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On $\sum$-nil-good rings

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Abstract. In this paper, we introduce a new concept of ring called $\sum$-nil-good ring as a generalization of the notion of nil-good ring, in the context of commutative rings. We study some basic properties of this ring. We investigate this notion to various context of commutative ring extensions such as homomorphic image, direct product, power series ring and amalgamation ring, with application on the transfer of this property to trivial ring extension. Our goal is to generate a new original class of rings satisfying the above property.
Key Words: $\sum$-nil-good, nil-good, 2-nil-good, homomorphic image, direct product, amalgamation, trivial ring extension.
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1 Introduction

Throughout the whole paper, all rings considered are commutative with unity and all modules are unital. In [32], Vámos introduced the concept of 2-good ring which is a ring for which any element can be written as a sum of two units. He showed that the endomorphism ring of a free module of rank at least 2 is 3-good and that every ring can be embedded in a 2-good ring. Answering a question of Skornjakov, he showed that a (right) self-injective von Neumann regular ring is 2-good provided it has no 2-torsion. This generalizes a result of Zelinsky's on the ring of linear transformations of a vector space. A significant work has been done on generalizations of 2-good rings. A more general concept of 2-good ring is the notion of k-good ring which is a ring for which any element can be written as a sum of k units. In [10], the authors defined a new class of unital rings called nil-good ring, as a ring for which each of its elements can be written as sum $u+b$ where u is a unit or equal 0 and b is a nilpotent. Also, the authors studied the structure as well as the behavior of this property with respect to subrings, matrix extensions and direct products. Later, in [1], a ring R is defined to be 2-nil-good if every element in R is the sum of two units and a nilpotent. Fundamental properties of such rings are obtained. Among other things, the authors proved that every strongly π-regular ring is 2-nil-good if and only if the identity is the sum of two units. One of the main results of their paper is that every square matrix ring over J-fine rings is 2-nil-good. Also, they established the 2-nil-good property for Morita contexts. This implies, in particular, that every matrix ring over a 2-nil-good ring is 2-nil-good. In [4], the authors introduced new classes of rings, namely, n^*-nil clean ring, $\sum$-nil clean ring and BB-ring, and they studied the behavior of the above notions to several commutative ring extensions such as homomorphic image, direct product, power series ring and amalgamation ring, with applications to the transfer of these properties in trivial ring extension. Motivated by these studies on generalizations of k-good rings, we introduce a new class of rings. We call a ring R to be $\sum$-nil-good if for every element x of R, there is an integer $n \geq 1$ such that x can be written as a sum of n units and a nilpotent element. We study some basic properties of these rings and establish

several characterizations. In this paper, we deal with a new concept of ring called $\sum$-nil-good ring as a generalization of the notion of nil-good ring, in the context of commutative rings. We study some basic properties of this ring. We investigate this notion to various context of commutative ring extensions such as homomorphic image, direct product, power series ring and amalgamation ring, with application to the transfer of this property to trivial ring extension. Our goal is to generate a new original class of rings satisfying the above property.
Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. In this setting, we can consider the following subring of $A \times B$: $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$, called *the amalgamation of A and B along J with respect to f*, introduced and studied by D'Anna, Finocchiaro and Fontana in [12, 13]. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [14] and [15]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations) (cf. [29, page 2]). For example, constructions such as the $A + XB[X]$, the $A + XB[[X]]$ and the $D + M$ constructions can be studied as particular cases of the amalgamation ([12, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [6]) are strictly related to it ([12, Example 2.7 and Remark 2.8]). In [12], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [13], they pursued the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.
Let A be a ring and E an A−module. The trivial ring extension of A by E (also called idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + ea')$. Trivial ring extensions have been studied extensively; and considerable work, part of is summarized in Glaz's book [18] and Huckaba's book [23], has been concerned with these extensions. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [2, 23].
For a ring R, we denote respectively by $Nilp(R)$, $Idem(R)$, $Jac(R)$, $U(R)$ and $Char(R)$, the ideal of all nilpotent elements of R, the set of all idempotent elements of R, the Jacobson radical of R, the multiplicative group of units of R and the characteristic of R.

