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New Eisenstein series identities for the Ramanujan–Göllnitz–Gordon continued fraction and combinatorial partition identities

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Abstract. In this paper, we establish several new Eisenstein series identities associated with Ramanujan–Göllnitz–Gordon continued fraction. We further derive inter relationships between Eisenstein series identities and combinatorial partition identities.
Key Words: Continued fraction, Eisenstein series, combinatorial partition identities.
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1 Introduction

Throughout this paper, we assume that q is a complex number such that $|q| < 1$ and use the standard product notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

For convenience, we sometimes use the multiple q-shifted factorial notation, which is defined as

$$(a_1, a_2, \ldots, a_m; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_m;q)_\infty.$$

The celebrated Ramanujan–Göllnitz–Gordon continued fraction is defined as

$$G(q) = \frac{q^{1/2}}{1+q} + \frac{q^2}{1+q^3} + \frac{q^4}{1+q^5} + \frac{q^6}{1+q^7} + \cdots. \tag{1}$$

An interesting product representation of $G(q)$ is recorded in [18]

$$G(q) = q^{\frac{1}{2}} \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty}. \tag{2}$$

On page 299 of his second notebook [18], Ramanujan recorded the two identities,

$$\frac{1}{G(q)} - G(q) = \frac{(-q^2;q^4)_\infty^2 (q^4;q^4)_\infty (q^4;q^8)_\infty}{q^{1/2}(q^8;q^8)_\infty}, \tag{3}$$

and

$$\frac{1}{G(q)} + G(q) = \frac{(-q;q^2)_\infty^2 (q^2;q^2)_\infty (q^4;q^8)_\infty}{q^{1/2}(q^8;q^8)_\infty}. \tag{4}$$

The identity (2) was rediscovered and proved independently by Göllnitz [12] and Gordon [13]. The identities (3) and (4) were first proved by Berndt [3] and then rediscovered by Chan and Huang [11].

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is equal to n. The integers in a partition are called *parts*. We denote the number of partitions of n by $p(n)$, and follow the convention $p(0) := 1$. For instance, since the partitions of 4 are

$$4\ ,\ 3+1\ ,\ 2+2\ ,\ 2+1+1 \text{ and } 1+1+1+1,$$

we have $p(4) = 5$. Euler established the identity

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{5}$$

Andrews et al. [2] investigated new double summation hypergeometric q-series representations for several families of partitions and further explored the role of double series in combinatorial partition identities by introducing the following general family:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn} r(l,u,v,w;n), \tag{6}$$

where

$$r(l,u,v,w;n) := \sum_{j=0}^{[\frac{n}{u}]} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j}. \tag{7}$$

The following interesting special cases of (6) are recalled [2, p. 106]

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty; \tag{8}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty; \tag{9}$$

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{10}$$

In [6], Chaudhary et al. have established the combinatorial partition identities for the identities (3) and (4) using (8)-(10). In [4, 5, 7, 8, 9, 10], Chaudhary et al. have established the several identities between q-product identities and combinatorial partition identities. Recently, Srivastava et al. [19, 20] established new results which depict the inter relationships between q-product identities, continued fraction identities and combinatorial partition identities.

The main purpose of this paper is to establish many new Eisenstein series identities involving the Ramanujan–Göllnitz–Gordon continued fraction. We further develop inter relationships between Eisenstein series identities and combinatorial partition identities for $G(q)$.

2 Definitions and Preliminary results

In this section, we present some basic definitions and preliminary results on Ramanujan's theta functions. Ramanujan's theta function $f(a,b)$ is defined, for $|ab|<1$, by

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}. \tag{11}$$

In Ramanujan's notation, Jacobi triple product identity is given by [1, Entry 19]

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{12}$$

Then it is easy to verify that

$$f(a,b)=f(b,a),\ f(1,a)=2f(a,a^3),\ f(-1,a)=0.$$

If n is an integer,

$$f(a,b)=a^{n(n+1)/2}\, b^{n(n-1)/2}\, f(a(ab)^n,b(ab)^{-n}). \tag{13}$$

The three most interesting special cases of $f(a,b)$ are [1, Entry 22]

$$\varphi(q):=f(q,q)=\sum_{n=-\infty}^{\infty} q^{n^2}=(-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{14}$$

