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Non-$\mathcal{P}$-Right Noetherian and Non-$\mathcal{P}$-Right-S-Noetherian Rings

Alaa Abouhalaka[1,2]

[1] *Department of mathematics, Çukurova University, 01330 Balcalı, Adana, Turkey.*
[2] *Ministry of Education and Higher Education, Doha, Qatar.*
e-mail: alaa1aclids@gmail.com

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Abstract. Let R be a noncommutative ring, and $\mathcal{P}(R)$ the Baer-McCoy radical of R. In this paper, we introduce the notions of non-$\mathcal{P}$-right ideals, and non-$\mathcal{P}$-right Noetherian rings, deriving corollaries that extend properties of Noetherian rings. Additionally, we introduce non-$\mathcal{P}$-right S-Noetherian rings and S-non-$\mathcal{P}$-ideals, where S is an m-system, providing further corollaries and a non-$\mathcal{P}$-S version of Cohen's Theorem, emphasizing the importance of the prime condition for $\mathcal{P}(R)$.

Key Words: Right Noetherian rings; right S-Noetherian rings; non-$\mathcal{P}$-ideals; non-$\mathcal{P}$–Noetherian rings; non-$\mathcal{P}$-right-S-Noetherian; noncommutative rings.
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1 Introduction

Throughout the paper, by R we mean a noncommutative ring without identity unless it is otherwise stated, and $\mathcal{P}$ will denote the Baer-McCoy radical, i.e., the (Baer's) lower nilradical or the prime radical. In referring to [15], the Baer-McCoy radical $\mathcal{P}$ is a complete idempotent Hoehnke radical, i.e.,

(i) $f(\mathcal{P}(R)) \subseteq \mathcal{P}(f(R))$ for any homomorphism $f\colon R \to R'$.
(ii) If $J \lhd R$ and $\mathcal{P}(J) = J$ then $J \subseteq \mathcal{P}(R)$.
(iii) $\mathcal{P}(R/\mathcal{P}(R)) = 0$.
(iv) For each ring R, $\mathcal{P}(\mathcal{P}(R)) = \mathcal{P}(R)$.

Let $S \subseteq R$ be a multiplicatively closed subset of a commutative ring R. The ideal $Nil^*(R)$ of all nilpotent elements in R is precisely the prime radical. In referring to [21], an ideal K of R is called an n-ideal if whenever $x, y \in R$ with $xy \in K$, then $x \in Nil^*(R)$ or $y \in K$. While K is called an S-n-ideal if $K \cap S = \emptyset$, and whenever $x, y \in R$ with $xy \in K$, then $xs \in Nil^*(R)$ or $ys \in K$ for some fixed $s \in S$, see [12]. The definition of nonnil ideal is given in [6]. The ideal K is said to be nonnil if $K \not\subseteq Nil^*(R)$ and the ring R is called Nonnil-Noetherian if each nonnil ideal of R is finitely generated. This concept is introduced by Badawi in [6] under the assumption that $Nil^*(R)$ is a divided prime ideal. On the other hand, R is called S-Noetherian if all the ideals of R are S-finite. Many of the properties of Noetherian rings are also true for Nonnil-Noetherian and S-Noetherian rings, as it is shown in [6], and [3], respectively. In addition, the notion of S-prime ideals is introduced in [17], with a Cohen's type theorem. Theorem 3 of [17] states that R is S-Noetherian if and only if every S-prime ideal is S-finite. The concept of S-prime ideals has attracted many researchers, prompting further developments in the area; see, for example, [5, 16, 18, 28]. Recently, in [30], the concept of Nonnil-S-Noetherian rings is introduced with many similar interesting properties.

In noncommutative rings, generalizations of theorems of Cohen and Kaplansky is shown in [24]. The concept of nonnil right Noetherian rings was first introduced in [27], as a generalization of the concept of nonnil Noetherian rings in commutative case. However, this generalization was done by using the upper nilradical ($Nil^*(R)$) which is the largest nil ideal of the ring. It is a well-known fact that for any ring R, $\mathcal{P}(R) \subseteq Nil^*(R)$ and if R is commutative, then $\mathcal{P}(R) = Nil^*(R)$. In addition, the concept of S-Noetherian rings has been generalized to the noncommutative case in two directions; the first generalization was done by keeping S as a multiplicatively closed subset, like in papers [7, 8, 9, 11], and the second generalization was done by the author by considering S as an m-system, see [1, 2].

In the second section of this paper, we introduce the notions of $\mathcal{P}$-right ideals, non-$\mathcal{P}$-right ideals and non-$\mathcal{P}$-right Noetherian rings. We obtain some corollaries that extend various properties of Noetherian rings.

In the third section, we introduce the notions of non-$\mathcal{P}$-right S-Noetherian rings, and S-non-$\mathcal{P}$-(right) ideals, where S is an m-system, in the sense of [1]. We also obtain a set of corollaries that extend various properties of right S-Noetherian rings. Proposition 3.13 shows that the condition $\mathcal{P}(R)$ is prime was necessary in [6] to get a proper generalization. Theorem 3.19 is the non-$\mathcal{P}$-S-version of Cohen's Theorem under the condition that $\mathcal{P}(R)$ is prime.

Throughout the paper, although if not explicitly stated, R is understood to denote a noncommutative ring. By (right) ideal we mean a proper (right) two-sided ideal unless otherwise specified. For an element $a \in R$, the ideals $(a\rangle$, $\langle a)$, and $\langle a\rangle$ are defined respectively as $(a\rangle = aR$, $\langle a) = Ra$, and $\langle a\rangle = RaR$.

