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ϕ-Classical 1-Absorbing Prime Submodules

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Abstract. In this paper, all rings are commutative with nonzero identity. Let R be a ring and M be an R-module. The purpose of this paper is to introduce and study the class of $\phi-$classical 1-absorbing prime submodules. Let $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. A proper submodule N of M is called a $\phi-$classical 1-absorbing prime submodule if whenever nonunits $a, b, c \in R$ and $m \in M$ with $abcm \in N \setminus \phi(N)$, then $abm \in N$ or $cm \in N$. Several characterizations of $\phi-$ classical 1-absorbing prime submodules are given. Among more, it is verified that if $N_1N_2N_3N_4 \subseteq N$ for some proper submodules N_1, N_2, N_3 and for some submodule N_4 of multiplication module M such that $N_1N_2N_3N_4 \nsubseteq \phi(N)$, then either $N_1N_2N_4 \subseteq N$ or $N_3N_4 \subseteq N$.

Key Words: Classical 1-absorbing prime submodules, weakly classical 1-absorbing prime submodules, $\phi-$ classical 1-absorbing prime submodules.
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1 Introduction

In this paper, all rings are commutative with nonzero identity and all modules are considered to be unitary. The concept of prime ideals and its generalizations have a significant place in commutative algebra since they are used in understanding the structure of rings. According to Anderson and Smith [2], a proper ideal I of a ring R is said to be weakly prime if for all $a, b \in R$, the condition $0 \neq ab \in I$ implies that either $a \in I$ or $b \in I$. In [12], in order to study unique factorization domain, Bhatwadekar and Sharma defined a new class of prime ideals. Let R be a ring, a proper ideal I of R is called almost prime ideal if $ab \in I \setminus I^2$ for some $a, b \in R$ implies that either $a \in I$ or $b \in I$. They investigated the relations among the prime ideals, pseudo prime ideals and almost prime ideals. Anderson and Batanieh [3] gave a generalization of prime ideals which covers mentioned definition above. Let $\phi : I(R) \longrightarrow I(R) \cup \{0\}$ be a function where $I(R)$ is the set of all ideals of R. A proper ideal I of R is said to be $\phi-$prime if for $a, b \in R$ with $ab \in I \setminus \phi(I)$, then $a \in I$ or $b \in I$. Recently, in [19], Koç et al. defined weakly 1-absorbing prime ideal. A proper ideal I of R is said to be a weakly 1-absorbing prime ideal if whenever nonunits $a, b, c \in R$ with $0 \neq abc \in I$ implies that $ab \in I$ or $c \in I$. For more details and related results, see [6, 13]. For further generalized versions of prime ideals, we refer the reader to [14, 15]. Several authors have extended the notion of prime ideals to modules.

See, for example, [16, 17, 21, 22, 29, 30, 31, 33]. Let M be a module over a commutative ring R. It is well known that a proper submodule N of M is called prime if for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M) = \{r \in R \mid rM \subseteq N\}$. A.Uğurlu [28], generalized the concept of prime submodules of a module over a commutative ring as follows: Let N be a proper submodule of an $R-$module M. Then N is said to be a 1-absorbing prime submodule of M if whenever nonunits $a, b \in R$ and $m \in M$ with $abm \in N$, then $m \in N$ or $ab \in (N :_R M)$. Zamani [36] introduced the concept of $\phi-$prime submodules. Let $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. A proper submodule N of an $R-$module M is called ϕ-prime if $a \in R$ and $m \in M$ with $am \in N \setminus \phi(N)$, then $m \in N$ or $a \in (N :_R M)$. He defined the map $\phi_\alpha : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ as follows:

1. $\phi_\emptyset :\ \phi(N) = \emptyset$ defines prime submodules.
2. $\phi_0 :\ \phi(N) = \{0\}$ defines weakly prime submodules.
3. $\phi_2 :\ \phi(N) = (N :_R M)N$ defines almost prime submodules.
4. $\phi_n (n \geq 2) :\ \phi(N) = (N :_R M)^{n-1}N$ defines $n-$almost prime submodules.
5. $\phi_\omega :\ \phi(N) = \cap_{n=1}^{\infty} (N :_R M)^n N$ defines $\omega-$prime submodules.
6. $\phi_1 :\ \phi(N) = N$ defines any submodule.

Also, Moradi and Azizi [23] investigated the notion of n-almost prime submodules. A proper submodule N of M is called a classical prime submodule, if for each $a, b \in R$ and $m \in M$, $abm \in N$ implies that $am \in N$ or $bm \in N$. This notion of classical prime submodules has been extensively studied by Behboodi in [8, 9, 10]. For more information on classical prime submodules, the reader is referred to [4, 5, 11, 24, 25, 34]. Recently, (resp. weakly) classical 1-absorbing prime submodules are presented and investigated in [32] and [35]. A proper submodule N of M is said to be a (resp. weakly) classical 1-absorbing prime submodule, if for each $m \in M$ and nonunits $a, b, c \in R$ with (resp. $0 \neq abcm \in N$) $abcm \in N$ implies that $abm \in N$ or $cm \in N$.

In this paper, we define and study on $\phi-$ classical 1-absorbing prime submodules which is a generalization of classical 1-absorbing prime submodules and also covers the concepts of weakly classical 1-absorbing prime and almost classical 1-absorbing prime submodules. Throughout this paper $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ denotes a function. Since $N \setminus \phi(N) = N \setminus (N \cap \phi(N))$ for any submodule N of M, without loss of generality we may assume that $\phi(N) \subseteq N$. For any two functions $\psi_1, \psi_2 : S(M) \longrightarrow S(M) \cup \{\emptyset\}$, we say $\psi_1 \leq \psi_2$ if $\psi_1(N) \subseteq \psi_2(N)$ for each $N \in S(M)$. Thus clearly we have the following order : $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$. Whenever $\psi_1 \leq \psi_2$, any ψ_1-classical 1-absorbing prime submodule is ψ_2-classical 1-absorbing prime. Among other results in this paper, we give several characterizations of $\phi-$classical 1-absorbing prime submodules in some modules over commutative rings (see, Theorems 2.2,2.3,2.4,2.5). Also, we investigate the stability of $\phi-$classical 1-absorbing prime submodules under homomorphism, in localization of modules, in cartesian product of modules (See, Propositions 3.3,3.4,3.14-3.17). Furthermore, we determine the $\phi-$classical 1-absorbing prime submodules of tensor product $F \otimes M$ for a flat (faithfully flat) $R-$module F and any $R-$module M (see Theorem 3.1, Corollary 3.2).

2 Characterizations of $\phi-$Classical 1-Absorbing Prime Submodules

Definition 2.1. Let R be a ring, M be an R-module and $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. Let N be a proper submodule of M. Then we say that N is a $\phi-$classical 1-absorbing prime submodule of M if whenever nonunits $a, b, c \in R$ and $m \in M$ with $abcm \in N \setminus \phi(N)$, then $abm \in N$ or $cm \in N$. We define the map $\phi_\alpha : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ as follows:

1. $\phi_\emptyset: \ \phi(N) = \emptyset$ defines classical 1-absorbing prime submodules.

2. $\phi_0: \ \phi(N) = \{0\}$ defines weakly classical 1-absorbing prime submodules.

3. $\phi_2: \ \phi(N) = (N :_R M)N$ defines almost classical 1-absorbing prime submodules.

4. $\phi_n(n \geq 2): \ \phi(N) = (N :_R M)^{n-1}N$ defines $n-$almost classical 1-absorbing prime submodules.

5. $\phi_\omega: \ \phi(N) = \cap_{n=1}^{\infty}(N :_R M)^n N$ defines $\omega-$classical 1-absorbing prime submodules.

6. $\phi_1: \ \phi(N) = N$ defines any submodule.

Let R be a ring and $\psi : I(R) \longrightarrow I(R) \cup \{\emptyset\}$ be a function. Clearly, I is a $\psi-1$-absorbing prime ideal of R if and only if I is a $\psi-$classical 1-absorbing prime submodule of R-module R.

Let M be an $R-$module and N a submodule of M. For every $a \in R$, $\{m \in M \mid am \in N\}$ is denoted by $(N :_M a)$. It is easy to see that $(N :_M a)$ is a submodule of M containing N. In the next theorem, we obtain several equivalent statements to characterize $\phi-$classical 1-absorbing prime submodules.

Theorem 2.2. Let M be an R-module, $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ a function. For a proper submodule N of M, the following conditions are equivalent.

