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Author(s):

Younes El Haddaoui, Hwankoo Kim & Najib Mahdou

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Younes El Haddaoui[1], Hwankoo Kim[2] and Najib Mahdou[3]
[1] Department of Mathematics, Faculty of Science and Technology, Fez, Morocco.
e-mail: *younes.elhaddaoui@usmba.ac.ma*
[2] Division of Computer Engineering, Hoseo University, Asan, Republic of Korea.
e-mail: *hkkim@hoseo.edu*
[3] Department of Mathematics, Faculty of Science and Technology, Fez, Morocco.
e-mail: *mahdou@hotmail.com*

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Abstract. The study of Gorenstein projective and injective modules has been a cornerstone in the field of Gorenstein homological algebra since these concepts were first introduced. This paper marks a significant advancement in the field by demonstrating the integration of ϕ-torsion theory into Gorenstein homological algebra. We take this exploration further by introducing and examining the novel concepts of nonnil-Gorenstein projective and injective modules. Our study also extends to the nonnil-Gorenstein projective and injective dimensions of a module, offering a deeper insight into their structure and implications. Furthermore, we delve into the concept of nonnil-Gorenstein global dimension of a ring, unveiling its significance and potential applications. A key application of these innovative concepts is their use in characterizing ϕ von Neumann regular rings. This approach not only adds a new dimension to our understanding of these rings but also highlights the versatility and depth of Gorenstein homological algebra.

Key Words: ϕ-torsion theory, ϕ-(weak) global dimension of rings, nonnil-Gorenstein global dimension of rings.
2020 MSC: 13A15, 13A18, 13F05, 13G05, 13C20.

Dedicated to our Professor David E. Dobbs for his 80$^{\text{th}}$ Birthday.

1 Introduction

We devote this introductory section to some conventions and a review of some standard background material. All rings considered in this paper are assumed to be commutative with non-zero identity and prime nilradical. We use $\mathrm{Nil}(R)$ to denote the set of nilpotent elements of R, and $Z(R)$ to denote the set of zero-divisors of R. A ring with $\mathrm{Nil}(R)$ being divided prime (i.e., $\mathrm{Nil}(R) \subset xR$, for every $x \in R \setminus \mathrm{Nil}(R)$) is called a ϕ-ring. Let $\mathcal{H}$ (resp., $\overline{\mathcal{H}}$) be the set of all rings with divided prime nilradical (resp., divided prime nilradical but not maximal). A ring R is called a strongly ϕ-ring if $R \in \mathcal{H}$ and $Z(R) = \mathrm{Nil}(R)$.
Let R be a ring and M be an R-module. We define

$$\phi\text{-tor}(M) = \{x \in M \mid sx = 0 \text{ for some s} \in R \setminus \mathrm{Nil}(R)\}.$$

If $\phi\text{-tor}(M) = M$, then M is called a ϕ-torsion module, and if $\phi\text{-tor}(M) = 0$, then M is called a ϕ-torsion-free module. An R-module M is said to be ϕ-uniformly torsion (ϕ-u-torsion for short) if $sM = 0$ for some $s \in R \setminus \mathrm{Nil}(R)$. An ideal I of R is called nonnil if $I \not\subseteq \mathrm{Nil}(R)$. A ring R is said to be self-injective if it is an injective module over itself; if, in addition, R is Noetherian, then R is said to be a quasi-Frobenius ring. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote the classical projective, injective, and flat dimension of M, respectively. For a Noetherian ring R, Auslander and Bridger [3] introduced the G-dimension, $\mathrm{Gdim}_R(M)$, for every finitely generated R-module M. They showed that $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for every finitely generated R-module M, and equality holds if $\mathrm{pd}_R(M)$ is finite. Several decades later, Enochs and Jenda [15, 16] introduced the notion of the Gorenstein projective dimension (G-projective dimension for short) as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of the Gorenstein projective dimension. To complete the analogy with the classical homological dimension, Enochs et al. [18] introduced the Gorenstein flat dimension. Some related references are [7, 12, 13, 15, 16, 18, 21, 22]. Recall from [15] that an R-module M is said to be Gorenstein projective (G-projective for short) if there exists an exact sequence of projective R-modules

$$\mathbf{P}: \quad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

such that $M \cong \mathrm{Im}\left(P_0 \to P^0\right)$ and such that the functor $\mathrm{Hom}_R(-, Q)$ leaves $\mathbf{P}$ exact whenever Q is projective. The complex $\mathbf{P}$ is called a complete projective resolution. It is known that M is G-projective if and only if M has a complete projective resolution $\mathbf{P}$ such that $\mathrm{Ext}^1_R(L, Q) = 0$ for any syzygy L of $\mathbf{P}$ and any module Q with finite projective dimension. The Gorenstein injective (G-injective for short) modules are defined dully.

On the other hand, in [32], Tang, Wang and Zhao introduced the class of ϕ-rings, called ϕ-von Neumann regular rings. An R-module M is said to be ϕ-flat if for every R-monomorphism $f : A \to B$ with $\mathrm{Coker}(f)$ ϕ-torsion, $f \otimes 1 : A \otimes_R M \to B \otimes_R M$ is an R-monomorphism [32, Definition 3.1]. An R-module M is ϕ-flat if and only if $M_{\mathfrak{p}}$ is ϕ-flat for every prime ideal $\mathfrak{p}$ of R, if and only if $M_{\mathfrak{m}}$ is ϕ-flat for every maximal ideal $\mathfrak{m}$ of R [32, Theorem 3.5]. A ϕ-ring R is called a ϕ-von Neumann regular ring if all R-modules are ϕ-flat. This is equivalent to stating that $R/\mathrm{Nil}(R)$ is a von Neumann regular ring [32, Theorem 4.1].

The authors of [14] introduced and defined the ϕ-(weak) global dimension of rings with prime nilradical. An R-module P is said to be ϕ-u-projective if $\mathrm{Ext}^1_R(P, N) = 0$ for each ϕ-u-torsion R-module N [14, Definition 3.1]. The ϕ-projective dimension of M over R, denoted by $\phi\text{-pd}_R M$, is defined to be at most n (where $n \geq 1$ and $n \in \mathbb{N}$) if either $M = 0$, or $M \neq 0$ and M is not ϕ-u-projective, and if it satisfies the condition $\mathrm{Ext}^{n+1}_R(M, N) = 0$ for any ϕ-u-torsion module N. If n is the least non-negative integer for which $\mathrm{Ext}^{n+1}_R(M, N) = 0$ for every ϕ-u-torsion module N, then we set $\phi\text{-pd}_R M = n$. If there is no such n, we set $\phi\text{-pd}_R M = \infty$. An R-module E is said to be nonnil injective if $\mathrm{Ext}^1_R(R/I, E) = 0$ for all nonnil ideals of R. The ϕ-injective dimension of M over R, denoted by $\phi\text{-id}_R M$, is defined to be at most n (where $n \geq 1$ and $n \in \mathbb{N}$) if either $M = 0$, or $M \neq 0$ and M is not nonnil injective, and if it satisfies the condition $\mathrm{Ext}^{n+1}_R(R/I, M) = 0$ for any nonnil ideal I of R. If n is the least non-negative integer for which $\mathrm{Ext}^{n+1}_R(R/I, M) = 0$ for every nonnil ideal I of R, then we set $\phi\text{-id}_R M = n$. If there is no such n, we set $\phi\text{-id}_R M = \infty$, and we easily have that an R-module M of ϕ-injective dimension 0 if and only if it is nonnil injective. For a ring R, its ϕ-global dimension is either 0 or the supremum of all $\phi\text{-pd}_R(R/I)$, where I is a nonnil ideal of R such that R/I is not ϕ-u-projective. In particular, for a ring R of $Z(R) = \mathrm{Nil}(R)$, its ϕ-global dimension is the supremum of $\phi\text{-pd}_R(R/I)$ for all nonnil ideals I of R.

In [2], Anderson and Badawi introduced the class of ϕ-rings called ϕ-Dedekind rings. A ϕ-ring R is said to be ϕ-Dedekind if $R/\mathrm{Nil}(R)$ is a Dedekind domain.

This paper is divided into four sections, including the introduction. In Section 2, we define nonnil-Gorenstein projective modules, nonnil-Gorenstein injective modules and nonnil-Gorenstein flat. We then present some characterizations of these modules and establish that every nonnil-Gorenstein projective (resp., nonnil-Gorenstein injective) module is Gorenstein projective (resp., Gorenstein injective). We also provide examples of Gorenstein projective (resp., Gorenstein injective) modules that are not nonnil-Gorenstein projective (resp., nonnil-Gorenstein injective). The section concludes by establishing analogs of the well-known behavior, as demonstrated in [21, 2.5. Theorem], showing more precisely that nonnil-Gorenstein projective (resp., nonnil-Gorenstein injective) R-modules are projectively (respectively, injectively) resolving.

In Section 3, we introduce analogs of Gorenstein projective and Gorenstein injective dimensions, termed nonnil-Gorenstein projective and injective dimensions.

