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Intermediate Rings and Prime Ideals in FIP Extensions of Commutative Rings

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Abstract. We consider extensions $R \subset S$ of commutative rings with only finitely many intermediate rings and establish several sharp inequalities involving the cardinality of the set of intermediate rings and constants related to the set of prime ideals of R.

Key Words: Integral domain, ring extension, intermediate ring, FO-domain, FIP extension, integrally closed, Prüfer domain, normal pair.
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1 Introduction

In this paper, we focus on FIP extensions of commutative rings, those ring extensions $R \subset S$ that admit only finitely many intermediate rings. Such extensions have attracted increasing interest in recent literature due to their connections with various finiteness conditions and structural properties of rings.

A range of characterizations for FIP extensions has been developed, especially in terms of the behavior of prime and maximal ideals, as well as other related finiteness criteria (see [10], [9], [17], [22], and [20]). When the prime spectrum $\mathrm{Spec}(R)$, the set of prime ideals of R, ordered by inclusion is explicitly known, it becomes possible to compute the exact number of intermediate rings in an extension and even give the list of all overrings as in [16] or intermediate rings as in [11]. However, in many practical settings, such detailed information is unavailable.

The main objective of this work is to approximate the cardinality of the lattice $[R, S]$, which denotes the set of all intermediate rings T such that $R \subseteq T \subseteq S$. We aim to identify and analyze the key factors that influence this cardinality, especially in contexts where complete knowledge of the prime spectrum is lacking. A special case that has already been studied in this context is when $S = qf(R)$, the field of fractions of the domain R, as detailed in [16], and [12]. In this paper, we extend those results to more general settings, encompassing arbitrary extensions of integral domains, and further investigate cases where R and S are not necessarily integral domains.

When an FIP extension $R \subset S$ is given, and the set $\mathrm{Spec}(R)$ is known along with its order structure, the number of intermediate rings can be computed precisely. See, for instance [21] for overrings of PIDs, [18] for integrally closed domains with only finitely many overrings (FO domains), [6] and [5] and [2] for normal pairs. Nevertheless, in cases where such data is incomplete or unavailable, our results demonstrate that it is still possible to derive sharp estimates for the number of intermediate rings. These estimates rely primarily on the number of maximal ideals of R and the height of an associated set of prime ideals ordered by inclusion.

Interestingly, the approximations obtained for the number of intermediate rings also yield insights into the structure of the ring, providing bounds for the number of maximal ideals and the height of the associated set of prime ideal. This direction of research was initiated in [1], where approximations were established for the number and lengths of chains of intermediate rings in normal pairs, as well as for the overrings of Prüfer domains.

For non-integrally closed rings, several recent advances have been made, as documented in [6]. Further approximations and exact computations concerning intermediate rings appear in [15], [14], and [13].

In Section 1, we present sharp upper bounds for the number of intermediate rings in an FIP integrally closed extension of integral domains, see Lemma 2.2 and Theorem 2.3. From this result, we deduce a lower bound for the number of intermediate rings in more general, not necessarily integrally closed extensions, see Corollary 2.6. We also provide several key estimates: Theorem 2.7 offers approximations for the number of maximal ideals and Corollary 2.9 gives an estimate for the height of the associated ordered set.

In Section 2, we provide several numerical characterizations highlighting several interesting particular cases.

Throughout this work, all rings are assumed to be commutative with unity. The set of prime ideals of a ring R is denoted by $\mathrm{Spec}(R)$, and maximal ideals by $\mathrm{Max}(R)$.

2 Intermediate rings and prime ideals

Let P and P' be two primes of R such that $P \subset P'$. The prime P' is said to cover P (in $\mathrm{Spec}(R)$), and we write P' cov P, if there is no prime Q of R such that $P \subset Q \subset P'$. A function α had been defined, in Theorem 2.3 of [18], on $\mathrm{Spec}(R)$ by

$$\begin{aligned} \alpha(P) \ &: \ = 1 \text{ if } P \text{ is a maximal ideal of } R, \text{ and} \\ \alpha(P) \ &: \ = \Pi_{P' \mathrm{cov}\, P}(1+\alpha(P')) \text{ if } P \text{ is not maximal.} \end{aligned}$$

The function α applied on the spectrum of an integrally closed FO domain R (domain with only finitely many overrings) gives the number of overrings of R, see Corollary 2.4 of [18], and to an appropriate subset of $\mathrm{Spec}(R)$ gives the number of intermediate rings, see Theorem 4.1 of [7]. For a subset I of R, $\mathcal{Z}(I)$ is usually defined to be the set of primes P of R such that $I \subseteq P$ and $\mathcal{M}(I)$ is the set of maximal ideals M of R such that $I \subseteq M$.