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty} q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \tag{15}$$

and

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_\infty. \tag{16}$$

Note that $\eta(\tau)=q^{1/24}f(-q)$ where $q=e^{2\pi i\tau}$, $\mathrm{Im}\tau>0$, and $\eta(\tau)$ is the Dedekind-eta function. Also, after Ramanujan, define

$$\chi(q):=(-q;q^2)_\infty. \tag{17}$$

The following q-serie identity:

$$(-q;q)_\infty=\frac{1}{(q;q^2)_\infty}=\frac{1}{\chi(-q)} \tag{18}$$

provides the analytical equivalence of Euler's theorem: The number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts. For convenience, we define, for a positive integer n, that

$$f_n:=f(-q^n)=(q^n;q^n)_\infty.$$

The following lemma is a consequence of the product representations (14)–(17).

Lemma 2.1. *[1] We have*

$$\varphi(q)=\frac{f_2^5}{f_1^2f_4^2},\quad \psi(q)=\frac{f_2^2}{f_1},\quad \varphi(-q)=\frac{f_1^2}{f_2},\quad \psi(-q)=\frac{f_1f_4}{f_2},$$

$$f(q)=\frac{f_2^3}{f_1f_4},\quad \chi(q)=\frac{f_2^2}{f_1f_4}\quad \text{and}\quad \chi(-q)=\frac{f_1}{f_2}.$$

The Jacobi theta function θ_1, which is defined as

$$\theta_1(x|z) = 2\sum_{n=0}^{\infty}(-1)^n q^{\frac{(2n+1)^2}{8}}\sin(2n+1)x = 2q^{\frac{1}{8}}\sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}}\sin(2n+1)x. \tag{19}$$

In [14], Chan et al. showed that

$$\begin{aligned} 2\sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}}\sin(2n+1)x &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(n+1)}{2}}\sin(2n+1)x \\ &= 2(\sin x)(q;q)_\infty(qe^{2ix};q)_\infty(qe^{-2ix};q)_\infty. \end{aligned} \tag{20}$$

Combining (20) and (19), we find the infinite product representation of θ_1:

$$\begin{aligned} \theta_1(x|z) &= 2q^{\frac{1}{8}}(\sin x)(q;q)_\infty(qe^{2ix};q)_\infty(qe^{-2ix};q)_\infty \\ &= iq^{\frac{1}{8}}e^{-ix}(q;q)_\infty(e^{2ix};q)_\infty(qe^{-2ix};q)_\infty. \end{aligned} \tag{21}$$

Differentiating both sides of the equation (21) and then setting $x=0$ yields

$$\theta_1'(0|z) = 2q^{1/8}(q;q)_\infty^3, \tag{22}$$

where θ_1' denotes the partial derivative of θ_1 with respect to x.

3 Eisenstein series identities associated with $G(q)$

In this section, we establish many new Eisenstein series identities associated with $G(q)$ by using the Ramanujan's ${}_1\Psi_1$ summation formula and the Jacobi theta function θ_1.

Theorem 3.1. Let $|q|<1$. Then, we have the Eisenstein series identities

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty}\frac{q^n-q^{3n}}{1-q^{8n}} + \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty}\frac{q^{5n}-q^{7n}}{1-q^{8n}} = \frac{\eta^4(16\tau)}{\eta^2(8\tau)}\left(\frac{1}{G(q^2)} - G(q^2)\right), \tag{23}$$

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty}\frac{q^n+q^{5n}}{1-q^{8n}} - \sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty}\frac{q^{3n}+q^{7n}}{1-q^{8n}} = \frac{\eta^2(32\tau)\eta(16\tau)}{\eta(8\tau)}\left(\frac{1}{G(q^4)} + G(q^4)\right), \tag{24}$$

and

$$\begin{aligned} &\sum_{\substack{n=1 \\ n\equiv 1 \pmod 8}}^{\infty}\frac{q^n-q^{5n}}{1-q^{8n}} - \sum_{\substack{n=1 \\ n\equiv 3 \pmod 8}}^{\infty}\frac{q^{3n}+q^{7n}}{1-q^{8n}} + \sum_{\substack{n=1 \\ n\equiv 5 \pmod 8}}^{\infty}\frac{q^n+q^{5n}}{1-q^{8n}} - \sum_{\substack{n=1 \\ n\equiv 7 \pmod 8}}^{\infty}\frac{q^{7n}-q^{3n}}{1-q^{8n}} \\ &= \frac{\eta^2(16z)\eta^2(64z)}{\eta(8z)\eta(32z)}\left(\frac{1}{G(q^8)} - G(q^8)\right). \end{aligned} \tag{25}$$

