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On quasi n-absorbing submodules

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Abstract. Let R be a commutative ring with $1 \neq 0$, M be an R-module, and n be a positive integer. The purpose of this article is to investigate quasi n-absorbing (resp., weakly quasi n-absorbing) submodules generalizing quasi n-absorbing ideals of rings. We will say a proper submodule N of M is quasi n-absorbing (resp., weakly quasi n-absorbing) submodule of M if whenever $a \in R$ and $x \in M$ with $a^n x \in N$, (resp., $0 \neq a^n x \in N$) then, either $a^n \in (N :_R M)$ or $a^{n-1}x \in N$.

Key Words: n-absorbing ideals, n-absorbing submodules, quasi n-absorbing ideals, quasi n-absorbing submodules.
2010 MSC: 13C13, 13A15.

Dedicated to our Professor David E. Dobbs for his 80th Birthday

1 Introduction

In this paper, all rings are commutative with non-zero identity and all modules are unital. Let R be a ring, M an R-module and N a submodule of M. We will denote by $(N :_R M)$ the residual of N by M, that is, the set of all $r \in R$ such that $rM \subseteq N$. The $\mathrm{ann}_R(M)$ is $(0 :_R M)$. An R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R. In this case, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$ [15].
A submodule N of M is called idempotent if $N = (N :_R M)N$, [3]. It is shown [4, Theorem 3] that if M is a multiplication module and $(N :_R M)$ is idempotent ideal of R then N is idempotent in M. The converse is true if we assume further that M is finitely generated and faithful. A submodule N of the R-module M is called a nilpotent submodule if $(N :_R M)^p N = 0$ for some positive integer p, and $m \in M$ is said to be nilpotent if Rm is a nilpotent submodule of M, [3]. Set $\mathrm{Nil}(M)$ the set of all nilpotent elements of M. Then, $\mathrm{Nil}(M)$ is a submodule of M provided that M is faithful module, and in addition M is multiplication, then $\mathrm{Nil}(M)$=$\mathrm{Nil}(R)M = \cap P$, where the intersection runs over all prime submodules of M, [3, Theorem 6]. A submodule N of M is prime (resp., primary) if whenever $ax \in N$ for some $a \in R$ and $x \in M$, then either $a \in (N :_R M)$ or $x \in N$ (resp., $a^p \in (N :_R M)$ for some positive integers p). If N is prime (resp., primary) submodule of M, then $P := (N :_R M)$ (resp., $P := \sqrt{(N :_R M)}$) is a prime ideal of R. In this case, we say that N is a P-prime (resp., P-primary) submodule of M. We say that M is *secondary* precisely when, for each $a \in R$, either $aM = M$ or there exists $p \in \mathbb{N}$ such that $a^p M = 0$. When this is the case, $P := \sqrt{(0 :_R M)}$ is a prime ideal of R : we say that M is P-secondary R-module.
In [7] Badawi introduced a new generalization of prime ideals in a commutative ring R. He defined a nonzero proper ideal I of R to be a 2-absorbing ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or

$ac \in I$ or $bc \in I$. This concept has a generalization, called weakly 2-absorbing ideals, which has been studied in [8]. A proper ideal I of R to be weakly 2-absorbing ideal of R if whenever $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Later Anderson and Badawi in [5], introduced the concept of n-absorbing ideals of R for a positive integer n. A proper ideal I of R is called n-absorbing(resp., strongly n-absorbing) ideal if whenever $a_1...a_{n+1} \in I$ for $a_1,...,a_{n+1} \in R$ (resp.,$I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1,\cdots,I_{n+1}$ of R), then there are n of the a_i's(resp., n of the I_i's) whose product is in I. Inspired by the study of absorbing ideals, in [14] Mukhtar et al. and later in [1] Ahmed et al. introduced and studied, respectively, the 2-absorbing factorization and n-absorbing factorization in commutative rings. In [13] H. Mostafanasab and A.Y. Darani introduced the notion of quasi-n-absorbing and semi n-absorbing ideals of a commutative ring. A proper ideal I of a commutative ring is said to be a quasi-n-absorbing (resp., semi n-absorbing) if whenever $a^n b \in I$ for $a,b \in R$, then either $a^n \in I$ or $a^{n-1}b \in I$ (resp., $a^{n+1} \in I$ for $a \in R$, then $a^n \in I$).
The concept of 2-absorbing (resp., weakly 2-absorbing) submodules was introduced and investigates in [10]. Let M be an R-module and N a proper submodule of M. N is said to be a 2-absorbing submodule (resp., weakly 2-absorbing submodule) of M if whenever $a,b \in R$ and $x \in M$ such that $abx \in N$ (resp., $0 \neq abx \in N$), then $ab \in (N :_R M)$ or $ax \in N$ or $bx \in N$. This concept has a generalization, called n-absorbing submodule introduced and investigates in [11]. Let n be a positive integer. A proper submodule N of an R-module M is called n-absorbing submodule if whenever $a_1 \cdots a_n x \in N$ for $a_1,...,a_n \in R$ and $x \in M$ (resp., $I_1 \cdots I_n L \subseteq N$ for ideals $I_1,...,I_n$ of R and submodule L of M), then either $a_1 \cdots a_n \in (N :_R M)$ (resp., $I_1 \cdots I_n \subseteq (N :_R M)$) or there are $n-1$ of a_i's (resp., I_i's) whose product with x (resp., with L) is in N. A proper submodule N of M is called an quasi n-absorbing (resp., weakly quasi n-absorbing) submodule of M if whenever $a^n x \in N$ for $a \in R$ and $x \in M$ (resp., $0 \neq a^n x \in N$) then either $a^n \in (N :_R M)$ or $a^{n-1}x \in N$. It is clear that if N is an n-absorbing submodule it is also a quasi n-absorbing submodule of M. But the converse is not true in general case, take for example $N = 30\mathbb{Z}$ as submodule of the $\mathbb{Z}$-module $\mathbb{Z}$, we have N is not 2-absorbing submodule, but is quasi 2-absorbing submodule of $\mathbb{Z}$ (Proposition 2.14) and $N = 0$ is weakly quasi 2-absorbing submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ which is not quasi 2-absorbing submodule.
Thus, if N is a submodule of the R-module M, then "N is a prime submodule of M" implies "N is an n-absorbing submodule of M" implies "N is a quasi n-absorbing submodule of M".
In this paper, we study the concept of quasi n-absorbing submodule and weakly quasi n-absorbing submodule, for a positive integer n.