2 On $\sum$-nil-good rings

We begin this section by recalling the notion of $\sum$-nil-good ring defined in the introduction.

Definition 2.1.
Let R be a ring. An element x of R is said to be $\sum$-nil-good if there exists a positive integer $n \geq 1$ such that $x = u_1 + u_2 + \cdots + u_n + w$ with $u_1, u_2, ..., u_n \in U(R)$ and $w \in Nilp(R)$. Then R is said to be $\sum$-nil-good if each of its elements is $\sum$-nil-good.

Example 2.2.
1. Every n-good ring is $\sum$-nil-good.
2. Every 2-nil-good ring is $\sum$-nil-good.

It is worthwhile noting from the definition that a 2-nil-good ring is $\sum$-nil-good. However, a $\sum$-nil-good ring need not be 2-nil-good, as shown in the following example:

Example 2.3. Let $R := \mathbb{Z}_6$ be the ring of integers modulo 6. Then:

1. R is $\sum$-nil-good.

2. R is not 2-nil-good.

Proof. (1) Notice first that $0 = 1 - 1$ and $1 = 1 + 0$ are $\sum$-nil-good elements of R. Since every element of $n \in \mathbb{Z}_6$ with $n \geq 2$, can be written as $n = \underbrace{1 + 1 + \cdots + 1}_{\text{n times}}$, then R is $\sum$-nil-good.
(2) R is not 2-nil-good since 5 can not be expressed as a sum of two unit elements and a nilpotent element. □

Recall that a ring R has unit stable range one if $\forall\ a \in R$, there is $u \in U(R)$ such that $a + u.1 \in U(R)$.

Proposition 2.4. *Let R be a ring. Then the following assertions hold:*

1. *If R is $\sum$-nil-good, then there exists a positive integer $n \geq 1$ such that $1 = v_1 + \cdots + v_n$ with $v_i \in U(R)$ for $i = 1, ..., n$.*

2. *Assume that R has unit stable range one with $1 = u_1 + \cdots + u_n$ and $u_i \in U(R)$ for all $i = 1, ..., n$. Then R is $\sum$-nil-good.*

Proof. (1) Assume that R is $\sum$-nil-good. Then there exist $u_1, u_2, ..., u_n \in U(R)$ and $w \in Nilp(R)$ such that $1 = u_1 + u_2 + \cdots + u_n + w$. So, $1 - w = u_1 + u_2 + \cdots + u_n$. Since $w \in Nilp(R) \subseteq Jac(R)$, $(1 - w) \in U(R)$ and so it follows that $1 = (1 - w)^{-1}u_1 + (1 - w)^{-1}u_2 + \cdots + (1 - w)^{-1}u_n$. One can easily check that $(1 - w)^{-1}u_i \in U(R)$ for every $i = 1, 2, ..., n$. Hence, there exist $v_1, ..., v_n \in U(R)$ such that $1 = v_1 + \cdots + v_n$, as desired.
(2) Assume that R is unit stable range one with $1 = v_1 + \cdots + v_n$ with $v_i \in U(R)$ for all $i = 1, ..., n$. We claim that R is $\sum$-nil-good. Indeed, let $a \in R$. Then there exists $u \in U(R)$ such that $a + 1.u \in U(R)$ (as R is unit stable range one). So, $a + (v_1 + \cdots + v_n)u = y \in U(R)$. Since $uv_i \in U(R)$ for every $i = 1, ..., n$, it follows that $a = v_1u + \cdots + v_nu + y$, which is a sum of $(n + 1)$ units. Hence, R is $\sum$-nil-good, as desired. □

Recall that a ring R is UU (i.e., every unit is unipotent) if every unit $u \in R$ is unipotent, that is, u can be expressed as $u = 1 + w$ for some nilpotent element w of R. In the next proposition, we show that in a UU-ring R containing the field of rational numbers $\mathbb{Q}$, the properties " $\sum$-nil-good" and "nil-good" collapse.