1. N is a $\phi-$classical 1-absorbing prime submodule of M.

2. For every nonunit $a, b, c \in R$, $(N :_M abc) = (\phi(N) :_M abc) \cup (N :_M ab) \cup (N :_M c)$.

3. For every nonunit $a, b \in R$ and $m \in M$ with $abm \notin N$; $(N :_R abm) = (\phi(N) :_R abm) \cup (N :_R m)$.

4. For every nonunit $a, b \in R$ and $m \in M$ with $abm \notin N$; $(N :_R abm) = (\phi(N) :_R abm)$ or $(N :_R abm) = (N :_R m)$.

5. For every nonunit $a, b \in R$, every ideal I of R and $m \in M$ with $abIm \subseteq N$ and $abIm \nsubseteq \phi(N)$, either $abm \in N$ or $Im \subseteq N$.

6. For every nonunit $a, b \in R$, $m \in M$ and every proper ideal I of R with $abIm \subseteq N$ and $abIm \nsubseteq \phi(N)$, either $aIm \subseteq N$ or $bm \in N$.

7. For every proper ideal I of R, nonunit $a \in R$ and $m \in M$ with $aIm \nsubseteq N$, $(N :_R aIm) = (\phi(N) :_R aIm)$ or $(N :_R aIm) = (N :_R m)$.

8. For every proper ideal I, J, K of R and $m \in M$ with $IJKm \subseteq N$ and $IJKm \nsubseteq \phi(N)$, either $IJm \subseteq N$ or $Km \subseteq N$.

Proof. (1) $\Rightarrow$ (2) Suppose that N is a $\phi-$classical 1-absorbing prime submodule of M. Let $m \in (N :_M abc)$. Then $abcm \in N$. If $abcm \in \phi(N)$, then $m \in (\phi(N) :_M abc)$. Assume that $abcm \notin \phi(N)$. Since N is $\phi-$ classical 1-absorbing prime, we have either $abm \in N$ or $cm \in N$. Hence $m \in (N :_M ab)$ or $m \in (N :_M c)$. Thus we conclude $(N :_M abc) \subseteq (\phi(N) :_M abc) \cup (N :_M ab) \cup (N :_M c)$. Since the reverse inclusion is always true, we have the required equality.

(2) $\Rightarrow$ (3) Suppose that $abm \notin N$ for some nonunits $a, b \in R$ and $m \in M$. Let $x \in (N :_R abm)$. Then $abxm \in N$, and since $abm \notin N$, x is a nonunit. Therefore $m \in (N :_M abx)$ and since $abm \notin N$, then $m \notin (N :_M ab)$. Thus by part (2) we have $m \in (\phi(N) :_M abx)$ or $m \in (N :_M x)$ whence $x \in (\phi(N) :_R abm)$ or $x \in (N :_R m)$. Then we get $(N :_R abm) = (\phi(N) :_R abm) \cup (N :_R m)$.

(3) $\Rightarrow$ (4) By the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.

(4) $\Rightarrow$ (5) Let $abIm \subseteq N$ and $abIm \nsubseteq \phi(N)$ for some nonunits $a, b \in R$, an ideal I of R and $m \in M$. Hence $I \subseteq (N :_R abm)$ and $I \nsubseteq (\phi(N) :_R abm)$. If $abm \in N$, then we are done. So, assume that $abm \notin N$. Therefore by part (4) we have that $I \subseteq (N :_R m)$, and then $Im \subseteq N$.

(5) $\Rightarrow$ (6) Let $abIm \subseteq N$ and $abIm \nsubseteq \phi(N)$ for some proper ideal I of R, nonunits $a, b \in R$ and $m \in M$. Assume that $aIm \nsubseteq N$. Then there exists $x \in I$ such that $axm \notin N$. Then note that $ax(Rb)m \subseteq N$. If $ax(Rb)m \nsubseteq \phi(N)$, then by part (5) we have $(Rb)m \subseteq N$ which completes the proof. So assume that $ax(Rb)m \subseteq \phi(N)$. Since $abIm \nsubseteq \phi(N)$, there exists $y \in I$ such that $ay(Rb)m \nsubseteq \phi(N)$. This implies that $a(x+y)(Rb)m \subseteq N$ and $a(x+y)(Rb)m \nsubseteq \phi(N)$. Since I is a proper ideal, we conclude that $x+y$ is a nonunit. As $ay(Rb)m \subseteq N$ and $ay(Rb)m \nsubseteq \phi(N)$, again by part (5), we have $aym \in N$ or $(Rb)m \subseteq N$. Let $aym \in N$. Then we have $a(x+y)m \notin N$. As $a(x+y)(Rb)m \subseteq N$ and $a(x+y)(Rb)m \nsubseteq \phi(N)$, by part (5), we get $(Rb)m \subseteq N$ which implies that $bm \in N$.

(6) $\Rightarrow$ (7) Let $aIm \nsubseteq N$ for some nonunit $a \in R$, some proper ideal I of R and $m \in M$. Choose $b \in (N :_R aIm)$. Then we have $abIm \subseteq N$. If $abIm \subseteq \phi(N)$, then we get $b \in (\phi(N) :_R aIm)$. Now assume that $abIm \nsubseteq \phi(N)$. Then by part (6), we get $bm \in N$ which implies that $b \in (N :_R m)$. Thus, we conclude that $(N :_R aIm) = (\phi(N) :_R aIm) \cup (N :_R m)$. This implies that $(N :_R aIm) = (\phi(N) :_R aIm)$ or $(N :_R aIm) = (N :_R m)$.

(7) $\Rightarrow$ (8) Let $IJKm \subseteq N$ and $IJKm \nsubseteq \phi(N)$ for some proper ideals I, J, K of R and $m \in M$. Assume that $IJm \nsubseteq N$. Then there exists $a \in J$ such that $aIm \nsubseteq N$. Since $aIKm \subseteq N$, we have $K \subseteq (N :_R aIm) = (\phi(N) :_R aIm)$ or $K \subseteq (N :_R aIm) = (N :_R m)$ by part (7). This gives $aIKm \subseteq \phi(N)$ or $Km \subseteq N$. In the latter case, we are done. So, we may assume that $aIKm \subseteq \phi(N)$. Since $IJKm \nsubseteq \phi(N)$, there exists $b \in J$ such that $IbKm \nsubseteq \phi(N)$. Since $K \subseteq (N :_R bIm)$, again by part (7), we have $bIm \subseteq N$ or $Km \subseteq N$. Now assume that $bIm \subseteq N$. Then we have $(a+b)Im \nsubseteq N$. Since J is a proper ideal, our conclusion is that $a+b$ is a nonunit. As $(a+b)IKm \subseteq N$, $(a+b)IKm \nsubseteq \phi(N)$ we obtain that $K \subseteq (N :_R (a+b)Im) = (N :_R m)$ by part (7), we conclude that $Km \subseteq N$.

(8) $\Rightarrow$ (1) Clear. □

In [26], Quartararo et al. said that a commutative ring R is a u-ring provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and um−ring is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal one of them. They show that every Bézout ring is a u−ring. Moreover, they proved that every Prüfer domain is a u−domain. Also, any ring which contains an infinite field as a subring is a u−ring [27, Exercise 3.63].

Theorem 2.3. Let R be a um-ring, M be an R-module and N be a proper submodule of M. Suppose that $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function. The following conditions are equivalent.

1. N is a ϕ−classical 1-absorbing prime submodule of M.
2. For every nonunit $a, b, c \in R$ and $m \in M$, $(N :_R abcm) = (\phi(N) :_R abcm)$ or $(N :_R abcm) = (N :_R abm)$ or $(N :_R abcm) = (N :_R cm)$.
3. For every nonunit $a, b, c \in R$, $(N :_M abc) = (\phi(N) :_M abc)$ or $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M c)$.
4. For every nonunit $a, b, c \in R$ and every submodule L of M ; $abcL \subseteq N$ and $abcL \nsubseteq \phi(N)$ implies that $abL \subseteq N$ or $cL \subseteq N$.
5. For every nonunit $a, b \in R$ and every submodule L of M with $abL \nsubseteq N$, $(N :_R abL) = (\phi(N) :_R abL)$ or $(N :_R abL) = (N :_R L)$.
6. For every nonunit $a, b \in R$, every ideal J of R and every submodule L of M with $abJL \subseteq N$ and $abJL \nsubseteq \phi(N)$ implies that $abL \subseteq N$ or $JL \subseteq N$.

7. For every nonunit $a \in R$, every proper ideal I, J of R and every submodule L of M with $aIJL \subseteq N$ and $aIJL \nsubseteq \phi(N)$ implies that $aIL \subseteq N$ or $JL \subseteq N$.

8. For every proper ideal I, J of R and every submodule L of M with $IJL \nsubseteq N$; $(N :_R IJL) = (\phi(N) :_R IJL)$ or $(N :_R IJL) = (N :_R L)$.

