ and $(BZ^\circ(R), \supseteq)$ are isomorphic.

Proof. Define $\phi : \{\mathrm{Ann}(f) : f \in R\} \to BZ^\circ(R)$, by $\phi(\mathrm{Ann}(f)) = P_f$, where $f \in R$. Let $\mathrm{Ann}(f) = \mathrm{Ann}(g)$. Then, $\mathrm{int}\, V(f) = \mathrm{int}\, V(g)$, by Lemma 3.8. This implies $P_f = P_g$, i.e., $\phi(\mathrm{Ann}(f)) = \phi(\mathrm{Ann}(g))$. If $\phi(\mathrm{Ann}(f)) = \phi(\mathrm{Ann}(g))$ for some $f, g \in R$, then $P_f = P_g$. This implies $f \in P_f = P_g$ and hence $\mathrm{int}\, V(g) \subseteq \mathrm{int}\, V(f)$. Also, $g \in P_g = P_f$ implies $\mathrm{int}\, V(f) \subseteq \mathrm{int}\, V(g)$. Thus, $\mathrm{int}\, V(f) = \mathrm{int}\, V(g)$. By Lemma 3.8, $\mathrm{Ann}(f) = \mathrm{Ann}(g)$.This says, ϕ is a one-one map. Clearly, ϕ is onto. It is enough to show that ϕ preserves $\vee$ and $\wedge$. Assume $f, g \in R$. Then, we have

$$\phi(\mathrm{Ann}(f) \vee \mathrm{Ann}(g)) = \phi(\mathrm{Ann}(fg)) = P_{fg} = P_f \vee P_g = \phi(\mathrm{Ann}(f)) \vee \phi(\mathrm{Ann}(g)),$$

and

$$\phi(\mathrm{Ann}(f) \wedge \mathrm{Ann}(g)) = \phi(\mathrm{Ann}(f^2 + g^2)) = P_{f^2+g^2} = P_f \wedge P_g = \phi(\mathrm{Ann}(f)) \wedge \phi(\mathrm{Ann}(g)).$$

□

Theorems 3.7 and 3.9 imply the next result, see also [11, Proposition 3.4].

Corollary 3.10. *The following statements are equivalent:*

1. *The lattice $\{\mathrm{Ann}(f) : f \in C(X)\}$ is a complemented lattice;*
2. *The lattice $BZ^\circ(C(X))$ is a complemented lattice;*
3. *The space X is a cozero complemented space.*

Acknowledgments.

The author is grateful to Professor Najib Mahdou for his comments and suggestions that helped improve the presentation of the paper.

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