Proposition 2.5. *Let R be a UU-ring such that $\mathbb{Q} \subseteq R$. Then R is $\sum$-nil-good if and only if R is nil-good.*

Proof. Assume that R is $\sum$-nil-good. Let $x \in R$. Then there exists $n \geq 1$ such that $x = u_1 + u_2 + \cdots + u_n + w$ with $u_i \in U(R)$ for every $i = 1, ..., n$ and $w \in Nilp(R)$. Since R is a UU-ring, $u_i = 1 + w_i$ with $w_i \in Nilp(R)$ for every $i = 1, ..., n$. So, $x = 1 + w_1 + 1 + w_2 + \cdots + 1 + w_n + w = n + (w_1 + w_2 + \cdots + w_n) + w$. Set $W = w_1 + w_2 + \cdots + w_n + w$. Clearly, $W \in Nilp(R)$ and $n \in R$. Therefore, $x = n + W$. Using the fact that $W \in Nilp(R)$ and n is a unit (as $\mathbb{Q} \subseteq R$), it follows that $n + W$ is a nil-good element. Hence, R is nil-good. The converse is trivial. □

Recall that in [28], the author defined the concept of strongly $\sum$-m-clean rings, as a ring for which each of its elements can be expressed as a sum of n units for some positive integer $n \geq 1$ and an m-potent element. The following result shows that the class of $\sum$-nil-good rings is included in the class of strongly $\sum$-m-clean rings.

Proposition 2.6. *Every $\sum$-nil-good ring is a strongly $\sum$-m-clean ring for every positive integer $m \geq 2$. The converse holds if R is an integral domain.*

Proof. Consider a $\sum$-nil-good ring R. Let $x \in R$. Then there exists $n \geq 1$ such that $x = u_1 + u_2 + \cdots + u_n + w$ with $u_i \in U(R)$ for each $i = 1, \ldots, n$ and $w \in Nilp(R)$. Observe that $u_n + w = u_n(1 + u_n^{-1}w) \in U(R)$, as $u_n^{-1}w \in Nilp(R)$. Therefore, $x = u_1 + u_2 + \cdots + u_{n-1} + v$ with $v = u_n + w \in U(R)$. Hence, $x = u_1 + u_2 + \cdots + u_{n-1} + v + 0$ is a strongly $\sum$-m-clean element. Thus, R is strongly $\sum$-m-clean.
Conversely, assume that R is a strongly $\sum$-m-clean domain. Let $x \in R$. Then there exists $n \geq 1$ such that $x = u_1 + u_2 + \cdots + u_n + f$ for some $u_1, u_2, \ldots, u_n \in U(R)$ and f is an m-potent element of R with $m \geq 2$. Since $f^m = f$, then $f^m - f = f(f^{m-1} - 1) = 0$. Two cases are then possible:
Case $1 : f = 0$, then $x = u_1 + u_2 + \cdots + u_n$ which is a sum of n units and so x is an n-good element.
Case $2 : f^{m-1} - 1 = 0$. Then $f^{m-1} = ff^{m-2} = 1$. Therefore, f is a unit of R and so $x = u_1 + u_2 + \cdots + u_n + f$ is a sum of $(n+1)$ units. Consequently, x is an $(n+1)$-good element.
Hence, in all cases, it follows that x is a $\sum$-nil-good element, making R, a $\sum$-nil-good ring. □

Proposition 2.7.
(1) *Let A be a ring and I be an ideal of A. If A is $\sum$-nil-good, then so is A/I. The converse holds if I is a nil ideal of A.*
(2) *Let I be a finite index set. If $A = \prod_{\alpha \in I} A_\alpha$ is $\sum$-nil-good, then so is A_α $\forall \alpha \in I$.*