9. For every proper ideal I, J, K of R and every submodule L of M with $IJKL \subseteq N$ and $IJKL \nsubseteq \phi(N)$ implies that $IJL \subseteq N$ or $KL \subseteq N$.

Proof. (1) $\Rightarrow$ (2) Let a, b, c be nonunits of R and $m \in M$. Suppose that $r \in (N :_R abcm)$. Then $abc(rm) \in N$. If $abc(rm) \in \phi(N)$, then $r \in (\phi(N) :_R abcm)$. Now, assume that $abc(rm) \notin \phi(N)$. Then we have $abc(rm) \in N \setminus \phi(N)$ and since N is a ϕ–classical 1-absorbing prime submodule we get $ab(rm) \in N$ or $c(rm) \in N$. Hence $r \in (N :_R abm)$ or $r \in (N :_R cm)$. Thus, we conclude that $r \in (\phi(N) :_R abcm) \cup (N :_R abm) \cup (N :_R cm)$, that is $(N :_R abcm) \subseteq (\phi(N) :_R abcm) \cup (N :_R abm) \cup (N :_R cm)$. Since the reverse inclusion is always true and R is a um-ring we have $(N :_R abcm) = (\phi(N) :_R abcm)$ or $(N :_R abcm) = (N :_R abm)$ or $(N :_R abcm) = (N :_R cm)$.

(2) $\Rightarrow$ (3) Let $m \in (N :_M abc)$. Then $abcm \in N$ and hence, $1_R \in (N :_R abcm) \subseteq (\phi(N) :_R abcm) \cup (N :_R abm) \cup (N :_R cm)$ by (2) which implies $1_R \in (\phi(N) :_R abcm)$ or $1_R \in (N :_R abm)$ or $1_R \in (N :_R cm)$. It follows that $m \in (N :_M abc) \subseteq (\phi(N) :_M abc) \cup (N :_M ab) \cup (N :_M c)$. Since R is um-ring, $(N :_M abc) = (\phi(N) :_M abc)$ or $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M c)$.

(3) $\Rightarrow$ (4) Let $abcL \subseteq N$ and $abcL \nsubseteq \phi(N)$ for some nonunits $a, b, c \in R$ and submodule L of M. Hence $L \subseteq (N :_M abc)$ and $L \nsubseteq (\phi(N) :_M abc)$. Therefore, part (2) implies that $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M c)$. So, either $L \subseteq (N :_M ab)$ or $L \subseteq (N :_M c)$, and then $abL \subseteq N$ or $cL \subseteq N$.

(4) $\Rightarrow$ (5) Let $abL \nsubseteq N$ for some nonunits $a, b \in R$ and submodule L of M. Assume that $c \in (N :_R abL)$. Then c is nonunit and $abcL \subseteq N$. If $abcL \subseteq \phi(N)$, then $c \in (\phi(N) :_R abL)$. Now, assume that $abcL \nsubseteq \phi(N)$. Thus by part (3) we have that $cL \subseteq N$, as $abL \nsubseteq N$. Hence, $c \in (N :_R L)$. Consequently, $(N :_R abL) = (\phi(N) :_R abL) \cup (N :_R L)$ and then $(N :_R abL) = (\phi(N) :_R abL)$ or $(N :_R abL) = (N :_R L)$.

(5) $\Rightarrow$ (6) Let for some nonunits $a, b \in R$, an ideal J of R and submodule L of M with $abJL \subseteq N$ and $abJL \nsubseteq \phi(N)$. Hence $J \subseteq (N :_R abL)$ and $J \nsubseteq (\phi(N) :_R abL)$. If $abL \subseteq N$, then we are done. So, assume that $abL \nsubseteq N$. Therefore by part (5) we have $J \subseteq (N :_R abL) = (N :_R L)$. Thus, $JL \subseteq N$.

(6) $\Rightarrow$ (7) Suppose that $aIJL \subseteq N$, $aIJL \nsubseteq \phi(N)$ and $JL \nsubseteq N$. Now, take $x \in I$. Since $aIJL \subseteq N$ and $aIJL \nsubseteq \phi(N)$, there exists $y \in I$ such that $ayJL \subseteq N$ $ayJL \nsubseteq \phi(N)$ and then by part (6), $ayL \subseteq N$. If $axJL \subseteq N$ and $axJL \nsubseteq \phi(N)$, then similarly we have $axL \subseteq N$. Hence, suppose that $axJL \subseteq \phi(N)$. Then we have $a(x+y)JL \subseteq N$, $a(x+y)JL \nsubseteq \phi(N)$ giving that $a(x+y)L \subseteq N$ which implies that $axL \subseteq N$ as $ayL \subseteq N$. Thus, we conclude that $aIL \subseteq N$.

(7) $\Rightarrow$ (8) Suppose that $IJL \nsubseteq N$ and $a \in (N :_R IJL)$. It is clear that a is nonunit. Then $aIJL \subseteq N$ and there exists $x \in J$ such that $xIL \nsubseteq N$. If $aIJL \subseteq \phi(N)$, then $a \in (\phi(N) :_R IJL)$. Now, assume that $aIJL \nsubseteq \phi(N)$. Then there exists $y \in J$ such that $ayIL = yI(Ra)L \subseteq N$ and $ayIL = yI(Ra) \nsubseteq \phi(N)$. Then by part (7) we conclude that $yIL \subseteq N$ or $(Ra)L \subseteq N$. In the latter case, we have $a \in (N :_R L)$. Assume the former case $yIL \subseteq N$. If $axIL = xI(Ra)L \subseteq N$ and $axIL = xI(Ra)L \nsubseteq \phi(N)$, then by part (7), we have $(Ra)L \subseteq N$ which implies that $a \in (N :_R L)$. So we may assume that $axIL \subseteq \phi(N)$. We conclude that $x+y$ is a nonunit, since J is a proper ideal. In this case, we conclude that $a(x+y)IL = (x+y)I(Ra)L \subseteq N$ and $a(x+y)IL = (x+y)I(Ra)L \nsubseteq \phi(N)$. Again by part (7), we conclude that $(x+y)IL \subseteq N$ or $(Ra)L \subseteq N$. If $(x+y)IL \subseteq N$, then we get $xIL \subseteq N$, as $yIL \subseteq N$. This is a contradiction. Thus we conclude that $(Ra)L \subseteq N$, that is, $a \in (N :_R L)$. By above arguments, we conclude that $(N :_R IJL) \subseteq (\phi(N) :_R IJL) \cup (N :_R L)$. Since reverse inclusion is always true, we have the equality $(N :_R IJL) = (\phi(N) :_R IJL) \cup (N :_R L)$. In this case, we have $(N :_R IJL) = (\phi(N) :_R IJL)$ or $(N :_R IJL) = (N :_R L)$.

(8) $\Rightarrow$ (9) Suppose that $IJKL \subseteq N$ and $IJKL \nsubseteq \phi(N)$. Then we have $K \subseteq (N :_R IJL)$ and $K \nsubseteq (\phi(N) :_R IJL)$. Then by part (8), we have $K \subseteq (N :_R IJL) = (N :_R L)$ which implies that $KL \subseteq N$.

(9) $\Rightarrow$ (1) Straightforward. $\square$

We recall that an R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R [7]. In this case, $N = (N :_R M)M$ [18]. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [1, Theorem 3.4], the product of N and K is independent of presentations from N and K. Let M be a multiplication R−module and K, L be submodules of M. Then there are ideals I, J of R such that $K = IM$ and $L = JM$. Thus $KL = IJM = IL$. In particular $KM = IM = K$. Also, for any $m \in M$ we define $Km := KRm$. Hence $Km = IRm = Im$. Next, we characterize $\phi-$ classical 1-absorbing prime submodules of multiplication modules in terms of some submodules of them.

Theorem 2.4. Let R be a um-ring, M be a finitely generated faithful multiplication R-module, N be a proper submodule of M and $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ is a function. Then the following conditions are equivalent.

1. N is a $\phi-$ classical 1-absorbing prime submodule of M.
2. If $N_1N_2N_3N_4 \subseteq N$ for some proper submodules N_1, N_2, N_3 and for some submodule N_4 of M such that $N_1N_2N_3N_4 \nsubseteq \phi(N)$, then either $N_1N_2N_4 \subseteq N$ or $N_3N_4 \subseteq N$.
3. If $N_1N_2N_3m \subseteq N$ for some proper submodules N_1, N_2, N_3 of M and $m \in M$ such that $N_1N_2N_3m \nsubseteq \phi(N)$, then either $N_1N_2m \subseteq N$ or $N_3m \subseteq N$.

