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Compactness of the numerical range of bounded operators on $H^p(\beta)$

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Abstract. In this paper we consider the numerical range of bounded linear operators on $H^p(\beta)$ and we prove that, by using the properties of the numerical range, the $ball(H^p(\beta))$ is smooth, i.e., there is one and only one semi-inner product $[.,.]$ on $H^p(\beta)$ such that $[f,f] = \|f\|^2$ for all $f \in H^p(\beta)$. Finally, we proved that the numerical range of bounded linear operators on $H^p(\beta)$ is compact if it is convex and contain the origin.

Key Words: Numerical Range, Weighted Hardy Space, Zero Inclusion, Smooth Space.
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1 Introduction

The numerical range of a linear operator on a normed linear space is a subset of the scalar field constructed in such a way that it is related both to the algebraic and the norm structures of the operator. In this it differs from the spectrum, which is related to the algebraic structure but independent of the norm (up to equivalence). For an operator on a Hilbert (or pre-Hilbert) space the numerical range has a very natural definition which was introduced, for finite dimensional spaces, by Toeplitz in 1918 [18], as follows. Let H denote a pre-Hilbert space with scalar product $<.,.>$ and norm $\|.\|$, and let $S(H)$ denote the unit sphere in H, $S(H) = \{h \in H : \|h\| = 1\}$. Then the numerical range of a linear operator $T : H \to H$ is the set $W(T)$ of scalars defined by

$$W(T) = \{<Th, h>: h \in S(H)\}.$$

The numerical range of an operator on a normed space and the numerical range of an element of a unitary Banach algebra, as developed by G. Lumer and F. F. Bonsall, are studied, and the theory of these numerical ranges is applied to Banach algebras. We are aware that our understanding of numerical range remains unbalanced, as we have not attempted to provide an understanding of its applications to initial value problems [4].

If T is a bounded linear operator on the normed space X, then the numerical range of T denoted by $V(T)$ and is defined as all complex numbers $x^*(Tx)$ where $x \in X, x^* \in X^*$, $\|x\| = \|x^*\| = x^*(x) = 1$ where X^* denotes the dual space of X [3].

The closure properties of the numerical range are studied. A construction of B. Berberian is extended to normed linear spaces, however because the numerical range need not be convex, the result obtained is weaker than that of Berberian for Hilbert spaces. A Hilbert space or an L^p-space, $1 < p < \infty$, is seen to be finite dimensional if and only if all the compact operators have closed numerical range. The numerical range of a compact operator, on a Hilbert space or an L^p-space, $1 < p < \infty$, have shown to contain all the nonzero extreme points of its closure. So, for a compact operator on a Hilbert space the numerical range is closed if and only if it contains the origin. In a Hilbert space the self-adjoint operators, which attain their numerical radius, showed to be dense among all the self-adjoint operators. This leads to a stronger form of a result by J. Lindenstrauss in the Banach space case of operators on a Hilbert space. see [14] and [15].

Let X be a Banach space and T is a bounded linear operator on X. The numerical range of T is denoted by $V(T)$ and defined by:

$$V(T) = \{< Tx, x^* >: x \in X, x^* \in X^*, \|x\| = \|x^*\| =< x, x^* >= 1\}$$

where X^* denotes the dual space of X.

In [11] Lumer defined the concept of a semi inner product on a linear space, and showed that every normed linear space $(X, \|.\|)$ has at least one semi-inner-product $[.,.]$ such that

$$[x, x] = \|x\|^2 \quad (x \in X) \quad (1)$$

In terms of a semi-inner-product satisfying (1), the definition of usual numerical range for Hilbert space operator at once generalizes to give the definition of the numerical range $W(T)$ for a linear operator on X,

$$W(T) = \{[Tx, x] : \|x\| = 1\}$$

one the face of it this definition has the serious defect that it is not an invariant of normed space $(X, \|.\|)$, since, except when the unit ball of X is smooth(i.e. for all x, $\|x\| = 1$, there is a unique $x^* \in X^*$ such that $\|x^*\| = 1$ and $< x, x^* >= 1$), there are infinitely many semi-inner products on X satisfying (1). However, this defect is more apparent that real, Lumer proved that $\overline{co}W(T)$, the closed convex hull of $W(T)$, is independent of the choice of semi-inner product satisfying (1). In fact, Lumer showed that $\overline{co}W(T)$ depends only on the norms of the operators. Let the numerical index of X be the real number $n(X)$ defined by