Proof. (1) If A is $\sum$-nil-good, then we claim that so is A/I. Indeed, this is due to the fact that any homomorphic image of a nilpotent (resp., unit) element is also a nilpotent (resp., unit) element.
Conversely, assume that A/I is $\sum$-nil-good and I is a nil ideal of A. We claim that A is $\sum$-nil-good. Let $a \in A$ such that $\bar{a} \in A/I$. Then there exists $n \geq 1$ such that $\bar{a} = \overline{u_1} + \cdots + \overline{u_n} + \overline{b} = (u_1 + \cdots + u_n + I) + b + I = (u_1 + \cdots + u_n + b + I)$ with $\overline{b} \in Nilp(A/I)$ and $\overline{u_i} \in U(A/I)$ for every $i = 1, \ldots, n$. Therefore, $a = u_1 + \cdots + u_n + b + b_1$, with $b_1 \in I$. Since $I \subseteq Nilp(A)$, then $b \in Nilp(A)$. The fact that I is nil ideal implies that $I \subseteq Jac(A)$ and so $u_i \in U(A)$ for every $i = 1, \ldots, n$. Consequently, there exists $n \geq 1$ such that $a = u_1 + \cdots + u_n + w$ with $u_i \in U(A)$ for every $i = 1, \ldots, n$ and $w \in Nilp(A)$. Hence, a is $\sum$-nil-good, making A, a $\sum$-nil-good ring.
(2) Assume that $A = \prod_{\alpha \in I} A_\alpha$ is a $\sum$-nil-good ring. Then by assertion (1) above, so is A_α $\forall \alpha \in I$ (as $\forall \alpha \in I$, A_α is the homomorphic image of A under the canonical projection over A_α). □

The next proposition examines the $\sum$-nil-good property in the formal power series ring $R[[X]]$.

Proposition 2.8. *Let R be a ring. Then $R[[X]]$ is $\sum$-nil-good if and only if so is R.*

Proof. Assume that $R[[X]]$ is $\sum$-nil-good. Then by assertion (1) of Proposition 2.7, $R[[X]]/X \simeq R$ is $\sum$-nil-good.
Conversely, assume that R is $\sum$-nil-good. Let $P \in R[[X]]$. We need to show that P is a $\sum$-nil-good element. We have $P = \sum_{i=0}^{\infty} a_i X^i$ where $a_i \in R$ for $i = 0, 1, 2, \ldots$ and so $a_0 \in R$ which is a $\sum$-nil-good ring. So, there exists $n \geq 1$ such that $a_0 = u_1 + u_2 + \cdots + u_n + w$ such that $u_i \in U(R)$ for $i = 1, \ldots, n$ and $w \in Nilp(R)$. Therefore, $P = u_1 + \sum_{i=1}^{\infty} a_i X^i + u_2 + \cdots + u_n + w$. Since $u_1 + \sum_{i=1}^{\infty} a_i X^i$ is a unit of $R[[X]]$ and $w \in Nilp(R) \subseteq Nilp(R[[X]])$), it follows that P is $\sum$-nil-good, making $R[[X]]$, a $\sum$-nil-good ring, as desired. □

The next result establishes the transfer of $\sum$-nil-good ring property to amalgamation ring.

Theorem 2.9. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then the following statements hold:

1. If $A \bowtie^f J$ is $\sum$-nil-good, then so are A and $f(A) + J$.
2. Assume that $J \subseteq Nilp(B)$. Then $A \bowtie^f J$ is $\sum$-nil-good if and only if so is A.
3. Assume that $\forall a \in U(A)$, $f(a) + j \in U(f(A) + J)$. Then $A \bowtie^f J$ is $\sum$-nil-good if and only if so is A and $\forall\ x \in J$, $x = l + h$ for some $l \in J \cap Jac(B)$ and $h \in J \cap Nilp(B)$.

Proof. (1) Assume that $A \bowtie^f J$ is $\sum$-nil-good. By [12, Proposition 5.1], A and $f(A)+J$ are homomorphic images of $A \bowtie^f J$, and so by Proposition 2.7(1), we obtain the desired result.