The final section briefly defines the nonnil-Gorenstein global dimension of a ring R as the supremum of all $\phi\text{-Gpd}_R M$, where M is an R-module. We note that $G\text{-gl.dim}(R) \leq \phi\text{-G.gl.dim}(R)$ for all rings R. We characterize rings of nonnil-Gorenstein global dimension 0 as those in which every nonnil-injective module is projective, equivalent to every ϕ-u-projective module being nonnil-injective. We also show that ϕ-rings of nonnil-Gorenstein global dimension 0 are fields. In the second part of this section, we introduce the closed nonnil-Gorenstein global dimension of rings as the supremum of all nonnil-Gorenstein projective dimensions of ϕ-u-torsion modules. We establish that ϕ-von Neumann regular rings are strongly ϕ-rings of closed nonnil-Gorenstein global dimension 0, equivalent to von Neumann regular rings being strongly ϕ-rings where every ϕ-u-projective module is nonnil-injective.

To provide examples, we discuss the trivial extension. Let R be a ring and E be an R-module. The trivial ring extension of R by E, denoted $R \propto E$, has the additive structure of the external direct sum $R \oplus E$ and multiplication defined by $(a,e)(b,f) := (ab, af + be)$ for all $a,b \in R$ and $e,f \in E$. This construction is also known by other terminology and notations, such as the idealization $R(+)E$ (see [6, 19, 23, 24]).

For any undefined terminology and notation, readers are referred to [12, 17, 19, 23, 33].

2 On nonnil-Gorenstein projective, injective and flat modules

We start this section by defining two new classes of modules, which we call nonnil-Gorenstein projective and nonnil-Gorenstein injective. These two classes are sub-classes of Gorenstein projective and Gorenstein injective, respectively.

Definition 2.1. An R-module M is said to be nonnil-Gorenstein projective (nonnil-G-projective for short) if there exists a complete projective resolution of M, that is an exact sequence of R-modules

$$\mathscr{P}: \quad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

in which all P_i, P^j are projective modules and $M \cong \text{Im}\left(P_0 \to P^0\right)$ and such that the functor $\text{Hom}_R(-,Q)$ leaves $\mathscr{P}$ exact whenever Q is a ϕ-u-projective module.

The nonnil-Gorenstein injective (nonnil-G-injective for short) modules are defined dually as follows: An R-module N is said to be nonnil-G-injective if there exists a complete injective resolution of M, that is

$$\mathscr{E}: \quad \cdots \longrightarrow E^1 \longrightarrow E^0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$$

such that all E_i, E^j are injective modules and such that both $N \cong \text{Im}(E^0 \to E_0)$ and $\text{Hom}(Q, \mathscr{E})$ is a complex exact of R-modules for every nonnil-injective module Q.

Remark 2.2. 1. It is easy to see from [17, Definition 10.2.1] that every nonnil-G-projective module is Gorenstein projective, and so the projective dimension of every nonnil-G-projective module is either zero or infinite by [17, Proposition 10.2.3]. Dually the nonnil-G-injective modules are Gorenstein injective by [17, Definition 10.1.1]. Thus, the injective dimension of any nonnil-G-injective module is either zero or infinite.

2. Every projective module P is nonnil-Gorenstein projective. In fact, the exact sequence $0 \to P \to P \to 0$ is a complete projective resolution of P, and for every ϕ-u-projective module M, we have the isomorphism of R-modules $0 \to \text{Hom}_R(P,M) \to \text{Hom}_R(P,M) \to 0$. In the same way, we establish that every injective module is nonnil-G-injective.

3. Note that a ϕ-u-projective module is not necessarily a nonnil-G-projective module. For instance, consider $R := K \propto K^n$, where $n \geq 2$ and K is a field. In this case, its Gorenstein global dimension is infinite, as shown in [27, Corollary 3.8]. Consequently, there exists no Gorenstein projective R-module M; in particular, M is not nonnil-G-projective by (1). However, R is a ϕ-von Neumann regular ring by [14, Theorem 5.16], which implies that M is ϕ-u-projective by [14, Corollary 5.34]. Similarly, we establish that there are nonnil-injective R-modules that are not nonnil-Gorenstein injective.

Next, we may write a complete projective resolution of an R-module M as follows:

$$\mathscr{P}: \quad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow P_{-2} \longrightarrow \cdots. \tag{1}$$

Note that $M \cong \mathrm{Im}(P_0 \to P_{-1})$. Set $K_i := \mathrm{Ker}(P_i \to P_{i-1})$ for the i-th syzygy of (1); in particular, $M \cong K_{-1}$.

In the same way, we may write a complete injective resolution of an R-module N as follows:

$$\mathscr{E}: \quad \cdots \longrightarrow E_{-1} \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots. \tag{2}$$

The i-th cosyzygy K_i of (2) is defined as $K_i := \mathrm{Im}(E_i \to E_{i+1})$; in particular, $N \cong K_{-1}$. Our first characterization of the nonnil-G-projective modules is as follows.

Theorem 2.3. The following are equivalent for an R-module M:

1. M is nonnil-Gorenstein projective,
2. M has a complete projective resolution (1) such that $\mathrm{Ext}^1_R(K_i, Q) = 0$ for every ϕ-u-projective module Q and every i-th syzygy K_i of (1),
3. M has a complete projective resolution (1) such that $\mathrm{Ext}^k_R(K_i, Q) = 0$ for every ϕ-u-projective module Q and every i-th syzygy K_i of (1) and every $k \geq 1$,
4. M has a complete projective resolution (1) such that $\mathrm{Ext}^k_R(K_i, Q) = 0$ for every R-module Q of finite ϕ-projective dimension and every i-th syzygy K_i of (1) and every $k \geq 1$,
5. M has a complete projective resolution (1) such that $\mathrm{Ext}^1_R(K_i, Q) = 0$ for every R-module Q of finite ϕ-projective dimension and every i-th syzygy K_i of (1),
6. M has a right projective resolution (i.e., $\mathscr{R}: 0 \to M \to P_{-1} \to P_{-2} \to \cdots$, where each P_i is projective) such that for every ϕ-u-projective module Q, both the complex $\mathrm{Hom}(\mathscr{R}, Q)$ is exact and $\mathrm{Ext}^k_R(M, Q) = 0$ for every $k > 0$.

Proof. (4) $\Rightarrow$ (6) & (4) $\Rightarrow$ (5) $\Rightarrow$ (1) They are obvious.

(1) $\Rightarrow$ (2) Assume that M is a ϕ-Gorenstein projective module and let $i \in \mathbb{Z}$ and Q be a ϕ-u-projective module. Then there exists a complete projective resolution of M as (1). Starting by the exact sequence $P_{i+2} \to P_{i+1} \to K_i \to 0$, we get the exact sequence $0 \to Hom_R(K_i, Q) \to Hom_R(P_{i+1}, Q) \xrightarrow{d} Hom_R(P_{i+2}, Q)$, this shows that $\ker(d) = Hom_R(K_i, Q)$. Now, considering the exact sequence $0 \to K_i \to P_i \to K_{i-1} \to 0$, we get the exact sequence $0 \to Hom_R(K_{i-1}, Q) \to Hom_R(P_i, Q) \xrightarrow{d} Hom_R(K_i, Q) \to Ext^1_R(K_{i-1}, Q) \to 0$. Since $\ker(d) = Im(d) = Hom_R(K_i, Q)$, we get immediately $Ext^1_R(K_{i-1}, Q) = 0$. Since $i \in \mathbb{Z}$, we get (2), as desired.

(2) $\Rightarrow$ (1) If (2) holds, then by considering the short exact sequences $0 \to K \to P_i \to K \to 0$ for every $i \in \mathbb{Z}$, so we have $0 \to Hom_R(K, Q) \to Hom_R(P_i, Q) \to Hom_R(K, Q) \to 0$ is exact for any ϕ-u-projective module Q. By linking these short exact sequences, we get a long exact sequence $\cdots \to Hom_R(P_{i+1}, Q) \to Hom_R(P_i, Q) \to Hom_R(P_{i-1}, Q) \to \cdots$, as desired M is ϕ-Gorenstein projective.

(2) $\Rightarrow$ (3) It follows from the short exact sequences $0 \to K \to P_i \to K \to 0$ that $Ext^{k+1}_R(K, Q) \cong Ext^k_R(K, Q)$ where K is the syzygies of the complete projective resolution. Now the assertion follows by induction on k.

(3) ⇒ (4) Let $n = \phi\text{-}pd_R Q$. If $n = 0$, then Q is ϕ-u-projective and so (4) holds by hypothesis. Now, assume that $n \neq 0$, i.e., Q is not ϕ-u-projective. Following [14, Theorem 3.10], there exists an exact sequence $0 \to Q_n \to Q_{n-1} \to \cdots \to Q_0 \to Q \to 0$ such that Q_i is projective for all $0 \leq i \leq n-1$ and Q_n is ϕ-u-projective. But considering these short exact sequences $0 \to Q_n \to Q_{n-1} \to K \to 0, \cdots,$ $0 \to K \to Q_0 \to Q \to 0$ where K is the syzygies of the above exact sequence, so we get for all $k \in \mathbb{N}^*$, $0 = Ext_R^k(K, Q_{n-1}) \to Ext_R^k(K,K) \to Ext_R^{k+1}(K, Q_n) = 0$. So $Ext_R^k(K,K) = 0$ for every K, in particular $Ext_R^k(K,Q) = 0$, as desired.

(6) ⇒ (1) Let $\mathscr{P}r : \cdots \to P_1 \to P_0 \to M \to 0$ be a projective resolution of M. Since $Ext_R^k(M,Q) = 0$ for every ϕ-u-projective module Q and every $k \geq 1$, we get $Hom(\mathscr{P}r, Q)$ is an exact complex. Set $\mathscr{P}r \cup \mathscr{R}$ is the linking between $\mathscr{P}r$ and $\mathscr{R}$ through M, we get a complete projective resolution of M such that $Hom(\mathscr{P}r \cup \mathscr{R}, Q)$ is an exact complex for every ϕ-u-projective module Q, as desired M is ϕ-Gorenstein projective. □

Dually with Theorem 2.3, the nonnil-G-injective modules are characterized as follows.

