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Abstract. Let G be a group with identity e and R a G -graded commutative ring with $1 \neq 0$. In this paper, we study the concepts of graded 1-absorbing primary and graded weakly 1-absorbing primary ideals within G -graded rings. We aim to compare them with other ideal types, such as graded primary (graded weakly primary) and graded 2-absorbing primary (graded weakly 2-absorbing primary). Additionally, we examine their primary properties and investigate their behavior under G -graded ring homomorphisms. We also provide characterizations of these classes of ideals in contexts such as localizations of graded rings, graded trivial ring extensions, and graded amalgamation algebras along an ideal.

Key Words: Graded ring, graded ideal, graded 1-absorbing primary ideal, graded weakly 1-absorbing primary, graded primary ideal, graded prime ideal, trivial ring extension, graded module, graded submodule.

2020 MSC: 13A15, 13A02, 13C05, 13H05.

Dedicated to our Professor David E. Dobbs for his 80th Birthday.

1 Introduction

Throughout this paper (unless otherwise stated), all rings are commutative with $1 \neq 0$. The concept of 1-absorbing primary ideals was introduced by Badawi and Yetkin Celikel in [6]. A proper ideal I of R is called a 1-absorbing primary ideal, if for any nonunit $a, b, c \in R$ with $abc \in I$, then either $ab \in I$ or $c \in \sqrt{I}$. The concept of weakly 1-absorbing primary ideal was introduced by Badawi and Yetkin Celikel [7]. A proper ideal I of R is called weakly 1-absorbing primary ideal if for any nonunit $a, b, c \in R$ with $0 \neq abc \in I$, then either $ab \in I$ or $c \in \sqrt{I}$.

In this paper, we study graded 1-absorbing primary ideals and graded weakly 1-absorbing primary ideals of graded rings. First, we review some basic properties of graded rings and modules that will be used in subsequently the sequel. For further details and more information on graded rings and modules, we refer to [17, 18]. Let G be a multiplication group with identity e . A ring R is called to be G -graded ring (or G -graded ring) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is equality, then the ring R is called strongly graded. The elements of R_g are called homogeneous of degree g and R_e is a subring of R and $1 \in R_e$. For $x \in R$, x can be written uniquely as $x = \sum_{g \in G} x_g$ where x_g is the component of x in R_g . Also we write $h(R) = \cup_{g \in G} R_g$ the set of all homogeneous elements. Moreover, if $r \in R_g$ and r is unit, then $r^{-1} \in R_{g^{-1}}$. A G -graded ring $R = \bigoplus_{g \in G} R_g$ is called a crossed product if R_g contains a unit for every $g \in G$. Note that a G -crossed product $R = \bigoplus_{g \in G} R_g$ is a strongly graded ring. Let R be a

G -graded ring and I be an ideal of R . Then I is called G -graded ideal if $I = \bigoplus_{g \in G} (I \cap R_g) = \bigoplus_{g \in G} I_g$, i.e, if $x \in I$ and $x = \sum_{g \in G} x_g$, then $x_g \in I$ for all $g \in G$. If $R = \bigoplus_{g \in G} R_g$ and $R' = \bigoplus_{g \in G} R'_g$ are two G -graded rings, then a mapping $f : R \rightarrow R'$ with $f(1_R) = 1_{R'}$ is called a gr-homomorphism if $f(R_g) \subseteq R'_g$ for all $g \in G$. Let $R = \bigoplus_{g \in G} R_g$ be a G -graded ring and let I be a graded ideal of R . Then the Quotient ring R/I is also a G -graded ring. Indeed, $R/I = \bigoplus_{g \in G} (R/I)_g$ where $(R/I)_g = \{x + I : x \in R_g\}$, with this grading, $(R/I)_e \cong R_e/I_e$. By the equality $I = \bigoplus_{g \in G} (I \cap R_g) = \bigoplus_{g \in G} I_g$, we obtain that an ideal I of R is a graded ideal if and only if I is generated by homogeneous elements of R . Recall that a G -graded ring is called *graded integral domain* if for element $a \in h(R)$, a is not a divisor of zero. In particular, R is said to be a *graded field* if every nonzero homogeneous element has a multiplicative inverse.

Let R be G -graded ring and let $S \subseteq h(R)$ be a multiplicatively closed subset of R . The the ring of fractions $S^{-1}R$ is a G -graded ring which is called the gr-ring of fractions. Indeed $S^{-1}R = \bigoplus_{g \in G} (S^{-1}R)_g$ where

$$(S^{-1}R)_g = \left\{ \frac{r}{s}, s \in R, s \in S \quad \text{and} \quad g = (degs)^{-1}(degr) \right\}.$$

Consider the ring gr-homomorphism $f : R \rightarrow S^{-1}R$ defined by $f(r) = \frac{r}{1}$. For any graded ideal I of R , the ideal of $S^{-1}R$ generated by $f(I)$ is denoted by $S^{-1}I$. Similar to non graded case, we can prove that

$$S^{-1}R = \{ \beta \in S^{-1}R : \beta = \frac{r}{s}, r \in I, s \in S \}.$$

Note that a proper graded ideal I of R is said to be graded prime if whenever $a, b \in h(R)$ such that $ab \in I$, then either $a \in I$ or $b \in I$. Let R be a G -graded ring. The *graded radical* of a graded ideal I , denoted by $\text{Gr}(I)$ is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in \text{Gr}(I)$ if and only if $r^n \in I$ for some positive integer n . In particular, we denotes $N(R) = \text{Gr}\{0\}$. A graded R -module is an R -module M such that $M = \bigoplus_{g \in G} M_g$ where M_g is an additive subgroup of M , and for every $g, h \in G$ we have $R_g M_h \subseteq M_{hg}$. Since $R_e M_h \subseteq M_h$, we can see that M_h is an R_e -submodule of M . The elements of $h(M) = \bigcup_{g \in G} M_g$ are called the homogeneous elements of M . A non zero element $m \in M_g$ is said to be a homogeneous element of degree g . Every $m \in M$ can be uniquely represented as a sum $m = \sum_{g \in G} m_g$ with $m_g \in M_g$ and finitely many nonzero m_g . The nonzero elements m_g in this sum are called the homogeneous components of m . An R -submodule N of M is said to be a graded submodule if for every $n \in N$, all its homogeneous components are also in N , i.e, $N = \sum_{g \in G} (N \cap M_g)$. If $I = \bigoplus_{g \in G} I_g$ is a graded ideal of R , then I_g is an R_e -module for every $g \in G$. A graded submodule N is called graded primary submodule if whenever $a \in h(R)$ and $m \in h(M)$ with $am \in N$, then either $m \in N$ or $a^k \in (N : M)$ for some positive integer. Therefore, if $P := (N : M)$ is a graded prime ideal, we say that N is a graded P -primary submodule [10, 19].

Let R be a commutative ring and M be a R -module. Then the trivial extension $R(+M)$ with the coordinate-wise addition and multiplication given by $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ is a commutative ring with unity $(1, 0)$ (even R -algebra) is called the idealization of M or the trivial extension of R by M . Note that R naturally embeds into $R(+M)$ by $r \mapsto (r, 0)$, if N is a submodule of M , then $0(+N)$ is an ideal of $R(+M)$, $0(+M)$ is a nilpotent ideal of $R(+M)$ of index 2. It is well known that if I is an ideal of R and N is a submodule of M such that $IM \subseteq N$, then $I(+N)$ is an ideal of $R(+M)$. We recall that $U(R(+M)) = U(R)(+M)$, see [3].

Suppose that $R = \bigoplus_{g \in G} R_g$ be a G -graded ring and $M = \bigoplus_{g \in G} M_g$ be a G -graded R -module. Then $R(+M)$ is a G -graded ring with $(M(+R))_g = R_g \oplus M_g$ for every $g \in G$. Consequently, $h(R(+M)) = \{(a, x) : a \in h(R), x \in h(M)\}$, see [3, 21].

In this paper, we aim to investigate the notions of graded 1-absorbing primary and graded weakly primary ideals in G -graded rings. In Section 2, we study the concept of graded primary, graded 2-absorbing primary, and graded 1-absorbing primary ideals. After examining the relationships

among these concepts, we provide a counterexample (see Examples [2.3](#)). In addition to some main properties of graded 1-absorbing primary in the Cartesian product of G -graded commutative rings. We also investigate the attitude of the notion of graded 1-absorbing primary ideals under a G -graded ring homomorphism. Moreover, we obtain some characterizations of this notion in the localization of graded rings, the graded trivial extension rings, and the graded amalgamation algebras along an ideal. In Section 3, we study the concept of graded weakly 1-absorbing primary ideals in G -graded rings. It is clear that every graded 1-absorbing primary ideal is a graded weakly 1-absorbing primary ideal, but the converse is not true, see Example [3.2](#). As well as some characterizations of graded weakly 1-absorbing primary ideals, we compare them with other classes of ideals, such as graded weakly primary, graded weakly 2-absorbing primary, and graded 1-absorbing primary ideals. Finally, similar to Section 2, we characterize the class of graded weakly 1-absorbing primary ideals in the localization of graded rings, the graded trivial extension rings, and the graded amalgamation algebras along an ideal.