2 Properties of quasi n-absorbing submodules

In this section we study some basic properties of quasi n-absorbing submodules of the R-module M. Let n be a positive integer. We recall that a proper submodule N of M is called quasi n-absorbing submodule of M, if whenever $a \in R$ and $x \in M$ with $a^n x \in N$, then $a^n \in (N :_R M)$ or $a^{n-1}x \in N$.
Let R be a ring, M be an R-module, N be a submodule of M, and I be an ideal of R. We set $(N :_M I) = \{x \in M \mid xI \subseteq N\}$.

Theorem 2.1. Let R be a ring, M an R-module and N be a submodule of M. Then the following statements are equivalent:

(1) N is a quasi n-absorbing submodule of M.

(2) For each $a \in R$ with $a^n \notin (N :_R M)$, we have $(N :_M a^n) = (N :_M a^{n-1})$.

(3) For each $a \in R$ and every submodule L of M with $a^n L \subseteq N$, either $a^n \in (N :_R M)$ or $a^{n-1}L \subseteq N$.

Proof. (1) $\Rightarrow$ (2) It is clear that $(N :_R a^{n-1}) \subseteq (N :_R a^n)$ for every $a \in R$. Let $a \in R$ such that $a^n \notin (N :_R M)$ and $x \in (N :_M a^n)$. Since N is a quasi n-absorbing submodule, $a^{n-1}x \in N$. Hence $x \in (N :_M a^{n-1})$.

(2) $\Rightarrow$ (3) Let $a \in R$ and L be a submodule of M such that $a^n L \subseteq N$. Suppose that $a^n \notin (N :_R M)$. By (2), for each $x \in L$, we have $a^{n-1}x \in N$. Thus, $a^{n-1}L \subseteq N$.
(3) $\Rightarrow$ (1) Assume that $a^n x \in N$ for some $a \in R$ and $x \in M$. By leading $L = Rx$, we obtain the desired result. □

Corollary 2.2. *Let R be a ring and M be an R-module. Then 0 is a quasi n-absorbing submodule of M if and only if for each $a \in R$, either $a^n \notin \mathrm{ann}_R(M)$ or $\mathrm{ann}_M(a^n) = \mathrm{ann}_M(a^{n-1})$.*

Corollary 2.3. *Let M be an R-module and N be a quasi n-absorbing submodule of M. Then $(N :_R M)$ is a quasi n-absorbing ideal of R.*

Proof. Let $a, b \in R$ such that $a^n b \in (N :_R M)$. If $a^n \notin (N :_R M)$, by Theorem 2.1, we have $b \in (N :_R a^{n-1}) = (N :_R a^n)$. So $a^{n-1}b \in N$, and hence $(N :_R M)$ is a quasi n-absorbing ideal of R. □

Corollary 2.4. *Let R be a ring and I be a proper ideal of R. Then the following statements hold:*

(1) I is a quasi n-absorbing ideal of R.

(2) For each $a \in R$ with $a^n \notin I$, we have $(I :_R a^n) = (I :_R a^{n-1})$.

(3) For each $a \in R$ and every ideal J of R with $a^n J \subseteq I$, either $a^n \in I$ or $a^{n-1}J \subseteq I$.

Example 2.5. The converse of the corollary 2.3 is not true in general. For example, consider $N = 8\mathbb{Z}$ as a submodule of $\mathbb{Z}$-module $\mathbb{Z}$. While $(N :_{\mathbb{Z}} \mathbb{Z}) = 8\mathbb{Z}$ is clearly a quasi 2-absorbing ideal of $\mathbb{Z}$, N is not quasi 2-absorbing. In fact $2^2.2 \in N$, but $2^2 \notin N$ and $2.2 \notin (N :_{\mathbb{Z}} \mathbb{Z})$.

