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On locally divided prime ideals of an integral domain and going-down to P

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Abstract. Let R be an integral domain with quotient field K and let $P \in \mathrm{Spec}(R)$. Several characterizations are given for P being a locally divided prime ideal of R. One of those characterizations specifies a subset $\Omega(R,P)$ of K such that P is a locally divided prime ideal of R if and only if P is straight with respect to $R \subseteq R[u]$ for each $u \in \Omega(R,P)$. Another characterization is the conjunction of a condition (α) (specifying that certain overring extensions of R satisfy GD to P) and a condition (β) (specifying a weakened version of P being straight with respect to certain overring extensions of R). It is shown that $R \subseteq T$ satisfies GD to P for each domain T having R as a subring if (and only if) $R \subseteq R[u]$ satisfies GD to P for each $u \in K$. Some sufficient conditions are given for the just-mentioned property to hold. While some of those sufficient conditions include aspects with the flavor of (β), others of those sufficient conditions have no overtly "straight" flavor. For instance: if R is quasi-local and treed and each element of PR_P is integral over R, then $R \subseteq T$ satisfies GD to P for each domain T that contains R as a subring. Nearly half the paper is devoted to proving a "one P at a time" theorem one of whose corollaries generalizes our 50-year-old result that a seminormal domain is a locally divided domain if (and only if) it is a going-down domain.

Key Words: Integral domain, prime ideal, locally divided prime ideal, going-down to P, P-unibranched ring extension, maximal ideal, integrality, prime ideal that is straight with respect to a ring extension, quotient field, CPI-overring, overring, seminormality, torsion-free module.
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1 Introduction

This paper is a sequel to [14]. All rings considered here are associative and commutative, and they are usually (integral) domains; all inclusions of rings denote subrings; and all rings, algebras, ring/algebra homomorphisms, subrings and modules are unital. If R is a domain with quotient field K, then by an *overring* of R, we mean any ring T such that $R \subseteq T \subseteq K$. For any ring A, $\mathrm{Spec}(A)$ denotes the set of prime ideals of A and $\mathrm{Max}(A)$ denotes the set of maximal ideals of A. In the spirit of [27], we let GD denote the going-down property of ring homomorphisms (for instance, of ring extensions).

Before summarizing the content of each of Sections 2-5 individually, we devote the next paragraph to a summary of some necessary background. Readers desiring additional background are directed to the Introduction of [14], [21] and [17].

Let R be a domain with quotient field K. Recall from [8] and [19] that R is said to be a *going-down domain* if $R \subseteq T$ satisfies GD for each overring T of R (equivalently, if $R \subseteq T$ satisfies GD for each domain T having R as a subring; equivalently, if $R \subseteq R[u]$ satisfies GD for each $u \in K$). The "GD to P" property of ring extensions ($A \subseteq B$ with $P \in \mathrm{Spec}(A)$) was characterized in several useful ways by Kaplansky in Exercise 37, pages 44-45 of [27]. (We will cite that exercise often in this paper.) It is clear that a ring extension $A \subseteq B$ satisfies GD if and only if $A \subseteq B$ satisfies GD to P for each $P \in \mathrm{Spec}(A)$. In particular, R is a going-down domain if and only if $R \subseteq R[u]$ satisfies GD to P for each