2 Graded 1-absorbing primary ideals

We remind the reader with the following definitions.

Definition 2.1. (cf. [\[15\]](#)) Let $R = \bigoplus_{g \in G} R_g$ be a G -graded ring, $I = \bigoplus_{g \in G} I_g$ be a graded ideal of R .

1. I is said to be a graded primary ideal of R if $I \neq R$, and whenever $r, s \in h(R)$ with $rs \in I$, then either $r \in I$ or $s \in Gr(I)$.
2. I is said to be a graded 2-absorbing primary ideal of R if $I \neq R$, and whenever $r, s, t \in h(R)$ with $rst \in I$, then either $rs \in I$ or $rt \in Gr(I)$ or $st \in Gr(I)$.
3. I is said to be a graded 1-absorbing primary ideal of R if $I \neq R$, and whenever $r, s, t \in h(R)$ nonunit homogeneous elements with $rst \in I$, then either $rs \in I$ or $t \in Gr(I)$.

Since the proof of the following result is elementary, we omit its proof.

Theorem 2.2. Let R be a G -graded ring and I be a proper graded ideal of R . Then:

1. If I is graded primary ideal of R , then I is graded 1-absorbing primary ideal of R .
2. If I is a graded 1-absorbing primary ideal of R , then I is graded 2-absorbing primary ideal of R .

The following example provide a graded 1-absorbing primary ideal that is not graded primary ideal.

Example 2.3. (A graded 1-absorbing primary ideal which is not graded primary) In Theorem [2.9](#), we will show that there exist a graded 1-absorbing ideal which is not a graded primary ideal. Precisely, if R is not a graded local ring, there exist a graded 1-absorbing primary ideal which is not graded primary ideal.

Theorem 2.4. Let R be a G -graded ring. If I is a graded 1-absorbing primary ideal of R , then $Gr(I)$ is a graded prime ideal of R .

Proof. Let r, s be homogeneous elements in $h(R)$ such that $rs \in Gr(I)$, we can assume that s, t are nonunits. Let $n \geq 2$ be a positive integer such that $(rs)^n \in I$. Then $n = 2m$ for some positive integer $m \geq 1$. Since $(rs)^n = r^m r^m s^n \in I$ and r^m, s^n are nonunit homogeneous elements of $h(R)$ and I is a graded 1-absorbing primary ideal, we conclude that either $r^m \cdot r^m = r^{2m} \in I$ or $s^n \in Gr(I)$. That is $r \in Gr(I)$ or $Gr(I)$. Hence $Gr(I)$ is a graded prime ideal of R . \square

Recall that for graded ideal M of a G -graded ring R is said to be a graded maximal ideal, if there is no graded ideal K such that $M \subset K \subset R$. Also, note that a G -graded commutative ring R is said to be a G -graded local ring, if R has a one graded maximal ideal M which is the ideal generated by noninvertible homogeneous elements.

Lemma 2.5. [12 Lemma 2.7] *Let R be a G -graded crossed product ring. Then R is a graded local ring if and only if R_e is a local ring.*

Lemma 2.6. [6 Lemma 1] *Let R be a ring. Suppose that for every nonunit element w of R and for every unit element u of R , we have $u + w$ is a unit element of R . Then R is a local ring.*

Let I be a graded ideal of a G -graded ring R . Then I is said to be a graded primary ideal if for every $a, b \in h(R)$ with $ab \in I$, then either $a \in I$ or $b \in Gr(I)$. It is clear that a graded primary ideal is a graded 1-absorbing primary ideal, but the converse is not true. In the next result, we give some examples of graded 1-absorbing primary which are not graded primary ideals. Also, we show that if R is G -graded ring which is not a graded local ring, then the concept of graded 1-absorbing primary ideals and graded primary ideals collapse.

Theorem 2.7. *Let R be a G -graded crossed product ring which admits a proper ideal which is graded 1-absorbing primary ideal that is not a graded primary. Then R is G -graded local ring.*

Proof. We will show that R_e is a local ring. Let c be a nonunit element of R_e . Suppose that I is a proper graded 1-absorbing primary ideal of R that is not a graded primary ideal of R . Hence there exist $a, b \in h(R)$ with $ab \in I$ and neither $a \in I$ nor $b \in Gr(I)$. Since c is homogeneous, $b \notin Gr(I)$ and $abc \in I$, then $ac \in I$. Now, let $d \in R_e$ be a unit homogeneous element of R . Assume that $d + c$ is nonunit which is in $R_e \subseteq h(R)$, we have $(d + c)ab \in I$. The fact that $b \notin Gr(I)$ and I is a graded 1-absorbing primary ideal give that $(d + c)a \in I$, since $ca \in I$ and I is an ideal. This implies $da \in I$. The fact that d is a unit element in R_e gives $a \in I$, a contradiction. Thus $d + c$ is unit. By Lemma 2.6, we deduce that R_e is local ring, and by Lemma 2.5, R is G -graded local ring. \square

Corollary 2.8. *Suppose that R is G -graded ring which is not a G -graded local ring. Then every proper graded ideal is a graded 1-absorbing primary ideal if and only if I is a graded primary ideal of R .*

Proof. Follows directly from the Theorem 2.7. \square

Let R_1 and R_2 be two G -graded rings, then $R_1 \times R_2$ is a G -graded ring with $(R_1 \times R_2)_g = (R_1)_g \times (R_2)_g$ for every $g \in G$. In the following result, we give a characterization of graded 1-absorbing primary ideals of a graded ring $R_1 \times R_2$.

Theorem 2.9. *Let R_1 and R_2 be G -graded commutative rings. Let $R = R_1 \times R_2$, and $K = I \times J$ be a nonzero proper graded ideal of R . Then the followings are equivalent:*

1. K is a graded 1-absorbing primary ideal of R ;
2. K is a graded primary ideal of R ;
3. $K = I \times R_2$, where I is a graded primary ideal of R_1 or $K = R_1 \times J$, where J is a graded primary ideal of R_2 .

Before proving Theorem 2.9, we first establish the following lemmas.

Lemma 2.10. *Let R_1 and R_2 be a G -graded commutative rings with identity. Let $K := I \times J$ a graded ideal of $R_1 \times R_2$, where I, J are graded ideals of R_1 and R_2 respectively. K is a graded prime ideal of $R_1 \times R_2$ if and only if $K = I \times R_2$ where I is a graded prime ideal of R_1 or $K = R_1 \times J$, where J is a graded prime ideal of R_2 .*

Proof. Assume that K is a graded prime ideal of $R_1 \times R_2$. Since $(1,0)(0,1) = (0,0) \in K$, then either $(1,0) \in I \times J$ or $(0,1) \in I \times J$. Thus $K = R_1 \times J$ or $K = I \times R_2$. Without loss the generality, assume that $K = I \times R_2$. Let $ab \in I$ for some homogeneous elements $a, b \in h(R)$, then $(a,0)(b,0) \in K$, with $(a,0), (b,0) \in h(R_1) \times h(R_2)$ and K is a graded prime ideal of $R_1 \times R_2$, we conclude that either $(a,0) \in K$ or $(b,0) \in K$. Hence $a \in I$ or $b \in I$. Thus I is a graded prime ideal of R_1 . Conversely assume that $K = I \times R_2$ and I is graded prime ideal of R_1 . Let $(a,a')(b,b') = (ab, a'b') \in I \times R_2$, for some $(a,a'), (b,b') \in h(R_1) \times h(R_2)$. Since $ab \in I$ and I is a graded prime of R_1 , we conclude either $a \in I$ or $b \in I$. Hence either $(a,a') \in K$ or $(b,b') \in K$. That is K is a graded prime ideal of $R_1 \times R_2$. Similarly, one can see the claim if $K = R_1 \times J$ and J is a graded prime ideal of R_2 . \square