Theorem 2.6. Let M be a cyclic R-module. Then for every submodule N of M the following statements are equivalent:

(1) N is a quasi n-absorbing submodule of M.

(2) $(N :_R M)$ is a quasi n-absorbing ideal of R.

Proof. Following Corollary 2.3, it suffices to prove (2) $\implies$ (1). Set $M = Rm$. Assume that $(N :_R M)$ an quasi n-absorbing ideal of R. Let $a \in R$ and $x \in M$ such that $a^n x \in N$. There exists $b \in R$ such that $x = bm$. Thus $a^n bm \in N$. So $a^n b \in (N :_R m) = (N :_R M)$. Since $(N :_R M)$ is a quasi n-absorbing ideal of R, we conclude that $a^n \in (N :_R M)$ or $a^{n-1}b \in (N :_R M)$. Hence $a^n \in (N :_R M)$ or $a^{n-1}bm = a^{n-1}x \in N$, that is, N is quasi n-absorbing submodule of M. □

Corollary 2.7. *Let R be a Von Neumann regular ring, M be a cyclic R-module and n be a positive integer. Then every submodule of M is quasi n-absorbing.*

Proof. Since every ideal of R is radical, it is clear that every ideal of R is quasi n-absorbing. Hence, the result follows from the theorem 2.6. □

Theorem 2.8. Let R be a ring and M be a multiplication non-torsion R-module. If every submodule of M is quasi n-absorbing, then $\dim(M) = 0$.

Proof. Suppose that $\dim(M) \geq 1$,. There exists a prime submodule N of M and a submodule L of M such that $N \subset L$. Since M is a multiplication non torsion module, then $(N :_R M)$ is a prime ideal of R and $(N :_R M) \subset (L :_R M)$. Let $a \in (L :_R M) \setminus (N :_R M)$. Since M is multiplication non-torsion R module, there exists a submodule K of M such that $J := (K :_R M) = a^{n+1}R$. Then $a^{n+1}R$ is quasi n-absorbing ideal of R, and $a^n.a \in J$ so $a^n \in J$ and there exists $b \in R$ with $a^n = a^{n+1}b$. Hence $a^n(1 - ab) = 0 \in (N :_R M)$ and $a^n \notin (N :_R M)$, thus $1 - ab \in (N :_R M) \subset (L :_R M)$ and $a \in (L :_R M)$ we conclude that $1 \in (L :_R M)$ a contradiction. Thus $\dim(M) = 0$. □

Corollary 2.9. *Let R be a ring. If every ideal of R is n-absorbing, then $dim R = 0$.*

Definition 2.10. Let $m > n$ be positive integers. A proper submodule N of a R-module M is called quasi (m,n)-absorbing if whenever $a^{m-1}x \in N$ for $a \in R$ and $x \in M$, then $a^n \in (N :_R M)$ or $a^{n-1}x \in N$.

Theorem 2.11. Let N be a proper submodule of a R-module M and $m > n$ positive integers. Then N is quasi (m,n)-absorbing if and only if it is quasi n-absorbing.

Proof. Assume that N is quasi (m,n)-absoring. Let $a^n x \in N$ for some $a \in R$ and $x \in M$. Since $n \leq m-1$, then $a^{m-1}x \in N$. Therfore $a^n \in (N :_R M)$ or $a^{n-1}x \in N$. Hence N is quasi n-absorbing submodule. Conversely, assume that N is quasi n-absorbing. Let $a^{m-1}x \in N$ for some $a \in R$ and $x \in M$. Therfore $a^n(a^{m-1-n}x) \in N$. Hence $a^n \in (N :_R M)$ or $a^{n-1}(a^{m-1-n}x) = a^{m-2}x \in N$. Repeationg the same argument we conclude that $a^n \in (N :_R M)$ or $a^{n-1}x \in N$. Thus N a is quasi (m,n)-absorbing submodule of M. □

Corollary 2.12. *[13, Proposition 3.8]. Let I be a proper ideal of a ring R and $m > n$ be positive integers. Then, I is a quasi (m,n)-absorbing ideal if and only if I is a quasi n-absorbing ideal of R.*

Proposition 2.13. *Let N be proper submodule of an R-module M.*

(1) N is a prime submodule if and only if it is a quasi 1-absorbing submodule of M.

(2) If N is a quasi n-absorbing submodule then it is an i-absorbing submodule for all $i \geq n$.

(3) If N is a prime submodule then it is a quasi n-absorbing for all $n \geq 1$.

(4) If N is an n-absorbing submodule then it is a quasi n-absorbing.