Proof. Changing n to $-n$ in the second summation, in the left-hand side of (23), we obtain

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty}\frac{q^n-q^{3n}}{1-q^{8n}} + \sum_{\substack{n=-\infty \\ n\equiv 1 \pmod 2}}^{-1}\frac{q^{-5n}-q^{-7n}}{1-q^{-8n}} = \sum_{n=-\infty}^{\infty}\frac{q^{2n+1}}{1-q^{16n+8}} - \sum_{n=-\infty}^{\infty}\frac{q^{6n+3}}{1-q^{16n+8}}. \tag{26}$$

Using a corollary of Ramanujan's ${}_1\Psi_1$ summation formula with $b = aq$ [1, Entry 17, p. 32]

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1-aq^n} = \frac{(az, q/az, q, q; q)_\infty}{(a, q/a, z, q/z; q)_\infty}, \qquad |q| < |z| < 1, \tag{27}$$

in (26), we find that

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty} \frac{q^n - q^{3n}}{1-q^{8n}} + \sum_{\substack{n=-\infty \\ n\equiv 1 \pmod 2}}^{-1} \frac{q^{-5n} - q^{-7n}}{1-q^{-8n}}$$
$$= \frac{(q^{16}; q^{16})_\infty^2}{(q^8; q^{16})_\infty^2} \left\{ q \frac{(q^6, q^{10}; q^{16})_\infty}{(q^2, q^{14}; q^{16})_\infty} - q^3 \frac{(q^2, q^{14}; q^{16})_\infty}{(q^6, q^{10}; q^{16})_\infty} \right\}. \tag{28}$$

Using (2) in (28), we obtain (23). Proofs of (24) and (25) are similar to the proof of (23), so we omit them here. □

Now we prove the following remarkable identity:

Lemma 3.2. *We have*

$$\sum_{n=1}^{\infty} \frac{q^n - q^{3n} - q^{5n} + q^{7n}}{1-q^{8n}} \sin 2nx = -\frac{\theta_1'(0|8z)\theta_1(2\pi z|8z)\theta_1(4\pi z|8z)\theta_1(2x|8z)}{4\theta_1(x+\pi z|8z)\theta_1(x-\pi z|8z)\theta_1(x+3\pi z|8z)\theta_1(x-3\pi z|8z)}. \tag{29}$$

Proof. For simplicity, we use $J(x|z)$ to denote the logarithmic derivative of θ_1 with respect to x. Logarithmically differentiating (21) with respect to x, after some manipulation, we have

$$J\left(x + \frac{\pi z}{2}\Big|z\right) = -i + 4\sum_{n=1}^{\infty} \frac{q^{n/2}}{1-q^n} \sin 2nx. \tag{30}$$

Replacing z by $8z$ in the above equation, we deduce that

$$J(x + 4\pi z|8z) = -i + 4\sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{8n}} \sin 2nx. \tag{31}$$

Replacing x by $x - 3\pi z$ in the above equation, we obtain

$$J(x + \pi z|8z) = -i + 4\sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{8n}} \sin 2n(x - 3\pi z).$$

Writing x as $-x$ in the above equation, we are led to the identity

$$J(x - \pi z|8z) = i + 4\sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{8n}} \sin 2n(x + 3\pi z).$$

Adding the above two equations together and using the trigonometric identity

$$\sin 2n(x - 3\pi z) + \sin 2n(x + 3\pi z) = 2\cos 6n\pi z \sin 2nx = (q^{3n} + q^{-3n}) \sin 2nx,$$

in the resulting equation, we immediately deduce that

$$J(x - \pi z|8z) + J(x + \pi z|8z) = 4\sum_{n=1}^{\infty} \frac{q^n + q^{7n}}{1-q^{8n}} \sin 2nx.$$

In a similar way, we find that

$$J(x-3\pi z|8z)+J(x+3\pi z|8z)=4\sum_{n=1}^{\infty}\frac{q^{3n}+q^{5n}}{1-q^{8n}}\sin 2nx.$$