$$n(X) = inf\{v(T) : T \in B(X), \|T\| = 1\}.$$

where $v(T) = \sup\{|\lambda| : \lambda \in V(T)\}$
If X is a complex normed linear space, then $1/e < n(X) < 1$. It has long been known that, for a complex Hilbert space X of dimension greater than one, $n(X) = 1/2$ [8]. Glickfeld [7] gives an example of a norm on $\mathbb{C}^2$ for which $n(\mathbb{C}^2) = 1/e$. Duncan, McGregor, Pryce and White [6] prove that if E is a compact Hausdorff space, then $n(C(E)) = 1$. They also prove that for every real number t with $1/e < t < 1$, there exists a norm on $\mathbb{C}$ which gives $n(\mathbb{C}^2) = t$; and for every real number t with $0 \leq t \leq 1$, there exists a norm on $\mathbb{R}^2$ which gives $n(\mathbb{R}^2) = t$. The numerical index of the weighted Hardy spaces as a Banach spaces, is an open problem. For the generalized case for elements of C^*-algebras and lattice, see the articles [2] and [17].

2 Main Results

Let $\{\beta(n)\}_n$ be a sequence of positive numbers with $\beta(0) = 1$ and let $1 < p < \infty$. Let $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ be such that

$$\|f\|^p = \|f\|^p_{H^p(\beta)} = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used weather or not the series converges for any value of z. The space of such formal power series is called the weighted Hardy space, which is denoted by $H^p(\beta)$. In the case $p = 2$, the classical Hardy space, Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1$, $\beta(n) = (n+1)^{\frac{-1}{2}}$ and $\beta(n) = (n+1)^{\frac{1}{2}}$, respectively. The space $H^2(\beta)$ becomes a Hilbert space with inner product

$$< f, g >= \sum_{n=0}^{\infty} a_n b_n \beta(n)^2$$

where $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ are the elements of $H^2(\beta)$[13]. Generally, the spaces $H^p(\beta)$ are reflexive Banach spaces with the norm $\|.\|_{H^p(\beta)}$, and the dual of $H^p(\beta)$ is $H^q(\beta^{\frac{p}{q}})$ where $1/p+1/q = 1$ and $\beta^{p/q} = \{\beta(n)^{p/q}\}$[12].

Definition 2.1. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let $f = \{\hat{f}(n)\}_{n=0}^{\infty} \in H^p(\beta)$ and define $\hat{f^*}(n) = |\hat{f}(n)|^{p-2}\hat{f}(n)$. Also, let $g = \{\hat{g}(n)\}_{n=0}^{\infty} \in H^q(\beta^{\frac{p}{q}})$ and define $\hat{{}^*g}(n) = |\hat{g}(n)|^{q-2}\hat{g}(n)$.*

Clearly $\|f^*\|_q^q = \|f^*\|^q_{H^q(\beta^{\frac{p}{q}})} = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p$. Hence $\|f\|_p^p = \|f\|^p_{H^p(\beta)} = \|f^*\|_q^q$. Also, by simple computation, we have $\|{}^*g\|_p^p = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p = \|g\|_q^q < \infty$ and so ${}^*g \in H^p(\beta)$. Obviously, one can see that ${}^*(f^*) = f$ for all $f \in H^p(\beta)$ and $({}^*g)^* = g$ for all g in $(H^p(\beta))^*$.

By simple computation we have the following consequences:

(a) If $\alpha \geq 0$ and $f \in H^p(\beta)$ then $(\alpha f)^* = \alpha^{\frac{p}{q}} f^*$

(b) If $f \in H^p(\beta)$, $< f, f^* >= \|f\|_p^p$

Recall that $V(T)$ is star shaped with respect to zero if $tz \in V(T)$ for $0 \leq t \leq 1$ and $z \in V(T)$.