Proof. Clear. □

Proposition 2.14. *Let R be a ring and M be an R-module. Let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a family of prime submodules of M. Then $\cap_{\lambda \in \Lambda} N_\lambda$ is a quasi i-absorbing submodule for every $i \geq 2$.*

Proof. Let $N = \cap_{\lambda \in \Lambda} N_\lambda$. By Proposition 2.13, it is sufficient to show that N is a quasi 2-absorbing submodule of M. Indeed, let $a \in R$ and $x \in M$ with $a^2 x = a(ax) \in N = \cap_{\lambda \in \Lambda} N_\lambda$. Since N_λ is prime submodule of M and $a^2 x = a(ax) \in N_\lambda$, then either $a \in (N_\lambda :_R M)$ or $ax \in N_\lambda$. In either case $ax \in N$. Hence N is a quasi 2-absorbing submodule of M. □

Remark 2.15. Let $p_1, ..., p_{n+1}$ distinct prime numbers. Set $N_i = p_i\mathbb{Z}$ for $1 \leq i \leq n+1$. It is clear that N_i is a prime submodule of the $\mathbb{Z}$-module $\mathbb{Z}$. By Proposition 2.14, we have $N = \cap_{i=1}^{n+1} N_i$ is a quasi i-absorbing submodule for every positive integers $i \geq 2$. But N is not a n-absorbing submodule of M by Corollary 2.3 since $(N :_{\mathbb{Z}} \mathbb{Z}) = \cap_{i=1}^{n+1} p_i\mathbb{Z}$ is a $n+1$-absorbing ideal but not a n-absorbing ideal of $\mathbb{Z}$, cf. [5, corollary 2.15].

Corollary 2.16. *Let R be a ring, M be a faithful and multiplication R-module. Then $Nil(M)$ is a quasi i-absorbing submodule of M for every positive integer $i \geq 2$.*

Proof. By [3, Theorem 6], we have $\text{Nil}(M) = \cap_{P \text{ prime}} P$. Now the results follows from Proposition 2.14. □

Theorem 2.17. Let $N_1, \cdots, N_t$ be submodules of an R-module M.

(1) If N_1 is quasi n-absorbing and N_2 is quasi m-absorbing for $m < n$, then $N_1 \cap N_2$ is quasi $n+1$-absorbing.

(2) If $N_1, \cdots, N_t$ are quasi n-absorbing, then $N_1 \cap N_2 \cdots \cap N_t$ is quasi $n+t$-absorbing.

(3) If $N_1, \cdots, N_t$ are quasi n_i-absorbing, for every $1 \leq i \leq t$ with $n_1 < n_2 < \cdots < n_t$ and $t > 2$, then $N_1 \cap N_2 \cdots \cap N_t$ is quasi $n_t + 2$-absorbing.

Proof. 1. Let $a \in R$ and $x \in M$ such that $a^{n+1}x \in N_1 \cap N_2$. We will show that either $a^{n+1} \in (N_1 \cap N_2 :_R M)$ or $a^n x \in N_1 \cap N_2$. Since N_1 is a quasi n-absorbing submodule of M, then by Theorem 2.11 it is quasi $(n+2, n)$-absorbing. Therefore either $a^n \in N_1$ or $a^{n-1}x \in N_1$. On the other hand N_2 is a quasi m-absorbing submodule of M and by Theorem 2.11 either $a^m \in (N_2 :_R M)$ or $a^{m-1}x \in N_2$. There are four cases:

Case 1. Assume that $a^n \in (N_1 :_R M)$ and $a^m \in (N_2 :_R M)$. Then $a^n \in (N_1 \cap N_2 :_R M)$.

Case 2. Assume that $a^n \in (N_1 :_R M)$ and $a^{m-1}x \in N_2$. Then $a^n x \in N_1 \cap N_2$.

Case 3. Assume that $a^{n-1}x \in N_1$ and $a^m \in (N_2 :_R M)$. Then $a^{n-1}x \in N_1 \cap N_2$.

Case 4. Assume that $a^{n-1}x \in N_1$ and $a^{m-1}x \in N_2$. Then $a^{n-1}x \in N_1 \cap N_2$.

In any case, we conclude that $a^n x \in N_1 \cap N_2$ and therefore $N_1 \cap N_2$ is a quasi $n+1$-absorbing submodule of M.

2. By induction on t : For $t = 1$ there is nothing to prove. Let $t > 1$ and suppose the claim holds for $t - 1$. Then $N_1 \cap \cdots \cap N_{t-1}$ is a quasi $n + t - 1$-absorbing submodule of M. Since N_t is a quasi n-absorbing submodule and $n + t - 2 \geq n$, then N_t is a quasi $n + t - 2$-absorbing submodule. Now the result follows from (1).

3. By induction on $t \geq 3$. The case of $t = 3$ is deduced from (1) and (2). Let $t > 3$ and suppose that for $t-1$ the claim holds. Then $N_1 \cap N_2 \cap \cdots \cap N_{t-1}$ is a quasi $n_{t-1}+2$-absorbing submodule. There are tree cases:

Case 1. Assume that $n_{t-1} + 2 < n_t$. Then $N_1 \cap N_2 \cap \cdots \cap N_t$ is quasi $n_t + 1$-absorbing submodule from (1). Therefore $N_1 \cap N_2 \cap \cdots \cap N_t$ is a quasi $n_t + 2$-absorbing submodule of M.