Lemma 2.11. *Let R_1 and R_2 be two G -graded commutative rings with identity. Let $K := I \times J$ be a graded ideal of $R_1 \times R_2$, where I, J are graded ideals of R_1 and R_2 , respectively. Then K is a graded primary ideal of $R_1 \times R_2$ if and only if $K = I \times R_2$, where I is a graded primary ideal of R_1 or $K = R_1 \times J$, where J is a graded primary ideal of R_2 .*

Proof. Assume that K is a graded primary ideal of $R_1 \times R_2$, then by [20, Lemma 1.8], $Gr(K)$ is a graded prime ideal of $R_1 \times R_2$. By Lemma 2.10, we have $Gr(K) = P \times R_2$, where P is a graded prime ideal of R_1 or $Gr(K) = R_1 \times Q$, where Q is a graded prime ideal of R_2 . Without loss the generality, we may assume that $Gr(K) = P \times R_2$. Let $a \in h(R_1) \setminus P$, so $(a,0) \notin Gr(K)$. Let $b \in R_2$, then $(0,b)(a,0) = (0,0) \in K$ and $(a,0) \notin Gr(K)$, this implies $(a,b) \in K$, thus $J = R_2$. Hence $K = I \times R_2$ with $Gr(I) = P$, and it is easy to see that I is a graded primary ideal of R_1 . Conversely, if $K = I \times R_2$, where I is a graded primary ideal of R_1 , it is clear that K is a graded primary ideal of $R_1 \times R_2$. \square

Proof of Theorem 2.9.

- (1) \Leftrightarrow (2) Follows from the Corollary 2.8 since $R_1 \times R_2$ is not a G -graded local ring.
- (2) \Leftrightarrow (3) Follows from the Lemma 2.11. \square

The following result gives us a method to construct a graded 1-absorbing primary ideal.

Theorem 2.12. *Let R be a graded local ring with a graded maximal ideal M . Let P a graded prime ideal such that $P \subset M$. Then PM is a graded 1-absorbing primary ideal of R .*

Proof. It is clear that PM is a graded ideal of R . On the other hand, we have $Gr(PM) = P$. Let r, s, t be nonunit homogeneous elements in $h(R)$ with $rst \in PM$. Since rs, t are homogeneous nonunit elements of $h(R)$ and P be a graded prime, we have either $rs \in P$ or $t \in P$. If $rs \in P$, we are done. Otherwise, $t \in P$. Since $P \subseteq Gr(P)$, we conclude that $t \in Gr(P) = Gr(PM)$. Thus, PM is a graded 1-absorbing primary ideal of R . \square

Theorem 2.13. *Let I be a graded 1-absorbing primary ideal of a G -graded ring R . Let $c \in h(R) \setminus I$ be a nonunit homogeneous element of $h(R)$. Then $(I : c) := \{r \in R : rc \in I\}$ is a graded primary ideal of R .*

Proof. Let $c \in h(R) \setminus I$. First, we prove that $(I : c)$ is a graded ideal of R . Let $r \in (I : c)$. Then $r \in R$ such that $rc \in I$. Since R is graded, $r = \sum_{g \in G} r_g$, where $r_g \in R_g$ and then $rc = \sum_{g \in G} r_g c$. Since $r_g c \in h(R)$ for all $g \in G$ and I is a graded ideal, we deduce that $r_g c \in I$, so $r_g \in (I : c)$ for all $g \in G$. Hence $(I : c)$ is a graded ideal of R . Now let $rs \in (I : c)$ for some nonunit homogeneous elements $r, s \in h(R)$, then $rs c \in I$. Assume that $s \notin (I : c)$. As I is graded 1-absorbing primary and $rc \notin I$, we have $s \in Gr(I)$. Since $Gr(I) \subseteq Gr((I : c))$ we obtain that $(I : c)$ is a graded primary ideal of R . \square

Recall that a G -graded ring R is called *graded divided* if for every graded prime ideal P and for every $x \in h(R) \setminus P$, we have $x|p$ for every $p \in h(R) \cap P$. Recall that a ring is called a *graded chained* if

for every $x, y \in h(R)$, either $x|y$ or $y|x$. Also, note that a G -graded ring is said to be a *graded valuation domain*, if for every nonzero homogeneous elements a, b of R , then either $a|b$ or $b|a$.

Theorem 2.14. Let R be a G -graded divided ring. Then a proper graded ideal I is a graded 1-absorbing primary ideal of R if and only if I is a graded primary ideal.

Proof. Let I be a graded primary ideal of R , and let $(rs).t = rst \in I$ for some nonunit homogeneous elements of $h(R)$ with $rs \notin I$, then $s \in Gr(I)$. Hence I is a graded 1-absorbing primary ideal of R . Conversely, assume that I is a graded 1-absorbing primary ideal of R and let $rs \in I$ for some nonunit homogeneous elements of R and $r \notin Gr(I)$. By Theorem 2.4, $Gr(I)$ is a graded prime ideal of R , thus $s \in Gr(I)$. Since R is a graded divided ring and $Gr(I)$ is a graded prime ideal and $r \notin Gr(I)$, we deduce that $r|s$. Thus $s = rt$ for some homogeneous element $t \in h(R)$. Since $rs = rtr \in I$ and $r \notin Gr(I)$ and I is a graded 1-absorbing primary ideal, we conclude $s = rt \in I$. Thus I is a graded primary ideal of R . \square

Corollary 2.15. Let R be a G -graded valuation domain. Then a proper graded ideal I is a graded 1-absorbing primary ideal of R if and only if I is a graded primary ideal.

Proof. Every graded valuation domain is a graded divided ring. Indeed, let P be a graded prime ideal and x be a homogeneous element in $h(R) \setminus P$. Assume that $y \in h(R) \cap P$. Since R is a graded valuation domain, then either $x|y$ or $y|x$. As $x \notin P$, we have $x|y$ for every $y \in h(R) \cap P$. Then R is a graded divided ring. Thus the result follows from the Theorem 2.14. \square

Theorem 2.16. Let R be a G -graded ring and $I = \bigoplus_{g \in G} (I \cap R_g) = \bigoplus_{g \in G} I_g$ be a graded ideal of R . Then the following statements hold:

1. If I is a graded 1-absorbing primary ideal of R , then I_e is a 1-absorbing primary ideal of R_e .
2. If I_e is a 1-absorbing primary ideal of R_e and R is a crossed product, then I is a graded 1-absorbing primary ideal of R .

Proof. (1) It is clear that I_e is an ideal of the ring R_e . Now let r, s, t nonunit elements in R_e such that $rst \in I_e \subseteq I$. Since I is a graded ideal and r, s, t are homogeneous nonunit elements, we get either $rs \in I$ or $t \in Gr(I)$. Assume $rs \in I$, then $rs \in R_e \cap I = I_e$. Now let $t \in Gr(I)$, there exists a positive integer $n \geq 1$ such that $t^n \in I$. Since $t^n \in R_e = R_e$, we conclude that $t^n \in R_e \cap I = I_e$, then $t \in Gr(I_e)$. Hence, I_e is a 1-absorbing primary ideal of R_e .

(2) Suppose that R is a crossed product and I_e is a 1-absorbing primary ideal of R_e . Let r, s, t be nonunit homogeneous elements in $h(R)$ such that $rst \in I$. There exist $g, h, k \in G$ such that $r \in R_g, s \in R_h, t \in R_k$. Since R is a crossed product, also $R_{g^{-1}}$ contains a unit, say s' and $R_{h^{-1}}$ contains a unit, say r' and $R_{k^{-1}}$ contains a unit, say t' . Thus $(rr')(ss')(tt') \in I_e$, since I_e is a 1-absorbing primary and rr', ss', tt' are nonunit elements in R_e , we conclude either $(rr')(ss') \in I_e$ or $tt' \in Gr(I_e)$. then either $rs \in I_{g^{-1}h^{-1}}$ or $tt' \in Gr(I_{k^{-1}})$. Hence either $rs \in I$ or $t \in Gr(I)$. Therefore, I is a graded 1-absorbing primary ideal of R . \square

For G -graded rings R and R' , a G -graded ring homomorphism $\varphi : R \rightarrow R'$ is a ring homomorphism such that $\varphi(R_g) \subseteq R'_g$ for every $g \in G$. The following result is an analogue of [6, Theorem 17].

Theorem 2.17. Let R and R' be two G -graded rings and $\varphi : R \rightarrow R'$ be a G -graded ring homomorphism with $\varphi(1) = 1$. Then the following statements hold:

1. Assume that for every nonunit homogeneous element r of $h(R)$, we have $\varphi(r)$ is nonunit elements of $h(R')$. Then if I' is graded 1-absorbing primary ideal of R' , then $\varphi^{-1}(I')$ is a graded 1-absorbing primary ideal of R .
2. If φ is a graded epimorphism and I is a graded 1-absorbing primary ideal of R containing $\ker(\varphi)$, then $\varphi(I)$ is a graded 1-absorbing primary ideal of R' .