2 Non-$\mathcal{P}$-right Noetherian rings

2.1 $\mathcal{P}$-right ideals and non-$\mathcal{P}$-right ideals

Many equivalent definitions have been provided in Proposition 1.15 of [20] for an ideal to qualify as a ρ-ideal, where ρ is a special radical. The one-sided version of Definition 1.2 of [20] for $\rho = \mathcal{P}$ is provided below.

Definition 2.1. A right ideal K of R is referred to as a $\mathcal{P}$-right ideal if for any right ideals I, J of R with $IJ \subseteq K$, either $I \subseteq \mathcal{P}(R)$ or $J \subseteq K$.

In commutative rings, what are referred to as $\mathcal{P}$-ideals in general ring theory are known as *n-ideals*. According to Definition 2.1 in [21], a proper ideal K of a commutative ring with unity is called an n-ideal if, for any $a, b \in R$ such that $ab \in K$, it follows that either $a \in K$ or $b \in \mathcal{P}(R) = \sqrt{0}$. It is important to note that in unitary commutative rings, this definition is equivalent to the definition of $\mathcal{P}$-ideals as given in Definition 2.1. This equivalence is demonstrated in the following corollary:

Corollary 2.2. *Let R be a commutative ring with identity. For an ideal K of R, the following statements are equivalent:*

(1) *For all $a, b \in R$, if $ab \in K$, then either $a \in K$ or $b \in \mathcal{P}(R) = \sqrt{0}$.*

(2) *For all ideals I and J of R, if $IJ \subseteq K$, then either $I \subseteq \mathcal{P}(R)$ or $J \subseteq K$.*

Proof. (1) $\Rightarrow$ (2): Let I and J be ideals of R such that $IJ \subseteq K$. Suppose $I \not\subseteq \mathcal{P}(R)$, and choose an element $a \in I \setminus \mathcal{P}(R)$. For every $b \in J$, we have $ab \in K$. Then by assumption (1), it follows that $b \in K$. Hence $J \subseteq K$.

(2) $\Rightarrow$ (1): Suppose $a, b \in R$ with $ab \in K$ and $a \notin \mathcal{P}(R)$. Then the product of the principal ideals $\langle a\rangle\langle b\rangle \subseteq K$. By assumption (2), it follows that $\langle b\rangle \subseteq K$, which implies $b \in K$. □

The zero ideal of a prime ring qualifies as a $\mathcal{P}$-(right) ideal. To provide more examples in a noncommutative case we need the following proposition which can be obtained by using the fact that $\mathbb{M}_n(\mathcal{P}(R)) = \mathcal{P}(\mathbb{M}_n(R))$, see [22].

Corollary 2.3. *An ideal K of a unitary ring R is a $\mathcal{P}$-ideal if and only if $\mathbb{M}_n(K)$ is a $\mathcal{P}$-ideal of $\mathbb{M}_n(R)$.*

Proof. Suppose K is a $\mathcal{P}$-ideal of R and let $\mathbb{M}_n(I)\mathbb{M}_n(J) \subseteq \mathbb{M}_n(K)$ for some ideals I, J of R, then $IJ \subseteq K$ and hence either $I \subseteq \mathcal{P}(R)$ or $J \subseteq K$ and hence either $\mathbb{M}_n(I) \subseteq \mathbb{M}_n(\mathcal{P}(R)) = \mathcal{P}(\mathbb{M}_n(R))$ or $\mathbb{M}_n(J) \subseteq \mathbb{M}_n(K)$, and vice versa. □

Example 3.2 of [21] shows that in the ring $\mathbb{Z}_{27}$, the ideal $\langle 9\rangle$ is an n-ideal, hence by Corollary 2.3 the ideal $\mathbb{M}_n(\langle 9\rangle)$ of $R = \mathbb{M}_n(\mathbb{Z}_{27})$ is a $\mathcal{P}$-ideal. Notice that $\mathcal{P}(R) = \mathbb{M}_n(\langle 3\rangle)$.

Proposition 2.4. *A right ideal K of a unitary ring R is a $\mathcal{P}$-right ideal if and only if whenever $r_1Rr_2 \subseteq K$ then either $r_1 \in \mathcal{P}(R)$ or $r_2 \in K$ for all $r_1, r_2 \in R$.*

Proof. Suppose K is a $\mathcal{P}$-right ideal and let $r_1, r_2 \in R$ with $r_1Rr_2 \subseteq K$. Then $\langle r_1\rangle\langle r_2\rangle \subseteq K$ and hence either $r_1 \in \langle r_1\rangle \subseteq \mathcal{P}(R)$ or $r_2 \in \langle r_2\rangle \subseteq K$. For the converse, let A, B be right ideals of R with $AB \subseteq K$. If $A \nsubseteq \mathcal{P}(R)$ then there exists $r_1 \in A \setminus \mathcal{P}(R)$ and thus for all $r_2 \in B$, $r_1Rr_2 \subseteq AB \subseteq K$. Consequently, $r_2 \in K$ and therefore $B \subseteq K$. Hence K is a $\mathcal{P}$-right ideal. □

Proposition 2.5. *If K is a $\mathcal{P}$-right ideal of a unitary ring R, then $K \subseteq \mathcal{P}(R)$.*

Proof. Assume K is a $\mathcal{P}$-right ideal. If $K \nsubseteq \mathcal{P}(R)$ then there exists $a \in K \setminus \mathcal{P}(R)$. Thus for all $b \in R$ we have $aRb \subseteq K$ which implies $b \in K$. Hence $K = R$, a contradiction. Thus $K \subseteq \mathcal{P}(R)$. □

In fact, many properties of $\mathcal{P}$-ideals which have been obtained in [20] can be generalized to get a one-sided version of them by using the same standard machinery and technique. However, we set Definition 2.1 just to show the difference between $\mathcal{P}$- and non-$\mathcal{P}$-(right) ideals, and Example 2.7 shows this clearly. The following definition is our definition of non-$\mathcal{P}$-(right) ideal.