Case 2. Assume that $n_{t-1} + 2 = n_t$. Then $N_1 \cap N_2 \cap \cdots \cap N_t$ is a quasi $n_t + 2$-absorbing submodule of M.

Case 3. Assume that $n_{t-1}+2 > n_t$, in this case $n_{t-1}+1 = n_t$ because $n_{t-1} < n_t$. Then $N_1 \cap N_2 \cap \cdots \cap N_t$ is a quasi $n_{t-1} + 3 = n_t + 2$-absorbing submodule of M.

□

Proposition 2.18. *Let $M = M_1 \oplus M_2$ where M_1 and M_2 are R-modules. Let N_1 a proper submodule of M_1 and N_2 a proper submodule of M_2. Then N_1 (resp. N_2) is a quasi n-absorbing submodule of M_1(resp. M_2) if and only if $N_1 \oplus M_2$ (resp. $M_1 \oplus N_2$) is a quasi n-absorbing submodule of M.*

Proof. $(\Rightarrow)$ Suppose that N_1 is a quasi n-absorbing submodule of M_1. Let $a^n(x, y) \in N_1 \oplus M_2$ for some $a \in R$ and $(x, y) \in M$. Therefore $a^n x \in N_1$, and so either $a^n \in (N_1 :_R M_1)$ or $a^{n-1}x \in N_1$. Consequently, either $a^n \in [N_1 \oplus M_2 :_R M]$ or $a^{n-1}(x, y) \in N_1 \oplus M_2$. Hence $N_1 \oplus M_2$ is a quasi n-absorbing submodule of M.
$(\Leftarrow)$ Assume that $N_1 \oplus M_2$ is a quasi n-absorbing submodule of M. Let $a^n x \in N_1$ for some $a \in R$ and $x \in M_1$. Then $a^n(x, 0) \in N_1 \oplus M_2$. Hence $a^n \in [N_1 \oplus M_2 :_R M] = [N_1 :_R M_1]$ or $a^{n-1}(x, 0) \in N_1 \oplus M_2$. Therfore $a^n \in [N_1 :_R M_1]$ or $a^{n-1}x \in N_1$. That is N_1 is quasi n-absorbing submodule of M_1. □

Theorem 2.19. Let $N_1, \cdots, N_t$ be submodules of R-modules $M_1, M_2, \cdots, M_t$ respectively.

(1) If N_1 is a quasi n-absorbing submodule of M_1 and N_2 is a quasi m-absorbing submodule of M_2 for $m < n$, then $N_1 \oplus N_2$ is a quasi $n + 1$-absorbing submodule of $M_1 \oplus M_2$.

(2) If $N_1, \cdots, N_t$ are quasi n-absorbing submodules of $M_1, M_2, \cdots, M_t$ respectively, then $N_1 \oplus N_2 \oplus \cdots \oplus N_t$ is a quasi $n + t$-absorbing submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_t$.

(3) If N_i is a quasi n_i-absorbing submodule of M_i for every $1 \le i \le t$ with $n_1 < n_2 < ... < n_t$ and $t > 2$, then $N_1 \oplus N_2 \oplus \cdots \oplus N_t$ is a quasi $n_t + t$-absorbing submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_t$.

Proof. 1. Let $a \in R$ and $(x,y) \in M_1 \oplus M_2$ such that $a^{n+1}(x,y) \in N_1 \oplus N_2$. Therefore $a^{n+1}x = a^n(ax) \in N_1$ and $a^{n+1}y \in N_2$. Since N_1 is a quasi n-absorbing submodule of M_1, then $a^n \in (N_1 :_R M_1)$ or $a^n x \in N_1$. Also, N_2 is a quasi m-absorbing submodule of M_2 and $a^{n+1}y = a^m(a^{n+1-m}x) \in N_2$, so $a^m \in (N_2 :_R M_2)$ or $a^{m-1}(a^{n+1-m}y) = a^n y \in N_2$.

Consider the following cases.

Case 1. Assume that $a^n \in (N_1 :_R M_1)$ and $a^m \in (N_2 :_R M_2)$. Then $a^n \in (N_1 \oplus N_2 :_R M_1 \oplus M_2)$.

Case 2. Assume that $a^n \in (N_1 :_R M_1)$ and $a^n y \in N_2$. Then $a^n(x,y) \in N_1 \oplus N_2$.

Case 3. Assume that $a^n x \in N_1$ and $a^m \in (N_2 :_R M_2)$. Then $a^n(x,y) \in N_1 \oplus N_2$.

Case 4. Assume that $a^n x \in N_1$ and $a^n y \in N_2$. Then $a^n(x,y) \in N_1 \oplus N_2$.

We conclude that $N_1 \oplus N_2$ is a quasi $n+1$-absorbing submodule of M.