Lemma 2.2. *If T is a bounded linear operator on $H^p(\beta)$, then $V(T)$ is star shaped with respect to zero if and only if*

$$V(T) = \{< Tf, f^* >: f \in H^p(\beta), \|f\|_p \leq 1\}.$$

Proof. For "if" part suppose $z \in V(T)$, there is $f \in H^p(\beta)$, $\|f\|_p \leq 1$ and $z = \sum(\widehat{Tf})(n)|\hat{f}(n)|^{p-1} sgn(\hat{f}(n))\beta(n)^p$. If $0 \leq t = \gamma^p \leq 1$ then

$$tz = \sum (\widehat{Tg})(n)|\hat{g}(n)|^{p-1} sgn(\hat{g}(n))\beta(n)^p,$$

where $\hat{g}(n) = \gamma \hat{f}(n)$. Since $\|g\| \leq 1$, $tz \in V(T)$ so $V(T)$ is star shaped with respect to zero. For "only if" part suppose that $\|f\| \leq 1$ and $f \neq 0$. Then

$$< Tf, f^* >= \|f\|_p^p < T(\frac{f}{\|f\|_p}), (\frac{f}{\|f\|_p})^* >\in V(T)$$

□

Theorem 2.3. If T is a bounded linear operator on $H^p(\beta)$, then

$$V(T) = \{< Tf, f^* >: \|f\| = 1, f \in H^p(\beta)\}.$$

Proof. Suppose $f \in H^p(\beta)$, $g \in (H^p(\beta))^*$, $\|f\| = \|g\| = 1$ and $< f, g >= 1$. Then

$$\begin{aligned} 1 &= < f, g > \\ &= \sum \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p \\ &= \sum \hat{f}(n)\beta(n)\overline{\hat{g}(n)}\beta(n)^{p/q} \\ &\leq \{\sum |\hat{f}(n)|^p \beta(n)^p\}^{1/p}\{\sum |\hat{g}(n)|^q \beta(n)^p\}^{1/q} \\ &= 1. \end{aligned}$$

So equality occurs in Holder inequality and then by [10] there is a complex number α such that

$$|\hat{f}(n)|^p \beta(n)^p = \alpha |\hat{g}(n)|^q \beta(n)^p$$

and

$$arg(\hat{f}(n)\overline{\hat{g}(n)}) = \eta \ (independent \ \ of \ \ n)$$

so

$$|\hat{f}(n)|^p = \alpha |\hat{g}(n)|^q.$$

But

$$\begin{aligned} 1 &= \|f\|_p^p \\ &= \sum |\hat{f}(n)|^p \beta(n)^p \\ &= \alpha \sum |\hat{g}(n)|^q \beta(n)^p \\ &= \alpha. \end{aligned}$$

Then

$$|\hat{f}(n)|^p = |\hat{g}(n)|^q.$$

On the other hand

$$\begin{aligned} 1 &= < f, g > \\ &= \sum \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p \\ &= \sum |\hat{f}(n)||\hat{g}(n)| \ e^{iarg(\hat{f}(n)\overline{\hat{g}(n)})}\beta(n)^p \\ &= e^{i\eta} \sum |\hat{f}(n)||\hat{f}(n)|^{p/q}\beta(n)^p \\ &= e^{i\eta}. \end{aligned}$$

Then

$$e^{iarg(\hat{f}(n)\overline{\hat{g}(n)})} = 1$$

or $e^{iarg(\hat{f}(n))} = e^{iarg(\hat{g}(n))}$ and then $\hat{g}(n) = |\hat{f}(n)|^{p/q} e^{iarg(\hat{f}(n))}$, or $g = f^*$. Then the unit ball of $H^p(\beta)$ is smooth and so there is one and only one semi-inner product on $H^p(\beta)$ satisfy (1). If $T \in H^p(\beta)$ and $W(T)$ is the numerical range of T resect to this semi-inner product, then $V(T) = W(T)$ [3]. Put

$$F_f := \|f\|^{2-p} f^*$$

and define

$$[f_1, f_2] := < f_1, F_{f_2} >$$

for f, f_1 and f_2 in $H^p(\beta)$. Then $[.,.]$ is a semi inner product on $H^p(\beta)$ and

$$[f, f] = \|f\|_p^2$$

so

$$\begin{aligned} W(T) &= \{< Tf, F_f >: \ \ \|f\|_p = 1 \ \ and \ \ f \in H^p(\beta)\} \\ &= \{< Tf, f^* >: \ \ \|f\|_p = 1 \ \ and \ \ f \in H^p(\beta)\} \end{aligned}$$

Then the proof is complete. □

The usual numerical range of a bounded linear operator on a Hilbert space is convex and for every bounded linear operator T on a normed space X, we know that $V_{ab}(T)$ is convex. By following example we show that the numerical range of linear operator T on $H^p(\beta)$ is not in general convex, even if T is compact.