Definition 2.6. A proper (right) ideal of a ring R is termed as a non-$\mathcal{P}$-(right) ideal if it is not contained in the Baer-McCoy radical of the ring.

The zero ideal serves as a counterexample of a non-$\mathcal{P}$-ideal. It is worthy to know that if a (right) ideal P is a non-$\mathcal{P}$-(right) ideal, then for any $n \in \mathbb{Z}^+$, P^n is also a non-$\mathcal{P}$-(right) ideal, due to the fact that $\mathcal{P}(R)$ is semiprime. In addition, obviously by the definitions and Proposition 2.5 any non-$\mathcal{P}$-(right) ideal is not a $\mathcal{P}$-(right) ideal. However, it is important to notice that if a (right) ideal is not a $\mathcal{P}$-(right) ideal, then it may not be a non-$\mathcal{P}$-(right) ideal. The following example provides an ideal which is neither a non-$\mathcal{P}$-ideal nor a $\mathcal{P}$-ideal.

Example 2.7. Let $R = M_2(\mathbb{Z}_{36})$ and $K = M_2(\langle 12\rangle)$. The Baer-McCoy radical of R is $\mathcal{P}(R) = M_2(\mathcal{P}(\mathbb{Z}_{36})) = M_2(\langle 6\rangle)$. Clearly $K \subseteq \mathcal{P}(R)$ and hence K is not a non-$\mathcal{P}$-ideal. In addition, consider $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right] R \left[\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right] \subseteq K$. However, neither $\left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right] \in \mathcal{P}(R)$ nor $\left[\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right] \in K$. Thus by Proposition 2.4, K is not a $\mathcal{P}$-ideal.

Corollary 2.8. *If I is a non-$\mathcal{P}$-right ideal, then every right ideal that contains I is a non-$\mathcal{P}$-right ideal. Specifically, the sum of non-$\mathcal{P}$-right ideals is again a non-$\mathcal{P}$-right ideal.*

Proof. The proof follows from the fact that if $I \nsubseteq \mathcal{P}(R)$, then each right ideal that contains I is not contained in $\mathcal{P}(R)$. □

In the next corollary the symbol $\mathcal{P}(K)$ is referring to the intersection of all prime ideals in R that contain the ideal K, i.e., the prime radical of the ideal K.

Corollary 2.9. *If K is a non-$\mathcal{P}$-ideal then $\mathcal{P}(R) \subsetneq \mathcal{P}(K)$.*

Proof. Clearly $\mathcal{P}(R) \subseteq \mathcal{P}(K)$. Then the proper inclusion can be obtained because $K \subseteq \mathcal{P}(K)$ and $K \nsubseteq \mathcal{P}(R)$. □

In the following, we show how the characteristic of being a non-$\mathcal{P}$-(right) ideal can be transferred to related rings.

Proposition 2.10. *Let $g : R \to A$ be a ring epimorphism.*
(1) *For a right ideal K of R, if $g(K)$ is a non-$\mathcal{P}$-right ideal of A, then K is a non-$\mathcal{P}$-right ideal of R.*
(2) *If B is a non-$\mathcal{P}$-right ideal of A, then $g^{-1}(B)$ is a non-$\mathcal{P}$-right ideal of R.*
(3) *Let $g(\mathcal{P}(R)) = \mathcal{P}(A)$ and $\mathrm{Ker}(g) \subseteq \mathcal{P}(R)$. If P is a non-$\mathcal{P}$-right ideal of R, then $g(P)$ is a non-$\mathcal{P}$-right ideal of A.*

Proof. (1) Suppose $g(K)$ is a non-$\mathcal{P}$-right ideal of A. If $K \subseteq \mathcal{P}(R)$, then $g(K) \subseteq g(\mathcal{P}(R)) \subseteq \mathcal{P}(A)$, contradiction.
(2) Let $K = g^{-1}(B)$. Then by (1), K is a non-$\mathcal{P}$-right ideal because $g(K) = B = g(g^{-1}(B))$ is a non-$\mathcal{P}$-right ideal of A.
(3) Suppose K is a non-$\mathcal{P}$-right ideal of R. If $g(K) \subseteq \mathcal{P}(A)$, then

$$K \subseteq g^{-1}(g(K)) \subseteq g^{-1}(\mathcal{P}(A)) = g^{-1}(g(\mathcal{P}(R)) = \mathcal{P}(R).$$

This is a contradiction, hence $g(K) \not\subseteq \mathcal{P}(A)$, and thus $g(K)$ is a non-$\mathcal{P}$-right ideal of A. □

2.2 Non-$\mathcal{P}$-right Noetherian rings

To be able to define the notion of non-$\mathcal{P}$-right-S-Noetherian rings, where S is an m-system, we have to first define non-$\mathcal{P}$-right Noetherian ring as in the subsequent definition. We use FG to denote the property of being finitely generated,

Definition 2.11. Let R be a ring.
(1) R is called non-$\mathcal{P}$-right Noetherian if every non-$\mathcal{P}$-right ideal is FG.
(2) R is called non-$\mathcal{P}$-left Noetherian if every non-$\mathcal{P}$-left ideal is FG.
(3) R is called non-$\mathcal{P}$-Noetherian if it is both right and left non-$\mathcal{P}$-Noetherian.

Evidently, every right Noetherian ring is also a non-$\mathcal{P}$-right Noetherian ring, and the two concepts coincide when $\mathcal{P}(R) = 0$. The following result presents some classes of ring in which the two concepts coincide.