$u \in K$ and each $P \in \mathrm{Spec}(R)$. This "one P at a time" style of characterizing several classes of domains has recently become popular, thanks mostly to a paper by Secord [30]. That paper showed, *i.a.*, that straight domains (in the sense of [21]) were the same as locally divided domains (in the sense of [9]), as a consequence of proving that a prime ideal P of R is straight with respect to $R \subseteq T$ (in the sense that T/PT is a torsion-free module over R/P) for each domain T containing R as a subring if and only if P is a locally divided prime ideal of R. (Some more background: In [1], Akiba studied domains R such that $PR_P = P$ for all $P \in \mathrm{Spec}(R)$ and named these AV-domains. In [9], these domains were called "divided domains"; and R was said to be a locally divided domain if R_M is a divided domain for each $M \in \mathrm{Max}(R)$. It was proved in Remark 2.7 (b) of [9] that each locally divided domain is a going-down domain; and in Corollary 2.8 of [9] that the converse holds for root closed domains (actually, the proof of the converse works for seminormal domains). Also, Example 2.9 of [9] showed that a going-down domain need not be a locally divided domain. Secord's point of view is that $P \in \mathrm{Spec}(R)$ is a *locally divided prime ideal* of R if and only if PR_M is a divided prime ideal of R_M for all $M \in \mathrm{Max}(R)$ containing P; that is, if and only if $PR_P = PR_M$ for all such M. It is clear that R is a locally divided domain if and only if each $P \in \mathrm{Spec}(R)$ is a locally divided prime ideal of R. Here is a third example of the "one P at a time" theme: it turns out that R is a straight domain if and only if each $P \in \mathrm{Spec}(R)$ is straight with respect to $R \subseteq T$ for each overring T of R.) As it was shown in [21] that all locally divided domains are straight (see its Corollary 3.8) and that all straight domains are going-down domains, Secord's results in [30] were quite timely. The above-mentioned exercise in [27] opened another direction for us to pursue in [14]. That exercise implies that if $P \in \mathrm{Spec}(R)$ and T is a domain containing R as a subring such that P is straight with respect to $R \subseteq T$, then $R \subseteq T$ satisfies GD to P. Since a ring extension $A \subseteq B$ satisfies GD if and only if $A \subseteq B$ satisfies GD to P for each $P \in \mathrm{Spec}(A)$, it seemed clear to the author that two interesting projects would be to find conditions such that, when "$R \subseteq T$ satisfies GD to P" is conjoined with such a condition, that conjunction is equivalent to P being straight with respect to $R \subseteq T$; and to find other conditions such that, when a suitable GD-theoretic condition is conjoined with such a condition, that conjunction is a sufficient condition for P to be a locally divided prime ideal of R. The paper [14] had two purposes: to give a short, direct proof that straight domains are locally divided domains; and then to begin the process of working "one P at a time" on the two just-mentioned projects. The present paper is intended to further the above two projects, in sympathy with the "one P at a time" approach, while being alert for applications to classical domains that can be obtained by applying universal quantification to some "one P at a time" results.

We proceed to summarize the content of Sections 2-5, devoting this paragraph to Section 2. In accordance with the view of Grothendieck-Dieudonné that the "GD to P" property should be studied in the context of arbitrary ring homomorphisms (not simply for inclusion maps of rings), Section 2 begins with a careful analysis of the action of the Spec functor on any ring of fractions. A consequence, in Proposition 2.3, is that, in the context of ring homomorphisms, the "going-down to P" property comports itself well with respect to local/global behavior. Returning to the context of inclusion maps of domains, Theorem 2.4 gives a new proof that if P is a locally divided prime ideal of a domain R, then $R \subseteq T$ satisfies GD to P for every domain T that contains R as a subring. (In Corollary 3.1.3 of [30], Secord proved the stronger result that the just-mentioned hypothesis implies that P is a straight prime ideal of R with respect to $R \subseteq T$ for any domain T that contains R as a subring. Proposition 2.6 uses Theorem 2.4 to give a new proof of Corollary 3.1.3 of [30].) Corollary 2.5 gives another noteworthy consequence of the "one P at a time" approach in Theorem 2.4, by providing a quick, new proof of a motivating result that was mentioned above, namely, the fact that every locally divided domain is a going-down domain. Readers who are interested in the "GD to P" property only for inclusions of domains may be able to skip Section 2 on a first reading or to refer to Section 2 only when necessary.

This paragraph summarizes the content of Section 3. Theorem 3.4 gives this paper's first characterization of a locally divided prime ideal of a domain. It states that if R is a domain and $P \in \mathrm{Spec}(R)$, then P is a locally divided prime ideal of R if and only if certain conditions (α) and (β) hold. Condition (α) states that $R \subseteq R[u^{-1}]$ satisfies GD to P for each nonzero element $u \in PR_P$; while we will leave the precise formulation of condition (β) to Section 3, we note here that (β) is constructed by using Lemma 3.1 (a result whose proof involves Nakayama's Lemma and was inspired by Secord's use of Nakayma's Lemma in [30]). The above-mentioned Proposition 2.3 and Corollary 2.4 play important roles in the proof of Theorem 3.4. Proposition 3.8 gives a companion for Secord's result that, for a domain R and a prime ideal P of R, P is a locally divided prime ideal of R if and only if P is a straight prime ideal of R with respect to $R \subseteq T$ for every domain T that contains R as a subring. Indeed, Proposition 3.8 shows that a certain subset $\Omega(R,P)$ of the quotient field of R (which was defined in [14]) provides "test domain extensions" for this property, as follows: P is a straight prime ideal of R with respect to $R \subseteq R[u]$ for each $u \in \Omega(R,P)$ (if and) only if P is a straight prime ideal of R with respect to $R \subseteq T$ for every domain T that contains R as a subring. Without explicitly stating it, Secord's reasoning in [30] essentially proved the case of Proposition 3.8 where R is quasi-local.