Let $R \subset S$ be a ring extension. If every intermediate ring T is integrally closed in S, then (R,S) is called a normal pair. Then for each maximal ideal M_i of R, there is, by Lemma 3.1 of [1], a prime Q_i of R such that

$$\begin{aligned} R_{Q_i} &= S_{M_i} \text{ and} \\ S &= \cap_{M_i \in \mathrm{Max}(R)} S_{M_i} = \cap_{M_i \in \mathrm{Max}(R)} R_{Q_i}. \end{aligned}$$

If Q is a prime of R then we let $\mathcal{U}(Q) = \{M \in \mathrm{Max}(R) : S_M = R_Q\}$.

The next result explains how the number of intermediate rings is obtained for FIP integrally closed extensions of domains.

Proposition 2.1. *Let $R \subset S$ be an extension of domains with only finitely many intermediate rings such that R is semilocal and integrally closed in S. Let $Q_1, Q_2, ..., Q_n$ be the maximal elements in the family $\{Q \in \mathrm{Spec}(R) : R_Q = S_M, M \in \mathrm{Max}(R)\}$. Then the following statements hold true.*

1. *$S = R_{Q_1} \cap R_{Q_2} \cap \ldots \cap R_{Q_n}$ is a minimal representation of S as an intersection of localizations of R.*
2. *The set $\bigcup_{i=1}^{n} \mathcal{Z}(Q_i)$ of primes is finite and the number of intermediate rings is given by $\|[R,S]\| = \prod_{i=1}^{n} \alpha(Q_i)$.*

Proof. 1. The FIP integrally closed extension $R \subset S$ is a normal pair by Corollary 3.3 of [17]. Hence, for each maximal ideal M_i of R, there is, by Lemma 3.1 of [1], a prime Q_i of R such that $R_{Q_i} = S_{M_i}$ and $S = \cap_{M_i \in \mathrm{Max}(R)} S_{M_i} = \cap_{M_i \in \mathrm{Max}(R)} R_{Q_i}$. Let $Q_1, Q_2, \ldots, Q_n$ be the maximal elements in the family $\{Q \in \mathrm{Spec}(R) : R_Q = S_M, M \in \mathrm{Max}(R)\}$. It is clear that

$$S = \cap_{M_i \in \mathrm{Max}(R)} R_{Q_i} \subseteq R_{Q_1} \cap R_{Q_2} \cap \ldots \cap R_{Q_n}.$$

Let Q be an element of the family $\{Q \in \mathrm{Spec}(R) : R_Q = S_M, M \in \mathrm{Max}(R)\}$. Then $Q \subseteq Q_j$ for some of the maximal elements $Q_1, \ldots, Q_n$. Hence

$$R_Q \supseteq R_{Q_j} \supseteq R_{Q_1} \cap R_{Q_2} \cap \ldots \cap R_{Q_n}.$$

Therefore,

$$S = \cap_{M_i \in \mathrm{Max}(R)} R_{Q_i} \supseteq R_{Q_1} \cap R_{Q_2} \cap \ldots \cap R_{Q_n}.$$

This completes the proof of the equality $S = R_{Q_1} \cap R_{Q_2} \cap \ldots \cap R_{Q_n}$.