2. We use induction on t. For $t = 1$ the claim is follows from (1). Let $t > 1$ and assume that for $t-1$ the claim holds. That is $N_1 \oplus N_2 \oplus \cdots \oplus N_{t-1}$ is a quasi $n+t-1$-absorbing submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_{t-1}$. As N_t is a quasi n-absorbing submodule of M_t, so it is a quasi $(n+t-2)$-absorbing submodule of M_t. Therefore $N_1 \oplus N_2 \oplus \cdots \oplus N_{t-1} \oplus N_t$ is a quasi $n+t$-absorbing submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_{t-1} \oplus M_t$.

3. We use induction on t. The results for $t = 3$ follows from (1) and (2). Let $t > 3$ and suppose that for $t-1$ the claim holds. Then $N_1 \oplus N_2 \oplus \cdots \oplus N_{t-1}$ is a quasi $n_{t-1}+2$-absorbing submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_{t-1}$. We consider the following cases:

Case 1. If $n_{t-1} + 2 < n_t$. In this case $N_1 \oplus \cdots \oplus N_t$ is a quasi $n_t + 1$-absorbing submodule of $M_1 \oplus \cdots \oplus M_t$ by (1). Hence $N_1 \oplus \cdots \oplus N_t$ is a quasi $n_t + 2$-absorbing submodule of $M_1 \oplus \cdots \oplus M_t$.

Case 2. If $n_{t-1}+2 = n_t$. Then $N_1 \oplus \cdots \oplus N_t$ is a quasi n_t+2-absorbing submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_t$ by (2).

Case 3. If $n_{t-1} + 2 > n_t$. Then $N_1 \oplus \cdots \oplus N_t$ is a quasi $n_{t-1} + 3$-absorbing submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_t$ by (1). Since $n_{t-1} + 1 = n_t$, because $n_{t-1} < n_t$. We conclude that $N_1 \oplus \cdots \oplus N_t$ is a quasi $n_t + 2$-absorbing submodule of $M_1 \oplus M_2 \oplus \cdots \oplus M_t$.

□

Theorem 2.20. Let M be an R-module and N a P-primary submodule of M such that $P^n M \subseteq N$. Then N is a quasi n-absorbing submodule of M. In particular, if M is a multiplication module and $P^n M$ is a P-primary submodule of M, then $P^n M$ is a quasi n-absorbing submodule of M.

Proof. Let $a \in R$ and $x \in M$ such that $a^n x = a.a^{n-1}x \in N$ and $a^{n-1}x \notin N$. Since N is a P-primary submodule, we conclude that $a \in P = \sqrt{(N :_R M)}$. Hence $a^n \in P^n \subseteq (N :_R M)$. Thus N is a quasi n-absorbing submodule of M. The in particular is clear. □

Theorem 2.21. If N is a secondary submodule of an R-module M and L is a quasi n-absorbing submodule of M. Then $N \cap L$ is also a secondary submodule of M.

Proof. Assume that N is P- secondary where $P := \sqrt{(0 :_R N)}$, and let $a \in R$. If $a \in P$ and since $(0 :_R N) \subseteq (0 :_R N \cap L)$ we conclude that $a \in \sqrt{(0 :_R N \cap L)}$. If $a \notin P$, than $aN = N$, so $a^n N = N$. Let $x \in N \cap L$, there exists $y \in N$ with $x = a^n y$. Thus $a^n y \in L$, but L is quasi n-absorbing submodule we get either $a^n \in (L :_R M)$ or $a^{n-1}y \in L$. If $a^n \in (L :_R M)$ then $a^n M \subseteq L$, on the other hand $N = a^n N \subseteq a^n M \subseteq L$ so $a(N \cap L) = aN = N = N \cap L$. If $a^{n-1}y \in L$, then $x = a^n y \in aL$, and $x \in a(N \cap L)$ hence $a(N \cap L) = N \cap L$. Which is the desired conclusion. □

Proposition 2.22. *Let $f : M \to M'$ be a homomorphism of R-modules. Then we have the following*

(1) If N' is a quasi n-absorbing submodule of M', then $f^{-1}(N')$ is a quasi n-absorbing submodule of M.

(2) If f is an epimorphism and N is a quasi n-absorbing submodule of M containing $\ker(f)$*, then $f(N)$ is quasi n-absorbing submodule of M'.*

Proof. 1. Let N' be a quasi n-absorbing submodule of M' and $a^n x \in f^{-1}(N')$ for some $a \in R$ and $x \in M$. Then $f(a^n x) = a^n f(x) \in N'$, and therefore, either $a^n \in (N' :_R M')$ or $a^{n-1} f(x) = f(a^{n-1}x) \in N'$. Thus $a^n \in (f^{-1}(N') :_R M)$ or $a^{n-1}x \in f^{-1}(N')$, as desired.

2. Let $a \in R$ and $x' \in M'$ such that $a^n x' \in f(N)$. Since f is an epimorphism, so there exists $x \in M$ such that $x' = f(x)$ and $a^n x' = a^n f(x) = f(a^n x) \in f(N)$. Also $\ker(f) \subseteq N$ implies that $a^n x \in N$. As N is a quasi n-absorbing submodule of M, either $a^n \in (N :_R M)$ or $a^{n-1}x \in N$. Therefore either $a^n \in (f(N) :_R M')$, as f is an epimorphism, or $a^{n-1}x' \in f(N)$. Thus $f(N)$ is a quasi n-absorbing submodule of M'.