Example 2.4. *Let $\beta(1) = 1$ and T be the linear operator on $H^p(\beta)$ given by*

$$\hat{Tf}(n) = \begin{cases} i\hat{f}(0) + \hat{f}(1) & n = 0 \\ -(\hat{f}(0) + i\hat{f}(1)) & n = 1 \\ 0 & n > 1 \end{cases}$$

Therefore we have conclude that $V(T)$ is not convex unless $p = 2$.

For more details see [1] .

Theorem 2.5. (a) There is a compact operator T on $H^p(\beta)$ such that $W(T)$ is not closed.

(b) the numerical range of a compact operator on $H^p(\beta)$ contains all nonzero extreme points of its closure.

(c) the numerical range of a compact operator on $H^2(\beta)$ is closed if and only if it contains the origin.

Proof. Since $H^p(\beta)$ is infinite dimensional (a) holds [14]. Let $\mu(K) = \sum_{n\in K} \beta(n)^p$, for $K \subseteq \mathbb{N}$. Then μ is a σ-finite measure on $\mathbb{N}$ and $H^p(\beta) = L^p(\mu)$[12]. Since (b) and (c) holds for L^p−spaces and Hilbert spaces [14], then holds for $H^p(\beta)$. □

Theorem 2.6. Let T be a compact operator on $H^p(\beta)$. If $V(T)$ is convex and contain the origin then is closed.

Proof. First we show that

$$V(T) = \{< Tf, f^* >: \|f\| \leq 1, f \in H^p(\beta)\}. \quad (2)$$

For this suppose that $\|f\| \leq 1$ then

$$< Tf, f^* >= \|f\|_p^p < T(\frac{f}{\|f\|_p}), (\frac{f}{\|f\|_p})^* > +(1 - \|f\|_p^p)0$$

since $V(T)$ is convex and $0 \in V(T)$ then $< Tf, f^* >\in V(T)$

Now let $\alpha \in \overline{V(T)}$ so there is a sequence h_n with $\|h_n\|_p = 1$ and $< Th_n, h_n^* >\to \alpha$, by reflexivity of $H^p(\beta)$ and also Alaogul's Theorem there is a sequence $\{n_k\}_{k=1}^{\infty}$ such that $h_{n_k} \to h$ in weak topology and $h_{n_k}^* \to g$ in weak* topology for some $h \in ball(H^p(\beta))$ and $g \in ball((H^p(\beta))^*)$.

We claim that: $g = h^*$.

To prove this claim, let $m \in \mathbb{N}$. Define the bounded linear functionals x, x^* by

$$x(f^*) := \hat{f}^*(m)$$

and

$$x^*(f) := \hat{f}(m)$$

respectively on $(H^p(\beta))^*$ and $H^p(\beta)$. Then

$$< h_{n_k}, x^* >\to< h, x^* >$$

and

$$< h_{n_k}^*, x >\to< g, x >$$

as $k \to \infty$. Then

$$\hat{h}_{n_k}(m) \to \hat{h}(m)$$

and

$$\hat{h^*}_{n_k}(m) \to \hat{g}(m)$$

as $k \to \infty$.But by definition $\hat{h^*}_{n_k}(m) = |\hat{h}_{n_k}(m)|^{\frac{p}{q}} e^{iarg(\hat{h}_{n_k}(m))}$ Then $\hat{g}(m) = |\hat{h}(m)|^{\frac{p}{q}} e^{iarg(\hat{h}(m))}$ or $g = h^*$ However,

$$\begin{aligned} |<Th_{n_k}, h^*_{n_k}> - <Th, h^*>| &\leq |<Th_{n_k}, h^*_{n_k}> - <Th, h^*_{n_k}>| \\ &+ |<Th, h^*_{n_k}> - <Th, h^*>| \\ &= |<T(h_{n_k} - h), h^*_{n_k}>| \\ &+ |<Th, (h^*_{n_k} - h^*)| \\ &\leq \|T(h_{n_k} - h)\| \|h^*_{n_k}\| \\ &+ |<Th, (h^*_{n_k} - h^*)|. \end{aligned}$$

Since T is completely continuous and $h_{n_k} \to h$ weakly then $\|T(h_{n_k} - h)\| \to 0$ [5]

Therefor $<Th_{n_k}, h^*_{n_k}> \to <Th, h^*>$ and so $\alpha = <Th, h^*>$. Then the proof is complete by (2). □

Corollary 2.7. *[9]. Let T be a compact operator on $H^p(\beta)$. If $0 \in V(T)$ then $co(V(T))$ is closed.*

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