Proof. (1) Suppose that I' is a graded 1-absorbing primary ideal of R' and let r, s, t be nonunit homogeneous elements of $h(R)$ such that $rst \in \varphi^{-1}(I')$, then $\varphi(rst) = \varphi(r)\varphi(s)\varphi(t) \in I'$. Since $\varphi(r), \varphi(s), \varphi(t)$ are nonunit homogeneous elements of $h(R')$ by assumption (1) and I' is a graded 1-absorbing primary ideal of R' . we conclude that either $\varphi(r)\varphi(s) \in I$ or $\varphi(t) \in Gr(I')$, and hence either $rs \in \varphi^{-1}(I')$ or $t \in \varphi^{-1}(Gr(I')) = Gr(\varphi^{-1}(I'))$. Therefore $\varphi^{-1}(I')$ is a graded 1-absorbing primary ideal of R .

(2) Suppose that I is a graded 1-absorbing primary ideal of R containing $\ker(\varphi)$ and $r's't' \in \varphi(I)$ for some nonunits $r', s', t' \in h(R')$. Since φ is a graded epimorphism, there exist $r, s, t \in h(R)$ such that $r' = \varphi(r), s' = \varphi(s)$ and $t' = \varphi(t)$. Since $\varphi(1) = 1$, we have $\varphi(r), \varphi(s), \varphi(t)$ are nonunit homogeneous elements of $h(R')$. Now we have $\varphi(rst) = r's't' \in \varphi(I)$. Since $\ker(\varphi) \subseteq I$, we have $rst \in I$. Since I is a graded 1-absorbing primary ideal of R , we obtain either $rs \in I$ or $t \in Gr(I)$. So either $r's' \in \varphi(I)$ or $t' \in \varphi(Gr(I)) = Gr(\varphi(I))$. Thus $\varphi(I)$ is a graded 1-absorbing primary ideal of R' . □

Let I be a proper graded ideal of a G -graded ring R . Then we define $G - Z_I(R) := \{r \in h(R) : rs \in I \text{ for some } s \in h(R) - Z_I(R)\}$. The following result is an analogue of [6, Theorem 18].

Theorem 2.18. Let I be a proper graded ideal of a G -graded ring R . Then the following statements hold:

1. If J is a proper graded ideal of R with $J \subseteq I$ and I is a graded 1-absorbing primary ideal of R , then I/J is a graded 1-absorbing primary ideal of a G -graded ring R/J .
2. Let J be a proper graded ideal of R with $J \subseteq I$ such that $U(R/J) \cap h(R/J) = \{r + J, r \in h(R) \cap U(R)\}$. If J is a graded 1-absorbing ideal of R and I/J is a graded 1-absorbing ideal of R/J . Then I is a graded 1-absorbing primary of R .

Proof. (1) Follows from Theorem 2.17(2).

(2) Suppose that $rst \in I$ for some nonunit homogeneous elements of R . If $rst \in J$, then either $rs \in J \subseteq I$ or $t \in Gr(J) \subseteq Gr(I)$. Now assume that $rst \notin J$. then $(r+J)(s+J)(t+J) \in I/J$, where $r+J, s+J, t+J$ are nonunit elements of R/J by hypothesis. Thus either $(r+J)(s+J) \in I/J$ or $t+J \in Gr(I/J)$. If $(r+J)(s+J) \in I/J$, then $rs \in I$. If $t+J \in Gr(I/J)$, there exists a positive integer n with $(t+J)^n \in I/J$, so $t^n \in I$ and hence $t \in Gr(I)$. □

Theorem 2.19. Let R be a G -graded ring, $S \subseteq h(R)$ be a multiplicatively closed subset of R , and I be a proper graded ideal of R . Then the followings hold:

1. If I is a graded 1-absorbing primary ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a graded 1-absorbing primary ideal of a graded ring $S^{-1}R$.
2. If $S^{-1}I$ is a graded 1-absorbing primary ideal of a graded ring $S^{-1}R$ such that $S \cap G - Z_I(R) = \emptyset$, then I is a graded 1-absorbing primary ideal of R .

Proof. (1) Suppose that $\frac{r_1 r_2 r_3}{s_1 s_2 s_3} \in S^{-1}R$ for some nonunit homogeneous elements of $S^{-1}R$. Then there exist $v \in S$ such that $0 \neq vr_1 r_2 r_3 \in I$. Since I is a graded 1-absorbing primary ideal of R and $vr_1 \in h(R)$, then either $vr_1 r_2 \in I$ or $r_3 \in Gr(I)$, then $\frac{vr_1 r_2}{vs_1 s_2} \in S^{-1}I$ or $\frac{r_3}{s_3} \in S^{-1}Gr(I) = Gr(S^{-1}I)$. Hence $S^{-1}I$ is a graded 1-absorbing primary ideal of $S^{-1}R$.

(2) Suppose that $r_1 r_2 r_3 \in I$ for some nonunit homogeneous elements of R . Then $\frac{r_1}{1} \frac{r_2}{1} \frac{r_3}{1} \in S^{-1}R$. Since $S^{-1}I$ is a graded 1-absorbing primary ideal of $S^{-1}R$, we deduce that either $\frac{r_1}{1} \frac{r_2}{1} \in S^{-1}I$ or $\frac{r_3}{1} \in Gr(S^{-1}I) = S^{-1}Gr(I)$. If $\frac{r_1}{1} \frac{r_2}{1} \in S^{-1}I$, then $vr_1 r_2 \in I$ for some $v \in S$. Since $S \cap G - Z_I(R) = \emptyset$ and $v \in S$, we conclude that $r_1 r_2 \in I$. If $\frac{r_3}{1} \in S^{-1}Gr(I)$, then $(tr_3)^n \in I$ for some $t \in S$ and some positive integer $n \geq 1$. The fact that $t \in S$, gives $t^n \notin G - Z_I(R)$ and hence $r_3^n \in I$. Thus $r_3 \in Gr(I)$. Therefore, I is a graded 1-absorbing primary ideal of R . \square

Definition 2.20. Let I be a graded 1-absorbing primary ideal of a G -graded ring. Then $Gr(I) = P$ is a graded prime ideal of R , by Theorem 2.4. Hence we call I as a graded P -1-absorbing primary ideal of R .

Theorem 2.21. Let I_1, \dots, I_n be graded P -1-absorbing primary ideals of a G -graded ring R . Then $I := \bigcap_{i=1}^n I_i$ is a graded P -1-absorbing primary ideal of R .

Proof. First observe that I is a graded ideal of R and $Gr(I) = P$. Let r, s, t nonunit homogeneous elements such that $rst \in I$ and $rs \notin I$. We may assume that $rs \notin I_1$. Since I_1 is a graded P -1-absorbing primary ideal, we obtain $t \in P$. Hence I is a graded P -1-absorbing primary ideal of R . \square

Lemma 2.22. (cf. [21] Proposition 3.3 and Corollary 3.5]) Let $R = \bigoplus_{g \in G} R_g$ be G -graded ring and $M = \bigoplus_{g \in G} M_g$ be a G -graded R -module. Let I be a graded ideal of R and N be a graded submodule of M such that $IM \subseteq N$. Then the following statements hold:

1. $I(+)N$ is a graded ideal of $R(+)M$.
2. $Gr(I(+)N) = Gr(I)(+)M$.

Theorem 2.23. Let $R = \bigoplus_{g \in G} R_g$ be G -graded ring and $M = \bigoplus_{g \in G} M_g$ be a G -graded R -module. Let I be a graded ideal of R . Then I is a graded 1-absorbing primary ideal of R if and only if $I(+)M$ is a graded 1-absorbing primary of $R(+)M$.

