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semi-J-ideals of commutative rings

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Abstract. In this paper, we introduce and study a new concept called semi-J-ideal which is a generalization of the notions of J-ideal and semi-n-ideal. A proper ideal I of a ring R is said to be a semi-J-ideal if whenever $a \in R$ with $a^2 \in I$ and $a \notin J(R)$, then $a \in I$. Several properties and characterizations of this class of ideals are established. We investigate the semi-J-ideal property under various contexts of ring theoretic constructions such as direct product, localization, homomorphic image, idealization and amalgamation rings. We also extend the notion of semi-J-ideal of rings to semi-J-submodules of modules and study some properties.

Key Words: J-ideal, semi-J-ideal, semi-J-submodule, trivial extension, amalgamated algebra.
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1 Introduction

Throughout this whole paper, all rings are assumed to be commutative with nonzero identity. Recall that a proper ideal I of a ring R is called semiprime if whenever $a \in R$ is such that $a^2 \in I$, then $a \in I$. It is well-known that an ideal I of a ring R is semiprime if and only if I is a radical ideal, that is $I = \sqrt{I} = \{x \in R : x^m \in I \text{ for some } m \in \mathbb{Z}\}$. In [7], Badawi defined an ideal I of a ring R to be weakly semiprime if whenever $a \in R$ and $0 \neq a^2 \in I$, then $a \in I$. In [22], Tekir , Koç and Oral introduced the concept of n-ideals of commutative rings as follows, a proper ideal I of a ring R is called an n-ideal if whenever $a, b \in R$ are such that $ab \in I$ and $a \notin \sqrt{0}$, then $b \in I$. Recently, in [16] Khashan and Bani-Ata generalized n-ideals by defining and studying the class of J-ideals. A proper ideal I of R is called a J-ideal if $ab \in I$ and $a \notin J(R)$ imply $b \in I$ for $a, b \in R$, where $J(R)$ denotes the Jacobson radical of R. As another generalization of n-ideals, in [20], Yetkin Çelikel and Khashan introduced the concept of semi-n-ideals of commutative rings. A proper ideal I of a ring R is called a semi-n-ideal if whenever $a \in R$ are such that $a^2 \in I$ and $a \notin \sqrt{0}$, then $a \in I$.

In this paper, we define a proper ideal I of a ring R to be a semi-J-ideal if whenever $a \in R$ with $a^2 \in I$ and $a \notin J(R)$, then $a \in I$. The class of semi-J-ideals is a generalization of semiprime ideals, semi-n-ideals and J-ideals. We start Section 2 by giving some examples showing that the semi-J-ideal is distinguishable of the other classes of ideals. Next, we establish several characterizations of semi-J-ideals (see for instance Theorem 2.6). Next we also study the stability of the notion of semi-J-ideals under various contexts, mainly the direct product, homomorphic image and localization. We also extend the notion of semi-J-ideal to submodule by introducing and studying the concept of semi-J-submodule. we establish some characterizations about this concept. We end this paper by examining the possible transfer of the concept of semi-J-ideal to to some commutative ring extensions such as idealization and amalgamation rings. For a ring R, we denote by $N(R)$, $Z(R)$, the ideal of all nilpotent elements of R, the zero divisor elements of R. For an integral domain R, we denote by $qf(R)$, the quotient field of R.

2 Main Results

This section deals with many properties of semi-J-ideals. We show the relationships between the notion of semi-J-ideal and the other related classes of ideals such as J-ideal, semi-n-ideal and semi-r-ideal and we establish several characterizations and examples.

Definition 2.1. A proper ideal I of a ring R is called a semi-J-ideal if whenever $a \in R$ with $a^2 \in I$ and $a \notin J(R)$, then $a \in I$.

In the following, we give some examples of semi-J-ideals.

Example 2.2. Let R be a ring and I be a proper ideal of R.

1. In a local ring all ideals are semi-J-ideals.
2. Every prime ideal is a semi-J-ideal.

Note that a J-ideal of R is a semi-J-ideal. However, the next example shows that the converse implication is not true in general.

Example 2.3. Consider the ring $R = \mathbb{Z}_{12}$ and let $I = (3)$ be an ideal of R. Clearly, I is a semi-J-ideal, since it is a prime ideal of R. Moreover, $3.2 \in I$ and $3 \notin J(R) = \{0, 6\}$ but $2 \notin I$, hence I is not a J-ideal.

The next proposition shows the relationship between the notions semi-J-ideal and semi-n-ideal.

Proposition 2.4. *Let R be ring. Then every semi-n-ideal I of R is a semi-J-ideal.*

Proof. Let $a \in R$ such that $a^2 \in I$ and $a \notin J(R)$. Since $N(R) \subseteq J(R)$, $a \notin N(R)$ and so $a \in I$ as I is a semi-n-ideal. Therefore, I is a semi-J-ideal of R. □

Recall that a ring R is called presimplifiable if $Z(R) \subseteq J(R)$. A subclass of presimplifiable rings is the class of domainlike rings where $Z(R) \subseteq N(R)$ ([4]). Recall also from [18] that a proper ideal I of a ring R is called a semi-r-ideal if whenever $a \in R$ such that $a^2 \in I$ and $Ann(a) = 0$, then $a \in I$.