Proof. (1) $\Rightarrow$ (2) Let $N_1N_2N_3N_4 \subseteq N$ for some proper submodules N_1, N_2, N_3 of M and a submodule N_4 of M such that $N_1N_2N_3N_4 \nsubseteq \phi(N)$. Since M is multiplication R-module, there are proper ideals I_1, I_2, I_3 of R such that $N_1 = I_1M$, $N_2 = I_2M$, $N_3 = I_3M$. Therefore $I_1I_2I_3N_4 \subseteq N$ and $I_1I_2I_3N_4 \nsubseteq \phi(N)$, so by Theorem 2.3, we have $I_1I_2N_4 \subseteq N$ or $I_3N_4 \subseteq N$. Hence, $N_1N_2N_4 \subseteq N$ or $N_3N_4 \subseteq N$.

(2) $\Rightarrow$ (3) Put $N_4 = Rm$, the desired result is obtained.

(3) $\Rightarrow$ (1) Suppose that $I_1I_2I_3m \subseteq N$ for some proper ideals I_1, I_2, I_3 of R and some $m \in M$. It is sufficient to set $N_1 := I_1M$, $N_2 := I_2M$ and $N_3 := I_3M$ in part (3). □

Let M be an R−module, K a submodule of M and $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ a function. Define $\phi_K : S(M/K) \longrightarrow S(M/K) \cup \{\emptyset\}$ by $\phi_K(N/K) = (\phi(N) + K)/K$ for every $N \in S(M)$ with $N \supseteq K$ (and $\phi_K(N/K) = \emptyset$ if $\phi(N) = \emptyset$).

Proposition 2.5. *Let M be an R-module and $K \subseteq N$ be proper submodules of M. Suppose that $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ is a function.*

1. *If N is a $\phi-$classical 1-absorbing prime submodule of M, then N/K is a ϕ_K-classical 1-absorbing prime submodule of M/K.*
2. *If $K \subseteq \phi(N)$ and N/K is a ϕ_K-classical 1-absorbing prime submodule of M/K, then N is a $\phi-$classical 1-absorbing prime submodule of M.*
3. *N is a $\phi-$classical 1-absorbing prime submodule of M if and only if $N/\phi(N)$ is a weakly classical 1-absorbing prime submodule of $M/\ \phi(N)$.*
4. *If $\phi(N) \subseteq K$ and N is a $\phi-$classical 1-absorbing prime submodule of M, then N/K is a weakly classical 1-absorbing prime submodule of M/K.*
5. *If $\phi(K) \subseteq \phi(N)$, K is a $\phi-$classical 1-absorbing prime submodule of M and N/K is a weakly classical 1-absorbing prime submodule of M/K, then N is a $\phi-$classical 1-absorbing prime submodule of M.*
6. *If N is a $\phi-$classical 1-absorbing prime submodule of M and $\phi(N)$ is a classical 1-absorbing prime submodule of M, then N is a classical 1-absorbing prime submodule of M.*

Proof. (1) Let a,b,c be nonunits in R and $m \in M$ such that $abc(m+K) \in N/K \setminus \phi_K(N/K)$. Then $abcm \in N \setminus \phi(N)$. Since N is a $\phi-$classical 1-absorbing prime submodule of M we have either $abm \in N$ or $cm \in N$. Therefore, $ab(m+K) \in N/K$ or $c(m+K) \in N/K$. Consequently, N/K is a ϕ_K-classical 1-absorbing prime submodule of M/K.

(2) Let a,b,c be nonunits in R and $m \in M$ such that $abcm \in N \setminus \phi(N)$. Then $abc(m+K) \in N/K \setminus (\phi(N)/K)$. Since N/K is a ϕ_K-classical 1-absorbing prime submodule we get either $ab(m+K) \in N/K$ or $c(m+K) \in N/K$, and so $abm \in N$ or $cm \in N$.

(3) Suppose that N is a $\phi-$classical 1-absorbing prime submodule of M. From (1), $N/\phi(N)$ is a $\phi_{\phi(N)}-$classical 1-absorbing prime submodule of $M/\phi(N)$. Since $\phi_{\phi(N)}(N/\phi(N)) = 0 + \phi(N)/\phi(N)$, it is a weakly classical 1-absorbing prime submodule of $M/\phi(N)$. Conversely, if $N/\phi(N)$ is a weakly classical 1-absorbing prime submodule of $M/\phi(N)$, that means that $N/\phi(N)$ is $\phi_{\phi(N)}-$classical 1-absorbing prime submodule of $M/\phi(N)$. Hence, by (2), we conclude that N is a $\phi-$classical 1-absorbing prime submodule of M.

(4) Let a,b,c be nonunits in R and $m \in M$ such that $0 \neq abc(m+K) \in N/K$. Hence $abcm \in N \setminus \phi(N)$ as $\phi(N) \subseteq K$. Since N is a $\phi-$classical 1-absorbing prime submodule, we have either $abm \in N$ or $cm \in N$. Therefore, $ab(m+K) \in N/K$ or $c(m+K) \in N/K$. Thus, N/K is a weakly classical 1-absorbing prime submodule of M/K.

(5) Let a,b,c be nonunits in R and $m \in M$ such that $abcm \in N \setminus \phi(N)$. Note that $\phi(K) \subseteq \phi(N)$ implies that $abcm \notin \phi(K)$. If $abcm \in K$, then we have $abcm \in K \setminus \phi(K)$. Since K is a $\phi-$classical 1-absorbing prime submodule we have either $abm \in K \subseteq N$ or $cm \in K \subseteq N$. Now, assume that $abcm \notin K$. So $0 \neq abc(m+K) \in N/K$. Since N/K is a weakly classical 1-absorbing prime submodule of M/K, we have either $ab(m+K) \in N/K$ or $c(m+K) \in N/K$. Thus $abm \in N$ or $cm \in N$. Consequently, N is a $\phi-$classical 1-absorbing prime submodule of M.

(6) Let N be a $\phi-$classical 1-absorbing prime submodule of M. Assume that $abcm \in N$ for some nonunits $a,b,c \in R$ and $m \in M$. If $abcm \in \phi(N)$, then since $\phi(N)$ is classical 1-absorbing prime, we conclude that $abm \in \phi(N) \subseteq N$ or $cm \in \phi(N) \subseteq N$, and so we are done. When $abcm \notin \phi(N)$ clearly the result follows. □

3 $\phi-$classical 1-absorbing prime submodules in some classes of modules

Recall from [20] that an $R-$module F is said to be a flat $R-$module if for each exact sequence $K \longrightarrow L \longrightarrow M$ of $R-$modules, the sequence $F \otimes K \longrightarrow F \otimes L \longrightarrow F \otimes M$ is also exact. Also, F is said to be a faithfully flat the sequence $K \longrightarrow L \longrightarrow M$ is exact if and only if $F \otimes K \longrightarrow F \otimes L \longrightarrow F \otimes M$ is exact. Azizi in [5, Lemma 3.2] showed that if M is a $R-$module, N is a submodule of M and F is a flat $R-$module, then $(F \otimes N :_{F\otimes M} a) = F \otimes (N :_M a)$ for every $a \in R$.

Theorem 3.1. Let R be a um-ring and N is a proper submodule of an R-module M and $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function. Suppose that $\psi : S(F \otimes M) \longrightarrow S(F \otimes M) \cup \{\emptyset\}$ is a function such that $\psi(F \otimes K) = F \otimes \phi(K)$ for all $K \in S(M)$. Then,

1. If F is flat R-module and N is a $\phi-$ classical 1-absorbing prime submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a $\psi-$ classical 1-absorbing prime submodule of $F \otimes M$.
2. Suppose that F is a faithfully flat R-module. Then N is a $\phi-$ classical 1-absorbing prime submodule of M if and only if $F \otimes N$ is a $\psi-$ classical 1-absorbing prime submodule of $F \otimes M$.

Proof. (1) Suppose that N is a ϕ-classical 1-absorbing prime submodule of M. Let $a,b,c \in R$ be nonunits. Then by Theorem 2.3, either $(N :_M abc) = (\phi(N) :_M abc)$ or $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M c)$. Assume that $(N :_M abc) = (\phi(N) :_M abc)$. Then by [5, Lemma 3.2] , $(F \otimes N :_{F\otimes M} abc) = F \otimes (N :_M abc) = F \otimes (\phi(N) :_M abc) = (F \otimes \phi(N) :_{F\otimes M} abc) = (\psi(F \otimes N) :_{F\otimes M} abc)$. Now, suppose that

$(N :_M abc) = (N :_M ab)$. Again by [8, Lemma 3.2], $(F \otimes N :_{F\otimes M} abc) = F \otimes (N :_M abc) = F \otimes (N :_M ab) = (F \otimes N :_{F\otimes M} ab)$. Similarly, we can show that if $(N :_M abc) = (N :_M c)$, then $(F \otimes N :_{F\otimes M} abc) = F \otimes (N :_M abc) = F \otimes (N :_M c) = (F \otimes N :_{F\otimes M} c)$. Consequently, by Theorem 2.3, we deduce that $F \otimes N$ is a $\phi-$ classical 1-absorbing prime submodule of $F \otimes M$.