Theorem 2.4. The following are equivalent for an R-module N:

1. N is nonnil-G-injective,
2. N has a complete injective resolution (2) such that $\text{Ext}_R^1(Q, K_i) = 0$ for every nonnil-injective module Q and every i-th cosyzygy K_i of (2),
3. N has a complete injective resolution (2) such that $\text{Ext}_R^k(Q, K_i) = 0$ for every nonnil-injective module Q and every i-th cosyzygy K_i of (2) and every $k \geq 1$,
4. N has a complete injective resolution (2) such that $\text{Ext}_R^k(Q, K_i) = 0$ for every R-module Q of finite ϕ-injective dimension, every i-th cosyzygy K_i of (2) and every $k \geq 1$,
5. N has a complete injective resolution (2) such that $\text{Ext}_R^1(Q, K_i) = 0$ for every R-module Q of finite ϕ-injective dimension and every i-th cosyzygy K_i of (2),
6. N has a left injective resolution (i.e., $\mathscr{L} : \cdots \to E_1 \to E_0 \to N \to 0$) such that for every nonnil-injective module Q, we get the exact complex $\text{Hom}(Q, \mathscr{L})$ and $\text{Ext}_R^k(Q,N) = 0$ for every $k > 0$.

Proof. This proof is analogous to that of Theorem 2.3 above. □

Remark 2.5. 1. If M is nonnil-G-projective with its complete projective resolution $\mathscr{P}$, then by symmetry all images and hence all kernels and cokernels of $\mathscr{P}$ are nonnil-G-projective modules.

2. If N is nonnil-G-injective with its complete projective resolution $\mathscr{E}$, then by by symmetry all images and hence all kernels and cokernels of $\mathscr{E}$ are nonnil-G-injective modules.

From Theorem 2.3 we can easily deduce the following.

Corollary 2.6. *1. If M is a nonnil-G-projective module, then* $\text{Ext}_R^k(M,Q) = 0$ *for every $k \geq 1$ and every R-module Q of finite ϕ-projective dimension.*

2. If N is a nonnil-G-injective module, then $\text{Ext}_R^k(Q,N) = 0$ *for every $k \geq 1$ and every R-module Q of finite ϕ-injective dimension.*

Recall from [32] that a ϕ-ring R is said to be ϕ-von Neumann regular if every R-module is ϕ-flat. It is shown in [14, Corollary 5.34] that every ϕ-von Neumann regular is characterized by the fact that every module over it is ϕ-u-projective and nonnil-injective, i.e., every module over a ϕ-von Neumann regular ring has both finite ϕ-projective and ϕ-injective dimension.

Corollary 2.7. *Let R be a ring in which every R-module has finite ϕ-projective dimension and M be an R-module. Then M is nonnil-G-projective if and only if M is projective.*

Corollary 2.8. *Let R be a ring in which every R-module has finite ϕ-injective dimension and M be an R-module.. Then N is nonnil-G-injective if and only if N is injective.*

Proof. This is easily done by Theorem 2.4. □

Remark 2.9. We now justify that the class of Gorenstein projective modules generalizes the class of nonnil-G-projective modules. In other words, the class of nonnil-G-projective modules is strictly contained in the class of Gorenstein projective modules. Let $R = K \propto K$, where K is a field. We claim that there exists a Gorenstein projective R-module which is not nonnil-G-projective. However, R is a ϕ-von Neumann regular ring. Thus, if every Gorenstein projective R-module is nonnil-G-projective, then R is a semisimple ring by Corollary 2.7 and [27, Corollary 3.8], a desired contradiction since $\mathrm{Nil}(R) \neq 0$. Therefore, R contains a Gorenstein projective module which is not nonnil-G-projective. Similarly, we can justify that there exists a Gorenstein injective R-module which is not nonnil-G-injective.

Example 2.10. Let $R = \mathbb{Z} \propto \mathbb{Q}$. Then every nonnil-G-injective R-module is injective. In fact, R is a strongly ϕ-ring and $\phi\text{-}gl.dim(R) = 1$ by [14, Example 6.5]. Therefore, every R-module has finite ϕ-injective dimension, a conclusion supported by the fact that every nonnil ideal of R is free (see [14, Theorem 4.1]).

The ϕ-von Neumann regular rings are characterized according to the nonnil-Gorenstein projectivity or nonnil-Gorenstein injectivity as follows.

Theorem 2.11. The following are equivalent for a ϕ-ring R:

1. R is a ϕ-von Neumann regular ring,
2. R is a strongly ϕ-ring such that every R-module has finite ϕ-projective dimension and every ϕ-torsion module is nonnil-G-projective,
3. R is a strongly ϕ-ring such that every R-module has finite ϕ-projective dimension and every ϕ-u-torsion module is nonnil-G-projective,
4. R is a strongly ϕ-ring such that every R-module has finite ϕ-injective dimension and every ϕ-torsion module is nonnil-G-injective,
5. R is a strongly ϕ-ring such that every R-module has finite ϕ-injective dimension and every ϕ-u-torsion module is nonnil-G-injective.

Proof. (2) $\Rightarrow$ (3) & (4) $\Rightarrow$ (5) These are straightforward.

(1) $\Rightarrow$ (2) & (1) $\Rightarrow$ (4) These follow directly from [14, Corollary 5.15, Theorem 5.29 and Corollary 5.33] and the fact that every ϕ-torsion R-module is zero.

(3) $\Rightarrow$ (1) Let I be a nonnil ideal of R. Then R/I is a ϕ-u-torsion R-module, and so R/I is nonnil-G-projective. By Corollary 2.7, we get that R/I is projective, and so I is generated by an idempotent. From [32, Theorem 4.1], we deduce that R is a ϕ-von Neumann regular ring.

(5) $\Rightarrow$ (1) Let $s \in R\backslash\mathrm{Nil}(R)$. Then R/sR is a ϕ-u-torsion R-module, and so R/sR is nonnil-G-injective. Thus R/sR is an injective R-module by Corollary 2.8. So we get $R = sR$, and so $s \in U(R)$. Therefore, R is a ϕ-von Neumann regular ring by [14, Theorem 5.14]. □

Proposition 2.12. *Every direct sums (resp., direct product) of nonnil-G-projective (resp., nonnil-G-injective) modules is nonnil-G-projective (resp., nonnil-G-injective).*

Proof. Let $\{M_i\}_{i\in I}$ be a family of nonnil-G-projective modules and $\mathscr{P}_i$ be a complete projective resolution of M_i for each $i \in I$. Then $\bigoplus_{i\in I} \mathscr{P}_i$ is a complete projective resolution of $\bigoplus_{i\in I} M_i$. By [33, Theorem 2.1.19], we get easily that $\bigoplus_{i\in I} M_i$ is nonnil-G-projective. Dually, we can establish the "nonnil-G-injective" case. □

Next, we establish an analog of the well-known result [21, 2.5. Theorem]. More precisely, the following theorem shows that the nonnil-G-projective (resp., injective) R-modules are projectively (resp., injectively) resolving.

Theorem 2.13. Let $0 \to A \to B \to C \to 0$ be a short exact sequence.

1. If C is a nonnil-G-projective module, then A is nonnil-G-projective if and only if B is nonnil-G-projective.
2. If A is a nonnil-G-injective module, then B is nonnil-G-injective if and only if C is nonnil-G-injective.

Proof. (1) This follows immediately from [10, Theorem 2.3 (1)].
(2) This follows immediately from [30, 2.10. Proposition]. □

Proposition 2.14. *(1) Every R-module is nonnil-Gorenstein projective if and only if every ϕ-u-projective R-module is injective.*
(2) Every R-module is nonnil-Gorenstein injective if and only if every nonnil-injective R-module is projective. In particular, when the above equivalent conditions are satisfied R is quasi-Frobenius.

Proof. (1) This follows immediately from [10, Proposition 2.4] by setting $\mathscr{X}$ to be the set of all ϕ-u-projective R-modules.
(2) This follows immediately from [30, 2.9. Proposition] by setting $\mathscr{Y}$ to be the set of all nonnil-injective R-modules. □

Next, we will define the nonnil-Gorenstein flat R-modules as follows.

Definition 2.15. Let R be a ring and M be an R-module. We say that M is nonnil-Gorenstein flat if there exists a complete flat resolution of M, which is an exact sequence of R-modules of the form

$$\mathscr{F} : \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots \to F^n \to \cdots$$

where each F_i is a flat R-module, and such that the complex $E \otimes_R \mathscr{F}$ is exact for every nonnil-injective R-module E. Furthermore, M is isomorphic to $\mathrm{im}(F_0 \to F^0)$.

Remark 2.16. It is easy to see that every nonnil-Gorenstein flat R-module is a Gorenstein-flat.

Proposition 2.17. *The class of nonnil-Gorenstein flat R-modules is closed under arbitrary direct sums.*

Proof. Simply note that a (degreewise) sum of complete flat resolutions again is a complete flat resolution (as tensorproducts commutes with sums). □

Theorem 2.18. For any left R-module M, we consider the following conditions:

1. M is a nonnil-Gorenstein flat R-module,
2. The Pontryagin dual $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a nonnil-Gorenstein injective R-module,

3. M admits a co-proper right flat resolution

$$\mathscr{F}_r:\ 0 \to M \to F^0 \to F^1 \to F^2 \to \cdots.$$

such that $E \otimes_R \mathscr{F}_r$ is an exact complex of R-modules and $\mathrm{Tor}_i^R(E,M) = 0$ for all nonnil-injective R-modules E, and all integers $i > 0$.