Proposition 2.5. *Let R be a presimplifiable (Domainlike) ring. Then every semi-r-ideal in R is a semi-J-ideal (semi-n-ideal).*

Proof. Suppose R is presimplifiable and let I be a semi-r-ideal of R. Let $a \in R$ such that $a^2 \in I$ with $a \notin J(R)$. Then $a \notin Z(R)$ and so clearly, $Ann(a) = 0$. Since I is a semi-r-ideal, $a \in I$ and so I is a semi-J-ideal. If R is Domainlike, then similarly, we conclude that any semi-r-ideal is a semi-n-ideal. □

Theorem 2.6. Let I be a proper ideal of a ring R. Then the following statements are equivalent:

1. I is a semi-J-ideal of R.
2. Whenever $a \in R$ with $0 \neq a^2 \in I$ and $a \notin J(R)$, then $a \in I$.

3. Whenever $a \in R$ with $a^m \in I$ for some positive integer m and $a \notin J(R)$, then $a \in I$.

4. $\sqrt{I} \subseteq J(R) \cup I$.

5. Whenever K is an ideal of R with $K^m \subseteq I$ for some positive integer m, then $K \subseteq J(R)$ or $K \subseteq I$.

Proof. (1) $\Leftrightarrow$ (2). Assume that (2) holds and let $a \in R$ such that $a^2 \in I$ and $a \notin J(R)$. If $a^2 \neq 0$, then we are done. If $a^2 = 0$, then $a \in \sqrt{0} \subseteq J(R)$. The converse is straightforward.
(1) $\Rightarrow$ (3). Let $a^m \in I$ and $a \notin J(R)$. To prove this implication, we use the mathematical induction method. If $m \leq 2$, then the claim is clear as I is a semi-J-ideal. Assume that the claim of (3) holds for all $2 < k < m$. We show that it is also true for m. Suppose that m is even, say $m = 2t$ for some positive integer t. Since I is a semi-J-ideal of R, then $a^m = (a^t)^2 \in I$ and $a^t \notin J(R)$ imply $a^t \in I$. By the induction hypothesis, we conclude that $a \in I$. Now, suppose m is odd. Then $m+1 = 2s$ for some $s < m$. Similarly, since $(a^s)^2 \in I$ and $a^s \notin J(R)$, we get $a^s \in I$ and again by the induction hypothesis, we conclude $a \in I$, so we are done.
(3)$\Rightarrow$(4). Let $a \in \sqrt{I}$. Then $a^k \in I$ for some $k \geq 1$ and so by (3), we get $a \in J(R)$ or $a \in I$. Thus, $\sqrt{I} \subseteq$J(R)$\cup I$.
(4)$\Rightarrow$(5). Assume that (4) holds. Let K be an ideal of R with $K^m \subseteq I$ for some positive integer m. Then $K \subseteq \sqrt{I} \subseteq I \cup J(R)$ and so $K \subseteq J(R)$ or $K \subseteq I$.
(5)$\Rightarrow$(1). Let $a^2 \in I$ for $a \in R$. The result follows by taking $K = (a)$ and $m = 2$ in (5). □

Corollary 2.7. *Let I be a semi-J-ideal of a ring R and k be a positive integer. If L is an ideal of R with $L^k \subseteq I$ for some positive integer k and $L \cap J(R) = \{0\}$, then $L \subseteq I$.*

Corollary 2.8. *Let I and L be two proper ideals of a ring R such that $I \cap J(R) = L \cap J(R) = \{0\}$.*

1. *If I and L are semi-J-ideals of R with $I^2 = L^2$, then $I = L$.*

2. *If I^2 is a semi-J-ideal, then $I^2 = I$.*

Proposition 2.9. *Let $f : R \to S$ be a ring epimorphism. Then the following assertions hold:*

1. *If I is a semi-J-ideal of R with $\ker(f) \subseteq I$, then $f(I)$ is a semi-J-ideal of S.*

2. *If K is a semi-J-ideal of S with $\ker(f) \subseteq J(R)$, then $f^{-1}(K)$ is a semi-J-ideal of R.*

Proof.

1. Let $b \in S$ such that $b^2 \in f(I)$ and $b \notin J(S)$. As f is a ring epimorphism, we can choose $x \in R$ such that $b = f(x)$. Then $b^2 = (f(x))^2 = f(x^2) \in f(I)$. Since $\ker f \subseteq I$, we conclude that $x^2 \in I$. We claim that $x \notin J(R)$. Deny. $x \in J(R)$, then $f(x) = b \in J(S)$ by [16, Lemma 2.22], which is a contradiction. Since I is a semi-J-ideal of R, we conclude that $x \in I$ and so $b = f(x) \in f(I)$, as desired.

2. Let $y \in R$ where $y^2 \in f^{-1}(K)$ and $y \notin J(R)$. Then $f(y^2) = (f(y))^2 \in K$. We prove that $f(y) \notin J(S)$. Suppose that $f(y) \in J(S)$ and let M be a maximal ideal of R. Then $f(M)$ is a maximal ideal of S since $\ker(f) \subseteq J(R) \subseteq M$. Thus, $f(y) \in f(M)$ and so $y \in M$ as $\ker(f) \subseteq M$. Hence, $y \in J(R)$ which is a contradiction. Since K is a semi-J-ideal, $f(y) \in K$ and so $y \in f^{-1}(K)$. It follows that $f^{-1}(K)$ is a semi-J-ideal of R. □

Corollary 2.10. *Let R be a ring and let I, K be two ideals of R with $K \subseteq I$. Then the followings hold.*

1. *If I is a semi-J-ideal of R, then I/K is a semi-J-ideal of R/K.*

2. *If I/K is a semi-J-ideal of R/K and $K \subseteq J(R)$, then I is a semi-J-ideal of R.*

Let I be a proper ideal of R. In the following, the notation $Z_I(R)$ denotes the set $\{r \in R \mid rs \in I \text{ for some } s \in R \setminus I\}$.