(2) Let N be a $\phi-$ classical 1-absorbing prime submodule of M and assume $F \otimes N = F \otimes M$. Then $0 \to F \otimes N \xrightarrow{\subseteq} F \otimes M \to 0$ is an exact sequence. Since F is a faithfully flat R-module, $0 \to N \xrightarrow{\subseteq} M \to 0$ is an exact sequence. This implies that $N = M$, which is a contradiction. So $F \otimes N \neq F \otimes M$. Then $F \otimes N$ is a $\psi-$ classical 1-absorbing prime submodule of $F \otimes M$ by part(1). Now for the converse, let $F \otimes N$ be a $\psi-$classical 1-absorbing prime submodule of $F \otimes M$. We have $F \otimes N \neq F \otimes M$ and so $N \neq M$. Let $a, b, c \in R$ be nonunits. Then by Theorem 2.3, $(F \otimes N :_{F\otimes M} abc) = (\psi(F \otimes N) :_{F\otimes M} abc)$ or $(F \otimes N :_{F\otimes M} abc) = (F \otimes N :_{F\otimes M} ab)$ or $(F \otimes N :_{F\otimes M} abc) = (F \otimes N :_{F\otimes M} c)$. Suppose that $(F \otimes N :_{F\otimes M} abc) = (\psi(F \otimes N) :_{F\otimes M} abc)$. Hence $F\otimes(N :_M abc) = (F \otimes N :_{F\otimes M} abc) = (\psi(F \otimes N) :_{F\otimes M} abc) = (F \otimes \phi(N) :_{F\otimes M} abc) = F \otimes (\phi(N) :_M abc)$. Thus $0 \to F \otimes (\phi(N) :_M abc) \xrightarrow{\subseteq} F \otimes (N :_M abc) \to 0$ is an exact sequence. Since F is a faithfully flat R-module $0 \to (\phi(N) :_M abc) \xrightarrow{\subseteq} (N :_M abc) \to 0$ is an exact sequence which implies that $(N :_M abc) = (\phi(N) :_M abc)$. Now, assume that $(F \otimes N :_{F\otimes M} abc) = (F \otimes N :_{F\otimes M} ab)$. Hence $F \otimes (N :_M abc) = (F \otimes N :_{F\otimes M} abc) = (F \otimes N :_{F\otimes M} ab) = F \otimes (N :_M ab)$. So $0 \to F \otimes (N :_M ab) \xrightarrow{\subseteq} F \otimes (N :_M abc) \to 0$ is an exact sequence. Since F is a faithfully flat R-module, $0 \to (N :_M ab) \xrightarrow{\subseteq} (N :_M abc) \to 0$ is an exact sequence which implies that $(N :_M abc) = (N :_M ab)$. Now, suppose that $(F \otimes N :_{F\otimes M} abc) = (F \otimes N :_{F\otimes M} c)$. Hence $F \otimes (N :_M abc) = (F \otimes N :_{F\otimes M} abc) = (F \otimes N :_{F\otimes M} c) = F \otimes (N :_M c)$. This gives $0 \to F \otimes (N :_M c) \xrightarrow{\subseteq} F \otimes (N :_M abc) \to 0$ is an exact sequence. Again since F is a faithfully flat R-module, $0 \to (N :_M c) \xrightarrow{\subseteq} (N :_M abc) \to 0$ is an exact sequence which implies that $(N :_M abc) = (N :_M c)$. Consequently, N is a $\phi-$ classical 1-absorbing prime submodule of M by Theorem 2.3. □

Corollary 3.2. *Let R be a um-ring, M be an R-module and X be an indeterminate. Suppose that $\psi : S(R[X] \otimes M) \longrightarrow S(F \otimes M) \cup \{\emptyset\}$ is a function such that $\psi(R[X] \otimes K) = F \otimes \phi(K)$ for all $K \in S(M)$. If N is a $\phi-$ classical 1-absorbing prime submodule of M, then $N[X]$ is a $\phi-$ classical 1-absorbing prime submodule of $M[X]$.*

Proof. Assume that N is a $\phi-$classical 1-absorbing prime submodule of M and $R[X] \otimes N = \psi(R[X] \otimes N)$. Notice that $R[X]$ is a flat R-module. So by Theorem 3.1, $R[X] \otimes N$ is a $\psi-$ classical 1-absorbing prime submodule of $R[X] \otimes M$. Since, $R[X] \otimes N \cong N[X]$, $R[X] \otimes M \cong M[X]$ and then $N[X]$ is a $\phi-$classical 1-absorbing prime submodule of $M[X]$. □

In general, we investigate the behaviour of $\phi-$classical 1-absorbing prime submodules under a homomorphism:

Proposition 3.3. *Let $f : M \longrightarrow M'$ be an epimorphism of R-modules, $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ and $\phi' : S(M') \longrightarrow S(M') \cup \{\emptyset\}$ be two functions. Then the following conditions hold.*

1. *If N' is a $\phi'-$classical 1-absorbing prime submodule of M' and $\phi\left(f^{-1}(N')\right) \supseteq f^{-1}(\phi'(N'))$, then $f^{-1}(N')$ is a $\phi-$classical 1-absorbing prime submodule of M.*

2. *If N is a $\phi-$classical 1-absorbing prime submodule of M containing $Ker(f)$ and $\phi'(f(N)) \supseteq f(\phi(N))$, then $f(N)$ is a $\phi'-$classical 1-absorbing prime submodule of M'.*

Proof. (1) Since f is an epimorphism, $f^{-1}(N')$ is a proper submodule of M. Let a, b, c be nonunits in R and $m \in M$ such that $abcm \in f^{-1}(N') \setminus \phi(f^{-1}(N'))$. Since $abcm \in f^{-1}(N')$, we have $f(abcm) = abcf(m) \in$

N'. Also, $\phi\left(f^{-1}(N')\right) \supseteq f^{-1}(\phi'(N'))$ implies that $abcf(m) \notin \phi'(N')$. This implies either $abf(m) \in N'$ or $cf(m) \in N'$. Hence $abm \in f^{-1}(N')$ or $cm \in f^{-1}(N')$ as needed.

(2) Let a,b,c be nonunit elements of R and $m' \in M'$ such that $abcm' \in f(N) \setminus \phi'(f(N))$. Since f is epimorphism, there exists $m \in M$ such that $m' = f(m)$. Then $abcf(m) = f(abcm) \in f(N) \setminus f(\phi(N))$. Since $Ker(f) \subseteq N$ and $\phi'(f(N)) \supseteq f(\phi(N))$ we have $abcm \in N \setminus \phi(N)$. Since N is a $\phi-$ classical 1-absorbing prime submodule of M, we have either $abm \in N$ or $cm \in N$. Then $abf(m) = abm' \in f(N)$ or $cf(m) = cm' \in f(N)$. Consequently, $f(N)$ is a ϕ'–classical 1-absorbing prime submodule of M'. □

Let T be a multiplicatively closed subset of R. It is well-known that each submodule of $T^{-1}M$ is in the form $T^{-1}N$ for some submodule N of M. Let $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ be a function and define $\phi_T : S(T^{-1}M) \longrightarrow S(T^{-1}M) \cup \{\emptyset\}$ by $\phi_T(T^{-1}N) = T^{-1}\phi(N)$ (and $\phi_T(T^{-1}N) = \emptyset$ when $\phi(N) = \emptyset$) for every submodule N of M.