(2) If $A \bowtie^f J$ is $\sum$-nil-good, then by assertion (1) above, so is A.
Conversely, assume that A is $\sum$-nil-good. We claim that $A \bowtie^f J$ is $\sum$-nil-good. Indeed, let $(a, f(a)+j) \in A \bowtie^f J$. Clearly, $a \in A$ which is $\sum$-nil-good. So, there exists $n \geq 1$ such that $a = u_1 + \cdots + u_n + w$ with $u_i \in U(A)$ for each $i = 1, \ldots, n$ and $w \in Nilp(A)$. Therefore, $(a, f(a)+j) = (u_1, f(u_1)) + \cdots + (u_n, f(u_n)) + (w, f(w)+j)$ with $(u_i, f(u_i)) \in U(A \bowtie^f J)$ for each $i = 1, \ldots, n$ and from [8, Lemma 2.10], $(w, f(w)+j) \in Nilp(A \bowtie^f J)$ as $w \in Nilp(A)$ and $j \in Nilp(B) \cap J$. Hence, $(a, f(a)+j)$ is a $\sum$-nil-good element, making $A \bowtie^f J$, a $\sum$-nil-good ring.

(3) If $A \bowtie^f J$ is $\sum$-nil-good, then by assertion (1) above, so is A. Let $x \in J$. Then $(0, x) \in A \bowtie^f J$ which is $\sum$-nil-good. So, there exists $n \geq 1$ such that $(0, x) = \sum_{i=1}^{n}(u_i, f(u_i)+l_i) + (w, f(w)+h)$, with $(w, f(w)+h) \in Nilp(A \bowtie^f J)$ and $(u_i, f(u_i)+l_i) \in U(A \bowtie^f J)$ for each $i = 1, 2, \ldots, n$. Therefore, $\sum_{i=1}^{n} u_i + w = 0$ and so $x = \sum_{i=1}^{n} l_i + h$, with $h \in Nilp(B) \cap J$ and $l_i \in J$ for each $i = 1, 2, \ldots, n$. On the other hand, for each $i = 1, 2, \ldots, n$, $(u_i, f(u_i)+l_i) \in U(A \bowtie^f J)$, then $f(u_i)+l_i \in U(f(A)+J)$. So, for each $i = 1, 2, \ldots, n$, there exists $f(v_i)+\alpha_i \in f(A)+J$ such that $(f(u_i)+l_i)(f(v_i)+\alpha_i) = 1$. It follows that $l_i(f(v_i)+\alpha_i) = 1 - f(u_i)(f(v_i)+\alpha_i)$ and so $1 - l_i(f(v_i)+\alpha_i) = f(u_i)(f(v_i)+\alpha_i) \in U(B)$. Consequently, $l_i \in J \cap Jac(B)$ for each $i = 1, 2, \ldots, n$. Set $l = \sum_{i=1}^{n} l_i$. Hence, $x = l + h$ with $l \in J \cap Jac(B)$ and $h \in J \cap Nilp(B)$.
Conversely, assume that A is $\sum$-nil-good and $\forall\ x \in J$, $x = l + h$ with $l \in J \cap Jac(B)$ and $h \in J \cap Nilp(B)$. Let $(a, f(a)+j) \in A \bowtie^f J$. Then $a \in A$ which is $\sum$-nil-good. So, there exists $n \geq 1$ such that $a = u_1 + u_2 + \cdots + u_n + w$ with $u_i \in U(A)$ for each $i = 1, 2, \ldots, n$ and $w \in Nilp(A)$. Therefore, $f(a) = f(u_1 + u_2 + \cdots + u_n) + f(w) = f(u_1) + f(u_2) + \cdots + f(u_n) + f(w)$ with $f(w) \in Nilp(B)$. Using the fact that $j \in J$, it follows that $j = l + h$ with $h \in J \cap Nilp(B)$ and $l \in J \cap Jac(B)$. Consequently, $(a, f(a)+j) = (u_1 + u_2 + \cdots + u_n + w, f(u_1) + f(u_2) + \cdots + f(u_n) + f(w) + l + h) = (u_1, f(u_1)+l) + (u_2, f(u_2)) + \cdots + (u_n, f(u_n)) + (w, f(w)+h)$. One can easily check that $(u_1, f(u_1)+l) \in U(A \bowtie^f J)$ as $l \in J \cap Jac(A)$ and $(u_i, f(u_i)) \in U(A \bowtie^f J)$ for each $i = 2, \ldots, n$. From [8, Lemma 2.10], it follows that $(w, f(w)+h) \in Nilp(A \bowtie^f J)$. Hence, $(a, f(a)+j)$ is a $\sum$-nil-good element of $A \bowtie^f J$, making $A \bowtie^f J$, a $\sum$-nil-good ring, as desired. □