Proposition 2.11.

Let S be a multiplicatively closed subset of a ring R such that $J(S^{-1}R) = S^{-1}J(R)$.

1. If I is a semi-J-ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a semi-J-ideal of $S^{-1}R$.
2. If $S^{-1}I$ is a semi-J-ideal of $S^{-1}R$ and $S \cap Z(R) = S \cap Z_I(R) = S \cap Z_{J(R)}(R) = \emptyset$, then I is a semi-J-ideal of R.

Proof.

1. Suppose for $\frac{a}{s} \in S^{-1}R$ that $(\frac{a}{s})^2 \in S^{-1}I$ and $\frac{a}{s} \notin S^{-1}I$. Then there exists $u \in S$ such that $ua^2 \in I$ and so $(ua)^2 \in I$. Since clearly $ua \notin I$ and I is a semi-J-ideal, we have $ua \in J(R)$. Thus, $\frac{a}{s} = \frac{ua}{us} \in S^{-1}J(R) = J(S^{-1}R)$. Therefore, $S^{-1}I$ is a semi-J-ideal of $S^{-1}R$.
2. Let $a^2 \in I$ for $a \in R \setminus I$. Then $(\frac{a}{1})^2 \in S^{-1}I$, since $S^{-1}I$ is a semi-J-ideal of $S^{-1}R$ and $(\frac{a}{1})^2 \in S^{-1}I$, we have either $\frac{a}{1} \in S^{-1}I$ or $\frac{a}{1} \in J(S^{-1}R) = S^{-1}J(R)$. If $\frac{a}{1} \in S^{-1}I$, then there exists $u \in S$ such that $ua \in I$. Since $S \cap Z_I(R) = \emptyset$, we conclude that $a \in I$. If $\frac{a}{1} \in J(S^{-1}R) = S^{-1}J(R)$, then there exists $u \in S$ with $ua \in J(R)$. Since $S \cap Z_{J(R)}(R) = \emptyset$, then $a \in J(R)$. Therefore, I is a semi-J-ideal of R. □

Note that in assertion (2) of Proposition 2, such a multiplicatively closed subset S exists. For instance, consider $S = \{\bar{1}, \bar{3}\}$ and $I = \langle \bar{2} \rangle = J(\mathbb{Z}_4)$. Then clearly $S \cap Z(R) = S \cap Z_I(R) = S \cap Z_{J(R)}(R) = \emptyset$.

Proposition 2.12. *Let R be a ring, and $\{I_i : i \in \Delta\}$ be a nonempty family of semi-J-ideals of R, then $\bigcap_{i \in \Delta} I_i$ is a semi-J-ideal of R.*

Proof. Let $a \in R$ such that $a^2 \in \bigcap_{i \in \Delta} I_i$ and $a \notin J(R)$. Then $a^2 \in I_i$ for all $i \in \Delta$. Since I_i is a semi-J-ideal for all $i \in \Delta$, we conclude that $a \in I_i$ for all i and so $a \in \bigcap_{i \in \Delta} I_i$. Therefore, $\bigcap_{i \in \Delta} I_i$ is a semi-J-ideal. □

If I and J are semi-J-ideals of a ring R, then IJ and $I + J$ need not be a semi-J-ideal as shown in the following example.

Example 2.13. Consider the ideals $I = \langle x \rangle$ and $J = \langle x - 4 \rangle$ of the ring $R = \mathbb{Z}[x]$. Then I and J are (semi) prime ideals and so are semi-J-ideals of R. On the other hand, $I + J = \langle x, x - 4 \rangle = \langle x, 4 \rangle$ is not a semi-J-ideal of R. Indeed, $(2 + x)^2 \in I + J$ and $2 + x \notin J(R) = \{0\}$, but $2 + x \notin I + J$. Also, $I^2 = \langle x^2 \rangle$ is not a semi-J-ideal of R as $x^2 \in I^2$ and $x \notin J(R) = \{0\}$, but $x \notin I^2$.

Proposition 2.14. *Let I and J be two proper ideals of R such that I is a semi-J-ideal and $J \subseteq J(R)$. Then $I + J$ is a semi-J-ideal of R.*

Proof. By (1) of Corollary 2.10, $I/I \cap J$ is a semi-J-ideal of $R/I \cap J$. Thus, $(I + J)/J \cong I/I \cap J$ is also a semi-J-ideal of R/J. Therefore, by (2) of Corollary 2.10, we conclude that $I + J$ is a semi-J-ideal of R. □

Proposition 2.15. *Let $R = R_1 \times R_2$ be a decomposable ring and $I = I_1 \times I_2$ be a proper ideal of R. If I is a semi-J-ideal of R, then I_1 is a semi-J-ideal of R_1 and I_2 is a semi-J-ideal of R_2.*

Proof. Assume that $I = I_1 \times I_2$ is a semi-J-ideal of R. Let $a \in R_1$ such that $a^2 \in I_1$ and $a \notin J(R_1)$. Then $(a, 0)^2 \in I_1 \times I_2$ and $(a, 0) \notin J(R)$ imply that $(a, 0) \in I$ and so $a \in I_1$. Therefore, I_1 is a semi-J-ideal of R_1. For the ideal I_2, the proof is similar to the previous one. □

In the following example, we show that the converse of Proposition 2.15 is not true in general.