Then (1) $\Rightarrow$ (2). If R is a coherent ring, then the previous conditions are equivalent.

Proof. (1) $\Rightarrow$ (2) Let $\mathscr{F} : \ldots \to F_1 \to F_0 \to F^0 \to F^1 \to \ldots$ be a complete flat resolution, such that $M \cong \mathrm{Im}\left(F_0 \to F^0\right)$. Then

$$\mathscr{F}^+ : \ldots \to F^{1+} \to F^{0+} \to F_0^+ \to F_1^+ \to \ldots$$

is an exact sequence of injective R-modules, such that $M^+ \cong \mathrm{Im}\left(F^{0+} \to F_0^+\right)$. On the other hand, we have for all nonnil-injective R-modules E,

$$\mathrm{Hom}_R(E,\mathscr{F}^+) = \mathrm{Hom}_R(E,\mathrm{Hom}_{\mathbb{Z}}(\mathscr{F},\mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(E \otimes_R \mathscr{F},\mathbb{Q}/\mathbb{Z})$$

which is exact. Then $\mathscr{F}^+$ is a complete injective resolution and M^+ is nonnil-Gorenstein injective.

Suppose now that R is a coherent ring.

(2) $\Rightarrow$ (3) From Remark 2.16 and [21, 3.6. Theorem], M admits a co-proper right flat resolution $0 \to M \to F^0 \to F^1 \to F^2 \to \cdots$.

Let's prove that $\mathrm{Tor}_i^R(E,M) = 0$ for all $i > 0$ and injective R-module E. Let $\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$ be a flat resolution of M, then $0 \to M^+ \to F_0^+ \to F_1^+ \to F_2^+ \to \ldots$ is an injective resolution. Let E be an R-module, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & Hom_R(E,F_1^+) & \longrightarrow & Hom_R(E,F_0^+) & \longrightarrow & Hom_R(E,M^+) & \longrightarrow 0 \\
& \downarrow & & \downarrow\cong & & \downarrow\cong & \\
\cdots \to & Hom_{\mathbb{Z}}(E \otimes_R F_1,\mathbb{Q}/\mathbb{Z}) & \to & Hom_{\mathbb{Z}}(E \otimes_R F_0,\mathbb{Q}/\mathbb{Z}) & \to & Hom_{\mathbb{Z}}(E \otimes_R M,\mathbb{Q}/\mathbb{Z}) & \to 0
\end{array}$$

such that the upper row of the diagram is exact as M^+ is nonnil-Gorenstein injective, then also the lower row is exact, which means that $\mathrm{Tor}_i^R(E,M) = 0$ for all $i > 0$ and every nonnil-injective R-module E. Now, let $\mathscr{F}_l :\ \cdots \to F_1 \to F_0 \to M \to 0$ be a flat resolution of M. By linking $\mathscr{F}_l$ with $\mathscr{F}_r$ trough M, we get the following complete flat resolution of M:

$$\cdots F_1 \to F_0 \to F^0 \to F^1 \to \cdots.$$

So, the following

$$\cdots F^{1+} \to F^{0+} \to F_0^+ \to F_1^+ \to \cdots$$

is a complete injective resolution of M^+. Since, M^+ is assumed nonnil-Gorenstein injective, we get the following exact sequence $\cdots \to Hom_R(E,F^{1+}) \to Hom_R(E,F^{0+}) \to Hom_R(E,M^+) \to 0$, that means the sequence $\cdots \to (E \otimes_R F^0)^+ \to (E \otimes_R F^0)^+ \to (E \otimes_R M)^+ \to 0$ is exact. Hence, we get that $E \otimes_R \mathscr{F}_r$ is exact, as desired.

(3) $\Rightarrow$ (1) By the same way as (2) $\Rightarrow$ (3), we get a complete flat resolution by linking $\mathscr{F}_l$ with $\mathscr{F}_r$ trough M. So, the assumption, $Tor_i^R(E,M) = 0$ for every nonnil-injective R-module E and every $i > 0$, implies that the complex $E \otimes_R \mathscr{F}_l$ is exact. By assumption, the complex $E \otimes_R \mathscr{F}_r$ is exact. Hence, by linking $E \otimes_R \mathscr{F}_r$ and $E \otimes_R \mathscr{F}_l$ through $E \otimes_R M$, we showed that M is a nonnil-Gorenstein flat R-module. □

Remark 2.19. Let K be a field and consider $R = K \propto K$. There exists a Gorenstein flat R-module M which is not nonnil-Gorenstein flat: in fact, from [27, Corollary 3.8] and [29, Proposition 2.3], every R-module is a Gorenstein flat. But, R is not a von Neumann regular ring, which means that, there exists an R-module M that not flat. If M is a nonnil-Gorenstein flat R-module, then $Tor_R^k(E, M) = 0$ for every R-module E and every $k > 0$ by Theorem 2.18, [14, Corollary 5.34] and Corollary 2.6. Hence, M is flat, a contradiction.

Corollary 2.20. *Let R be a coherent ring and $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. If C is a nonnil-Gorenstein flat R-module, then A is a nonnil-Gorenstein flat R-module if and only if B is a nonnil-Gorenstein flat R-module.*

Proof. The short exact sequence of R-modules $0 \to A \to B \to C \to 0$ induces the short exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. By Theorem 2.18, if C is a nonnil-Gorenstein flat R-module then C^+ is a nonnil-Gorenstein injective R-module. So, from Theorem 2.13, A^+ is a nonnil-Gorenstein injective R-module if and only if B^+ is a nonnil-Gorenstein injective R-module. Again Theorem 2.18, A is a nonnil-Gorenstein flat R-module if and only if B is a nonnil-Gorenstein flat R-module. □

Theorem 2.21. Let R be a coherent ring and consider the short exact sequence of R-modules $0 \to A \to B \to C \to 0$, where A and B are nonnil-Gorenstein flat R-module. If $Tor_1^R(E, C) = 0$ for every (nonnil)-injective R-module E, then C is a nonnil-Gorenstein flat R-module.

Proof. The proof follows from [30, 4.6. Proposition] if we consider E as an injective R-module. Now, assuming that E is nonnil-injective. Considering the short exact sequence of R-modules $0 \to C^+ \to B^+ \to A^+ \to 0$. By Theorem 2.18, both A^+ and B^+ are nonnil-Gorenstein injective R-modules. By assumption, we get $Ext_R^1(E, C^+) = 0$ for every nonnil-injective R-module E. But for every $k > 1$, we get the isomorphism $Ext_R^k(E, A^+) \cong Ext_R^{k+1}(E, C^+) = 0$. Therefore, C^+ is a nonnil-Gorenstein injective R-module. But R is a coherent ring, and so, C is a nonnil-Gorenstein flat R-module. □

Proposition 2.22. *If R is coherent, then the class of nonnil-Gorenstein flat left R-modules is closed under extensions, kernels of epimorphisms, direct sums and direct summands.*

Proof. It follows immediately from [30, 4.5. Proposition] by setting $\mathscr{Y}$ as the set of all nonnil-injective R-modules. □

3 On nonnil-G-projective, nonnil-G-injective dimensions and nonnil-G-flat dimensions

In this section, we introduce the analogs of the Gorenstein injective dimension and the Gorenstein projective dimension.

Definition 3.1. Let M be an R-module. Then M is said to have a finite nonnil-G-projective dimension at most $n \in \mathbb{N}$ (we denote $\phi\text{-Gpd}_R M \leq n$) if there exists a finite nonnil-G-projective resolution, that is

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0, \tag{3}$$

in which every P_i is nonnil-G-projective. If M does not have a finite length of nonnil-G-projective resolution, then we set $\phi\text{-Gpd}_R M = \infty$.

Dually, let N be an R-module. Then N is said to have a finite nonnil-G-injective dimension at most $n \in \mathbb{N}$ if there exists a nonnil-G-injective resolution, that is

$$0 \to N \to E_0 \to E_1 \to \cdots \to E_n \to 0, \tag{4}$$

in which every E_j is nonnil-G-injective. If N does not have a finite length of nonnil-G-injective resolution, then we set $\phi\text{-Gid}_R N = \infty$.

Remark 3.2. Obviously, an R-module M is nonnil-G-projective if and only if its nonnil-G-projective dimension is zero. Similarly, an R-module N is nonnil-G-injective if and only if its nonnil-G-injective dimension is zero.

Next, the following theorem is an analog of the well-known result [33, Proposition 11.3.4].

Theorem 3.3. Let $0 \to A \to B \to C \to 0$ be an exact sequence.

1. If $\phi\text{-Gpd}_R B > \phi\text{-Gpd}_R C$, then $\phi\text{-Gpd}_R A = \phi\text{-Gpd}_R B$.
2. If $\phi\text{-Gid}_R B > \phi\text{-Gid}_R A$, then $\phi\text{-Gid}_R B = \phi\text{-Gid}_R C$.