Proof. Assume that $I(+)M$ is a graded 1-absorbing primary ideal of $R(+)M$. Let $abc \in I$ for some nonunit homogeneous elements of R . Since $U(R(+)M) = U(R)(+)M$, we get $(a, 0), (b, 0), (c, 0)$ are nonunit homogeneous elements of $R(+)M$. Then $(a, 0)(b, 0)(c, 0) \in I(+)M$ and $I(+)M$ is a graded 1-absorbing primary of $R(+)M$ imply that either $(a, 0)(b, 0) \in I(+)M$ or $(c, 0) \in Gr(I(+)M) = Gr(I)(+)M$. Thus, we have either $ab \in I$ or $c \in Gr(I)$, that is, I is a graded 1-absorbing primary ideal of R . Conversely, let $(a, e)(b, f)(c, g) \in I(+)M$ for some nonunit homogeneous elements of $R(+)M$. Since $U(R(+)M) = U(R)(+)M$, we have a, b, c are nonunit homogeneous elements of R and $abc \in I$. Since I is a graded 1-absorbing primary ideal of R , either $ab \in I$ or $c \in Gr(I)$. Hence we conclude either $(a, e)(b, f) \in I(+)M$ or $(c, g) \in Gr(I)(+)M = Gr(I(+)M)$. Thus, $I(+)M$ is a graded 1-absorbing primary ideal of $R(+)M$. \square

Theorem 2.24. Let $R = \bigoplus_{g \in G} R_g$ be G -graded ring and $M = \bigoplus_{g \in G} M_g$ be a G -graded R -module. Let I be a graded ideal of R and N be a graded submodule of M . Then $I(+)N$ is a graded 1-absorbing primary ideal of $R(+)M$ if and only if either:

1. $N = M$ and I is a graded 1-absorbing primary ideal of R or
2. $N \subsetneq M, IM \subseteq N, I$ is a graded 1-absorbing primary of R and N is graded P -primary submodule of M , where $P = Gr(I)$ is a graded prime ideal of R .

Proof. (\Rightarrow). Assume that $I(+)N$ is a graded 1-absorbing primary ideal. Let $abc \in I$ for some nonunit homogeneous elements of R . Since $(a, 0), (b, 0), (c, 0)$ are nonunit homogeneous elements of $R(+)M$ and $(a, 0)(b, 0)(c, 0) \in I(+)N$, we conclude that either $(a, 0)(b, 0) \in I(+)M$ or $(c, 0) \in Gr(I(+)N) = Gr(I)(+)M$. Thus, we obtain either $ab \in I$ or $c \in Gr(I)$, so I is a graded 1-absorbing primary ideal of R . Now by

the Theorem 2.4, we have $Gr(I(+)N)$ is a graded prime ideal of $R(+)M$. By [9 Proposition 3.1], we get $Gr(I(+)N) = P(+)M$ for some graded prime ideal P of R , and with Lemma 2.22, we deduce $Gr(I) = P$.

(\Leftarrow). If $N = M$ see Theorem 2.23. Assume that $N \subsetneq M$. Since $IM \subseteq N$, and I is a graded ideal of R , we conclude that $I(+)N$ is a graded ideal of $R(+)M$. Now let $(a, e), (b, f), (c, g)$ be nonunit homogeneous elements of $R(+)M$ such that $(a, e)(b, f)(c, g) = (abc, abg + acf + bce) \in I(+)N$. Since $U(R(+)M) = U(R)(+)M$ we have $a, b, c \in h(R) \setminus U(R)$ and $abc \in I$. As I is a graded 1-absorbing primary ideal of R , we conclude either $ab \in I$ or $c \in Gr(I)$. If $c \in Gr(I)$, then $(c, f) \in Gr(I(+)N) = Gr(I)(+)M$ and so we are done. We may assume that $c \notin Gr(I) = P$. On the one hand, we have $ab \in I$ and $acf + bce \in N$ since $IM \subseteq N$, on the other hand, $c(af + be) \in N$ and $c \notin P = (M : N)$, hence $af + be \in N$ since N is a graded P -primary submodule of M . Thus $(a, e)(b, f) = (ab, af + be) \in I(+)N$. This implies that $I(+)N$ is a graded ideal of $R(+)M$. \square

3 Graded weakly 1-absorbing primary ideals

Definition 3.1. Let $R = \bigoplus_{g \in G} R_g$ be a G -graded ring, $I = \bigoplus_{g \in G} I_g$ be a graded ideal of R .

1. I_g is called a prime subgroup of R_g , if $I_g \neq R_g$; and whenever $a, b \in R_g$ with $ab \in I_g$, then either $a \in I_g$ or $b \in I_g$.
2. I is called a graded weakly 1-absorbing primary ideal of R if $I \neq R$ and whenever $r, s, t \in h(R)$ nonunit homogeneous elements with $0 \neq rst \in I$, then either $rs \in I$ or $t \in Gr(I)$.

It is clear that every graded 1-absorbing primary ideal is a graded weakly 1-absorbing primary ideal. The following example shows that the converse is not true.

Example 3.2. (cf. [8 Example 2.2]) **(A graded weakly 1-absorbing primary ideal which is not graded 1-absorbing primary)** Let $R = \mathbb{Z}_6[i]$ and $G = \mathbb{Z}_2$. Then R is graded by $R_0 = \mathbb{Z}_6$ and $R_1 = i\mathbb{Z}_6$. Let $I = \{0\}$ which is a graded ideal of R . We have $2.2.3 \in I$ but neither $2.2 \in I$ nor $3 \in Gr(I)$. On the other hand, I is graded weakly 1-absorbing primary ideal by the definition.

Now, let us continue with the following result:

Theorem 3.3. Let $R = \bigoplus_{g \in G} R_g$ be a G -graded ring, $I = \bigoplus_{g \in G} I_g$ be a graded ideal of R .

1. If I is a graded weakly primary ideal of R , then I is a graded weakly 1-absorbing primary ideal.
2. If I is a graded weakly 1-absorbing primary ideal, then I is a graded weakly 2-absorbing primary ideal.
3. If $\{0\}$ is a graded 1-absorbing primary ideal, then I is a graded weakly 1-absorbing primary ideal if and only if I is a graded 1-absorbing primary ideal.
4. If R is a graded integral domain, then I is graded weakly 1-absorbing primary ideal if and only if I is a graded 1-absorbing primary ideal.
5. Let R be a graded local ring with graded maximal ideal $Gr(0)$. Then every graded proper ideal of R is a graded weakly 1-absorbing primary ideal of R .

Proof. (1) Straightforward.

(2) Let $0 \neq rst$ for some homogeneous elements $r, s, t \in h(R)$. If one of r, s, t is unit, we are done. We can assume that r, s, t nonunits elements, as I is graded weakly 1-absorbing, we get either $rs \in I$ or $t \in Gr(I)$, this implies $rs \in I$ or $st \in Gr(I)$ or $tr \in Gr(I)$. Hence I is a graded weakly 2-absorbing primary ideal of R .

Let $rst \in I$ for some nonunit homogeneous elements of $h(R)$. If $0 \neq rst$, as I is graded weakly 1-absorbing primary ideal of R , we get that either $rs \in I$ or $t \in Gr(I)$. If $0 = rst \in \{0\}$, by assumption, we have either $0 = rs \in I$ or $t \in Gr(0) \subset Gr(I)$. Hence I is a graded 1-absorbing primary ideal of R .

(3) It is clear that R is graded integral domain if and only if $\{0\}$ is graded prime ideal. Thus, the result follows from (3).

(4) Let I be a graded ideal of R . Since R is graded local ideal with graded maximal ideal $Gr(0)$, then $Gr(I) = Gr(0)$. Then by the [20, Proposition 1.11], we get I is graded primary ideal, and so graded weakly 1-absorbing primary ideal of R . \square

Theorem 3.4. Let $R = \bigoplus_{g \in G} R_g$ be a graded ring and I be a graded weakly 1-absorbing primary which is not graded 1-absorbing primary ideal of R . Then $Gr(I)$ is not graded maximal ideal of R .

Proof. Let I be graded weakly 1-absorbing primary ideal such that $Gr(I) = M$ is a graded maximal ideal, by the [20, Proposition 1.11], we get I is graded primary ideal. Thus I is a graded 1-absorbing primary ideal, a contradiction. \square

Recall that R is called a graded reduced ring if R is a G -graded ring and $Gr(0) = 0$.

Theorem 3.5. Let R be a G -graded reduced ring. If I is a nonzero graded weakly 1-absorbing primary ideal of R , then $Gr(I)$ is a graded weakly prime ideal of R and there exists $g \in G$ such that I_g is not prime subgroup of R_g .