Example 2.16. Consider the ring $R = \mathbb{Z}_8 \times \mathbb{Z}_{10}$ and let $I_1 = (4)$ be an ideal of $\mathbb{Z}_8$ and $I_2 = (5)$ an ideal of $\mathbb{Z}_{10}$. Clearly, I_1 and I_2 are semi-J-ideals. However, $I = I_1 \times I_2$ is not a semi-J-ideal, since $(2,5)^2 = (4,5) \in I$ and $(2,5) \notin J(R) = (2) \times (0)$ but $(2,5) \notin I$.

Proposition 2.17. *Let $R = R_1 \times R_2$ where R_1 and R_2 are two rings and I_1, I_2 be proper ideals of R_1 and R_2, respectively. Then $I_1 \times R_2$ (resp., $R_1 \times I_2$) is a semi-J-ideal of R if and only if I_1 is a semi-J-ideal of R_1 (resp., I_2 is a semi-J-ideal of R_2).*

Proof. Let $I_1 \times R_2$ be a semi-J-ideal of R and $a \in R_1$ with $a^2 \in I_1$ and $a \notin J(R_1)$. Then $(a,1)^2 \in I_1 \times R_2$ and $(a,1) \notin J(R)$ imply that $(a,1) \in I_1 \times R_2$ and so $a \in I_1$. Thus I_1 is a semi-J-ideal of R_1. Conversely, suppose that I_1 is a semi-J-ideal of R_1, and let $(a,b)^2 \in I_1 \times R_2$ with $(a,b) \in R$ and $(a,b) \notin J(R)$, hence $a \notin J(R_1)$ and so $a \in I_1$, since I_1 is a semi-J-ideal of R_1, thus $(a,b) \in I_1 \times R_2$. Therefore $I_1 \times R_2$ is a semi-J-ideal of R. The proof of the case $R_1 \times I_2$ is similar.

□

Recall that for an ideal I of a ring R, the Jacobson radical of I (denoted by $J(I)$) is defined as the intersection of all maximal ideals of R containing I. The following properties can be easily verified for any ideals I and K of R:

(1) $I \subseteq \sqrt{I} \subseteq J(I)$.

(2) $I \subseteq K$ implies that $J(I) \subseteq J(K)$.

(3) $J(R) \subseteq J(I)$.

(4) $J(J(I)) = J(I)$.

(5) $J(R)/I = J(R/I)$.

Definition 2.18. Let I be a proper ideal of a ring R. Then, I is called semi-J-primary if whenever $a \in R$, $a^2 \in I$ implies that $a \in J(I)$ or $a \in I$.

In the following proposition, we show that semi-J-ideals and semi-J-primary ideals that contained in the Jacobson radical are the same.

Proposition 2.19. *Let I be an ideal of a ring R such that $I \subseteq J(R)$. Then I is a semi-J-ideal if and only if I is semi-J-primary.*

Proof. Let I be a proper ideal of R and $a \in R$. Assume that I is a semi-J-ideal of R and let $a \in I$ with $a \notin J(I)$. Since $J(R) \subseteq J(I)$, we conclude that $a \notin J(R)$. Now, I is a semi-J-ideal implies that $a \in I$ and so, I is a semi-J-primary ideal of R. Conversely, let I be a semi-J-primary ideal of R and let $a^2 \in I$ with $a \notin J(R)$. Now, $I \subseteq J(R)$ implies that $J(I) \subseteq J(J(R)) = J(R)$. Thus, $a \notin J(I)$ and so $a \in I$ as I is semi-J-primary. Therefore, I is a semi-J-ideal. □

The following example shows the failure of Proposition 2.19 beyond the context $I \subseteq J(R)$.

Example 2.20. Let $R = \mathbb{Z}$ be the ring of integers, $I = 9\mathbb{Z}$ be an ideal of R, then $I \not\subset J(R) = (0)$. We claim that I is a semi-J-primary ideal of R. Indeed, let $a^2 \in I$ with $a \in Z$, then $9|a^2$ and so $3|a$, therefore $a \in 3\mathbb{Z} = J(I)$. However, I is not a semi-J-ideal, indeed, for $a = 3 \in R$, we have $a^2 = 3^2 = 9 \in I$ and $a \notin J(R) = (0)$, but $a \notin I$.

3 semi-J-submodules over commutative rings

In this section, we generalize the concept of semi-J-ideal to submodule of R-module. We first clarify some analogous results to those of J-ideals with a focus on multiplication modules.

For an R-module M and a submodule N of M, the residual ideal in R of N by M is defined as $(N : M) = \{r \in R : rm \in N \text{ for all } m \in M\}$. Moreover, for an ideal I of R and a submodule N of M, the set $(N :_M I) = \{m \in M : am \in N \text{ for all } a \in I\}$ is a submodule of M.

Definition 3.1. Let M be an R-module. Then a proper submodule N of M is called a semi-J-submodule if for all $a \in R$ and $m \in M$, whenever $a^2 m \in N$ with $a \notin (J(R)M : M)$, then $am \in N$.