Proof. We will only prove (1); the case of (2) can be dually proved with (1). Write $m = \phi\text{-Gpd}_R A$, $n = \phi\text{-Gpd}_R B$, and $s = \phi\text{-Gpd}_R C$. First let $s = 0$, that is, C is a nonnil-G-projective module. If $m < \infty$, consider a nonnil-G-projective resolution

$$\mathscr{P} : \cdots \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$$

of A. For $i > m$, set $P_i = 0$. Since C is nonnil-G-projective, there is an exact sequence

$$\mathscr{Q} : \cdots \to Q_m \to Q_{m-1} \to \cdots \to Q_0 \to C \to 0$$

such that each Q_i is projective and each syzygy C_i is nonnil-G-projective. So, we can prove that there exists a complex exact sequence $0 \to \mathscr{P} \to \mathscr{F} \to \mathscr{Q} \to 0$, where each term $F_i = P_i \oplus Q_i$ of $\mathscr{F}$ is a nonnil-G-projective module. Denote by B_i the i-th syzygy of the complex $\mathscr{F}$. Then there is an exact sequence $0 \to P_m \to B_m \to C_m \to 0$. By Theorem 2.13(1), B_m is nonnil-G-projective. Thus $n \leqslant m$. Hence if $n = \infty$, then $m = \infty$.

Let $n < \infty$. Then there is an exact sequence $0 \to F_n \to \cdots \to F_0 \to B \to 0$, where each F_i is nonnil-G-projective. Let K_0 be the kernel of the homomorphism $F_0 \to B$. Then $\phi\text{-Gpd}_R K_0 = n-1$. Thus we have the following commutative diagram with all exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_0 & = & K_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & F_0 & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By Theorem 2.13 (1), L is nonnil-G-projective, and so $m \leq (n-1)+1 = n$. Therefore $m = n$. For $s > 0$ in the general case, we can examine a nonnil-Gorenstein projective resolution of A and a complete projective resolution of C. Now the assertion follows by applying the above discussion. □

Corollary 3.4. *Let M be a nonnil-G-projective (resp., nonnil-G-injective) module and N be an R-module. Then $\phi\text{-Gpd}_R(M \oplus N) = \phi\text{-Gpd}_R N$ (resp., $\phi\text{-Gid}_R(M \oplus N) = \phi\text{-Gid}_R N$).*

Proof. This follows from the short exact sequence $0 \to N \to M \oplus N \to M \to 0$. If $M \oplus N$ is nonnil-G-projective, then N is also nonnil-G-projective by Theorem 2.13(1). If $\phi\text{-Gpd}_R(M \oplus N) = n > 0$, then we get $\phi\text{-Gpd}_R N = n$ by Theorem 3.3. We can prove the "nonnil-G-injective dimension" case dully. □

Proposition 3.5. *Let M be an R-module with finite nonnil-Gorenstein injective dimension n. Then there exist exact sequences*

$$0 \longrightarrow M \longrightarrow I \longrightarrow F \longrightarrow 0$$

with I nonnil-Gorenstein injective, $\mathrm{id}(F) \leq n-1$, *and*

$$0 \longrightarrow I' \longrightarrow F' \longrightarrow M \longrightarrow 0$$

with I' nonnil-Gorenstein injective, $\mathrm{id}\,(F') \leq n$.

Proof. It follows immediately from [30, 2.12. Proposition] by setting $\mathscr{Y}$ as the set of all nonnil-injective R-modules. □

Corollary 3.6. *Let $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$ be a short exact sequence of R-modules, where I_0 and I_1 are nonnil-Gorenstein injective modules and $\mathrm{Ext}^1_R(I,M) = 0$ for all injective R-modules I. Then M is nonnil-Gorenstein injective.*

Proof. It follows immediately from [30, 2.13. Corollary] by setting $\mathscr{Y}$ as the set of all nonnil-injective R-modules. □

The following theorem is the main ingredient of the important functorial description of the nonnil-G-projective dimension.

Theorem 3.7. The following are equivalent for an R-module M of finite nonnil-G-projective dimension:

1. $\phi\text{-Gpd}_R M \leq n$.
2. $\mathrm{Ext}^k_R(M,Q) = 0$ for every R-module Q with finite ϕ-projective dimension and every $k > n$.
3. $\mathrm{Ext}^k_R(M,Q) = 0$ for every ϕ-u-projective R-module Q and every $k > n$.
4. If $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ is an exact sequence such that every P_i, where $0 \leq i \leq n-1$, is a projective module, then P_n is nonnil-G-projective.
5. If $0 \to Q_n \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0$ is an exact sequence such that every Q_i, where $0 \leq i \leq n-1$, is a nonnil-G-projective module, then Q_n is nonnil-G-projective.
6. $\mathrm{Ext}^k_R(M,Q) = 0$ for every R-module Q with finite projective dimension and every $k > n$.
7. $\mathrm{Ext}^k_R(M,Q) = 0$ for every projective R-module Q and every $k > n$.

Proof. Set $m = \phi\text{-Gpd}_R M$. Then there exists a nonnil-G-projective resolution

$$0 \to G_m \to G_{m-1} \to \cdots \to G_1 \to G_0 \to M \to 0 \tag{5}$$

such that each G_i is nonnil-G-projective. We set K_i to be its i-th syzygy of (5). In particular, $K_{-1} := M$. Note that if $k > m+1$, then $K_{k-2} \cong G_{k-1}$.

(5) ⇒ (1) This is straightforward.

(1) $\Rightarrow$ (2) Let $k > 0$ and $i \geq -1$. Then we have the short exact sequence $0 \to K_{i+1} \to G_i \to K_i \to 0$. By Theorem 2.3, we get $\mathrm{Ext}_R^k(G_i, Q) = 0$, and so we have the isomorphism $\mathrm{Ext}_R^k(K_{i+1}, Q) \cong \mathrm{Ext}_R^{k+1}(K_i, Q)$. Since ϕ-$\mathrm{Gpd}_R M \leq n$, it follows that for every $k > n$, $\mathrm{Ext}_R^k(M, Q) \cong \mathrm{Ext}_R^{k-1}(K_0, Q) \cong \cdots \cong \mathrm{Ext}_R^1(K_{k-2}, Q) = \mathrm{Ext}_R^1(G_{k-1}, Q) = 0$.

(2) $\Rightarrow$ (3) This is straightforward.

(3) $\Rightarrow$ (1) If $m = 0$, naturally we have $m \leqslant n$. Let $m > 0$ and set $L := \mathrm{Ker}(G_0 \to M)$. Then ϕ-$\mathrm{Gpd}_R L = m - 1$. Take a projective module P and an epimorphism $P \to M$. Set $K := \mathrm{Ker}(P \to M)$. By [33, Theorem 2.3.12], there is an exact sequence $0 \to K \to L \oplus P \to G_0 \to 0$. By Corollary 3.4, ϕ-$\mathrm{Gpd}_R(L \oplus P) = m - 1$. By Proposition 3.3, ϕ-$\mathrm{Gpd}_R K = m - 1$. Since $\mathrm{Ext}_R^{k-1}(K, Q) = \mathrm{Ext}_R^k(M, Q) = 0$ for any $k > n$ and any ϕ-u-projective module Q, by induction $m - 1 \leqslant n - 1$. Thus $m \leqslant n$.

(1) $\Rightarrow$ (4) Assume that ϕ-$\mathrm{Gpd}_R M \leq n$. Then there exists a nonnil-Gorenstein projective resolution of M as follows:

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

in which every G_i is nonnil-G-projective. By [33, Theorem 3.2.1], we get the following commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

By [33, Lemma 11.3.3], there exists an exact sequence $0 \to P_n \to P_{n-1} \oplus G_n \to \cdots \to P_1 \oplus G_2 \to P_0 \oplus G_1 \to G_0 \to 0$. Since G_0, G_1, $\ldots$, G_n are nonnil-Gorenstein projective modules, repeated application of Theorem 2.13 implies that P_n is also nonnil-G-projective.

(4) $\Rightarrow$ (5) Let $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be an exact sequence of R-modules such that $P_0, P_1, \ldots, P_{n-1}$ are projective modules. Then P_n is nonnil-G-projective. By [33, Theorem 3.2.1], we get the following commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & Q_n & \longrightarrow & Q_{n-1} & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Therefore, by [33, Lemma 11.3.3], there exists an exact sequence $0 \to P_n \to P_{n-1} \oplus Q_n \to \cdots \to P_1 \oplus Q_2 \to P_0 \oplus Q_1 \to Q_0 \to 0$. Decompose this exact sequence into two exact sequences

$$0 \to P_n \to P_{n-1} \oplus Q_n \to K \to 0$$

and

$$0 \to K \to P_{n-2} \oplus Q_{n-1} \to \cdots \to P_0 \oplus Q_1 \to Q_0 \to 0.$$

By Theorem 2.13, we deduce that K is nonnil-G-projective, since all Q_i are nonnil-G-projective modules. Therefore, Q_n is nonnil-G-projective, as desired.