Combining the above two equations, we find that

$$4\sum_{n=1}^{\infty}\frac{q^{n}-q^{3n}-q^{5n}+q^{7n}}{1-q^{8n}}\sin 2nx$$
$$=J(x-\pi z|8z)+J(x+\pi z|8z)-J(x-3\pi z|8z)-J(x+3\pi z|8z). \tag{32}$$

Recall the following identity, which can be found in [15, Theorem 5], [17, Corollary 2]:

$$J(x_1|z)+J(x_2|z)+J(x_3|z)-J(x_1+x_2+x_3|z)$$
$$=\frac{\theta_1'(0|z)\theta_1(x_1+x_2|z)\theta_1(x_2+x_3|z)\theta_1(x_1+x_3|z)}{\theta_1(x_1|z)\theta_1(x_2|z)\theta_1(x_3|z)\theta_1(x_1+x_2+x_3|z)}. \tag{33}$$

Replacing z by $8z$ in the above equation and then letting x_1 to $x-\pi z, x_2$ to $x+\pi z, x_3$ to $-x+3\pi z$, we obtain

$$J(x-\pi z|8z)+J(x+\pi z|8z)-J(x-3\pi z|8z)-J(x+3\pi z|8z)$$
$$=-\frac{\theta_1'(0|8z)\theta_1(2\pi z|8z)\theta_1(4\pi z|8z)\theta_1(2z|8z)}{\theta_1(x+\pi z|8z)\theta_1(x-\pi z|8z)\theta_1(x+3\pi z|8z)\theta_1(x-3\pi z|8z)}. \tag{34}$$

Combining (34) and (32), we get (29). □

Using identity (29), we can obtain the following Eisenstein series identities:

Theorem 3.3. Let $|q|<1$. Then, we have the Eisenstein series identities

$$\sum_{n=1}^{\infty}\frac{n(q^{n}-q^{3n}-q^{5n}+q^{7n})}{1-q^{8n}}=\frac{\eta^4(8z)\eta^2(4z)}{\eta^2(2z)}\left(\frac{1}{G(q)}+G(q)\right), \tag{35}$$

and

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^{n}-q^{3n}-q^{5n}+q^{7n}}{1-q^{8n}}=\frac{\eta(z)\eta^2(2z)\eta(6z)\eta^3(8z)\eta(24z)}{\eta(3z)\eta^5(4z)}\left(\frac{1}{G(q)}-G(q)\right). \tag{36}$$

Proof. Dividing both sides of (29) by z and then letting $z\to 0$, we get

$$\sum_{n=1}^{\infty}\frac{n(q^{n}-q^{3n}-q^{5n}+q^{7n})}{1-q^{8n}}=-\frac{\theta_1'(0|8z)^2\theta_1(2\pi z|8z)\theta_1(4\pi z|8z)}{4\theta_1^2(\pi z|8z)\theta_1^2(3\pi z|8z)}. \tag{37}$$

Using (21), we easily find that

$$\theta_1(\pi z|8z)=iq^{1/2}(q,q^7,q^8;q^8)_\infty,\quad \theta_1(2\pi z|8z)=i(q^2,q^6,q^8;q^8)_\infty, \tag{38}$$
$$\theta_1(3\pi z|8z)=iq^{-1/2}(q^3,q^5,q^8;q^8)_\infty,\ \theta_1(4\pi z|8z)=iq^{-1}(q^4;q^8)_\infty^2(q^8;q^8)_\infty. \tag{39}$$

Employing (38) and (39) in (37), after simplification, we obtain

$$\sum_{n=1}^{\infty}\frac{n(q^{n}-q^{3n}-q^{5n}+q^{7n})}{1-q^{8n}}=\frac{q(q^2;q^2)_\infty(q^4;q^4)_\infty(q^8;q^8)_\infty^2}{(q;q^2)_\infty^2}. \tag{40}$$

Using Lemma 2.1 and (4) in the right-hand side of the (40), then after some simplifications, we obtain our desire identity (35).