Proposition 3.4. *Let M be an $R-$module, T be a multiplicatively closed subset of R such that $T^{-1}N \neq T^{-1}M$ and N be a $\phi-$classical 1-absorbing prime submodule of M. Then $T^{-1}N$ is a ϕ_T-classical 1-absorbing prime submodule of $T^{-1}M$.*

Proof. Let N be a $\phi-$ classical 1-absorbing prime submodule of M such that $T^{-1}N \neq T^{-1}M$. Suppose that $\frac{a_1}{t_1}\frac{a_2}{t_2}\frac{a_3}{t_3}\frac{m}{t_4} \in T^{-1}N \setminus \phi_T(T^{-1}N)$ for some nonunits $\frac{a_1}{t_1}, \frac{a_2}{t_2}, \frac{a_3}{t_3} \in T^{-1}R$ and $\frac{m}{t_4} \in T^{-1}M$. Then there exists $t \in T$ such that $t(a_1a_2a_3m) = a_1a_2a_3(tm) \in N$. If $ta_1a_2a_3m \in \phi(N)$, then $\frac{a_1}{t_1}\frac{a_2}{t_2}\frac{a_3}{t_3}\frac{m}{t_4} = \frac{ta_1a_2a_3m}{tt_1t_2t_3t_4} \in T^{-1}\phi(N) = \phi_T(T^{-1}N)$, a contradiction. Since N is a $\phi-$classical 1-absorbing prime submodule, then we have $a_1a_2(tm) \in N$ or $a_3(tm) \in N$.Thus $\frac{a_1}{t_1}\frac{a_2}{t_2}\frac{m}{t_4} = \frac{ta_1a_2m}{tt_1t_2t_4} \in T^{-1}N$ or $\frac{a_3}{t_3}\frac{m}{t_4} = \frac{ta_3m}{tt_3t_4} \in T^{-1}N$. Consequently, $T^{-1}N$ is a ϕ_T-classical 1-absorbing prime submodule of $T^{-1}M$. □

Proposition 3.5. *Let M be an $R-$module and N be a proper submodule of M. Suppose that $\phi : S(M) \longrightarrow S(M) \cup \{\emptyset\}$ and $\psi : I(R) \longrightarrow I(R) \cup \{\emptyset\}$ be two functions.*

1. *If N is a $\phi-$classical 1-absorbing prime submodule of M, then $(N :_R m)$ is a $\psi-$1-absorbing prime ideal of R for every $m \in M \setminus N$ satisfying $(\phi(N) :_R m) \subseteq \psi((N :_R m))$.*
2. *If $(N :_R m)$ is a $\psi-$1-absorbing prime ideal of R for every $m \in M \setminus N$ with $\psi((N :_R m)) \subseteq (\phi(N) :_R m)$, then N is a $\phi-$classical 1-absorbing prime submodule of M.*
3. *Let $\psi((N :_R m)) = (\phi(N) :_R m)$. Then N is a $\phi-$classical 1-absorbing prime submodule of M if and only if $(N :_R m)$ is a $\psi-$1-absorbing prime ideal of R for every $m \in M \setminus N$.*

Proof. (1) Suppose that N is a $\phi-$classical 1-absorbing prime submodule. Let $m \in M \setminus N$ with $(\phi(N) :_R m) \subseteq \psi((N :_R m))$ and $abc \in (N :_R m) \setminus \psi((N :_R m))$ for some nonunits $a,b,c \in R$. Then $abcm \in N \setminus \phi(N)$. Since N is a $\phi-$classical 1-absorbing prime submodule, we have either $abm \in N$ or $cm \in N$. Hence $ab \in (N :_R m)$ or $c \in (N :_R m)$. Consequently, $(N :_R m)$ is a $\psi-1$-absorbing prime ideal of R.

(2) Assume that $(N :_R m)$ is a $\psi-1$-absorbing prime ideal of R for every $m \in M \setminus N$ with $\psi((N :_R m)) \subseteq (\phi(N) :_R m)$. Let $abcm \in N \setminus \phi(N)$ for some $m \in M$ and nonunits $a,b,c \in R$. If $m \in N$, then we are done. So we assume that $m \notin N$. Hence $abc \in (N :_R m) \setminus (\phi(N) :_R m)$. Since $\psi((N :_R m)) \subseteq (\phi(N) :_R m)$ we have $abc \in (N :_R m) \setminus \psi((N :_R m))$. Since $(N :_R m)$ is a $\psi-1$-absorbing prime ideal, we have either $ab \in (N :_R m)$ or $c \in (N :_R m)$. Hence $abm \in N$ or $cm \in N$ and so N is a $\phi-$classical 1-absorbing prime submodule of M.

(3) Combine (1) and (2). □

Proposition 3.6. *Let M be an $R-$module and a be an element of R such that $aM \neq M$. Suppose that $(0 :_M a) \subseteq aM$. Then aM is an almost classical 1-absorbing prime submodule of M if and only if it is a classical 1-absorbing prime submodule of M.*

Proof. Assume that aM is an almost classical 1-absorbing prime submodule of M. Let x,y,z be nonunit elements of R and $m \in M$ such that $xyzm \in aM$. We will show that $xym \in aM$ or $zm \in aM$. If $xyzm \notin (aM :_R M)aM$, then there is nothing to prove. So, assume that $xyzm \in (aM :_R M)aM$. Note that $xy(z+a)m \in aM$. If $z+a$ is unit, then we have $xym \in aM$, as required. So, assume that $z+a$ is a nonunit. If $xy(z+a)m \notin (aM :_R M)aM$, then $xym \in aM$ or $(z+a)m \in aM$ and we are done. Therefore, suppose that $xy(z+a)m \in (aM :_R M)aM$. Hence $xyzm \in (aM :_R M)aM$ gives $xyam \in (aM :_R M)aM$. Then, there exists $m' \in (aM :_R M)M$ such that $xyam = am'$ and so $xym - m' \in (0 :_M a) \subseteq aM$ which shows $xym \in aM$, as $m' \in aM$. Consequently, aM is a classical 1-absorbing prime. The converse is straightforward. □

Now, we introduce the following definition:

Definition 3.7. Let N be a proper submodule of M and a,b,c be nonunits in R and $m \in M$. If N is a $\phi-$ classical 1-absorbing prime submodule and $abcm \in \phi(N)$, $abm \notin N$ and $cm \notin N$, then we say that (a,b,c,m) is called a $\phi-$classical 1-quadruple-zero of N.

Theorem 3.8. Let N be a $\phi-$classical 1-absorbing prime submodule of an $R-$module M and suppose that $abcK \subseteq N$ for some nonunits $a,b,c \in R$ and some submodule K of M. If (a,b,c,k) is not a $\phi-$classical 1-quadruple-zero of N for any $k \in K$, then $abK \subseteq N$ or $cK \subseteq N$.

Proof. Suppose that (a,b,c,k) is not a $\phi-$classical 1-quadruple-zero of N for every $k \in K$. Assume on the contrary that $abK \nsubseteq N$ and $cK \nsubseteq N$. Then there are $k_1, k_2 \in K$ such that $abk_1 \notin N$ and $ck_2 \notin N$. If $abck_1 \notin \phi(N)$, then we have $ck_1 \in N$. If $abck_1 \in \phi(N)$, then since $abk_1 \notin N$ and (a,b,c,k_1) is not a $\phi-$ classical 1-quadruple-zero of N, we conclude again that $ck_1 \in N$. By a similar argument, since (a,b,c,k_2) is not a $\phi-$classical 1-quadruple-zero of N and $ck_2 \notin N$, then we deduce that $abk_2 \in N$. By our hypothesis, $abc(k_1+k_2) \in N$ and (a,b,c,k_1+k_2) is not a $\phi-$classical 1-quadruple-zero of N. Hence we have either $ab(k_1+k_2) \in N$ or $c(k_1+k_2) \in N$. If $ab(k_1+k_2) \in N$, then since $abk_2 \in N$, we have $abk_1 \in N$, which is a contradiction. If $c(k_1+k_2) = ck_1 + ck_2 \in N$, then since $ck_1 \in N$, we have $ck_2 \in N$, which again is a contradiction. Thus, we deduce $abK \subseteq N$ or $cK \subseteq N$. □

We introduce the next definition to give a further characterization of this class of submodules in terms of some ideals and submodules.

Definition 3.9. Let N be a $\phi-$classical 1-absorbing prime submodule of an R-module M and suppose that $HIJK \subseteq N$ for some proper ideals H,I,J of R and some submodule K of M. We say that N is a free $\phi-$classical 1-quadruple-zero with respect to $HIJK$ if (a,b,c,k) is not a $\phi-$classical 1-quadruple-zero of N for every $a \in H$, $b \in I$, $c \in J$ and $k \in K$.

Let N be a $\phi-$classical 1-absorbing prime submodule of M and suppose that $HIJK \subseteq N$ for some proper ideals H,I,J of R and some submodule K of M such that N is a free $\phi-$classical 1-quadruple-zero with respect to $HIJK$. Hence, if $a \in H$, $b \in I$, $c \in J$ and $k \in K$, then $abk \in N$ or $ck \in N$.