Corollary 2.12. *Let A be a unitary ring. The classes of non-$\mathcal{P}$-right Noetherian rings and right Noetherian rings coincide if A is one of the following rings:*
(a) *A is prime.*
(b) *A is semiprime.*
(c) *A is Jacobson semisimple (semiprimitive).*
(d) *$A = R/rad(R)$ for some ring R.*
(e) *For some ring R, $A = R/\mathcal{P}(R)$.*
(f) *A is von Neumann regular.*

Proof. (a) ((b)) is true because the zero ideal is prime (semiprime).
(c) and (f) are true because Jacobson radical ($rad(R)$) is zero and $\mathcal{P}(R) \subseteq rad(R)$.
(d) ((e)) is true because $rad(R/rad(R)) = 0$ ($\mathcal{P}(R/\mathcal{P}(R)) = 0$). □

The following theorem is the noncommutative version of Proposition 1.2 of [10] on page 394.

Theorem 2.13. The following are equivalent:
(1) R is a non-$\mathcal{P}$-right Noetherian ring.
(2) R satisfies the ACC on the non-$\mathcal{P}$-right ideals.
(3) R satisfies the ACC on the non-$\mathcal{P}$-FG right ideals.
(4) Every nonempty set of non-$\mathcal{P}$-right ideals of R has a maximal element with respect to inclusion.

Proof. (1) ⇒ (2): Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \subseteq \cdots$ be an ascending chain of non-$\mathcal{P}$-right ideals of R and let $K = \bigcup_{k=1}^{\infty} I_k$, then by Corollary 2.8, $K \nsubseteq \mathcal{P}(R)$, hence K is a non-$\mathcal{P}$-right ideal. Consequently, K is FG. Then a standard argument shows that $K = I_n$ for some $n \in \mathbb{Z}^+$.

(2) ⇒ (3): Clear.

(3) ⇒ (1): Suppose R satisfies the ACC on the non-$\mathcal{P}$-FG right ideals and let J be a non-$\mathcal{P}$-right ideal of R. Since J is a non $\mathcal{P}$-right ideal, there exists $x_1 \in J - \mathcal{P}(R)$. Put $J_1 = x_1 R$. Let

$$J_1 \subseteq J_1 + J_2 \subseteq \cdots \subseteq J_1 + J_2 + \cdots + J_k \subseteq \cdots$$

be an ascending chain of right ideals such that $J_i = (x_i\rangle$ where $x_i \in J$, and $x_i = x_j$ if and only if $i = j$ for all $i \in \{1, 2, \cdots\}$. Then $J_i \subseteq J$ for all $i \in \{1, 2, \cdots\}$. Since J_1 is a non $\mathcal{P}$-right ideal, then by Corollary 2.8, $J_1, J_1 + J_2, \cdots, J_1 + J_2 + \cdots + J_k, \cdots$ are all non-$\mathcal{P}$-right ideals. Consequently, the above mentioned chain is an ascending chain of non-$\mathcal{P}$-FG right ideals, and hence the chain eventually stabilizes, which implies that J is FG. Thus R is a non-$\mathcal{P}$-right Noetherian ring.

(2) ⇒ (4): Consider $\emptyset \neq \mho$ as a set of non-$\mathcal{P}$-right ideals of R. If $\mho$ does not have a maximal element, then for any $I_1 \in \mho$, we can find $I_2 \in \mho$ with $I_1 \subset I_2$. Also I_2 is not maximal in $\mho$, hence we can find $I_3 \in \mho$ with $I_1 \subset I_2 \subset I_3$ and so we obtain a strictly ascending chain of non-$\mathcal{P}$ right ideals of R, which is a contradiction.

(4) ⇒ (1): Let I be a non-$\mathcal{P}$-right ideal of R, and $\mho$ the set of non-$\mathcal{P}$-FG right ideals contained in I. Since $I \nsubseteq \mathcal{P}(R)$, there exists $x \in I - \mathcal{P}(R)$, hence $(x\rangle \nsubseteq \mathcal{P}(R)$, consequently, $(x\rangle$ is a non-$\mathcal{P}$-right ideal and $(x\rangle \in \mho \neq \emptyset$. Thus $\mho$ has a maximal element K. For all $r \in I$, $K + (r\rangle \nsubseteq \mathcal{P}(R)$, thus $K + (r\rangle \in \mho$, so by the maximality of K, $K + (r\rangle = K$, and hence $r \in K$. Consequently, $K = I$. Thus the non-$\mathcal{P}$-right ideal I is FG and R is a non-$\mathcal{P}$-right Noetherian ring. □

Recall that a ring R is called a duo ring (a right duo ring) if $aR = Ra$ ($Ra \subseteq aR$) for all $a \in R$, see [14].

Proposition 2.14. *Let R be a ring with $Rr \subseteq rR$ for all $r \in R$. Then R is a non-$\mathcal{P}$-right Noetherian ring, if and only if R/I is a right Noetherian ring for every non-$\mathcal{P}$-ideal I of R.*

Proof. Suppose R is a non-$\mathcal{P}$-right Noetherian ring and let $\bar{P} = P/I$ be a right ideal of R/I, then $I \subseteq P$ and hence P is a non-$\mathcal{P}$-right ideal of R, consequently P is FG and R/I is a right Noetherian ring. Conversely, suppose R/I is right Noetherian for each non-$\mathcal{P}$-ideal I of R, and let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \subseteq \cdots$ be an ascending chain of non-$\mathcal{P}$-right ideals of R. Then R/I_1 is right Noetherian and hence the ascending chain $I_2/I_1 \subseteq I_3/I_1 \subseteq \cdots \subseteq I_i/I_1 \subseteq \cdots$ eventually stabilizes, thus $I_n/I_1 = I_{n+1}/I_1$ for some $n \in \mathbb{Z}^+$, consequently $I_n = I_{n+1}$. Hence, by Theorem 2.13 R is a non-$\mathcal{P}$-right Noetherian ring. □