Proof. Suppose that $0 \neq rs \in Gr(I)$ for some nonunit homogeneous elements of $h(R)$. Then there exists a positive integer $n = 2m$ such that $(rs)^n \in I$. Since R is a graded reduced ring and $I \neq 0$, we have $0 \neq r^n s^n \in I$. Hence $0 \neq r^m r^m s^n \in I$. Since r^m, s^n are nonunit homogeneous elements of R and I is a graded weakly 1-absorbing primary ideal of R , we obtain either $r^m \cdot r^m = r^n \in I$ or $s^n \in Gr(I)$. Thus $Gr(I)$ is a graded weakly prime ideal of R . Now, suppose by way of contradiction that for every $g \in G$ that I_g is prime subgroup of R_g . Thus by [4, Theorem 2.4], we get $Gr(I) = Gr(0)$. Since R is graded reduced, we conclude that $I = 0$, a contradiction. \square

A G -graded ring $R = \bigoplus_{g \in G} R_g$ is called a crossed product if R_g contain a unit for every $g \in G$. Note that a G -crossed product $R = \bigoplus_{g \in G} R_g$ is a strongly graded ring, that is, $R_{gh} = R_g R_h$ for every $g, h \in G$.

Theorem 3.6. Let R be a G -graded ring and $I = \bigoplus_{g \in G} (I \cap R_g) = \bigoplus_{g \in G} I_g$ be a graded ideal of R . Then the following statements hold:

1. If I is a graded weakly 1-absorbing primary ideal of R , then I_e is a weakly 1-absorbing primary ideal of R_e .
2. If I_e is a weakly 1-absorbing primary ideal of R_e and R is a crossed product, then I is a graded weakly 1-absorbing primary ideal of R .

Proof. (1) It is clear that I_e is an ideal of the ring R_e . Now let r, s, t nonunit elements in R_e such that $0 \neq rst \in I_e \subseteq I$. Since I is a graded weakly 1-absorbing primary ideal and r, s, t are homogeneous nonunit elements, we get either $rs \in I$ or $t \in Gr(I)$. Let $rs \in I$, then $rs \in R_e \cap I = I_e$. Now, assume that $t \in Gr(I)$, then $t^n \in I$ for some positive integers n , thus $t^n \in I \cap R_e = I_e$. This yields $t \in \sqrt{I_e}$. Hence I_e is a weakly 1-absorbing prime ideal of R_e .

(2) Suppose that R is a crossed product and I_e is a weakly 1-absorbing primary ideal of R_e . Let r, s, t be nonunit homogeneous elements in $h(R)$ such that $0 \neq rst \in I$. There exist $g, h, k \in G$ such that $r \in R_g, s \in R_h, t \in R_k$. Since R is a crossed product, also $R_{g^{-1}}$ contains a unit, say s' and $R_{h^{-1}}$ contains a unit, say r' and $R_{k^{-1}}$ contains a unit, say t' . Thus $0 \neq (rr')(ss')(tt') \in I_e$, since I_e is a weakly 1-absorbing primary ideal and rr', ss', tt' are nonunit elements in R_e , we conclude either $(rr')(ss') \in I_e$

or $tt' \in \sqrt{I_e}$. Then either $rs \in I_{g^{-1}h^{-1}}$ or $t^n \in I_{k^{-n}}$ for some positive integer n . Hence either $rs \in I$ or $t \in Gr(I)$. Therefore, I is a graded weakly 1-absorbing primary ideal of R . \square

Definition 3.7. Let $R = \bigoplus_{g \in G} R_g$ be a G -graded ring, $I = \bigoplus_{g \in G} I_g$ be a graded ideal of R and $g \in G$.

1. I is called a g -1-absorbing primary (resp., g weakly 1-absorbing primary) ideal of R , if $I_g \neq R_g$ and whenever r, s, t nonunit elements in R_g with $rst \in I$ (resp., $0 \neq rst \in I$), then $rs \in I$ or $t \in Gr(I)$.
2. Let $r, s, t \in R_g$. We say that (r, s, t) is a g -1-triple-zero of I if $rst = 0, rs \notin I$ and $t \notin Gr(I)$.

Theorem 3.8. Let $R = \bigoplus_{g \in G} R_g$ be a G -graded ring and $I = \bigoplus_{g \in G} I_g$ be a graded weakly 1-absorbing primary ideal of R . Let $g \in G$. If (r, s, t) is a g -1-triple-zero of I , then

1. $rsI_g = 0$.
2. If $rt \notin I$ and $st \notin I$, then $rtI_g = stI_g = rI_g^2 = sI_g^2 = tI_g^2 = 0$.
3. If $rt \notin I$ and $st \notin I$, then $I_g^3 = 0$.

Proof. (1) Suppose that $rsI_g \neq 0$. Then $rsu \neq 0$ for some nonunits $u \in I_g$. Hence $0 \neq rs(t + u) \in I$. Since $rs \notin I$, we conclude that $t + u \in R_g$ nonunit element. Since I is graded weakly 1-absorbing and $rs \notin I$, we conclude that $u + t \in Gr(I)$. Since $u \in I_g \subseteq I \subseteq Gr(I)$, we have $t \in Gr(I)$, a contradiction. Thus, we have $rsI_g = 0$.

(2) Suppose that $rtI_g \neq 0$. Then $rtw \neq 0$ for some $w \in I_g$. Hence $0 \neq rtw = r(s+w)t \in I$. Since $rt \notin I$, we conclude that $s+w$ is nonunit element of R_g . Since I is a graded weakly 1-absorbing primary ideal of R and $rs \notin I$ and $rw \in I$. We conclude that $r(s+w) \notin I$, and hence $t \in Gr(I)$, a contradiction. Thus $rtI_g = 0$. By a similarly argument, we obtain that $stI_g = 0$. Now, we prove that $rI_g^2 = 0$. Suppose $ruv \neq 0$ for some nonunit elements u, v of I_g . Since $rsI_g = 0$ by (1) and $rtI_g = 0$ by (2), $0 \neq auv = r(u+s)(v+t) \in I$. As $ab \notin I$, we conclude that $(v+t)$ is nonunit element of R_g . Since I is graded weakly 1-absorbing primary of R , we have either $r(u+s) \in I$ or $(v+t) \in Gr(I)$. By $u, v \in I_g \subseteq I$, we conclude that either $rs \in I$ or $t \in Gr(I)$, a contradiction. Thus $rI_g^2 = 0$. By the same argument above, we conclude that $sI_g^2 = tI_g^2 = 0$.

(3) Assume that $uvw \neq 0$ for some nonunits $u, v, w \in I_g$. Then $0 \neq uvw = (r+u)(s+v)(t+w) \in I$, by (1) and (2). Since $rs \notin I$, we conclude that $(t+w)$ is nonunit element of R_g . Since neither $rt \in I$ nor $st \in I$, we conclude that $r+u$ and $s+v$ are nonunit of R_g . As I is a graded weakly 1-absorbing primary ideal of R , we have $(r+u)(s+v) \in I$ or $s+w \in Gr(I)$. Since $u, v, w \in I_g \subseteq I$, we conclude that $rs \in I$ or $t \in Gr(I)$, a contradiction. Thus $I_g^3 = 0$. \square

Theorem 3.9. Let $R = \bigoplus_{g \in G} R_g$ be a G -graded ring, $I = \bigoplus_{g \in G} I_g$ be a graded weakly 1-absorbing primary ideal of R , which is not g -1-absorbing primary ideal of R for every $g \in G$. If For each $g \in G$, I admits a g -1-triple-zero $(r, s, t) \in (R_g)^3$ such that neither $rt \in I$ nor $st \in I$. Then $Gr(I) = Gr(0)$.

Proof. Suppose for every $g \in G$, there exists $(r, s, t) \in (R_g)^3$ which is a g -1-triple-zero of I such that neither $rt \in I$ nor $st \in I$. By Theorem 3.6 we have $I_g^3 = 0$ for every $g \in G$. Let $x \in Gr(I)$. For every $g \in G$, we have $x_g^3 = 0$, then $x_g \in Gr(0)$ and hence $x \in Gr(0)$. Thus $Gr(I) \subseteq Gr(0)$. Since $Gr(0) \subseteq Gr(I)$, we conclude that $Gr(I) = Gr(0)$. \square

Corollary 3.10. Let $R = \bigoplus_{g \in G} R_g$ be a G -graded reduced ring, $I = \bigoplus_{g \in G} I_g$ be a graded weakly 1-absorbing primary ideal of R wich is not g -1-absorbing primary ideal of R for every $g \in G$. If For each $g \in G$, I admits a g -1-triple-zero $(r, s, t) \in (R_g)^3$ such that neither $rt \in I$ nor $st \in I$. Then $I = 0$.

Proof. Since R is a graded reduced ring, we get $Gr(0) = 0$. Thus $Gr(I) = 0$ and $I = 0$. \square

Let R and R' be two G -graded rings, $\varphi : R \rightarrow R'$ be a G -graded ring homomorphism such that $\varphi(R_g) \subseteq R'_g$ for every $g \in G$. The following result is an analogue of [6] Theorem 17].