Clearly, if $(J(R)M : M) = R$, then from the above definition, it follows that any proper submodule of M is a semi-J-submodule. However, by Nakayama's Lemma, the only finitely generated R-module M such that $(J(R)M : M) = R$ is the trivial module $\{0\}$.

Proposition 3.2. *Let M be an R-module, N be a submodule of M and I be an ideal of R.*

1. *If N is a semi-J-submodule of M and $(J(R)M : M) = J(R)$, then $(N : M)$ is a semi-J-ideal of R.*
2. *If N is a semi-J-submodule of M, then $(N :_M I)$ is also a semi-J-submodule of M.*

Proof.
1. Let $a^2 \in (N : M)$ where $a \in R$ and $a \notin J(R)$. Then we have $a^2 M \subseteq N$ and so $a^2 m \in N$ for all $m \in M$. Since N is a semi-J-submodule of M and $a \notin (J(R)M : M)$, $am \in N$ for all $m \in M$. Thus, $aM \subseteq N$ and so $a \in (N : M)$. Therefore, $(N : M)$ is a semi-J-ideal of R.
2. Let $r^2 m \in (N :_M I)$ where $r \in R$ and $m \in M$ with $r \notin (J(R)M : M)$. Then we have $r^2 mi \in N$ for all $i \in I$. Since N is a semi-J-submodule of M and $r \notin (J(R)M : M)$, $rmi \in N$ for all $i \in I$. Thus, $rmI \subseteq N$ and so $rm \in (N :_M I)$. Therefore, $(N :_M I)$ is a semi-J-submodule of M. ☐

In the following proposition, we give a characterization of semi-J-submodules.

Proposition 3.3. *Let M be an R-module and N be a proper submodule of M. The following statements are equivalent:*

1. *N is a semi-J-submodule of M.*
2. *For any ideal I of R and any submodule K of M, if $I^2 K \subseteq N$ with $I \nsubseteq (J(R)M : M) = J_M$, then $IK \subseteq N$.*

Proof. Assume that N is a semi-J-submodule. Let I be an ideal of R and K be a submodule of M such that $I^2 K \subseteq N$ and $I \nsubseteq J_M$. Let $i \in I$, then $i^2 \in I^2$, which means $i^2 k \in I^2 K$ for all $k \in K$. Since $I^2 K \subseteq N$, we have $i^2 k \in N$. We also know that since $I \nsubseteq J_M$, there exists an element $a \in I$ such that $a \notin J_M$. We consider two cases for the element i. If $i \notin J_M$, using the fact that $i^2 k \in N$ and the definition of a semi-J-submodule, it follows that $ik \in N$. Now we may assume that $i \in J_M$. Consider the element $x = a + i$. Since a and i are in I, x is also in I. The fact that $a \notin J_M$ and $i \in J_M$, it follows that $x \notin J_M$. Deny. $a = x - i \in J_M$, which is a contradiction. Now, since $x \in I$ and $k \in K$, we have $x^2 k \in I^2 K \subseteq N$. As N is a semi-J-submodule and $x \notin J_M$, it follows that $xk \in N$. Consequently, $(a + i)k = ak + ik \in N$. We also know $a^2 k \in I^2 K \subseteq N$. Since $a \notin J_M$, the semi-J-submodule property implies $ak \in N$. From $ak + ik \in N$ and $ak \in N$, it follows that $ik \in N$. In both cases, $ik \in N$. Therefore, $IK \subseteq N$. Conversely, assume that for any ideal I and submodule K, $I^2 K \subseteq N$ with $I \nsubseteq J_M$ implies $IK \subseteq N$. Let $a \in R$ and $m \in M$ such that $a^2 m \in N$ and $a \notin J_M$. Let I be the principal ideal $< a >$ and K be the cyclic submodule $< m >$. Since $a \notin J_M$, the ideal $I =< a >$ is not a subset of J_M. The submodule $I^2 K$ is generated by $a^2 m$. Since $a^2 m \in N$, it follows that $I^2 K \subseteq N$. Now, since $I^2 K \subseteq N$ and $I \nsubseteq J_M$, then $IK \subseteq N$. Now since, IK is generated by am, so $am \in N$. This shows that N is a semi-J-submodule. ☐

Recall from [11] that an R-module M is called multiplication, if each submodule of M is of the form IM for some ideal I of R. Equivalently, M is multiplication if and only if $N = (N : M)M$ for every submodule N of M. Multiplication modules have played a central role in theory of modules and it has been widely studied in several authors. See, for example [21] and [5]. Moreover, M is called faithful if $(0 : M) = \{0\}$, the zero ideal of R. It is well known that if M is a finitely generated faithful multiplication R-module, then $(IM : M) = I$ for any ideal I of R.