3 Non-$\mathcal{P}$ right S-Noetherian rings

In referring to [3], an ideal I of a commutative ring R is called S-finite (where S is a multiplicatively closed subset of R) if there exists a finitely generated ideal K such that $Is \subseteq K \subseteq I$ for some $s \in S$. Clearly, any ideal which meets S is an S-finite ideal. Thus, being I disjoint from S is essential. Moreover, in all the works [13, 17, 23, 25, 29, 30] concerning S-prime (-weakly prime, -primary, -weakly primary, -radical) ideals, it is essential to assume that there is no intersection between S and the ideal in each definition of the above previous notions; otherwise all the definitions would be trivial. Recently, in [30], the notion of nonnil-S-Noetherian commutative rings has been studied by considering S as a multiplicatively closed subset of R. In this context, a commutative ring R is said to be a nonnil-S-Noetherian ring if each nonnil ideal of R is S-finite. Notice that if $s \in S \cap Nil^*(R)$, then by the definitions of multiplicatively closed subset and the prime radical (in commutative case), there exists $k \in \mathbb{Z}^+$ such that $0 = s^k \in S$. Consequently, every ideal is S-finite, and R is a nonnil-S-Noetherian ring. Hence it is necessary in [30] to put the condition $S \cap Nil^*(R) = \emptyset$, as in the previously mentioned works.

In this section, by S we mean an m-system unless we stated otherwise. SF denotes the property of being S-finite, and SN means S-Noetherian ring.

3.1 Properties of non-$\mathcal{P}$ right S-Noetherian rings

Definition 3.1. [Definition 3.1 of [1]] A (right) ideal K of R is said to be S-finite (SF) if there exists a finitely generated (FG) (right) ideal F such that $K\langle s\rangle \subseteq F \subseteq K$ for some $s \in S$.

The following is our definition of a non-$\mathcal{P}$-right-S-Noetherian ring.

Definition 3.2. A ring R is called a non-$\mathcal{P}$-right-SN ring if every non-$\mathcal{P}$-right ideal is SF.

Remark 3.3. Clearly, by definitions, every non-$\mathcal{P}$-right-Noetherian ring is a non-$\mathcal{P}$-right-SN ring. In addition, notice that in case $S \cap \mathcal{P}(R) \neq \emptyset$, then since every m-system meets $\mathcal{P}(R)$ must meet the zero ideal, then $0 \in S$. Consequently, if I is a non-$\mathcal{P}$-right ideal, then $I\langle 0\rangle \subseteq \langle 0\rangle \subseteq I$, hence every non-$\mathcal{P}$-right ideal is SF and thus R is non-$\mathcal{P}$-right-SN. Hence we will assume that $S \cap \mathcal{P}(R) = \emptyset$ even if we do not refer to this. Now, it is clear that the upper nilradical ($Nil^*(R)$) is incompatible with the m-system S, and using them together will be different from the previous counterparts and odd and may not give satisfactory results.

In the following, we present a theorem that is the non-$\mathcal{P}$-version of Theorem 3.5 of [1]. Recall that $(E : D)^* = \{r \in R \mid Dr \subseteq E\}$ and $(E : D) = \{r \in R \mid rD \subseteq E\}$ for some subsets E, D of R.

Theorem 3.4. Let K be a non-$\mathcal{P}$-right ideal of a unitary ring R which is maximal with respect to the property of being non-S-finite among all non-$\mathcal{P}$-right ideals of R. If $K \subset RK$, then K is a prime right ideal.

Proof. Assume K is not prime, then there exist $x, y \in R - K$ such that $xRy \subseteq K$. Thus, $K \subsetneq K + \langle x\rangle$. In addition, $Rxy \subseteq RK$ and thus $y \in (RK : \langle x\rangle)^*$. Since K is contained in the ideal RK then $K \subsetneq (RK : \langle x\rangle)^*$. Notice that by Corollary 2.8, both of $K + \langle x\rangle$ and $(RK : \langle x\rangle)^*$ are non-$\mathcal{P}$-right ideals and hence due to the maximality of K the right ideals $K + \langle x\rangle$ and $(RK : \langle x\rangle)^*$ are both S-finite. Consequently, there exist elements $s_1, s_2 \in S$ and FG right ideals I, J of R with $(RK : \langle x\rangle)^*\langle s_1\rangle \subseteq I \subseteq (RK : \langle x\rangle)^*$ and $[K + \langle x\rangle]\langle s_2\rangle \subseteq J \subseteq K + \langle x\rangle$. Let $J = \langle k_1 + xr_1, \cdots, k_n + xr_n\rangle \subseteq \langle k_1, \cdots, k_n\rangle + \langle x\rangle$ for some $k_i \in K$ and $r_i \in R$ and hence for any $a \in K\langle s_2\rangle \subseteq J$ we have $a = \Sigma k_i\gamma_i + x\beta$ for some $\beta, \gamma_1, \cdots, \gamma_n \in R$, thus $x\beta \in RK$ and $\beta \in (RK : \langle x\rangle)^*$, so $a \in \langle k_1, \cdots, k_n\rangle + x(RK : \langle x\rangle)^*$ and hence $K\langle s_2\rangle \subseteq \langle k_1, \cdots, k_n\rangle + x(RK : \langle x\rangle)^*$. Therefore,

$$K\langle s_2\rangle\langle s_1\rangle \subseteq \langle k_1, \cdots, k_n\rangle\langle s_1\rangle + x(RK : \langle x\rangle)^*\langle s_1\rangle \subseteq \langle k_1, \cdots, k_n\rangle + xI \subseteq RK.$$