(5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) These follow immediately from [30, 3.14. Proposition] by setting $\mathscr{X}$ as the set of all ϕ-u-projective R-modules. □

Corollary 3.8. *Let M be an R-module with finite nonnil-Gorenstein projective dimension. Then $\mathrm{Gpd}_R M = \phi$-$\mathrm{Gpd}_R M$. In particular, an R-module M is Gorenstein projective if and only if M is nonnil-Gorenstein projective.*

Proof. It follows immediately from [30, 3.15. Corollary] by setting $\mathscr{X}$ as the set of all ϕ-u-projective R-modules. □

Remark 3.9. We define $\mathscr{P}r(R)$ (resp., $\overline{\mathscr{P}r(R)}$) for the class of all ϕ-u-projective modules (resp., all R-modules of finite ϕ-projective dimension). By Theorem 3.7, the nonnil-G-projective dimension of an R-module M is given as follows:

$$\begin{aligned}\phi\text{-}\mathrm{Gpd}_R M &= \sup\left\{i \in \mathbb{N} \mid \exists\ Q \in \mathscr{P}r(R) \text{ such that } \mathrm{Ext}^i_R(M,Q) \neq 0\right\}\\ &= \sup\left\{i \in \mathbb{N} \mid \exists\ Q \in \overline{\mathscr{P}r(R)} \text{ such that } \mathrm{Ext}^i_R(M,Q) \neq 0\right\}.\end{aligned}$$

Dually with Theorem 3.7, we establish the following theorem.

Theorem 3.10. Let $n \in \mathbb{N}$. Then the following are equivalent for an R-module N of finite nonnil-G-projective dimension:

1. ϕ-$\mathrm{Gid}_R N \leq n$,
2. $\mathrm{Ext}^k_R(Q,N) = 0$ for every R-module Q with finite ϕ-injective dimension and every $k > n$,
3. $\mathrm{Ext}^k_R(Q,N) = 0$ for every nonnil-injective R-module Q and every $k > n$,
4. If $0 \to N \to E_0 \to E_1 \cdots \to E_n \to 0$ such that every E_i, where $0 \leq i \leq n-1$, is an injective module, then E_n is nonnil-G-injective.
5. If $0 \to N \to Q_0 \to Q_1 \cdots \to Q_n \to 0$ such that every Q_i, where $0 \leq i \leq n-1$, is nonnil-G-injective module, then Q_n is nonnil-G-injective.
6. $\mathrm{Ext}^k_R(Q,N) = 0$ for every injective R-module Q and every $k > n$,
7. $\mathrm{Ext}^k_R(Q,N) = 0$ for every R-module Q with finite injective dimension and every $k > n$,

Proof. The proof the first statements is similar of to that of Theorem 3.7.
(5) ⇔ (6) ⇔ (7) These follow immediately from [30, 2.15. Proposition]. □

Remark 3.11. We define $\mathscr{I}(R)$ (resp., $\overline{\mathscr{I}(R)}$) for the class of all nonnil-injective modules (resp., all R-modules of finite ϕ-injective dimension). By Theorem 3.10, the nonnil-G-injective dimension of an R-module N is given as follows:

$$\begin{aligned}\phi\text{-}\mathrm{Gid}_R M &= \sup\left\{i \in \mathbb{N} \mid \exists\ Q \in \mathscr{I}(R) \text{ such that } \mathrm{Ext}^i_R(Q,M) \neq 0\right\}\\ &= \sup\left\{i \in \mathbb{N} \mid \exists\ Q \in \overline{\mathscr{I}(R)} \text{ such that } \mathrm{Ext}^i_R(Q,M) \neq 0\right\}.\end{aligned}$$

Corollary 3.12. *Let N be an R-module with finite nonnil-Gorenstein injective dimension. Then* $\mathrm{Gid}_R N = \phi$-$\mathrm{Gid}_R N$. *In particular, an R-module is nonnil-Gorenstein injective if and only if it is Gorenstein injective.*

Proof. It follows immediately from [30, 2.16. Corollary] by setting $\mathscr{Y}$ as the class of all nonnil-injective R-modules. □

Corollary 3.13. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules and let $n \in \mathbb{N}$. Then the following hold.*

1. *If ϕ-$\mathrm{Gpd}_R B \leq n$ and ϕ-$\mathrm{Gpd}_R C \leq n$, then ϕ-$\mathrm{Gpd}_R A \leq n$. In particular, if both B and C are nonnil-G-projective, then so is A.*
2. *If ϕ-$\mathrm{Gid}_R A \leq n$ and ϕ-$\mathrm{Gid}_R B \leq n$, then ϕ-$\mathrm{Gid}_R C \leq n$. In particular, if both A and B are nonnil-G-injective, then so is C.*

Proof. (1) By Theorem 3.7(4) and [33, Theorem 2.6.6 (Horseshoe Lemma)], we get that ϕ-$\mathrm{Gpd}_R A < \infty$. So, for every $k > n$ and every ϕ-u-projective module Q, we get the exact sequence $\mathrm{Ext}^k_R(B,Q) \to \mathrm{Ext}^k_R(A,Q) \to \mathrm{Ext}^{k+1}_R(C,Q)$. By Theorem 3.7, $\mathrm{Ext}^k_R(A,Q) = 0$, i.e., ϕ-$\mathrm{Gpd}_R A \le n$.
(2) The proof is similar to that of (1). □

Corollary 3.14. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then:*

1. *If both A and B are nonnil-G-projective modules, then C is projective if and only if $\mathrm{Ext}^1_R(C,Q) = 0$ for every nonnil-G-projective module Q.*
2. *If both B and C are nonnil-G-injective modules, then A is injective if and only if $\mathrm{Ext}^1_R(Q,A) = 0$ for every nonnil-G-injective module Q.*

Proof. (1) By Theorem 2.7, ϕ-$\mathrm{Gpd}_R C \le 1$, and so there exists a short exact sequence $0 \to M \to F \to C \to 0$, where F is projective and M is nonnil-G-projective. Since $\mathrm{Ext}^1_R(C,M) = 0$, it follows that $0 \to M \to F \to C \to 0$ is split, and so C is projective. The converse is straightforward.
(2) This proof is similar to that of (1). □

Proposition 3.15. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules and let $n \in \mathbb{N}$. Then the following hold.*

1. *If ϕ-$\mathrm{Gpd}_R B < \phi$-$\mathrm{Gpd}_R C$, then ϕ-$\mathrm{Gpd}_R A = \phi$-$\mathrm{Gpd}_R C - 1$. In general, we get ϕ-$\mathrm{Gpd}_R C \le 1 + max\{\phi$-$\mathrm{Gpd}_R A, \phi$-$\mathrm{Gpd}_R B\}$.*
2. *If ϕ-$\mathrm{Gid}_R B < \phi$-$\mathrm{Gid}_R A$, then ϕ-$\mathrm{Gid}_R C = \phi$-$\mathrm{Gid}_R A - 1$. In general, we get ϕ-$\mathrm{Gid}_R A \le 1 + max\{\phi$-$\mathrm{Gid}_R A, \phi$-$\mathrm{Gid}_R C\}$.*

Proof. It follows immediately from [34, Corollary 3.7]. □

Proposition 3.16. *Let R be a ring and $n \in \mathbb{N}$. Then the following properties hold:*

1. *If ϕ-$\mathrm{Gpd}_R M \le n$ for every R-module M, then the injective dimension of any ϕ-u-projective module is at most n.*
2. *If ϕ-$\mathrm{Gid}_R N \le n$ for every R-module N, then the projective dimension of any nonnil-injective module is at most n.*

Proof. (1) Let Q be a ϕ-u-projective module and M be an R-module. Since ϕ-$\mathrm{Gpd}_R M \le n$, there exists an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

where $P_0, P_1, \ldots, P_{n-1}$ are projective modules and P_n is nonnil-G-projective. Then $\mathrm{Ext}^{n+1}_R(M,Q) \cong \mathrm{Ext}^1_R(P_n,Q) = 0$. Therefore, $\mathrm{id}_R Q \le 1$.
(2) This can be proved dually to (1). □

Next, we give an analog of the well-known result [33, Theorem 11.3.14 (Holm) & Theorem 11.3.15 (Christensen-Frankild-Holm)].

Proposition 3.17. *Let M be an R-module with finite nonnil-Gorenstein projective dimension n, then there exist exact sequences*

$$0 \longrightarrow H \longrightarrow G \longrightarrow M \longrightarrow 0$$

with nonnil-Gorenstein projective and $pd(H) \le n-1$ and

$$0 \longrightarrow M \longrightarrow H' \longrightarrow G' \longrightarrow 0.$$

Proof. It follows immediately from [30, 3.11. Proposition] by setting $\mathscr{X}$ as the set of all ϕ-u-projective R-modules. □

We next define the nonnil-Gorenstein flat dimension of modules as follows.

Definition 3.18. Let R be a ring and M be an R-module. Then M is said to have a ϕ-Gorenstein flat dimension at most $n \in \mathbb{N}$, and we write $\phi\text{-}G\text{-}fd_R M \leq n$, if there exists a resolution of nonnil-Gorenstein flat R-modules as follows

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

If no such resolution exists, we set $\phi\text{-}G\text{-}fd_R M = \infty$.

Proposition 3.19. *Let R be a ring and M be an R-module. The following hold.*

1. $\phi\text{-}G\text{-}id_R M^+ \leq \phi\text{-}G\text{-}fd_R M$.
2. *If R is a coherent ring, then* $\phi\text{-}G\text{-}id_R M^+ = \phi\text{-}G\text{-}fd_R M$.

Proof. (1) If $\phi\text{-}G\text{-}fd_R M = \infty$, the inequality holds. Assuming that $\phi\text{-}G\text{-}fd_R M \leq n$, where $n \in \mathbb{N}$. Then, there exists a resolution of Gorenstein flat R-modules as follows

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

The above resolution induces

$$0 \to M^+ \to F_0^+ \to \cdots \to F_1 \to F_{n-1}^+ \to F_n^+ \to 0$$

which is a resolution of nonnil-Gorenstein injective R-modules. So, it follows that $\phi\text{-}G\text{-}id_R M^+ \leq n$, as desired.