The next corollary is a consequence of assertion (2) of Theorem 2.9 on the transfer of $\sum$-nil-good property to trivial ring extension.

Corollary 2.10.
Let A be a ring, E be an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then R is $\sum$-nil-good if and only if so is A.

Proof. Consider $f : A \hookrightarrow R$ to be the injective ring homomorphism defined by $f(a) = (a, 0)$, for every $a \in A$ and let $J := 0 \propto E$ be an ideal of R. Clearly, $f^{-1}(J) = 0$. Therefore, by [12, Proposition 5.1 (3)], $f(A) + J = A \propto 0 + 0 \propto E = A \propto E = R \simeq A \bowtie^f J$. On the other hand, $J := 0 \propto E \subseteq Nilp(B)$ and so by an application to assertion (2) of Theorem 2.9, we have the desired result. □

The following corollary is an immediate consequence of assertion (3) of Theorem 2.9.

Corollary 2.11. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B such that $J \subseteq Jac(B)$. Then $A \bowtie^f J$ is $\sum$-nil-good if and only if so is A and $\forall\ x \in J$, $x = l + h$ for some $h \in J \cap Nilp(B)$ and $l \in J \cap Jac(B)$.*

The following example illustrates Theorem 2.9 by providing a new class of $\sum$-nil-good rings which are not 2-nil-good.

Example 2.12. Let $A := \mathbb{Z}_6$ be the ring of integers modulo 6, E be an A-module, $B := A \propto E$ be the trivial ring extension of A by E. Consider the injective ring homomorphism $f : A \hookrightarrow B$ defined by $f(a) = (a, 0)$ for every $a \in A$ and let $J := 0 \propto E$ be an ideal of B. Then:

1. $A \bowtie^f J$ is $\sum$-nil-good.

2. $A \bowtie^f J$ is not 2-nil-good.

Proof. (1) By assertion (2) of Theorem 2.9, $A \bowtie^f J$ is $\sum$-nil-good since A is $\sum$-nil-good and $J \subseteq Nilp(B)$.
(2) By Proposition 2.7(1), $A \bowtie^f J$ is not 2-nil-good since the homomorphic image A of $A \bowtie^f J$ is not 2-nil-good (as 5 can not be written as a sum of two unit elements and a nilpotent element). □

As another application of Theorem 2.9, we generate a new original class of $\sum$-nil-good rings which are not 4-good.

Example 2.13. Let $A := \mathbb{Z}$, which is a $\sum$-nil-good ring, $B := \mathbb{Z}_4$ be the ring of integers modulo 4, $f : A \longrightarrow B$ be the canonical surjection and $J := 2\mathbb{Z}_4$ is the maximal ideal of B. Then:

1. $A \bowtie^f J$ is $\sum$-nil-good.

2. $A \bowtie^f J$ is not a 4-good ring.

Proof. (1) Since A is a $\sum$-nil-good ring and $J^2 = 0 \subseteq Nilp(B)$, then by assertion (2) of Theorem 2.9, it follows that $A \bowtie^f J$ is $\sum$-nil-good.
(2) By [4, Proposition 4.3(1)], $A \bowtie^f J$ is not a 4-good ring since the homomorphic image A of $A \bowtie^f J$ is not a 4-good ring (as the element 6 can not be expressed as a sum of 4 units). □

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