Corollary 3.10. *Let N be a $\phi-$ classical 1-absorbing prime submodule of an R-module M and suppose that $HIJK \subseteq N$ for some proper ideals H,I,J of R and some submodule K of M. If N is a free $\phi-$classical 1-quadruple-zero with respect to $HIJK$, then $HIK \subseteq N$ or $JK \subseteq N$.*

Proof. Suppose that N is a free $\phi-$classical 1-quadruple-zero with respect to $HIJK$. Assume that $HIK \nsubseteq N$ and $JK \nsubseteq N$. Then there are $a \in H$, $b \in I$, $c \in J$ with $abK \nsubseteq N$ and $cK \nsubseteq N$. Since $abcK \subseteq N$ and N is a free $\phi-$classical 1-quadruple-zero with respect to $HIJK$, then Theorem 3.8 implies that $abK \subseteq N$ and $cK \subseteq N$, which is a contradiction. Therefore, $HIK \subseteq N$ or $JK \subseteq N$. □

Theorem 3.11. Let N be a $\phi-$classical 1-absorbing prime submodule of M and suppose that (a,b,c,m) is a $\phi-$classical 1-quadruple-zero of N for some nonunits $a,b,c \in R$ and $m \in M$. Then

1. $abcN \subseteq \phi(N)$.

2. $ab(N :_R M)m \subseteq \phi(N)$.

3. If $acm \notin N$ and $bcm \notin N$, then $ac(N :_R M)m \subseteq \phi(N)$ and $bc(N :_R M)m \subseteq \phi(N)$.

4. If $acm \notin N$ and $bcm \notin N$, then $a(N :_R M)^2m \subseteq \phi(N)$, $b(N :_R M)^2m \subseteq \phi(N)$ and $c(N :_R M)^2m \subseteq \phi(N)$.

5. If $acm \notin N$ and $bcm \notin N$, then $(N :_R M)^3m \subseteq \phi(N)$.

Proof. (1) Suppose that $abcN \not\subseteq \phi(N)$. Then there exists $n \in N$ with $abcn \notin \phi(N)$. Hence $abc(m+n) \in N \setminus \phi(N)$, so we conclude that $ab(m+n) \in N$ or $c(m+n) \in N$. Thus $abm \in N$ or $cm \in N$, which contradicts the assumption that (a,b,c,m) is $\phi-$classical 1-quadruple-zero. Thus $abcN \subseteq \phi(N)$.

(2) Assume that $abxm \notin \phi(N)$ for some $x \in (N :_R M)$. Then $ab(c+x)m \notin \phi(N)$. Since $xm \in N$, $ab(c+x)m \in N$. If $c+x$ is a unit, then $abm \in N$ which is a contradiction. Thus $c+x$ is nonunit and since N is a $\phi-$classical 1-absorbing prime submodule we have either $abm \in N$ or $(c+x)m \in N$. Thus $abm \in N$ or $cm \in N$, again a contradiction. Thus, $ab(N :_R M)m \subseteq \phi(N)$.

(3) Assume that $acm \notin N$ and $bcm \notin N$. Suppose that $ac(N :_R M)m \not\subseteq \phi(N)$. Then there exists an element $x \in (N :_R M)$ such that $acxm \notin \phi(N)$. This implies that $a(b+x)cm \in N \setminus \phi(N)$. If $(b+x)$ is a unit, then $acm \in N$ which is a contradiction. Thus $b+x$ is nonunit. As $a(b+x)cm \in N \setminus \phi(N)$, we get either $a(b+x)m \in N$ or $cm \in N$ which implies that $abm \in N$ or $cm \in N$, again a contradiction. Thus, $ac(N :_R M)m \subseteq \phi(N)$. Likewise, we have $bc(N :_R M)m \subseteq \phi(N)$.

(4) Let $acm \notin N$ and $bcm \notin N$. We want to show $a(N :_R M)^2m \subseteq \phi(N)$ and to see this, suppose the contrary. Then there exist $x, y \in (N :_R M)$ such that $axym \notin \phi(N)$. It implies $a(b+x)(c+y)m \in N \setminus \phi(N)$. By (2) and (3), $b+x$ and $c+y$ are nonunits, as $acm \notin N$ and $abm \notin N$. Since N is a $\phi-$classical 1-absorbing prime submodule, we have either $a(b+x)m \in N$ or $(c+y)m \in N$. Which implies that $abm \in N$ or $cm \in N$. This is a contradiction. Thus, we have $a(N :_R M)^2m \subseteq \phi(N)$. Similarly we can prove that $b(N :_R M)^2m \subseteq \phi(N)$ and $c(N :_R M)^2m \subseteq \phi(N)$.

(5) Let $acm \notin N$ and $bcm \notin N$. Suppose that $(N :_R M)^3m \not\subseteq \phi(N)$. Then there exist $x, y, z \in (N :_R M)$ such that $xyzm \notin \phi(N)$. Then we have $(a+x)(b+y)(c+z)m \in N \setminus \phi(N)$. If $(a+x)$ is unit, then we obtain $(b+y)(c+z)m \in N$. Thus we get $bcm \in N$, which is a contradiction. Similarly, we can show $(b+y)$ and $(c+z)$ are nonunits. Then we conclude that $(a+x)(b+y)m \in N$ or $(c+z)m \in N$. This gives $abm \in N$ or $cm \in N$, again a contradiction. Hence, $(N :_R M)^3m \subseteq \phi(N)$. □

Proposition 3.12. *Let N be a $\phi-$classical 1-absorbing prime submodule of an R-module M that is not classical 1-absorbing prime. Then there exists a $\phi-$classical 1-quadruple-zero (a,b,c,m) of N. Suppose that $acm, bcm \notin N$. Then $(N :_R M)^3N \subseteq \phi(N)$. Additionally, if $\phi(N) \subseteq (N :_R M)^4N$, then N is $\omega-$classical 1-absorbing prime. Furthermore, if M is multiplication, then $N^4 \subseteq \phi(N)$.*

Proof. Assume that $(N :_R M)^3N \not\subseteq \phi(N)$, then there are $x_1, x_2, x_3 \in (N :_R M)$ and $n \in N$ such that $x_1x_2x_3n \notin \phi(N)$. By Theorem 3.11, $(a+x_1)(b+x_2)(c+x_3)(m+n) \in N \setminus \phi(N)$. Since N is a $\phi-$classical 1-absorbing prime submodule we have that either $(a+x_1)(b+x_2)(m+n) \in N$ or $(c+x_3)(m+n) \in N$. Therefore, $abm \in N$ or $cm \in N$, a contradiction. This shows that $(N :_R M)^3N \subseteq \phi(N)$.

Now suppose that $\phi(N) \subseteq (N :_R M)^4N$. Then we have $(N :_R M)^3N \subseteq \phi(N) \subseteq (N :_R M)^4N \subseteq (N :_R M)^3N$, that is, $\phi(N) = (N :_R M)^3N = (N :_R M)^4N$. Therefore, $\phi(N) = (N :_R M)^jN$ for all $j \geq 3$ and the result is obtained. If M is multiplication, then $N = (N :_R M)M$. Thus, $N^4 = (N :_R M)^3N \subseteq \phi(N)$. □

Let N be a proper submodule of a nonzero R-module M. Then M-radical of N, denoted by M-*rad*(N), is defined to be intersection of all prime submodules of M containing N. If M has no prime submodule containing N, then M-*rad*$(N) = M$. It is shown in [18, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then M-rad$(N) = \sqrt{(N :_R M)}M$.

Proposition 3.13. *Let N be a ϕ–classical 1-absorbing prime submodule of an R-module M which is not classical 1-absorbing prime. If there exists a ϕ–classical 1-quadruple-zero (a,b,c,m) of N satisfying $acm, bcm \notin N$, then we have the following:*

1. $\sqrt{(N :_R M)} = \sqrt{(\phi(N) :_R M)}$
2. *If M is multiplication, then $M - rad(N) = M - rad(\phi(N))$.*

Proof. (1)Assume that N is not classical 1-absorbing prime. By Proposition 3.12, $(N :_R M)^3 N \subseteq \phi(N)$. Then $(N :_R M)^4 = (N :_R M)^3 (N :_R M) \subseteq \left((N :_R M)^3 N :_R M\right) \subseteq (\phi(N) :_R M)$ and so $(N :_R M) \subseteq (\phi(N) :_R M)$. Hence we have $\sqrt{(N :_R M)} = \sqrt{(\phi(N) :_R M)}$

(2)By part (1), $M - rad(N) = \sqrt{(N :_R M)}M = \sqrt{(\phi(N) :_R M)}M = M - rad(\phi(N))$. □

Next, we will discuss the ϕ– classical 1-absorbing prime submodules of cartesian product of modules.

Theorem 3.14. Let M_1, M_2 be R-modules and N_1 be a proper submodule of M_1. Suppose that $\psi_i : S(M_i) \longrightarrow S(M_i) \cup \{\emptyset\}$ be functions (for $i = 1, 2$) and let $\phi = \psi_1 \times \psi_2$.Then the following conditions are equivalent.