Hence, for some $r \in R$, $s = s_2 r s_1 \in S$ and $K\langle s\rangle \subseteq K\langle s_2\rangle\langle s_1\rangle \subseteq F \subseteq RK$. Where $F = \langle k_1, \cdots, k_n\rangle + xI$ is an FG right ideal. Thus,

$$RK\langle s\rangle \subseteq RF \subseteq RK$$

Because RF is an FG ideal, the ideal RK is SF, and since $K \subseteq RK$ then by Corollary 2.8, RK is a non-$\mathcal{P}$-right ideal. Hence $K = RK$, contradiction. Thus, K is prime. □

The following theorem is essentially Theorem 3.4 under another condition. The proof follows similar steps to those in Theorem 3.4, but we include it here for completeness.

Theorem 3.5. Let K be a non-$\mathcal{P}$-ideal of a duo unitary ring R which is maximal with respect to the property of being non-SF among all non-$\mathcal{P}$-ideals of R. Then K is a prime ideal.

Proof. Assume K is not prime, then there exist $x, y \in R - K$ such that $xRy \subseteq K$. Thus $K \subsetneq K + \langle x \rangle$ and $K \subsetneq (K : \langle x \rangle)$. Notice that by Corollary 2.8 both of $K + \langle x \rangle$ and $(K : \langle x \rangle)$ are non-$\mathcal{P}$-right ideals, and hence due to the maximality of K, the right ideals $K + \langle x \rangle$ and $(K : \langle x \rangle)$ are both SF. Consequently, there exist elements $s_1, s_2 \in S$ and FG right ideals I, J of R such that $(K : \langle x \rangle)\langle s_1 \rangle \subseteq I \subseteq (K : \langle x \rangle)$ and $[K + \langle x \rangle]\langle s_2 \rangle \subseteq J \subseteq K + \langle x \rangle$. Let $J = \langle k_1 + xr_1, \cdots, k_n + xr_n \rangle \subseteq \langle k_1, \cdots, k_n \rangle + \langle x \rangle$ for some $k_i \in K$ and $r_i \in R$, then for any $a \in K\langle s_2 \rangle \subseteq J$ we have $a = \Sigma k_i \gamma_i + x\beta$ for some $\beta, \gamma_1, \cdots, \gamma_n \in R$, thus $x\beta \in K$ and $\beta \in (K : \langle x \rangle)$, so $a \in \langle k_1, \cdots, k_n \rangle + x(K : \langle x \rangle)$ and hence $K\langle s_2 \rangle \subseteq \langle k_1, \cdots, k_n \rangle + x(K : \langle x \rangle)$. Therefore,

$$K\langle s_2 \rangle \langle s_1 \rangle \subseteq \langle k_1, \cdots, k_n \rangle \langle s_1 \rangle + x(K : \langle x \rangle)\langle s_1 \rangle \subseteq \langle k_1, \cdots, k_n \rangle + xI \subseteq K.$$

Thus for some $r \in R$ we have $s = s_2 r s_1 \in S$ and $K\langle s \rangle \subseteq K\langle s_2 \rangle \langle s_1 \rangle \subseteq F \subseteq K$, where $F = (k_1, \cdots, k_n \rangle + xI$ is an FG right ideal. Hence K is SF, contradiction. Thus K is prime. □

Corollary 3.6. *Let P be a non-$\mathcal{P}$-ideal of a duo unitary ring R which is maximal with respect to the property of being non FG among all non-$\mathcal{P}$-ideals of R. Then P is a prime ideal.*

Proof. The proof follows by Theorem 3.5 by taking $S = \{1\}$. Recall that, in this case, an ideal is SF if and only if it is FG. □

Similar to the proof of Theorem 3.5, one can obtain the following corollary.

Corollary 3.7. *Let P be an ideal of a duo unitary ring R, which is maximal with respect to the property of being non-SF. Then P is a prime ideal.*

By using the above corollary we can give another version of Theorem 3.11 of [1] as follows:

Theorem 3.8. In a duo unitary ring R, the following are equivalent:
(1) Every right S-prime ideal is SF.
(2) Every prime ideal is SF.
(3) R is an SN ring.

The following theorem is another version of Cohen's type theorem, which can be proved by applying Theorem 3.4. Recall that for any right ideals I, J and P of a ring R, if $P \subseteq I \cup J$ then either $P \subseteq I$ or $P \subseteq J$.

Theorem 3.9. Let R be a unitary ring. If $K \subset RK$ for all non-SF right ideals K of R, then the following are equivalent:
(1) Every non-$\mathcal{P}$-prime right ideal of R is SF.
(2) R is a non-$\mathcal{P}$-right-SN ring.

Proof. (1) $\Rightarrow$ (2): Suppose K is a non-$\mathcal{P}$-right ideal. If K is not SF, then let:

$$\Omega = \{J : J \text{ is a non-}\mathcal{P}\text{-right ideal of } R \text{ that is not } S\text{F}\}.$$

Then $K \in \Omega \neq \phi$. Thus Ω is nonempty, and partially ordered under inclusion. Now we show that every chain in Ω has an upper bound. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_i \subseteq \cdots$ be an ascending chain of non-$\mathcal{P}$-right ideals of R that is not SF, and let $P = \bigcup_{i \in \mathbb{A}} I_i$ for some index set $\mathbb{A}$. By Corollary 2.8, P is a non-$\mathcal{P}$-right ideal. If there exists an element $s \in S$ such that $P\langle s \rangle \subseteq J \subseteq P = \bigcup_{i \in \mathbb{A}} I_i$ for some FG right ideal J, then by the remark made in the paragraph above the theorem, there exists I_i for some $i \in \mathbb{A}$ such that:

$$I_i \langle s \rangle \subseteq P\langle s \rangle \subseteq J \subseteq I_i.$$

Hence I_i is an SF non-$\mathcal{P}$-right ideal, which is a contradiction. Thus P is not SF and so $P \in \Omega$. Applying Zorn's lemma, we find that Ω possess a maximal element P. Hence by Theorem 3.4, P is prime, contradiction. Hence every non-$\mathcal{P}$-right ideal of R is SF. Therefore, R is a non-$\mathcal{P}$-right-SN ring.