(2) This is obvious from Theorem 2.18. □

Corollary 3.20. *Let R be coherent and consider the short exact sequence of R-modules $0 \to A \to B \to C \to 0$, where B is nonnil-Gorenstein flat. If C is nonnil-Gorenstein flat, then so is A. If otherwise $n > 0$, then:*

$$\phi\text{-}G\text{-}fd_R A = \phi\text{-}G\text{-}fd_R C - 1.$$

Proof. Considering the short exact sequence of R-modules $0 \to C^+ \to B^+ \to A^+ \to 0$, and applying Proposition 3.19 (2) in conjunction with Proposition 3.15. □

Corollary 3.21. *Let R be a coherent ring. If $(M_\lambda)_{\lambda \in \Lambda}$ is any family of R-modules, then we have an equality:*

$$\phi\text{-}G\text{-}fd_R\Big(\bigoplus M_\lambda\Big) = \sup\{\phi\text{-}G\text{-}fd_R M_\lambda \mid \lambda \in \Lambda\}.$$

Proof. This follows immediately from Proposition 3.19 (2). □

Corollary 3.22. *Let M be an R-module of finite ϕ-Gorenstein flat dimension with R is coherent, and let n be an integer. Then the following conditions are equivalent:*

1. $\phi\text{-}G\text{-}fd_R \leq n$,
2. $Tor_i^R(L, M) = 0$ *for all $i > n$, and all R-modules L with finite ϕ-injective dimension.*
3. $\mathrm{Tor}_i^R(E, M) = 0$ *for all $i > n$, and all nonnil-injective R-modules E.*

4. *for every exact sequence* $0 \to K_n \to G_{n-1} \to \ldots \to G_0 \to M \to 0$, *where* $G_0, \ldots, G_{n-1}$ *are nonnil-Gorenstein flats, then also* K_n *is nonnil-Gorenstein flat.*

5. $Tor_i^R(L, M) = 0$ *for all* $i > n$*, and all R-modules L with finite injective dimension.*

6. $\mathrm{Tor}_i^R(E, M) = 0$ *for all* $i > n$*, and all injective R-modules E.*

Consequently, the nonnil-Gorenstein flat dimension of M is determined by the following formulas:

$$\begin{aligned}\phi\text{-}\mathrm{Gfd}_R M &= \sup\left\{i \in \mathbb{N} \mid \exists L \in \overline{\phi\text{-}\mathcal{I}(R)} : \mathrm{Tor}_i^R(L, M) \neq 0\right\},\\ &= \sup\left\{i \in \mathbb{N} \mid \exists E \in \phi\text{-}\mathcal{I}(R) : \mathrm{Tor}_i^R(E, M) \neq 0\right\},\end{aligned}$$

where, $\phi\text{-}\mathcal{I}(R)$ *(resp.,* $\overline{\phi\text{-}\mathcal{I}(R)}$*) is the set of all nonnil-injective R-modules (resp., all R-modules with finite* ϕ*-injective dimension).*

Proof. The equivalences of the first four statements follow immediately from Proposition 3.19 (2) and Theorem 3.10.

(4) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) These follow immediately from [30, 4.9. Proposition] by setting $\mathscr{Y}$ as the set of all nonnil-injective R-modules. □

4 On nonnil-Gorenstein global dimension

In this section, we denote $\mathscr{M}_R$ (resp., $\mathscr{T}_R$) for the class of all R-modules (resp., for all ϕ-u-torsion R-modules). We start with the following definition of the nonnil-Gorenstein global dimension as follows.

Definition 4.1. For a ring R, define

$$\phi\text{-G-gl.dim}(R) := \sup\{\phi\text{-Gpd}_R M \mid M \in \mathscr{M}_R\}, \tag{6}$$

which are called the nonnil-Gorenstein global dimension of R.

Our next goal is to characterize the rings of finite nonnil-Gorenstein global dimension at most $n \in \mathbb{N}$.

Theorem 4.2. Let $n \in \mathbb{N}$. Then the following are equivalent for a ring R of finite nonnil-Gorenstein global dimension R:

1. ϕ-G-gl.dim$(R) \leq n$.

2. The injective dimension of any ϕ-u-projective module is at most n.

3. The projective dimension of any nonnil-injective module is at most n.

4. $\phi\text{-Gpd}_R M \leq n$ for any finitely generated R-module M.

5. $\phi\text{-Gpd}_R(R/I) \leq n$ for any ideal I of R.

6. $\mathrm{Ext}_R^k(M, N) = 0$ for any module M, any module N of finite ϕ-projective dimension, and any $k > n$.

7. $\phi\text{-Gid}_R N \leq n$ for every R-module N.

Proof. (3) $\Rightarrow$ (7) Let N be an R-module and E be a nonnil-injective R-module. Then for every $k > n$, we have $\mathrm{Ext}_R^k(E,N) = 0$, since $\mathrm{pd}_R E \leq n$. Then ϕ-$\mathrm{Gid}_R N \leq n$ by Theorem 3.10.

(7) $\Rightarrow$ (3) Let E be a nonnil-injective module and N be an R-module. Then $\mathrm{Ext}_R^{n+1}(E,N) = 0$ by hypothesis and Theorem 3.10. Therefore, $\mathrm{pd}_R E \leq n$.

(1) $\Rightarrow$ (4) $\Rightarrow$ (5) These are direct.

(5) $\Rightarrow$ (2) Let P be a ϕ-u-projective module. Then by Theorem 3.7, we get $\mathrm{Ext}_R^k(R/I,P) = 0$ for every $k > n$ and every ideal I of R. Therefore, $\mathrm{id}_R P \leq n$.

(1) $\Leftrightarrow$ (6) This follows from Theorem 3.7.

(1) $\Leftrightarrow$ (2) The necessity follows immediately from Proposition 3.16. We claim the sufficiency. Let M be an R-module and P be a ϕ-u-projective module. Then for all $k > n$, we get that $\mathrm{Ext}_R^k(M,P) = 0$, since the injective dimension of P is at most n. By Theorem 3.7, we deduce that ϕ-$\mathrm{Gpd}_R M \leq n$.

(1) $\Leftrightarrow$ (3) This is the dual of the proof of (1) $\Leftrightarrow$ (2). □

Proposition 4.3. *Let R be a ring. Then*

$$\begin{aligned}\phi\text{-G-gl.dim}(R) &= \sup\{\phi\text{-}\mathrm{Gid}_R N \mid N \in \mathscr{M}_R\}\\ &= \sup\{\mathrm{Gid}_R N \mid N \in \mathscr{M}_R\}\\ &= \sup\{\phi\text{-}Gp_R M \mid M \text{ is a finitely generated } R\text{-module}\}\\ &= \sup\{\phi\text{-}Gp_R(\frac{R}{I}) \mid I \text{ is an ideal of } R\}.\end{aligned}$$

Proof. It follows immediately from Theorem 4.2 and [34, Theorem 4.5]. □

Proposition 4.4. *Let R be a ring. Then* G-gl. dim$(R) \leq \phi$-G-gl. dim(R), *with equality if ϕ-G-gl.dim$(R) < \infty$.*

Proof. This follows immediately from Remark 2.2, Corollary 3.8, [21, 2.20. Theorem], and [33, Definition 11.4.1]. □

Recall that a ring R is said to be quasi-Frobenius (QF for short) if every projective module is injective.

Definition 4.5. A ring R is said to be strongly quasi-Frobenius (strongly QF for short) if its nonnil-Gorenstein global dimension is zero.

Remark 4.6. It is easy to see that every strongly QF ring is QF. The converse is not true by Remark 2.9.

The following theorem is an analog of the well-known result [33, Theorem 4.6.10 (Faith-Walker)].

Theorem 4.7. The following are equivalent for a ring R of finite nonnil-Gorenstein global dimension:

1. R is strongly QF,
2. Every nonnil-injective module is projective,
3. Every ϕ-u-projective module is injective,
4. R is QF such that every nonnil-injective module is injective,
5. R is QF such that every ϕ-u-projective module is projective,
6. Every R-module is nonnil-G-injective.

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (6) These follow immediately from Theorem 4.2.

(2) ⇔ (4) This follows immediately from [33, Theorem 4.6.10 (Faith-Walker)].

(3) ⇒ (5) & (5) ⇒ (1) These follow immediately from [33, Theorem 4.6.10 (Faith-Walker)]. □

Remark 4.8. Note that a QF ring is not necessarily a strongly QF ring. In fact, for every field K, it is easy to see that the ring $R = K \propto K$ is a QF ring by [27, Corollary 3.8]. However, R is never a strongly QF ϕ-ring, since its nilradical is a ϕ-u-projective R-module but not projective by [33, Proposition 6.7.12].

It is natural to ask: what are the ϕ-rings of nonnil-Gorenstein global dimension 0? The following theorem answers this question.

Theorem 4.9. If $R \in \mathcal{H}$, then R is a field if and only if its nonnil-Gorenstein global dimension equals zero.

Proof. The necessity is trivial. Now we prove the sufficiency. Let $R \in \mathcal{H}$ of nonnil-Gorenstein global dimension 0. By Proposition 4.4 and [7, Proposition 2.6], R is a QF ring. In particular, $\mathrm{Nil}(R)$ is finitely generated. By [5, Lemma 2.3], either R is an integral domain or a local Artinian ring with maximal ideal $\mathrm{Nil}(R) \neq 0$. But if R is not an integral domain, then R is a ϕ-von Neumann regular ring. So every R-module is projective by Corollary 2.7, i.e., R is a semisimple ring, and so $\mathrm{Nil}(R) = 0$, a desired contradiction. Therefore R is an integral domain. Consequently, R is a field. □

We can give a second proof of Theorem 4.9 as follows.