Proposition 3.4. *Let N be a submodule of a multiplication R-module M such that $(N : M)$ is a semi-J-ideal of R. Then N is a semi-J-submodule of M.*

Proof. Let $r \in R$ and $m \in M$ such that $r^2m \in N$ and $r \notin (J(R)M : M)$. Then $r(\langle rm\rangle : M) \subseteq (\langle r^2m\rangle : M) \subseteq (N : M)$. Since $J(R) \subseteq (J(R)M : M)$, $r \notin J(R)$ and so $(\langle rm\rangle : M) \subseteq (N : M)$ as $(N : M)$ is a semi-J-ideal of R. Now, M is a multiplication module implies that $\langle rm\rangle = (\langle rm\rangle : M)M \subseteq (N : M)M = N$. Thus, $rm \in N$ and so N is a semi-J-submodule of M. □

Corollary 3.5. *Let M be a finitely generated faithful multiplication R-module and N be a proper submodule of M. The following are equivalent.*

1. *N is a semi-J-submodule of M.*
2. *$(N : M)$ is a semi-J-ideal of R.*
3. *$N = LM$ where L is a semi-J-ideal of R.*

Theorem 3.6. Let $f : M \longrightarrow N$ be an R-module epimorphism. Then:

1. If P is a semi-J-submodule of M with $\ker(f) \subseteq P$, then $f(P)$ is a semi-J-submodule of N.
2. If Q is a semi-J-submodule of N with $\ker(f) \subseteq J(R)M$, then $f^{-1}(Q)$ is a semi-J-submodule of M.

Proof.

1. Let $r \in R$ and $n \in N$ such that $r^2n \in f(P)$ with $r \notin (J(R)N : N)$. Choose $m \in M$ such that $f(m) = n$. Then $r^2n = r^2f(m) = f(r^2m) \in f(P)$. Thus, $f(r^2m-a) = 0$ for some $a \in P$ and so $r^2m-a \in \ker(f) \subseteq P$. It follows that $r^2m \in P$. Moreover, we have $r \notin (J(R)M : M)$. Indeed, if $rM \subseteq J(R)M$, then $rN = rf(M) = f(rM) \subseteq f(J(R)M) = J(R)f(M) = J(R)N$ which is a contradiction. Since P is a semi-J-submodule, $rm \in P$ and so $rn = rf(m) \in f(P)$ as desired.
2. Let $r \in R$ and $m \in M$ such that $r^2m \in f^{-1}(Q)$ and $r \notin (J(R)M : M)$. Then $r^2f(m) = f(r^2m) \in Q$. We prove that $r \notin (J(R)N : N)$. Assume by the way of a contradiction that $rN \subseteq J(R)N$. Then $rf(M) \subseteq J(R)f(M)$ and so $f(RM) \subseteq f(J(R)M)$. Now, if $x \in rM$, then $f(x) \in f(RM) \subseteq f(J(R)M)$ and hence $x-t \in \ker(f) \subseteq J(R)M$ for some $t \in J(R)M$. It follows that $x \in J(R)M$ and $rM \subseteq J(R)M$, which is a contradiction. Since Q is a semi-J-submodule of N, $rf(m) \in Q$, then $rm \in f^{-1}(Q)$. □

4 Ring extension of semi-J-ideals

Let R be a ring and M an R-module. The idealization $R \propto M = \{(r,m) : r \in R, m \in M\}$ is a ring with addition and multiplication defined respectively by: $(r,m)+(s,m') = (r+s, m+m')$ and $(r,m)(s,m') = (rs, rm'+sm)$ for each $r,s \in R$ and $m,m' \in M$. Additionally, $I \propto N$ is an ideal of $R \propto M$ where I is an ideal of R and N is a submodule of M if and only if $IM \subseteq N$ (see [14] and [15]). In this case, $I \propto N$ is called a homogeneous ideal of $R \propto M$.

Theorem 4.1. Let I be an ideal of a ring R and N be a submodule of an R-module M, then.

1. If $I \propto N$ is a semi-J-ideal of $R \propto M$, then I is a semi-J-ideal of R.

2. $I \propto M$ is a semi-J-ideal of $R \propto M$ if and only if I is a semi-J-ideal of R.

Proof. 1. Let $a \in R$ with $a^2 \in I$ and $a \notin J(R)$. Then $(a,0)^2 \in I \propto N$ and $(a,0) \notin J(R \propto M) = J(R) \propto M$. Since $I \propto N$ is a semi-J-ideal, then $(a,0) \in I \propto N$ and $a \in I$. Therefore, I is a semi-J-ideal of R.

2. Assume that $I \propto M$ is a semi-J-ideal of $R \propto M$. Then by assertion (1) above, I is a semi-J-ideal of R. Conversely, suppose that $(a,m)^2 \in I \propto M$ and $(a,m) \notin J(R \propto M)$ for $(a,m) \in R \propto M$. Then $a^2 \in I$ and $a \notin J(R)$, then $a \in I$ as I is a semi-J-ideal so $(a,m) \in I \propto M$. Therefore, $I \propto M$ is a semi-J-ideal of $R \propto M$. ☐

It is worthwhile noting that the converse of assertion (1) of Theorem 4.1 is not true in general, as shown in the following example.

Example 4.2. Consider the ring $R = \mathbb{Z}_6$, let $I = (3)$ be an ideal of R, $M = R$ and $N = (2)$ be a submodule of an R-module M. Clearly, I is a semi-J-ideal. Moreover, $(3,1)^2 = (3,0) \in I \propto N$ and $(3,1) \notin J(R \propto M) = \{0\} \propto M$ but $(3,1) \notin I \propto N$. Thus, $I \propto N$ is not a semi-J-ideal.