(2) $\Rightarrow$ (1): Clear. □

Theorem 3.10. In a duo unitary ring R the following are equivalent:
(1) Every non-$\mathcal{P}$-prime right ideal of R is SF.
(2) R is a non-$\mathcal{P}$-right-SN ring.

Proof. The proof follows by Theorem 3.9 and Theorem 3.5. □

In the next theorem we show a necessary and sufficient condition for a non-$\mathcal{P}$-right-SN ring to be classified as right-SN.

Theorem 3.11. If $I \subset RI$ for all non-SF right ideals I of a ring R with identity, then the following are equivalent:
(1) R is a right-SN ring.
(2) R is a non-$\mathcal{P}$-right-SN ring and $\mathcal{P}(R)$ is SF.

Proof. (1) $\Rightarrow$ (2): Clear.
(2) $\Rightarrow$ (1): Due to Theorem 3.11 of [1], It is sufficient to show that every prime right ideal of R is SF. Let P be a prime right ideal of R, then $\mathcal{P}(R) \subseteq P$. Now if $\mathcal{P}(R) = P$, then by assumption, P is SF. If $\mathcal{P}(R) \subset P$, then $P \nsubseteq \mathcal{P}(R)$, consequently, P is a non-$\mathcal{P}$-right ideal, and again by assumption, P is SF. Thus R is a right-SN ring. □

Corollary 3.12. *If R is a von Neumann regular unitary ring, then R is a non-$\mathcal{P}$-right-SN ring if and only if R is a right-SN ring.*

Proof. Follows from the fact that $\mathcal{P}(R) = 0$. □

Proposition 3.13. *If R is a non-$\mathcal{P}$-SN duo unitary ring but not SN, then $\mathcal{P}(R)$ is prime.*

Proof. Assume that $\mathcal{P}(R)$ is not prime. Let $P_1 \in Spec(R)$ such that $P_1 \subseteq \mathcal{P}(R)$. Then $P_1 \subseteq \mathcal{P}(R) = \cap_{P \in Spec(R)} P \subseteq P_1$ contradicts with the initial assumption. Thus all the prime ideals are non-$\mathcal{P}$-ideals, and hence all the prime ideals are SF, consequently, by Theorem 3.8, R is SN, contradiction. Hence $\mathcal{P}(R)$ must be prime. □

Proposition 3.14. *If R is a non-$\mathcal{P}$-Noetherian duo unitary ring but not Noetherian, then $\mathcal{P}(R)$ is prime.*

Proof. The proof follows by Proposition 3.13 and by Theorem 3.8 by taking $S = \{1\}$. □

As we mentioned in the introduction, Badawi in [6] was the first who introduced the notion of nonnil Noetherian rings in commutative rings, given the stringent assumption that $\mathcal{P}(R)$ is prime. But now it turns out that it was needed to get a proper generalization.

In the following theorems we give the necessary and sufficient condition for the upper triangular matrix ring $R = T_2(R_1)$, over R_1 to be classified as a non-$S_{T_2(R_1)}$-$\mathcal{P}$-right Noetherian ring, where:

$$S_{T_2(R_1)} = \{\bar{s} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}; s \in S\}, \text{ is an } m\text{-system of } T_2(R_1).$$

Recall that from [26], the prime radical of $T_2(R_1)$ (when R_1 is unitary) is

$$\mathcal{P}(T_2(R_1)) = \begin{bmatrix} \mathcal{P}(R_1) & R_1 \\ 0 & \mathcal{P}(R_1) \end{bmatrix}.$$

Theorem 3.15. If $R = T_2(R_1)$ is a non-$\mathcal{P}$-right $S_{T_2(R_1)}$N ring, then the unitary ring R_1 is a non-$\mathcal{P}$-right-SN ring.

Proof. Let F be a right ideal of R_1 such that $F \not\subseteq \mathcal{P}(R_1)$. Then for some $f \in F$, $\begin{bmatrix} F & F+\langle f\rangle \\ 0 & 0 \end{bmatrix}$ is a right ideal of $T_2(R_1)$, and

$$\bar{F} = \begin{bmatrix} F & F+\langle f\rangle \\ 0 & 0 \end{bmatrix} \not\subseteq \begin{bmatrix} \mathcal{P}(R_1) & R_1 \\ 0 & \mathcal{P}(R_1) \end{bmatrix} = \mathcal{P}(T_2(R_1)).$$

Thus $\bar{F}$ is $S_{T_2(R_1)}$F, consequently, there exist an FG right ideal $\bar{J} = \begin{bmatrix} J_1 & J_2 \\ 0 & J_3 \end{bmatrix}$ for some FG right ideals J_1, J_2, and J_3 of R_1 with $J_1 \subseteq J_2$, and $\bar{s} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \in S_{T_2(R_1)}$ such that $\bar{F}\langle \bar{s}\rangle \subseteq \bar{J} \subseteq \bar{F}$, and hence

$$\begin{bmatrix} F\langle s\rangle & (F+\langle f\rangle)\langle s\rangle \\ 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} J_1 & J_2 \\ 0 & J_3 \end{bmatrix} \subseteq \begin{bmatrix} F & F+\langle f\rangle \\ 0 & 0 \end{bmatrix}.$$

Hence, $F\langle s\rangle \subseteq J_1 \subseteq F$, and thus F is SF. □

Theorem 3.16. If R_1 is a non-$\mathcal{P}$-right-SN unitary ring and $\mathcal{P}(R_1)$ is a weakly prime ideal but not prime, then:

(1) R_1 is right SN.