2. The primes $Q_1 S, Q_2 S, \ldots, Q_n S$ are the maximal ideals of S. Then, using Theorems 3.6 and 4.1 of [7], we obtain

$$\|[R,S]\| = |\prod_{i=1}^{n} [R_{\mathcal{M}(Q_i)}, R_{Q_i}]\| = \prod_{i=1}^{n} \alpha(Q_i).$$

□

The set of primes $\{N \cap R : N$ is a maximal ideal of $S\}$ will be denoted $\mathcal{A}(R,S)$. Let $\mathcal{P}$ be a finite ordered set. The height $h(\mathcal{P})$ of $\mathcal{P}$ is the cardinality of the longest chain in $\mathcal{P}$. Notice that the height of $\mathrm{Spec}(R)$ ordered by the usual set inclusion satisfies $h(\mathrm{Spec}(R)) = 1 + \dim R$, where $\dim R$ is the Krull dimension of R.

We establish in this section sharp upper bounds for the number of intermediate rings in FIP extensions $R \subset S$, in several situations. The bounds depend on the height of the ordered set $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)$. We start with the case where the upper ring S is local.

In this case $\mathcal{A}(R,S)$ consists of exactly one prime $N \cap R$, where N is the maximal ideal of S.

Lemma 2.2. *Let $R \subset S$ be a FIP extension of integral domains such that R is integrally closed in S. If R is semilocal with m maximal ideals, S is local with maximal ideal N, and $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)$ is of finite height h, then*

$$2^m + h - 2 \leq \|[R,S]\| \leq h^m.$$

Proof. The set $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)$ is equal to $\mathcal{Z}(N \cap R)$ since S is local. The height h of $\mathcal{Z}(N \cap R)$ is necessarily larger than 1, as $h = 1$ would imply that $S_M = R_M$ for each maximal ideal M and $R = S$. If $h = 2$ the inequalities are trivially satisfied since

$$2^m = 2^m + h - 2 \leq \|[R,S]\| \leq h^m = 2^m.$$

Assume by induction that the stated inequalities are satisfied for every h such that $2 \leq h \leq n$ and let S such that the height of $\mathcal{Z}(N \cap R)$ is $n+1$. The set $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P) \setminus \{N \cap R\}$ is equal to $\bigcup_{i=1}^{i=k} \mathcal{Z}(P_i)$, where $P_1, P_2, \ldots, P_k$ are the primes covering $N \cap R$. Each set $\mathcal{Z}(P_i)$ is order isomorphic to $\mathrm{Spec}(R/P_i)$. The pair $(R_{\mathcal{M}(P_i)}, R_{P_i})$ is normal by Remark 15 of [11] and Lemma 2.6 of [7], and $h(\mathcal{Z}(P_i)) = h_i \leq n$. Let $m_i = |\mathcal{M}(P_i)|$. Then,

$$2^{m_i} + h_i - 2 \leq |[R_{\mathcal{M}(P_i)}, R_{P_i}]| = \alpha(P_i) \leq h_i^{m_i}.$$

Hence,

$$2^{m_i} + h_i - 1 \leq 1 + \alpha(P_i) \leq 1 + h_i^{m_i}.$$

We also have

$$|[R, S]| = \alpha(N \cap R) = \prod_{i=1}^{k} (1 + \alpha(P_i)).$$

Hence,

$$\begin{aligned} \prod_{i=1}^{k} (2^{m_i} + h_i - 1) &\leq |[R, S]| = \prod_{i=1}^{k} (1 + \alpha(P_i)) \\ &\leq \prod_{i=1}^{k} (1 + h_i^{m_i}). \end{aligned}$$

For the left inequality and since $1 \leq h_i \leq n$ for each i, $h_i = n$ for at least one i, and $\sum m_i = m$, we have

$$\begin{aligned} 2^m + n - 1 &= \prod_{i=1}^{k} (2^{m_i}) + n - 1 \\ &\leq \prod_{i=1}^{k} (2^{m_i} + h_i - 1) \\ &\leq |[R, S]|. \end{aligned}$$

For the right inequality and since $\sum m_i = m$, we have

$$\begin{aligned} |[R, S]| &\leq \prod_{i=1}^{k} (1 + n^{m_i}) \\ &\leq \prod_{i=1}^{k} (1 + n)^{m_i} \\ &\leq (1 + n)^m. \end{aligned}$$

Therefore, we obtain the desired result:

$$2^m + n - 1 = 2^m + (n + 1) - 2 \leq |[R, S]| \leq (1 + n)^m.$$

□

The following main result concerns the case where S is not necessarily local.