Theorem 3.11. Let R and R' be two G -graded rings and $\varphi : R \rightarrow R'$ be a G -graded ring homomorphism with $\varphi(1) = 1$. Then the following statements hold:

1. Suppose that φ is injective. Then if I' is graded weakly 1-absorbing primary ideal of R' , then $\varphi^{-1}(I')$ is a graded weakly 1-absorbing primary ideal of R .
2. If φ is a graded epimorphism and I is a graded weakly 1-absorbing primary ideal of R containing $\ker(\varphi)$, then $\varphi(I)$ is a graded weakly 1-absorbing primary ideal of R' .

Proof. (1) Suppose that I' is a graded weakly 1-absorbing primary ideal of R' and let r, s, t be nonunit homogeneous elements of $h(R)$ such that $0 \neq rst \in \varphi^{-1}(I')$. Since $\ker(\varphi) = 0$, then $0 \neq \varphi(rst) = \varphi(r)\varphi(s)\varphi(t) \in I'$. As $\varphi(r), \varphi(s), \varphi(t)$ are nonunit homogeneous elements of $h(R')$ and I' is a graded weakly 1-absorbing primary ideal of R' , we conclude that either $\varphi(r)\varphi(s) \in I'$ or $\varphi(t) \in Gr(I')$, and hence either $rs \in \varphi^{-1}(I')$ or $t \in \varphi^{-1}(Gr(I')) = Gr(\varphi^{-1}(I'))$. Therefore $\varphi^{-1}(I')$ is a graded 1-absorbing primary ideal of R .

(2) Follows immediately from [8] Proposition 2.22]. \square

Theorem 3.12. Let R be a G -graded ring, $S \subseteq h(R)$ be a multiplicatively closed subset of R , and I be a proper graded ideal of R . Then the following statements hold:

1. If I is a graded 1-absorbing primary ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a graded 1-absorbing primary ideal of a G -graded ring $S^{-1}R$.
2. If $S^{-1}I$ is a graded weakly 1-absorbing primary ideal of a G -graded ring $S^{-1}R$ and $S \cap G - Z_I(R) = \emptyset$ and $S \cap G - Z(R) = \emptyset$, then I is a graded weakly 1-absorbing primary ideal of R .

Proof. (1) Suppose that $0 \neq \frac{r_1}{s_1} \frac{r_2}{s_2} \frac{r_3}{s_3} \in S^{-1}I$ for some $\frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3}$ of $S^{-1}R$ as $S \cap G - Z(R) = \emptyset$. Then there exists $v \in S$ such that $0 \neq vr_1r_2r_3 \in I$. Since I is a graded weakly 1-absorbing primary ideal of R and $vr_1 \in h(R)$, then either $vr_1r_2 \in I$ or $r_3 \in Gr(I)$, then $\frac{vr_1r_2}{vs_1s_2} \in S^{-1}I$ or $\frac{r_3}{s_3} \in S^{-1}Gr(I) = Gr(S^{-1}I)$. Hence $S^{-1}I$ is a graded weakly 1-absorbing primary ideal of $S^{-1}R$.

(2) Assume that $0 \neq r_1r_2r_3 \in I$ for some nonunit homogeneous elements of R . Then $\frac{r_1}{1} \frac{r_2}{1} \frac{r_3}{1} \in S^{-1}I$. Since $S^{-1}I$ is a graded weakly 1-absorbing primary ideal of $S^{-1}R$, we deduce that either $\frac{r_1}{1} \frac{r_2}{1} \in S^{-1}I$ or $\frac{r_3}{1} \in G(S^{-1}I) = S^{-1}Gr(I)$. If $\frac{r_1}{1} \frac{r_2}{1} \in S^{-1}I$, then $vr_1r_2 \in I$ for some $v \in S$. Since $S \cap G - Z_I(R) = \emptyset$ and $v \in S$, we conclude that $r_1r_2 \in I$. If $\frac{r_3}{1} \in S^{-1}Gr(I)$, then $(tr_3)^n \in I$ for some $t \in S$ and some positive integer $n \geq 1$. The fact that $t \in S$ gives $t^n \notin G - Z_I(R)$ and hence $r_3^n \in I$. Thus $r_3 \in Gr(I)$. Therefore I is a graded weakly 1-absorbing primary ideal of R . \square

Definition 3.13. Let I be a graded weakly 1-absorbing ideal of a graded ring A . Let a, b, c be nonunit homogeneous elements of $h(A)$. We say that (a, b, c) is a 1-triple-zero of I , if $abc = 0$ but neither $ab \in I$ nor $c \in Gr(I)$.

Theorem 3.14. Let $R = \bigoplus_{g \in G} R_g$ be G -graded ring and $M = \bigoplus_{g \in G} M_g$ be a G -graded R -module. Let I be a graded ideal of R . Then the followings are equivalent:

1. $I(+M)$ is a graded weakly 1-absorbing primary of $R(+M)$;
2. I is a graded weakly 1-absorbing primary ideal of R and for 1-triple zero (a, b, c) nonunit homogeneous of R , we have $ab, bc, ac \in \text{ann}(M)$.

Proof. \Rightarrow . Assume that $I(+)$ M is graded weakly 1-absorbing primary, and let $0 \neq abc \in I$ for some nonunit homogeneous elements of R , then $0 \neq (a, 0)(b, 0)(c, 0) \in I(+)$ M. Thus either $(a, 0)(b, 0) = (ab, 0) \in I(+)$ M or $(c, 0) \in Gr(I(+))M = Gr(I(+))M$. Thus either $ab \in I$ or $c \in Gr(I)$, that is, I is a graded weakly 1-absorbing primary ideal of R . Let $a, b, c \in H(R) \setminus U(R)$ with $abc = 0$, and neither $ab \in I$ nor $c \in Gr(I)$. Without loss the generality assume that $ab \notin \text{ann}(M)$, so there exists $e \in M$ such that $abe \neq 0$, since M is a graded R module, there exists $g \in G$ with $abe_g \neq 0$. Hence $0 \neq (a, 0)(b, 0)(c, e_g) = (0, abe_g) \in I(+)$ M, the fact that $I(+)$ M is a graded weakly 1-absorbing gives that either $(ab, 0) \in I(+)$ M or $(c, e_g) \in Gr(I(+))M$. Consequently, we obtain that either $ab \in I$ or $c \in Gr(I)$, a contradiction.

\Leftarrow . Let $(a, e), (b, f), (c, g)$ are nonunit homogeneous elements of $R(+)$ M with $(a, e)(b, f)(c, g) = (abc, abg+acf+bce) \in I(+)$ M. If $abc \neq 0$, then I is a graded weakly 1-absorbing primary ideal of R . Then either $ab \in I$ or $c \in Gr(I)$. Thus we have either $(a, e)(b, f) \in I(+)$ M or $(c, g) \in Gr(I(+))M = Gr(I(+))M$. Now assume that $abc = 0$, if we have either $ab \in I$ or $c \in Gr(I)$ we are done, otherwise we have $ab, ac, bc \in \text{ann}(M)$ by assumption, then $(a, e)(b, f)(c, g) = (0, 0)$, a contradiction. Thus $I(+)$ M is a graded weakly 1-absorbing primary ideal of $R(+)$ M. \square

Let $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ be two rings graded by an arbitrary group G . Also, let $J = \bigoplus_{g \in G} J_g$ be a graded ideal of B and let $f : A \rightarrow B$ be a graded ring homomorphism. Then

$$R := A \bowtie^f J := \{(a, f(a) + j) / a \in A, j \in J\}$$

called amalgamation of A with B along J respect to f , see [10], [11]. Therefore, according to [13], $R = \bigoplus_{g \in G} R_g$ is a graded ring, with $h(R) = \bigcup_{\alpha \in \Gamma} R_\alpha$, where for every $g \in G$,

$$R_g = (A \bowtie^f J)_g = \{(a_g, f(a_g) + j_g) / a_g \in A, j_g \in J\}.$$

Also, note that the set $h(R) = \bigcup_{g \in G} R_g$ is the homogeneous element of the graded ring $R := A \bowtie^f J$. This construction, first introduced and studied in [13, 14], allows us to meet the challenges posed by providing rich examples that enrich the current literature of graded ring theory. One of the key tools for studying $A \bowtie^f J$ is based on the fact that the graded amalgamation can be studied in the frame of the pullback constructions [14, Section 3]. Other classical constructions, such as the graded amalgamated duplication of a ring along an ideal denoted by $A \bowtie I$ and the graded trivial ring extension of A by E [21] ($A \bowtie E$), can be interpreted as a particular cases of the general graded amalgamation construction [13, Examples 3.3 & 3.4].