Second proof of Theorem 4.9. By Theorem 4.7 and [31, Theorem 1.6], we deduce that if R is a strongly QF ϕ-ring, then R is a QF integral domain. Therefore, R must be a field. □

Also, there is a third proof as follows.

Third proof of Theorem 4.9. First, if $R \in \mathcal{H}$, then $R/\mathrm{Nil}(R)$ is a ϕ-u-projective R-module. In fact, we claim that $\mathrm{Ext}^1_R(R/\mathrm{Nil}(R), X) = 0$ for any ϕ-u-torsion R-module X. Let X be a ϕ-u-torsion R-module. We first show that $\mathrm{Hom}_R(\mathrm{Nil}(R), X) = 0$. Let $\gamma \in \mathrm{Hom}_R(\mathrm{Nil}(R), X)$. Since X is a ϕ-u-torsion R-module, $sX = 0$ for some $s \in R \setminus \mathrm{Nil}(R)$. Since $R \in \mathcal{H}$, it follows that $\mathrm{Nil}(R)$ is a ϕ-divisible R-module, and so $\mathrm{Nil}(R) = s\mathrm{Nil}(R)$. Then for every $n \in \mathrm{Nil}(R)$, we can write $n = sn'$ for some $n' \in \mathrm{Nil}(R)$, and so $\gamma(n) = s\gamma(n') \in sX = 0$. It follows that $\mathrm{Hom}_R(\mathrm{Nil}(R), X) = 0$. However, it follows from the short exact sequence $0 \to \mathrm{Nil}(R) \to R \to R/\mathrm{Nil}(R) \to 0$ that $0 = \mathrm{Hom}_R(\mathrm{Nil}(R), X) \to \mathrm{Ext}^1_R(R/\mathrm{Nil}(R), X) \to 0$. Therefore, $R/\mathrm{Nil}(R)$ is a ϕ-u-projective R-module.

Now, if the nonnil-Gorenstein global dimension of R is zero, then by Theorem 4.7, R is a *QF* ring such that $R/\mathrm{Nil}(R)$ is a projective R-module, and so $\mathrm{Nil}(R)$ is a projective ideal of R. It follows that $\mathrm{Nil}(R) = 0$ by [33, Proposition 6.7.12], and so R is both an integral domain and a QF-ring, that is a field. The converse is obvious. □

As shown in Theorem 4.9, the ϕ-rings with nonnil-Gorenstein global dimension 0 are precisely the fields. Therefore, the class of ϕ-rings of nonnil-Gorenstein global dimension 0 is well-established. To extend this class, let us consider the following definition.

Definition 4.10. Let R be a ring. Define its closed nonnil-Gorenstein global dimension, denoted by $\overline{\phi\text{-}\mathrm{G.gl.dim}(R)}$, as follows:

$$\overline{\phi\text{-}\mathrm{G.gl.dim}(R)} := \sup\{\phi\text{-}\mathrm{Gpd}_R M \mid M \in \mathcal{T}\}. \tag{7}$$

Theorem 4.11. The following are equivalent for a ring R of finite closed nonnil-Gorenstein global dimension:

1. $\overline{\phi\text{-G.gl.dim}(R)} \leq n$,
2. The ϕ-injective dimension of any ϕ-u-projective module is at most n,
3. The ϕ-projective dimension of nonnil-injective module is at most n,
4. $\phi\text{-Gpd}_R M \leq n$ for any finitely generated ϕ-u-torsion R-module M,
5. $\phi\text{-Gpd}_R(R/I) \leq n$ for any nonnil ideal I of R,
6. $\text{Ext}_R^k(M,N) = 0$ for any ϕ-u-torsion module M, any module N of finite ϕ-projective dimension, and any $k > n$.

Proof. (1) ⇒ (4) ⇒ (5) These are straightforward.

(5) ⇒ (2) Let P be a ϕ-u-projective module. Then by Theorem 3.7, we get $\text{Ext}_R^k(R/I,P) = 0$ for every $k > n$ and every nonnil ideal I of R. Therefore, $\phi\text{-id}_R P \leq n$.

(1) ⇔ (6) This follows from Theorem 3.7.

(1) ⇔ (2) The necessity follows immediately from Proposition 3.16. We claim the sufficiency. Let M be a ϕ-u-torsion R-module and P be a ϕ-u-projective module. Then for any $k > n$, we get $\text{Ext}_R^k(M,P) = 0$, since the ϕ-injective dimension of P is at most n. By Theorem 3.7, we deduce that $\phi\text{-Gpd}_R M \leq n$.

(1) ⇔ (3) This can be proved dually with (1) ⇔ (2). □

Next, a ϕ-ring R is said to be nonnil-self-injective if R is a nonnil injective module over itself. The following theorem characterizes the ϕ-von Neumann regular rings in terms of the closed nonnil-Gorenstein global dimension.

Theorem 4.12. The following are equivalent for a strongly ϕ-ring R:

1. R is a nonnil self-injective ring,
2. $\text{Nil}(R)$ and $R/\text{Nil}(R)$ are nonnil injective R-modules,
3. $\text{Nil}(R)$ is a nonnil-injective ideal of R and $R/\text{Nil}(R)$ is a self-injective ring,
4. $R/\text{Nil}(R)$ is a self injective ring,
5. R is a ϕ-von Neumann regular ring,
6. $\overline{\phi\text{-G.gl.dim}(R)} = 0$.

Proof. (1) ⇒ (2) Assume that R is a nonnil self-injective ring. Then $\text{Nil}(R)$ is a nonnil injective ideal. In fact, it is easy to see that $R/\text{Nil}(R)$ is a ϕ-torsion-free R-module, and so for any nonnil ideal I of R, we get $\text{Hom}_R(R/I, R/\text{Nil}(R)) = 0$ by [32, Theorem 2.3]. Using the sequence $0 \to \text{Nil}(R) \to R \to R/\text{Nil}(R) \to 0$, we obtain that $\text{Ext}_R^1(R/I,\text{Nil}(R)) = 0$ since R is nonnil self-injective, and so $\text{Nil}(R)$ is a nonnil-injective ideal. Since $\text{Ext}_R^1(R/I, R/\text{Nil}(R)) \cong \text{Ext}_R^2(R/I,\text{Nil}(R))$, it follows that $\text{Ext}_R^1(R/I, R/\text{Nil}(R)) = 0$, and so $R/\text{Nil}(R)$ is a nonnil injective R-module.

(2) ⇒ (1) If $\text{Nil}(R)$ and $R/\text{Nil}(R)$ are nonnil injective R-modules, then it is straightforward to see that R is a nonnil self-injective ring.

(2) ⇔ (3) This follows from [31, Proposition 1.4].

(3) ⇒ (5) Let $s \in R \setminus \text{Nil}(R)$. Since $R/\text{Nil}(R)$ is a nonnil-injective R-module, it is a divisible R-module by [14, Theorem 2.9], and so there exists $r \in R$ such that $1 + \text{Nil}(R) = sr + \text{Nil}(R)$. Hence $sr \in 1 + \text{Nil}(R) \subset U(R)$. Therefore, $R \setminus \text{Nil}(R) \subset U(R)$. It follows that R is a local ϕ-ring with maximal ideal $\text{Nil}(R)$. Therefore, R is a ϕ-von Neumann regular ring by [14, Theorem 5.14].

(5) $\Rightarrow$ (1) This follows from [14, Remarks 4.9 and Theorem 5.32].

(4) $\Leftrightarrow$ (5) This follows immediately from [32, Theorem 4.1] and the fact that every self-injective integral domain is a field.

(5) $\Rightarrow$ (6) If R is a ϕ-von Neumann regular ring, then each ϕ-u-torsion R-module is equal zero. Then $\overline{\phi\text{-}\mathrm{G.gl.dim}(R)} = 0$.

(6) $\Rightarrow$ (1) Assume that R is a ϕ-ring of the closed nonnil-Gorenstein global dimension 0. By Theorem 4.11, we deduce easily that R is nonnil-self-injective. □

By combining Theorem 2.11 and Theorem 4.12, we give the following corollary.

Corollary 4.13. *The following are equivalent for a ϕ-ring R:*

1. *R is a ϕ-von Neumann regular ring,*
2. *R is a strongly ϕ-ring such that every R-module has finite ϕ-projective dimension and every ϕ-torsion module is nonnil-G-projective,*
3. *R is a strongly ϕ-ring such that every R-module has finite ϕ-projective dimension and every ϕ-u-torsion module is nonnil-G-projective,*
4. *R is a strongly ϕ-ring such that every R-module has finite ϕ-injective dimension and every ϕ-torsion module is nonnil-G-injective,*
5. *R is a strongly ϕ-ring such that every R-module has finite ϕ-injective dimension and every ϕ-u-torsion module is nonnil-G-injective,*
6. *R is a strongly ϕ-ring such that every ϕ-u-projective module is nonnil-injective,*
7. *R is a strongly ϕ-ring such that every nonnil-injective module is ϕ-u-projective,*
8. *R is a strongly ϕ-ring of $\overline{\phi\text{-}\mathrm{G.gl.dim}(R)} = 0$.*

Statements and Declarations

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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