Let R and S be two rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. The set $R \bowtie^f J = \{(r, f(r)+j) \mid r \in R, j \in J\}$ is a subring of $R \times S$ (with identity element $(1_R, 1_S)$) called the amalgamation of R and S along J with respect to f. In particular, if $Id_R : R \to R$ is the identity homomorphism on R, then $R \bowtie J = R \bowtie^{Id_R} J = \{(r, r+j) \mid r \in R, j \in J\}$ is the amalgamated duplication of a ring along an ideal J. This construction has been first defined and studied by D'Anna and Fontana, [9]. Many properties of this ring have been investigated and analyzed over the last two decades, see for example [8, 12]. Let I be an ideal of R and K be an ideal of $f(R)+J$. Then $I \bowtie^f J = \{(i, f(i)+j) \mid i \in I, j \in J\}$ and $\overline{K}^f = \{(a, f(a)+j) \mid a \in R, j \in J, f(a)+j \in K\}$ are ideals of $R \bowtie^f J$, see for instance [12].

Lemma 4.3. *[13, Proposition 2.6] Let (R,S) be a pair of rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. Then the set of maximal ideals of $R \bowtie^f J$ is $Max(R \bowtie^f J) = \{M \bowtie^f J \mid M \in Max(R)\} \cup \{\overline{Q}^f \mid Q \in Max(S) \setminus V(J)\}$, where $V(J)$ denotes the set of all prime ideals containing J.*

Theorem 4.4. Let (R,S) be a pair of rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. Let I be an ideal of R. Consider the following two assertions:

1. $I \bowtie^f J$ is a semi-J-ideal of $R \bowtie^f J$.
2. I is a semi-J-ideal of R.

Then (1) implies (2) and the converse holds if $J \subseteq J(S)$.

Proof. Assume that $I \bowtie^f J$ is a semi-J-ideal of $R \bowtie^f J$. Let $a \in R$ such that $a^2 \in I$ and $a \notin J(R)$. Then $(a, f(a))^2 \in I \bowtie^f J$ and $(a, f(a)) \notin J(R \bowtie^f J)$. Deny. $a \in M$ for each $M \in \text{Max}(R)$ by Lemma 4.3, which is a contradiction. It follows that $(a, f(a)) \in I \bowtie^f J$ and so $a \in I$. Thus I is a semi-J-ideal of R. Conversely, let $(a, f(a)+j) \in R \bowtie^f J$ such that $(a, f(a)+j)^2 = (a^2, f(a^2) + 2jf(a) + j^2) \in I \bowtie^f J$ and $(a, f(a)+j) \notin J(R \bowtie^f J) = J(R) \bowtie^f J$ as $J \subseteq J(S)$, then $a^2 \in I$ and $a \notin J(R)$, and so $a \in I$. Therefore, $(a, f(a)+j) \in I \bowtie^f J$. Hence, $I \bowtie^f J$ is a semi-J-ideal of $R \bowtie^f J$. ☐

Theorem 4.4 generates a new original class of semi-J-ideals which are not J-ideals.

Example 4.5. Let $R = \mathbb{Z}_{12}$, $S = \mathbb{Z}_6$ and $f : \mathbb{Z}_{12} \to \mathbb{Z}_6$ be the ring homomorphism defined by $f(0) = f(6) = 0$, $f(1) = f(7) = 1$, $f(2) = f(8) = 2$, $f(3) = f(9) = 3$, $f(4) = f(10) = 4$ and $f(5) = f(11) = 5$. Consider the semi-J-ideal $I := 3\mathbb{Z}_{12}$ of R and $J := m_1 \cap m_2 = J(S)$ is the Jacobson radical of S, where $m_1 := 2\mathbb{Z}_6$ and $m_2 = 3\mathbb{Z}_6$ are the maximal ideals of $\mathbb{Z}_6$. Then:

1. $I \bowtie^f J$ is a semi-J-ideal of $R \bowtie^f J$.

2. $I \bowtie^f J$ is not a J-ideal of $R \bowtie^f J$.

Proof. 1. By Theorem 4.4, $I \bowtie^f J$ is a semi-J-ideal of $R \bowtie^f J$ since I is a semi-J-ideal of $R \bowtie^f J$, as it is a prime ideal of R.

2. From [16, Corollary 2.24(1)], $I \bowtie^f J$ is not a J-ideal of $R \bowtie^f J$, as $\frac{I \bowtie^f J}{\{0\} \times J} \simeq I$ is not a J-ideal (see Example 2.3). ☐

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Example 4.6. Let (R, m) be a valuation domain, $k = qf(R)$ be the quotient field of R, $S = k[[x]]$ be the ring of formal power series in one indeterminate x with coefficients in k. Consider the injective ring homomorphism $f : R \hookrightarrow S$ defined by $f(a) = \frac{a}{1}$ and $J = (x)$ is the maximal ideal of S. Then by Theorem 4.4, $m \bowtie^f J$ is a semi-J-ideal of $R \bowtie^f J$, since m is a semi-J-ideal of R and $J = J(S) = (x)$. ☐

As a particular case of Theorem 4.4, we have the following immediate corollary.