(2) $T_2(R_1)$ is non-$\mathcal{P}$-right $S_{T_2(R_1)}$-Noetherian.

Proof. (1) Let F be a right ideal of R. If $F \not\subseteq \mathcal{P}(R_1)$, then F is SF. If $F \subseteq \mathcal{P}(R_1)$, then for some $s \in S$, $F \subseteq (\mathcal{P}(R_1) : \langle s\rangle)$. Now since $\mathcal{P}(R_1)$ is a weakly prime then by Theorem 1.14 of [19], either $\mathcal{P}(R_1) = (\mathcal{P}(R_1) : \langle s\rangle)$ or $(\mathcal{P}(R_1) : \langle s\rangle) = (0 : \langle s\rangle)$, and since $\mathcal{P}(R_1)$ is not prime, then by Proposition 3.13 of [1], $\mathcal{P}(R_1) \neq (\mathcal{P}(R_1) : \langle s\rangle)$. Consequently, $F \subseteq (0 : \langle s\rangle)$, and hence $F\langle s\rangle = 0$, and F is SF. Thus R_1 is right SN.

(2) By (1) and Theorem 3.9 of [1]. □

3.2 S-non-$\mathcal{P}$-(right) ideal

The next definition introduces a new type of ideals.

Definition 3.17. A (right) ideal P of R disjoint from S is called an S-non-$\mathcal{P}$-(right) ideal if $P \not\subseteq (\mathcal{P}(R) : \langle s\rangle)$ for all $s \in S$.

Since $\mathcal{P}(R) \subseteq (\mathcal{P}(R) : \langle s\rangle)$, then every S-non-$\mathcal{P}$-(right) ideal is a non-$\mathcal{P}$-(right) ideal. However, the reverse implication does not hold as demonstrated in the following example. Recall that for polynomial ring $R[x]$, $\mathcal{P}(R[x]) = \mathcal{P}(R)[x]$, see Theorem 3 of [4].

Example 3.18. Let $R_1 = \mathbb{Z}_{48}[x]$, and $R = \begin{bmatrix} R_1 & R_1 \\ R_1 & R_1 \end{bmatrix}$. Notice that $I = \mathcal{P}(\mathbb{Z}_{48}) = \langle 6\rangle$. Then by the remark in the above paragraph and Theorem 10.21 of [22], we obtain $\mathcal{P}(R) = \begin{bmatrix} \mathcal{P}(R_1) & \mathcal{P}(R_1) \\ \mathcal{P}(R_1) & \mathcal{P}(R_1) \end{bmatrix} = \begin{bmatrix} I[x] & I[x] \\ I[x] & I[x] \end{bmatrix}$. Let us consider the ideal $J = \begin{bmatrix} \langle 3x\rangle & \langle 3x\rangle \\ \langle 3x\rangle & \langle 3x\rangle \end{bmatrix}$, then clearly $J \not\subseteq \mathcal{P}(R)$, and hence J is a non-$\mathcal{P}$-ideal. On the other hand, take the m-system $S = \{s, s^2, s^4, s^8, \ldots\}$, where $s = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then $J \cap S = \emptyset$ and $J\langle s\rangle \subseteq \mathcal{P}(R)$, hence $J \subseteq (\mathcal{P}(R) : \langle s\rangle)$, and thus J is not an S-non-$\mathcal{P}$-ideal.

The following theorem, is the non-$\mathcal{P}$-S-version of Cohen's Theorem under the condition that $\mathcal{P}(R)$ is prime.

Theorem 3.19. Let R be a unitary ring. If $\mathcal{P}(R)$ is prime, then the following are equivalent:

(1) Each non-$\mathcal{P}$-right ideal of R is SF.

(2) Each S-non-$\mathcal{P}$-right ideal of R is SF.

(3) R is non-$\mathcal{P}$-right-SN.

Proof. (1) $\Rightarrow$ (2): Let K be an S-non-$\mathcal{P}$-right ideal of R, then $K \cap S = \emptyset$. Since $K \not\subseteq (\mathcal{P}(R) : \langle s\rangle)$ and for all $s \in S$, $\mathcal{P}(R) \subseteq (\mathcal{P}(R) : \langle s\rangle)$, we obtain that $K \not\subseteq \mathcal{P}(R)$, and hence K is a non-$\mathcal{P}$-right ideal of R. Thus K is SF.

(2) $\Rightarrow$ (3): Let K be a non-$\mathcal{P}$-right ideal of R, then $K \not\subseteq \mathcal{P}(R)$. If $K \subseteq (\mathcal{P}(R) : \langle s\rangle)$, then $K\langle s\rangle \subseteq \mathcal{P}(R)$, and hence $(RK)\langle s\rangle \subseteq \mathcal{P}(R)$ and by assumption, either $K \subseteq RK \subseteq \mathcal{P}(R)$ or $\langle s\rangle \subseteq \mathcal{P}(R)$, contradiction in

both cases. Thus $K \not\subseteq (\mathcal{P}(R) : \langle s \rangle)$. Thus K is an S-non-$\mathcal{P}$-right ideal and by (2), K is SF. Consequently, R is non-$\mathcal{P}$-right-SN.

(3) ⇒ (1): Clear. □

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Conflict of interest

The author declares no conflicts of interest.

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