Theorem 2.3. Let $R \subset S$ be a FIP extension of integral domains such that R is integrally closed in S. If R is semilocal with m maximal ideals and $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)$ is of finite height h, then

$$2^m + h - 2 \leq |[R, S]| \leq h^m.$$

Proof. We first notice that (R,S) is a normal pair by Corollary 3.3 of [17] since $R \subset S$ is a FIP extension of integral domains and R is integrally closed in S. Then $|\mathrm{Max}(S) \leq \mathrm{Max}(R)|$ by Theorem 3.10 of [1]. Let $N_1,\dots,N_k$ be the maximal ideals of S, and $P_i = N_i \cap R$. Since (R,S) is a normal pair, then using Proposition 2.1 we obtain

$$\begin{aligned} |[R,S]| &= \left|\prod_{N\in \mathrm{Max}(S)} [R_{\mathcal{M}(N\cap R)}, R_{N\cap R}]\right| \\ &= \prod_{i=1}^{k} |[R_{\mathcal{M}(P_i)}, R_{P_i}]| \\ &= \prod_{i=1}^{k} \alpha(P_i). \end{aligned}$$

Each pair $(R_{\mathcal{M}(P_i)}, R_{P_i})$ is also a normal pair with $\bigcup_{P\in\mathcal{A}(R_{\mathcal{M}(P_i)}, R_{P_i})} \mathcal{Z}(P) = \mathcal{Z}(P_i)$ and is of height $h_i = h(\mathcal{Z}(P_i))$ such that $2 \leq h_i \leq n$. Let $m_i = |\mathcal{M}(P_i)|$. Then,

$$h(\mathcal{Z}(P_i)) + 2^{m_i} - 2 \leq \alpha(P_i) \leq (h(\mathcal{Z}(P_i)))^{m_i}.$$

Therefore,

$$\begin{aligned} \prod_{i=1}^{k} (h(\mathcal{Z}(P_i)) + 2^{m_i} - 2) &\leq |[R,S]| = \prod_{i=1}^{k} \alpha(P_i) \\ &\leq \prod_{i=1}^{k} (h(\mathcal{Z}(P_i)))^{m_i}. \end{aligned}$$

In one hand for the right inequality, we have

$$\prod_{i=1}^{k} (h(\mathcal{Z}(P_i)))^{m_i} \leq \prod_{i=1}^{k} \left(h(\bigcup_{P\in\mathcal{A}(R,S)} \mathcal{Z}(P)) \right)^{m_i} = \left(h(\bigcup_{P\in\mathcal{A}(R,S)} \mathcal{Z}(P)) \right)^{m}.$$

On the other hand for the left inequality, let P_t be a prime in $\{P_1, P_2, \dots, P_k\}$ with $h(\bigcup_{P\in\mathcal{A}(R,S)} \mathcal{Z}(P)) = h(P_t)$. We have $h(\mathcal{Z}(P_i)) - 2 \geq 0$ for every $i \in \{1,2,\dots,k\}$. Hence,

$$\begin{aligned} \prod_{i=1}^{k} (h(\mathcal{Z}(P_i)) + 2^{m_i} - 2) &\geq (h(\mathcal{Z}(P_t)) + 2^{m_t} - 2) \prod_{i\neq t} (2^{m_i}) \\ &\geq (h(\mathcal{Z}(P_t) - 2) \prod_{i\neq t} (2^{m_i}) + 2^{m_t} \prod_{i\neq t} (2^{m_i}) \\ &\geq (h(\mathcal{Z}(P_t) - 2) + 2^{m}. \end{aligned}$$

Combining the two inequalities, we obtain:

$$\begin{aligned} (h(\mathcal{Z}(P_t) - 2) + 2^{m} &\leq \prod_{i=1}^{k} (h(\mathcal{Z}(P_i)) + 2^{m_i} - 2) \\ &\leq |[R,S]| \\ &\leq \prod_{i=1}^{k} (h(\mathcal{Z}(P_i)))^{m_i} \\ &\leq \left(h(\bigcup_{P\in\mathcal{A}(R,S)} \mathcal{Z}(P)) \right)^{m}. \end{aligned}$$

This implies the desired result:

$$h\left(\bigcup_{P\in\mathcal{A}(R,S)}\mathcal{Z}(P)\right)+2^m-2\leq|[R,S]|\leq\left(h\left(\bigcup_{P\in\mathcal{A}(R,S)}\mathcal{Z}(P)\right)\right)^m.$$

□

We notice that the lower bound provided by Theorem 2.3 improves the one provided in Corollary 3.5 of [19]. The following example highlights this improvement.