Corollary 4.7. *Let R be a ring and I, J be two proper ideals of R. If $I \bowtie J$ is a semi-J-ideal of $R \bowtie J$, then I is a semi-J-ideal of R. The converse holds if $J \subseteq J(R)$.*

Proposition 4.8. *Consider the amalgamation ring $R \bowtie^f J$ with respect to the ring homomorphism $f : R \to S$ and $J \subseteq J(S)$ is an ideal of S. The semi-J-ideals of $R \bowtie^f J$ containing $\{0\} \times J$ are of the form $I \bowtie^f J$ where I is a semi-J-ideal of R.*

Proof. First, note that $I \bowtie^f J$ is a semi-J-ideal of $R \bowtie^f J$ for any semi-J-ideal I of R by Theorem 4.4. Let K be a semi-J-ideal of $R \bowtie^f J$ containing $\{0\} \times J$. Consider the surjective ring homomorphism $\varphi : R \bowtie^f J \to R$ defined by $\varphi((a, f(a) + j)) = a$ for all $(a, f(a) + j) \in R \bowtie^f J$. Since $\mathrm{Ker}(\varphi) = \{0\} \times J \subseteq K$, then $I := \varphi(K)$ is a semi-J-ideal of R by Proposition 2.9. Since $\{0\} \times J \subseteq K$, we conclude that K is of the form $I \bowtie^f J$. ☐

Theorem 4.9. Let (R, S) be a pair of rings, J be a maximal ideal of S, $f : R \to S$ be a ring epimorphism and K be an ideal of S. Consider the following two assertions:

1. If $\overline{K}^f$ is a semi-J-ideal of $R \bowtie^f J$.

2. K is a semi-J-ideal of S.

Then (1) implies (2) and the converse holds if $f(J(R)) = J(S) + J$ and $Ker(f) \subseteq J(R)$.

Proof. Assume that $\overline{K}^f$ is a semi-J-ideal of $R \bowtie^f J$. Let $x \in S$, say, $x = f(a)$ for $a \in R$ such that $x^2 \in K$ and $x \notin J(S)$. Then $(a, f(a)) \in R \bowtie^f J$ such that $(a, f(a))^2 = (a^2, f(a^2)) \in \overline{K}^f$. If $(a, f(a)) \in J(R \bowtie^f J)$, then $(a, f(a)) \in \overline{Q}^f$ for all $Q \in \mathrm{Max}(S) \setminus V(J)$. Moreover, since J is maximal in S, then $f^{-1}(J)$ is maximal in R and so $(a, f(a)) \in f^{-1}(J) \bowtie^f J$. Thus, $f(a) \in J = V(J)$ and so $f(a) \in Q$ for all $Q \in \mathrm{Max}(S)$, a contradiction. Therefore, $(a, f(a)) \notin J(R \bowtie^f J)$ and so $(a, f(a)) \in \overline{K}^f$ as $\overline{K}^f$ is a semi-J-ideal of $R \bowtie^f J$. Hence, $x = f(a) \in K$ and we are done. Conversely, assume that $f(J(R)) = J(S) + J$, $\mathrm{Ker}(f) \subseteq J(R)$ and K is a semi-J-ideal of S. Let $(a, f(a) + j) \in R \bowtie^f J$ such that $(a, f(a) + j)^2 = (a^2, (f(a) + j)^2) \in \overline{K}^f$ and $(a, f(a) + j) \notin J(R \bowtie^f J)$. We claim that $f(a) + j \notin J(S)$. Deny. then $f(a) \in J(S) + J = f(J(R))$ and so $a \in J(R)$ as $\mathrm{Ker}(f) \subseteq J(R)$. Thus, $(a, f(a) + j) \in \{M \bowtie^f J : M \in \mathrm{Max}(R)\}$. Moreover, $f(a) + j \in Q$ for all $Q \in \mathrm{Max}(S)$ implies that $(a, f(a) + j) \in \bigcap \{\overline{Q}^f : Q \in \mathrm{Max}(S) \setminus V(J)\}$. It follows by Lemma 4.3 that $(a, f(a) + j) \in J(R \bowtie^f J)$, which is a contradiction. Since $(f(a) + j)^2 \in K$ and K is a semi-J-ideal of S, then

$f(a)+j \in K$. Hence, $(a, f(a)+j) \in \overline{K}^f$ and the result follows. ☐

Corollary 4.10. *Let R be a ring, K be a proper ideal of R and J is a maximal ideal of R. If $\overline{K} = \{(a, a+j) : a \in R, j \in J, a+j \in K\}$ is a semi-J-ideal of $R \bowtie J$, then K is a semi-J-ideal of R. Moreover, the converse is true if $J \subseteq J(R)$.*

It is worthwhile noting that a semiprime ideal is a semi-J-ideal, however a semi-J-ideal need not be a semiprime ideal. The next example illustrates Corollary 4.10 by generating a new original example of a semi-J-ideal which is not semiprime.

Example 4.11. Consider the ring $R = \mathbb{Z}_8$. Let $I = (4)$ be an ideal of R, $J = (2)$ is the maximal ideal of R. Then:

1. $\bar{I}$ is a semi-J-ideal of $R \bowtie J$.
2. $\bar{I}$ is not a semiprime ideal of $R \bowtie J$.

Proof. 1. By Corollary 4.10, $\bar{I}$ is a semi-J-ideal of $R \bowtie J$, since I is a semi-J-ideal of R (as R is a local ring).

2. We claim that $\bar{I}$ is not a semiprime ideal of $R \bowtie J$. Indeed, consider the element $(2,2) \in R \bowtie J$ such that $(2,2)^2 = (4,4) \in \bar{I}$. We have $(2,2) \notin \bar{I}$ since $2 \notin I$. Hence, $\bar{I}$ is not a semiprime ideal of $R \bowtie J$. ☐

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