□

Let R be a ring and M an R-module. Then $R(M) := R \propto M = \{(a,x)/a \in R, x \in M\}$, with multiplication $(a,x)(b,y) = (ab, ay + bx)$ and with point wise addition, is a commutative ring with identity, called the idealization of an R-module M. Let I be an ideal of R and N be a submodule of M, then $I \propto N$ is an ideal of $R(M)$ if and only if $IM \subseteq N$, i.e. $I \subseteq (N :_R M)$.

Theorem 2.23. Let R be a ring, M an R-module and $n > 1$ a positive integer.

(1) For an ideal I of R and a submodule N of M. If $I \propto N$ is a quasi n-absorbing ideal of $R(M)$, then I is a quasi n-absorbing ideal of R and N is a quasi n-absorbing submodule of M. In particular I is quasi n-absorbing ideal of R if and only if $I \propto M$ is a quasi n-absorbing ideal of $R(M)$.

(2) Assume that $\mathbb{Q} \subseteq R$. If N is a quasi $n-1$-absorbing submodule of M and I is a quasi n-absorbing ideal of R, then $I \propto N$ is a quasi $n+1$-absorbing ideal of $R(M)$.

(3) Assume that R is an integral domain, then $0 \propto N$ is a quasi n-absorbing ideal of $R(M) := R \propto M$ if and only if N is a quasi n-absorbing submodule of M and $(N :_R M)$ is a semi $n-1$-absorbing ideal of R.

(4) Assume that R is an integral domain and M is a K-vector space where K is the quotient field of R. Then $0 \propto N$ is a quasi n-absorbing ideal of $R(M)$ if and only if N is a K-subspace of M. Furthermore, the quasi n-absorbing ideals of $R(M)$ are either of the form $I \propto M$, where I is a quasi n-absorbing ideal of R, or $0 \propto N$, where N is a K-subspace of M.

Proof. 1. Assume that $I \propto N$ is an quasi n-absorbing ideal of $R(M)$. Let $a, b \in R$ such that $a^n b \in I$, then $(a,0)^n(b,0) \in I \propto N$. Since $I \propto N$ is a quasi n-absorbing ideal, therefore $(a,0)^n \in I \propto N$ or $(a,0)^{n-1}(b,0) \in I \propto N$. Hence $a^n \in I$ or $a^{n-1}b \in I$, that is, I is quasi n-absorbing ideal of R. Now let $a \in R$ and $x \in M$ such that $a^n x \in N$, then $(a,0)^n(0,x) \in I \propto N$. As $I \propto N$ is quasi n-absorbing ideal of $R(M)$, so $(a,0)^n \in I \propto N$ or $(a,0)^{n-1}(0,x) \in I \propto N$. Consequently, $a^n \in I \subseteq (N :_R M)$ or $a^{n-1}x \in N$, that is N is a quasi n-absorbing submodule of M.

2. Let $(a,e),(b,f) \in R(M)$ such that $(a,e)^{n+1}(b,f) = (a^{n+1}b, (n+1)a^n be + a^{n+1} f) \in I \propto N$. Then $a^{n+1}b \in I$ and $(n+1)a^n be + a^{n+1} f \in N$. Therefore, since I is a quasi n-absorbing ideal of R, $a^n \in I$ or $a^{n-1}b \in I$. If $a^n \in I$, then $(a,e)^{n+1} = (a^{n+1}, (n+1)a^n e) \in I \propto N$ because $IM \subseteq N$. If $a^n b \in I$, then $a^{n+1} f = a^{n-1}(a^2 f) \in N$. Consequently, as N is quasi $n-1$-absorbing submodule of M, $a^n f \in N$. In the mean time, $(n+1)a^n be \in N$ and $\mathbb{Q} \subseteq R$, and the fact that $[N :_R \{e\}]$ is quasi $(n-1)$-absorbing ideal of R, we conclude that $a^{n-1}be \in N$. Finally $(a,e)^n(b,f) = (a^n b + na^{n-1}be + a^n f) \in I \propto N$. That is $I \propto N$ is a quasi $n-1$- absorbing ideal of $R(M)$.

3. Assume that $0 \propto N$ is a quasi n-absorbing ideal of $R(M)$. Then from (1) we conclude that N is a quasi n-absorbing submodule of M. Now, let $0 \neq a \in R$ such that $a^n \in (N :_R M)$. Then $(a,0)^n(0,x) \in 0 \propto N$ for every $x \in M$. Since $0 \propto N$ is quasi n-absorbing ideal and $(a,0)^n \notin 0 \propto N$, we conclude that $(a,0)^{n-1}(0,x) = (0, a^{n-1}x) \in 0 \propto N$. Then $a^{n-1}x \in N$ for each $x \in M$. Hence $a^{n-1} \in (N :_R M)$, that is $(N :_R M)$ is a semi $n-1$-absorbing ideal of R.