If p is a prime, we use $\left(\frac{\cdot}{p}\right)$ to denote the Legendre symbol modulo p. Setting $x = \frac{\pi}{3}$ in (29) and noting that

$$\sin\frac{2n\pi}{3} = \frac{\sqrt{3}}{2}\left(\frac{n}{3}\right), \quad \theta_1(\frac{2\pi}{3}|z) = \sqrt{3}q^{1/8}(q^3;q^3)_\infty,$$

we find that

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n-q^{3n}-q^{5n}+q^{7n}}{1-q^{8n}} = -\frac{q\theta_1'(0|8z)\theta_1(2\pi z|8z)\theta_1(4\pi z|8z)(q^{24};q^{24})_\infty}{2\theta_1(\pi/3+\pi z|8z)\theta_1(\pi/3-\pi z|8z)\theta_1(\pi/3+3\pi z|8z)\theta_1(\pi/3-3\pi z|8z)}. \tag{41}$$

Recall the beautiful identity [16, Eq. (3. 1)]

$$\theta_1\left(\frac{\pi}{3}-x|z\right)\theta_1\left(\frac{\pi}{3}+x|z\right) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}\frac{\theta_1(3x|3z)}{\theta_1(x|z)}.$$

Using the above identity in (41), after some simplifications, we obtain

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n-q^{3n}-q^{5n}+q^{7n}}{1-q^{8n}} = \frac{q(q;q)_\infty(q^4;q^4)_\infty(q^{24};q^{24})_\infty}{(q^8;q^8)_\infty(q^3;q^6)_\infty}. \tag{42}$$

Using Lemma 2.1 and (3) in the right-hand side of the (42), then after some simplifications, we obtain our desire identity (36). □

4 Combinatorial partition identities

In this section, we give certain inter relationships between Eisenstein series identities and combinatorial partition identities.

Theorem 4.1. Let $|q| < 1$. Then, we have the identities

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty}\frac{q^n-q^{3n}}{1-q^{8n}} + \sum_{\substack{n=1 \\ n\equiv 1 \pmod 2}}^{\infty}\frac{q^{5n}-q^{7n}}{1-q^{8n}} = q^2R^2(8,8,1,1,1,2)\left(\frac{1}{G(q^2)}-G(q^2)\right), \tag{43}$$

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 4}}^{\infty}\frac{q^n+q^{5n}}{1-q^{8n}} - \sum_{\substack{n=1 \\ n\equiv 3 \pmod 4}}^{\infty}\frac{q^{3n}+q^{5n}}{1-q^{8n}} = q^3R(8,8,1,1,1,2)R(16,16,1,1,1,2)\left(\frac{1}{G(q^4)}+G(q^4)\right), \tag{44}$$

and

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 8}}^{\infty}\frac{q^n-q^{5n}}{1-q^{8n}} - \sum_{\substack{n=1 \\ n\equiv 3 \pmod 8}}^{\infty}\frac{q^{3n}+q^{7n}}{1-q^{8n}} + \sum_{\substack{n=1 \\ n\equiv 5 \pmod 8}}^{\infty}\frac{q^n+q^{5n}}{1-q^{8n}} - \sum_{\substack{n=1 \\ n\equiv 7 \pmod 8}}^{\infty}\frac{q^{7n}-q^{3n}}{1-q^{8n}} = q^5R(8,8,1,1,1,2)R(32,32,1,1,1,2)\left(\frac{1}{G(q^8)}-G(q^8)\right), \tag{45}$$

Proof. Using the identity of (10) with $m = 8$ in (23), we are led to the desire identity (43). Next, using the identity of (10) with $m = 8$ and $m = 16$ in (24), we are led to the desire identity (44). Further, using the identity of (10) with $m = 8$ and $m = 32$ in (25), we are led to the desire identity (45). □

Theorem 4.2. Let $|q| < 1$. Then, we have the identities

$$\sum_{n=1}^{\infty} \frac{n(q^n - q^{3n} - q^{5n} + q^{7n})}{1 - q^{8n}} = q(q^8;q^8)_\infty^2 R(2,2,1,1,2,2) R^2(1,1,1,1,1,2), \tag{46}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n - q^{3n} - q^{5n} + q^{7n}}{1 - q^{8n}} = \frac{q(q;q)_\infty (q^8;q^8)_\infty (q^{24};q^{24})_\infty}{(q^3;q^6)_\infty R(4,4,1,1,1,2)}. \tag{47}$$

Proof. Using the identities (9) and (10) with $m = 1$ in (35), we are led to the desire identity (46). Similarly, using the identity of (10) with $m = 4$ in (36), we are led to the desire identity (47). □

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