Example 2.4. *Let R be a Prüfer semilocal domain with finite Krull dimension $\dim R$, and let P a prime of R that is not maximal. Assume that $\dim R\geq 2$, P is of height $h=\dim R-1$ and is contained in all maximal ideals of R. The extension $R\subset S=R_P$ is a FIP extension of integral domains and R is integrally closed in S. Thus, according to Lemma 2.2 and Theorem 2.3, we have $|[R,S]|\geq 2^m+h-2$, where $m=|\mathrm{Max}(R)|$. This clearly improves the available lower bound $|[R,S]|\geq|\mathrm{Spec}(R)|-|\mathrm{Spec}(S)|+1=m+1$, provided by Corollary 3.5 of [19]. As a concrete example, we can consider the ring extension defined in Example 2.5.*

Example 2.5. *Let $p_1,p_2,\ldots,p_m$ be m distinct prime numbers, and as in Example 9 of [12], consider the domain $T=\mathbb{Z}_{p_1\mathbb{Z}}\cap\mathbb{Z}_{p_2\mathbb{Z}}\cap\ldots\cap\mathbb{Z}_{p_m\mathbb{Z}}$ and define R with the following pullback construction of commutative rings:*

$$\begin{array}{ccc} R\simeq T+x\mathbb{Q}[x]_{(x)} & \rightarrow & T\\ \downarrow & & \downarrow\\ \mathbb{Q}[x]_{(x)} & \rightarrow & \mathbb{Q}[x]_{(x)}/x\mathbb{Q}[x]_{(x)}\simeq\mathbb{Q}\end{array}$$

The integral domain T is a PID with $\mathrm{Spec}(T)=\{0,p_1T,\ldots,p_mT\}$, which is ordered as in Fig. 1 below. The domain T is Prüfer as it is an overring of $\mathbb{Z}$. By Theorem 2.1 of [3], we can see that the integral domain $R\simeq T+x\mathbb{Q}[x]_{(x)}$ is also a Prüfer domain with a spectrum consisting of the prime ideals:

$$\{0\},M=x\mathbb{Q}[x]_{(x)},\text{and } P_i=M+p_iT,1\leq i\leq m.$$

The integral domain R is a Prüfer domain with a spectrum obtained by gluing $\mathrm{Spec}(T)$ over $\mathrm{Spec}(Q[x]_{(x)})$ and is ordered as in Fig. 1. The extension $R\subset S=R_M$ satisfies the conditions of Lemma 2.2 and Theorem 2.3. Therefore,

$$|[R,S]|\geq 2^m+h-2=2^m.$$

This largely improves the available lower bound provided by Corollary 3.5 of [19] which is:

$$|[R,S]|\geq|\mathrm{Spec}(R)|-|\mathrm{Spec}(S)|+1=m+1.$$

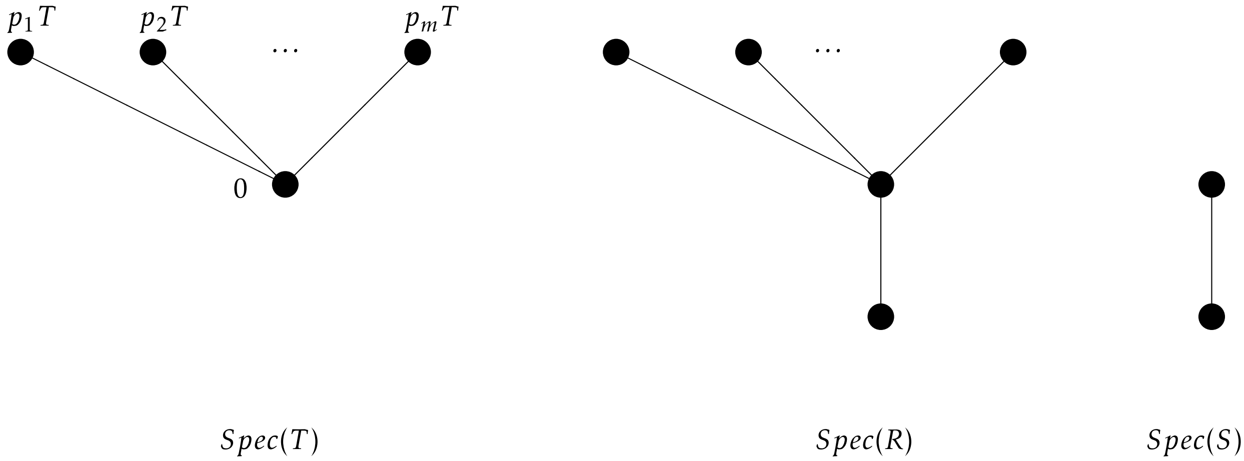

Fig. 1

Corollary 2.6. *Let $R_i \subseteq S_i$ be extensions of integral domains for $i = 1,\dots,n$, $n \geq 1$, $R = R_1 \times R_2 \times \dots \times R_n$ and $S = S_1 \times S_2 \times \dots \times S_n$. Assume R semilocal with m maximal ideals and $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)$ is of finite height h. If $R \subseteq S$ is a FIP extension, and (R,S) is a normal pair, then the cardinality of the set $[R,S]$ of intermediate rings satisfies:*

$$2^m + h - 2 \leq \|[R,S]\| \leq h^m.$$

Proof. The equalities are satisfied for $n = 1$ by Theorem 2.3. Let m_i be the number of maximal ideals of R_i, and h_i be the height of $\bigcup_{P \in \mathcal{A}(R_i,S_i)} \mathcal{Z}(P)$. Using the fact that $\|[R,S]\| = \prod_{i=1}^{n} \|[R_i,S_i]\|$, see [8], and that the inequalities are satisfied for each $\|[R_i,S_i]\|$, we obtain from [2.3] that for each i we have

$$2^{m_i} + h_i - 2 \leq \|[R_i,S_i]\| \leq h_i^{m}.$$

Therefore,

$$\prod_{i=1}^{n}(2^{m_i} + h_i - 2) \leq \prod_{i=1}^{n}\|[R_i,S_i]\| \leq \prod_{i=1}^{n} h_i^{m_i} \leq h^m.$$

For the left hand-side term, we have

$$\begin{aligned}
\prod_{i=1}^{n}(h_i + 2^{m_i} - 2) &\geq (h_k + 2^{m_k} - 2)\prod_{i=1, i\neq k}^{n}(h_i + 2^{m_i} - 2) \\
&\geq (h_k + 2^{m_k} - 2)\prod_{i=1, i\neq k}^{n}(2^{m_i}) \\
&\geq (h_k - 2) + \prod_{i=1}^{n}(2^{m_i}) \\
&= (h_k - 2) + 2^m
\end{aligned}$$

In particular for $h_k = h$, we have

$$|[R,S]| = \prod_{i=1}^{n} |[R_i, S_i]| \geq h - 2 + 2^m.$$

Finally

$$h - 2 + 2^m \leq |[R,S]| \leq \prod_{i=1}^{n} h_i^{m_i} \leq h^m.$$

□

The following result provide sharp bounds for the number of maximal ideals. For the particular case where R is an integral domain and S is the field of fractions of R see Theorem 2.5 of [23] and Theorem 8 of [12].