   Conversely assume that $(N :_R M)$ is a semi $n-1$-absorbing ideal of R and N is a quasi n-absorbing submodule of M. Let $(a,x),(b,y) \in R(M)$ such that $(a,x)^n(b,y) \in 0 \propto N$. Then $a^n b = 0$ and $a^n y + nba^{n-1}x \in N$. So, since R is an integral domain, $a = 0$ or $b = 0$. If $a = 0$, then $(0,x)^n = (0,0) \in I \propto N$. If $b = 0$, then $a^n y \in N$. Consequently, since N is a quasi n-absorbing submodule of M, $a^n \in (N :_R M)$ or $a^{n-1}y \in N$. Suppose that $a^n \in (N :_R M)$, the fact that $(N :_R M)$ is semi $(n-1)$-absorbing ideal of R, implies $a^{n-1} \in [N :_R M]$. Hence $(a,x)^{n-1}(0,y) = (0, a^{n-1}y) \in 0 \propto N$. Now suppose that $a^{n-1}y \in N$, then $(a,x)^{n-1}(0,y) = (0, a^{n-1}y) \in 0 \propto N$. Thus,$0 \propto N$ is quasi n-absorbing ideal of $R(M)$, as desired.

4. Assume that N is a K-subspace of M, then by [9, Theorem 3.2] $0 \propto N$ is n-absorbing ideal of $R(M)$ henceforth quasi n-absorbing ideal of $R(M)$. Now assume that $0 \propto N$ is a quasi n-absorbing ideal of $R(M)$. Let a be a nonzero element of R, we show that $\frac{1}{a}f \in N$ for every $f \in N$. For this purpose, consider $f \in N$ and assume that $x = (a,0), y = (0, \frac{f}{a^n})$, then $x^n y = (0,f) \in 0 \propto N$. Since $a \neq 0$ and $0 \propto N$ is a quasi n-absorbing ideal of $R(M)$, we conclude that $x^{n-1}y = (0, \frac{f}{a}) \in 0 \propto N$. Thus $\frac{f}{a} \in N$. Now, if $f \in N$ and $b \in K$. Then $b = \frac{c}{a}$ for some $c, a \in R$ with $a \neq 0$. Therefore, since $\frac{1}{a}f \in N$ and N is a R submodule of M, we conclude that $bf \in N$. Hence N is a K-subspace of M. Furthermore, every ideal of $R(M)$ has either of the form $I \propto M$ or $0 \propto N$, where I is an ideal of R and N is a submodule of M, since M is R-module divisible. Now the result follows from (3) and (1).

□

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References

[1] M. T. Ahmed, T. Dumitrescu, and M. A. Khadam, *Commutative rings with absorbing factorization*, Comm. Algebra 48 (2020), 5067-5075.

[2] M. M. Ali, *Invertibility of multiplication modules*, New Zealand J. Math 35 (2006). 17-29.

[3] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Comm. Algebra 36 (2008), 4620-4642.

[4] M. M. Ali, *Invertibility of multiplication modules III*, New Zealand J. Math . 39 (2009), 193-213.

[5] D. F. Anderson, A. Badawi, *On the n-absorbing ideals of commutative rings*, Comm. Algebra 39 (2011), 1646-1672.

[6] D.D. Anderson and M. Winderes, *Idealisation of a module*, J. Comm. Algebra 1 (2009), 3-56.

[7] A. Badawi, *On the 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. 75 (2007), 417-429.

[8] A. Badawi, A.Y. Darani, *On weakly 2-absorbing ideals of commutative rings*, Houston J. Math.

[9] A. Badawi, M. Issoual and N. Mahdou, *On n-absorbing ideals and (m, n)-closed ideals in trivial extension rings*, J. Algebra Appl. 18(7), 2019.

[10] A.Y.Darani and F. Soheilnia, *2-absorbing and weakly 2-absorbing submodules*, Thai J. Math. 9 (2011), 577-584.

[11] A.Y.Darani and F. Soheilnia, *On n-absorbing submodules*, Math, commune.17 (2012), 547-557

[12] S. Kabbaj, *Matlis' semi-regularity and semi-coherence in trivial ring extensions: a survey*, Moroccan J. Algebra, Geom. Appl. 1(1) (2022), 1–17

[13] H. Mostafanasab, A. Y. Darani, *On n-absorbing ideals and two generalizations of semiprime ideals*, Thai J. Math 15 (2017) 387-408.

[14] M. Mukhtar, M.T. Ahmed and T. Dumitrescu, Commutative rings with two-absorbing factorization, Comm. Algebra 46 (2018), 970-978.

[15] P. F. Smith, Z. EL-Bast, *Multiplication modules*, Comm. Algebra 16 (1988), 755-799.

[16] E. A. Ugurlu, *On 1-absorbing prime submodules*, Moroccan J. Algebra, Geom. Appl. 2(1) (2023), 162–173