Lemma 3.15. *Let $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ be two rings graded by an arbitrary group G . Let $J = \bigoplus_{g \in G} J_g$ be a graded ideal of B and let $f : A \rightarrow B$ be a graded ring homomorphism. Let K be a graded ideal of $A \bowtie^f J$ which contains $0 \times J$. Then $Gr(K) = Gr(I) \bowtie^f J$, for a graded ideal*

$$I := \{a \in A : (a, f(a) + j) \in K \text{ for some } j \in J\}.$$

Proof. Note that I is clearly an ideal of A . Now, we will show that I is a graded ideal of A . Take $a = \sum_{g \in G} a_g$ for $a_g \in A_g$. Then there exists $j \in J$ such that $(a, f(a) + j) \in K$. Since $(a, f(a) + j) = \sum_{g \in G} (a_g, f(a_g) + j_g)$ and K is a graded ideal of $A \bowtie^f J$, we have $(a_g, f(a_g) + j_g) \in K$ for all $g \in G$. This implies $a_g \in I$ for all $g \in G$. Hence I is a graded ideal of A . Now, take $(a, f(a) + j) \in Gr(K)$. Then for all $g \in G$ there exists $n_g > 0$ such that $(a_g, f(a_g) + j_g)^{n_g} = (a_g^{n_g}, f(a_g^{n_g}) + \sum_{k=0}^{n_g-1} \binom{n_g}{k} f(a_g^k) j_g^{n_g-k})$. By definition of I , we get $a_g^{n_g} \in I$ and this yields $a \in Gr(I)$. Consequently, we have $(a, f(a) + j) \in Gr(I) \bowtie^f J$.

Conversely, assume $(a, f(a) + j) \in Gr(I) \bowtie^f J$ for some $a \in A$ and $j \in J$. Since $0 \times J \subseteq K \subseteq Gr(K)$, it is sufficient to show that $(a, f(a)) \in Gr(K)$. As $a \in Gr(I)$, then for all $g \in G$, there exists $n_g > 0$ such

that $a_g^{n_g} \in I$. Then we get $(a_g^{n_g}, f(a_g^{n_g}) + j') \in K$ for some $j' \in J$. Moreover, $(a_g^{n_g}, f(a_g^{n_g})) = (a_g, f(a_g))^{n_g} \in Gr(K)$ and this yields $(a_g, f(a_g)) \in Gr(Gr(K)) = Gr(K)$. Hence, we conclude that $(a, f(a)) \in Gr(K)$, which completes the proof. \square

Corollary 3.16. Let $A = \bigoplus_{g \in G} A_g$ and $B = \bigoplus_{g \in G} B_g$ be two rings graded by an arbitrary group G . Let $J = \bigoplus_{g \in G} J_g$ be a graded ideal of B and let $f : A \rightarrow B$ be a graded ring homomorphism. Let I be a graded ideal of A . Then $Gr(I \bowtie^f J) = Gr(I) \bowtie^f J$.

Proof. It follows from Lemma 3.15 since $I \bowtie^f J$ is a graded ideal of $A \bowtie^f J$ such that $0 \times J \subseteq I \bowtie^f J$. \square

Theorem 3.17. Let A and B be two rings graded by an arbitrary group G . Let J a graded ideal of B and let $f : A \rightarrow B$ be a graded ring homomorphism. Let I be a proper ideal of A . Then the following statements hold:

1. If $I \bowtie^f J$ is a graded weakly 1-absorbing primary not a graded-1-absorbing primary ideal of $A \bowtie^f J$, then I is a graded weakly 1-absorbing primary not a graded 1-absorbing primary ideal of A .
2. If I is a graded weakly 1-absorbing primary not a graded weakly 1-absorbing primary ideal of A and for every 1-triple-zero (a, b, c) nonunit homogeneous elements, we have $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$ and $J^3 = 0$. Then $I \bowtie^f J$ is a graded weakly 1-absorbing primary which is not a graded 1-absorbing primary ideal of $A \bowtie^f J$.

Proof. (1) Assume that $I \bowtie^f J$ is a graded weakly 1-absorbing primary not graded 1-absorbing primary ideal of $A \bowtie^f J$. Let $0 \neq abc \in I$ for some nonunit homogeneous elements $a, b, c \in h(A)$. Then $(0, 0) \neq (a, f(a))(b, f(b))(c, f(c)) \in I \bowtie^f J$. Since f is a graded homomorphism of graded rings, we have $(a, f(a)), (b, f(b))$ and $(c, f(c))$ are nonunit homogeneous elements of $A \bowtie^f J$. By assumption, we get either $(a, f(a))(b, f(b)) \in I \bowtie^f J$ or $(c, f(c)) \in Gr(I \bowtie^f J)$. By the Lemma 3.15 we have either $ab \in I$ or $c \in Gr(I)$. Hence I is a graded weakly 1-absorbing primary ideal of A . By [15, Theorem 2.18], I is not a graded 1-absorbing primary ideal. Thus I is a graded weakly 1-absorbing primary ideal which is not a graded 1-absorbing primary ideal of a graded ring A . Then there exists a 1-triple-zero $(r, s, t) \in h(A)^3$ of I , that is, $rst = 0$ but neither $rs \in I$ nor $t \in Gr(I)$. Without loss the generality we can assume that $0 \neq f(r)f(s)J$. Then $f(r)f(s)j_g \neq 0$ for some homogeneous element j_g of J . Thus $0 \neq (r, f(r))(s, f(s))(t, f(t) + j_g) = (0, f(r)f(t)j_g) \in I \times J_g \subseteq I \bowtie^f J$, which implies either $rs \in I$ or $t \in Gr(I)$, a contradiction. Hence $f(r)f(s)J = f(r)f(t)J = f(s)f(t)J = 0$.

(2) Let $0 \neq (a, f(a) + j)(b, f(b) + k)(c, f(c) + l) \in I \bowtie^f J$ for some nonunit homogeneous elements $(a, f(a) + j), (b, f(b) + k), (c, f(c) + l)$ of $h(A \bowtie^f J)$. It is clear that $a, b, c \in h(A)$ are nonunit homogeneous elements of A . Let $0 \neq abc$. By assumption we get either $ab \in I$ or $c \in Gr(I)$, which implies $(a, f(a) + j)(b, f(b) + k) \in I \bowtie^f J$ or $(c, f(c) + l) \in Gr(I) \bowtie^f J = Gr(I \bowtie^f J)$. Then $I \bowtie^f J$ is graded weakly 1-absorbing primary ideal of $A \bowtie^f J$. We can assume that $abc = 0$. By the way of contradiction, suppose that neither $(a, f(a) + j)(b, f(b) + k) \in I \bowtie^f J$ nor $(c, f(c) + l) \in Gr(I \bowtie^f J)$. We have necessarily ab is not in I and c is not in $Gr(I)$. Thus (a, b, c) is a 1-triple-zero of I . Then $f(a)f(b)J = f(a)f(c)J = f(b)f(c)J = 0$ by assumption. We conclude that $(a, f(a) + j)(b, f(b) + k)(c, f(c) + l) = (0, jkl) = (0, 0)$. Since $J^3 = 0$, we deduce $jkl = 0$, and hence $(a, f(a) + j)(b, f(b) + k)(c, f(c) + l) = (0, 0)$, a contradiction. On the other hand, I is not a graded 1-absorbing primary ideal of A , then by [13, Theorem 3.5(2)], we have $I \bowtie^f J$ is not a graded 1-absorbing primary ideal. This completes the proof. \square

Corollary 3.18. Let $A = \bigoplus_{g \in G} A_g$ be a graded ring and $I = \bigoplus_{g \in G} I_g$ be a graded proper ideal of A . Let K be a proper graded ideal of A . Then the followings are equivalent:

1. If $K \bowtie I$ is a graded weakly 1-absorbing primary which is not graded-1-absorbing primary ideal of $A \bowtie I$, then K is graded weakly 1-absorbing primary which is a not graded 1-absorbing primary ideal of A ;
2. If K is a graded weakly 1-absorbing primary which is not graded weakly 1-absorbing primary ideal of A and for every 1-triple-zero (a, b, c) nonunit homogeneous elements, we have $ab, ac, bc \in \text{ann}(I)0$ and $I^3 = 0$. Then $K \bowtie^f I$ is a graded weakly 1-absorbing primary which is not graded 1-absorbing primary ideal of $A \bowtie I$.

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