Theorem 2.7. Let $R_i \subseteq S_i$ be extensions of integral domains for $i = 1, \ldots, n$, $n \geq 1$, $R = R_1 \times R_2 \times \ldots \times R_n$ and $S = S_1 \times S_2 \times \ldots \times S_n$. Assume R semilocal with m maximal ideals and $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)$ is of finite height h. If $R \subseteq S$ is a FIP extension, and (R,S) is a normal pair, then the number of maximal ideals satisfies the inequalities:

$$\log_h |[R,S]| \leq m \leq \log_2(|[R,S]| - h + 2).$$

Proof. We have by Theorem 2.6,

$$2^m + h - 2 \leq |[R,S]|.$$

Hence,

$$2^m \leq |[R,S]| + 2 - h.$$

This yields

$$m \leq \log_2(|[R,S]| + 2 - h).$$

This establishes the right hand inequality.
On the other hand and using the same theorem, we obtain,

$$|[R,S]| \leq h^m.$$

Therefore,

$$m \geq \log_h |[R,S]|.$$

This establishes the left hand inequality and concludes the proof for both stated inequalities.
Finally $\log_h |[R,S]| \leq m \leq \log_2(|[R,S]| + 2 - h)$ as required. □

Example 2.8. *Let T, R and S as in Example 2.4 or Example 2.5. Thus, according to Theorem 2.7, we have*

$$\log_h |[R,K]| \leq m = |\mathrm{Max}(R)| \leq \log_2(|[R,K]| + 2 - h).$$

However the known upper bound is provided by Theorem 2.5 of [23],which is:

$$|\mathrm{Max}(R)| \leq |[R,K]| - \dim(R) - 1 = |[R,K]| - h,$$

This is clearly larger than $\log_2(|[R,K]| + 2 - h)$ for sufficiently large values of $|[R,K]|$.

The following result gives lower and upper bounds for the height of the set $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)$ of prime ideals related to the extension $R \subset S$. For the particular case where R is an integral domain and S is the field of fractions of R see Corollary of [12].

Corollary 2.9. *Let $R_i \subseteq S_i$ be extensions of integral domains for $i = 1, \ldots, n$, $n \geq 1$, $R = R_1 \times R_2 \times \ldots \times R_n$ and $S = S_1 \times S_2 \times \ldots \times S_n$. Assume R semilocal with m maximal ideals and $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)$ is of finite height h. If $R \subseteq S$ is a FIP extension, and (R,S) is a normal pair, then*

$$\sqrt[m]{|[R,S]|} \leq h\left(\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)\right) \leq |[R,S]| + 2 - 2^m.$$

Proof. Let $h = h\left(\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)\right)$. Using the left inequality of Theorem 2.6, we have

$$(2^m + h - 2) \leq |[R,S]|.$$

This trivially implies that

$$h \leq |[R,S]| + 2 - 2^m.$$

Then the required right hand inequality is proven. Now we use the right hand inequality $|[R,S]| \leq h^m$ of the same theorem. We obtain

$$\sqrt[m]{|[R,S]|} \leq h.$$

This concludes the proof for both stated inequalities. □

3 Numerical Characterizations

The inequalities of Theorem 2.6 contain several interesting particular cases.

Remark 3.1. *Let $R_i \subseteq S_i$ be extensions of integral domains for $i = 1, \ldots, n$, $n \geq 1$, $R = R_1 \times R_2 \times \ldots \times R_n$ and $S = S_1 \times S_2 \times \ldots \times S_n$. Assume R semilocal with m maximal ideals and $\bigcup_{P \in \mathcal{A}(R,S)} \mathcal{Z}(P)$ is of finite height h. If $R \subseteq S$ is a FIP extension, and (R,S) is a normal pair. Then,*

1. *$m = 1$ if and only if $n = 1$ and $[R,S]$ is a chain of integral domains.*
2. *$h = 2$ in Theorem 2.3 if and only if S is an intersection of localizations of R at ideals of dimension ≤ 1.*

The case where the number of intermediate rings is a prime number is particularly interesting and is presented in the next result.

Proposition 3.2. *Let $R \subset S$ be a FIP extension of integral domains such that R is integrally closed in S. If R is semilocal and $|[R,S]|$ is a prime number, then there is a prime Q of R such that $S = R_Q$.*

Proof. Since $|[R,S]| = \Pi_{P \in \mathcal{A}(R,S)} \alpha(P)$ is a prime number, then there is a unique term in this product. Hence $\mathcal{A}(R,S)$ contains a unique element. Hence S is local with a unique maximal ideal N. Then $S = S_N = R_{N \cap R}$. □

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