1. $N = N_1 \times M_2$ is a ϕ– classical 1-absorbing prime submodule of $M = M_1 \times M_2$.
2. N_1 is a ψ_1– classical 1-absorbing prime submodule of M_1 and for each nonunits $r,s,t \in R$ and $m_1 \in M_1$ if $rstm_1 \in \psi_1(N_1)$, $rsm_1 \notin N_1$, $tm_1 \notin N_1$, then $rst \in (\psi_2(M_2) :_R M_2)$.

Proof. (1) ⇒ (2) Suppose that $N = N_1 \times M_2$ is a ϕ– classical 1-absorbing prime submodule of $M = M_1 \times M_2$. Let r,s,t be nonunit elements of R and $m_1 \in M_1$ be such that $rstm_1 \in N_1 \setminus \psi_1(N_1)$. Then $rst(m_1, 0) \in N \setminus \phi(N)$. Since N is a ϕ–classical 1-absorbing prime submodule, we have either $rs(m_1, 0) \in N$ or $t(m_1, 0) \in N$, and so $rsm_1 \in N_1$ or $tm_1 \in N_1$. Consequently, N_1 is a ϕ– classical 1-absorbing prime submodule of M_1. Now, assume that $rstm_1 \in \psi_1(N_1)$ for some nonunits $r,s,t \in R$ and $m_1 \in M_1$ such that $rsm_1 \notin N_1$ and $tm_1 \notin N_1$. Suppose that $rst \notin (\psi_2(M_2) :_R M_2)$. Therefore there exists $m_2 \in M_2$ such that $rstm_2 \notin \psi_2(M_2)$. Hence $rst(m_1, m_2) \in N \setminus \phi(N)$, and so $rs(m_1, m_2) \in N$ or $t(m_1, m_2) \in N$. Thus $rsm_1 \in N_1$ or $tm_1 \in N_1$ which is a contradiction. Consequently, $rst \in (\psi_2(M_2) :_R M_2)$.

(2) ⇒ (1) Let r,s,t be nonunit elements of R and $(m_1, m_2) \in M = M_1 \times M_2$ be such that $rst(m_1, m_2) \in N \setminus \phi(N)$. First assume that $rstm_1 \notin \psi_1(N_1)$. Since N_1 is ψ_1– classical 1-absorbing prime and $rstm_1 \in N_1 \setminus \psi_1(N_1)$, we have either $rsm_1 \in N_1$ or $tm_1 \in N_1$. So, $rs(m_1, m_2) \in N$ or $t(m_1, m_2) \in N$ and thus we are done. If $rstm_1 \in \psi_1(N_1)$, then $rstm_2 \notin \psi_2(M_2)$. Therefore, $rst \notin (\psi_2(M_2) :_R M_2)$ and so part (2) implies that either $rsm_1 \in N_1$ or $tm_1 \in N_1$. Again we have that $rs(m_1, m_2) \in N$ or $t(m_1, m_2) \in N$ which shows N is a ϕ– classical 1-absorbing prime submodule of M. □

Theorem 3.15. Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module where M_1 is an R_1-module and M_2 is an R_2-module. Let $\psi_i : S(M_i) \longrightarrow S(M_i) \cup \{\emptyset\}$ be functions for $i = 1, 2$ and let $\phi = \psi_1 \times \psi_2$. If $N = N_1 \times M_2$ is a proper submodule of M, then the following conditions are equivalent.

1. N_1 is a classical 1-absorbing prime submodule of M_1.
2. N is a classical 1-absorbing prime submodule of M.
3. N is ϕ– classical 1-absorbing prime submodule of M where $\psi_2(M_2) \neq M_{2.}$.

Proof. (1) $\Rightarrow$ (2) Follows from [32, Theorem 10].

(2) $\Rightarrow$ (3) It is clear that every classical 1-absorbing prime submodule is a $\phi-$classical 1-absorbing prime submodule.

(3) $\Rightarrow$ (1) Let $abcm_1 \in N_1$ for some nonunits $a, b, c \in R_1$ and $m_1 \in M_1$. By assumption, there exists $m_2 \in M_2 \setminus \psi_2(M_2)$. Thus $(a,1)(b,1)(c,1)(m_1,m_2) \in N \setminus \phi(N)$. So, we have $(a,1)(b,1)(m_1,m_2) \in N$ or $(c,1)(m_1,m_2) \in N$. Hence, $abm_1 \in N_1$ or $cm_1 \in N_1$. Therefore, N_1 is a classical 1-absorbing prime submodule of M_1. □

Proposition 3.16. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module where M_1 is an R_1-module and M_2 is an R_2-module. Let $\psi_i : S(M_i) \longrightarrow S(M_i) \cup \{\emptyset\}$ be functions for $i = 1,2$ where $\psi_2(M_2) = M_2$ and let $\phi = \psi_1 \times \psi_2$. If $N = N_1 \times M_2$ is a $\phi-$classical 1-absorbing prime submodule of M, then N_1 is a ψ_1-classical 1-absorbing prime submodule of M_1.*

Proof. Suppose that N is a $\phi-$classical 1-absorbing prime submodule of M. First we show that N_1 is a ψ_1-classical 1-absorbing prime submodule of M_1 independently whether $\psi_2(M_2) = M_2$ or $\psi_2(M_2) \neq M_2$. Let $a_1b_1c_1m_1 \in N_1 \setminus \psi_1(N_1)$ for some nonunits $a_1, b_1, c_1 \in R_1$ and $m_1 \in M_1$. Then $(a,1)(b,1)(c,1)(m_1,m_2) \in (N_1 \times M_2) \setminus (\psi_1(N_1) \times \psi_2(M_2)) = N \setminus \phi(N)$ for any $m_2 \in M_2$. Since N is a $\phi-$classical 1-absorbing prime submodule of M, we have either $(a,1)(b,1)(m_1,m_2) \in (N_1 \times M_2)$ or $(c,1)(m_1,m_2) \in (N_1 \times M_2)$. So, clearly we conclude $abm_1 \in N_1$ or $cm_1 \in N_1$. Therefore, N_1 is a ψ_1-classical 1-absorbing prime submodule of M_1. □

Proposition 3.17. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module where M_1 is an R_1-module and M_2 is an R_2-module. Suppose N_1, N_2 are proper submodules of M_1, M_2 respectively. $\psi_i : S(M_i) \longrightarrow S(M_i) \cup \{\emptyset\}$ be functions for $i = 1,2$ and let $\phi = \psi_1 \times \psi_2$. If $N = N_1 \times N_2$ is a $\phi-$classical 1-absorbing prime submodule of M, then N_1 is a ψ_1-classical 1-absorbing prime submodule of M_1 and N_2 is a ψ_2-classical 1-absorbing prime submodule of M_2.*

Proof. Suppose that $N = N_1 \times N_2$ is a $\phi-$ classical 1-absorbing prime submodule of M. Let $abcm \in N_1 \setminus \psi_1(N_1)$ for some nonunits $a, b, c \in R_1$ and $m \in M_1$. Choose an element $n \in N_2$. We have $(a,1)(b,1)(c,1)(m,n) \in N \setminus \phi(N)$. Then $(a,1)(b,1)(m,n) \in N$ or $(c,1)(m,n) \in N$. Thus $abm \in N_1$ or $cm \in N_1$ and thus N_1 is a ψ_1-classical 1-absorbing prime submodule of M_1. By a similar argument we can show that N_2 is a ψ_2-classical 1-absorbing prime submodule of M_2. □

In general, if N_1 and N_2 are ψ_i-classical 1-absorbing prime submodules $(i = 1,2)$ of M_1, M_2 respectively, then $N_1 \times N_2$ need not to be $\psi_1 \times \psi_2$-classical 1-absorbing prime. Consider $\mathbb{Z}$-module $\mathbb{Z}_{12}$ and $N_1 = 4\mathbb{Z}$, $N_2 = \{0\}$. It is easy to see that N_1 and N_2 are ϕ_0-classical 1-absorbing prime submodule of $\mathbb{Z}_{12}$ (where $\phi_0(N) = 0$ for each $N \in S(\mathbb{Z}_{12})$. However, $N_1 \times N_2$ is not a $\phi_0 \times \phi_0$-classical 1-absorbing prime submodule of $\mathbb{Z} \times \mathbb{Z}$-module $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ as $(2,2)(1,3)(2,1)(1,2) \in 4\mathbb{Z} \times \{0\} \setminus \phi_0 \times \phi_0(4\mathbb{Z} \times \{0\})$ but neither $(2,2)(1,3)(1,2) \in 4\mathbb{Z} \times \{0\}$ nor $(2,1)(1,2) \in 4\mathbb{Z} \times \{0\}$.

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