We next summarize Section 4. For a prime ideal P of a domain R, Proposition 4.2 provides a sufficient condition that P be a locally divided prime ideal of R, by augmenting the above-mentioned condition (β) with a condition that is motivated by a result (Proposition 2.1 of [10]) that characterized quasi-local going-down domains. (That just-mentioned result from [10] is itself sharpened, in the "one P at a time" manner, in Corollary 4.9.) A result that finds use later in Section 4 and in Section 5 is Proposition 4.3, which, for a domain R with a prime ideal P contained in a maximal ideal M, gives a sufficient condition that the set of prime ideals of R that are contained in M is pinched at P. The main sufficient conditions for the "GD to P" property in Section 4 flow from Theorem 4.6 (a) [see, especially Corollaries 4.7-4.9]. No part of those sufficient conditions has an apparent "straight"-related flavor. For instance, Theorem 4.6 (a) states the following: if R is a domain, P a prime ideal of R and M a maximal ideal of R that contains P such that the set of prime ideals of R contained in M is linearly ordered by inclusion and if also each element of PR_P is integral over R_M, then $R_M \subseteq T_{R\setminus M}$ satisfies GD to PR_M for each domain T that contains R as a subring. (Note that the "linear ordering" hypothesis in Theorem 4.6 (a), while stronger than the "pinched conclusion" in Proposition 4.3, is weaker than a hypothesis that R is a treed domain.) Because of our interest in finding a reasonably "small" set of simple overrings that suffices as a set of "test domain extensions" for certain properties, we would like to also highlight Theorem 4.6 (b) here. It states the following: if R is a domain, P a prime ideal of R and M a maximal ideal of R that contains P such that the set of prime ideals of R contained in M is linearly ordered by inclusion, then $R_M \subseteq R_M[v^{-1}]$ satisfies GD to PR_M for each nonzero element $v \in PR_P$ (if and) only if $R_M \subseteq T_{R\setminus M}$ satisfies GD to PR_M for each domain T that contains R as a subring. Also, we note that Theorem 4.10 sharpens, in the "one P at a time" manner, part of Theorem 1 of [19], as follows: if R is a domain with quotient field K and $P \in \mathrm{Spec}(R)$, then $R \subseteq R[u]$ satisfies GD to P for each $u \in K$ (if and) only if $R \subseteq T$ satisfies GD to P for each domain T that contains R as a subring.

Finally, we come to a summary of the most consequential (and longest) section, Section 5. Whereas much of the earlier part of this paper needed to use information about how "GD to P" and related "straight" properties behave with respect to integrality, Section 5 is aimed at the more delicate behavior when "integrality" is replaced by "seminormality". Put differently: whereas much of the earlier part of this paper exploits the first part of [9] which sharpened a result of McAdam (Corollary 11 of [29]) which established, in effect, that an integrally closed domain is a locally divided domain if and only if it is a going-down domain, Section 5 is aimed at a "one P at a time" sharpening of the result in Corollary 2.6 of [9] which established, in effect, that a seminormal domain is a locally divided domain if and only if it is a going-down domain (see also Corollary 2.8 of [9]). That sharpening is given in Corollary 5.15. That result is one of several corollaries of Theorem 5.10, which is the main result in Section 5. The next paragraph summarizes most of the main theorems of Section 5 while explaining how those results and a newly introduced concept in Section 5 lead to a successful adaptation of the proof of Corollary 2.6 in [9] to what becomes the proof of the most difficult part of Theorem 5.10.

To appreciate the nature and extent of the novelty that was needed to adapt the proof of Corollary 2.6 of [9] into a proof of Theorem 5.10 (c), readers are invited to (eventually) peruse the first paragraph, the second paragraph and, especially, the sixth paragraph of the proof of Theorem 5.10 (c), paying special attention to the results and concepts that are used in those paragraphs. As Remark 5.18 (b) reiterates, the essential concepts that figure in the proof of Theorem 5.10 (c) but do not appear at all in [9] are the concepts of a P-unibranched extension (which I believe to be new here) and a contraction map $c_A^B : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ that gives an order isomorphism of prime spectra. These concepts and our results about them in this paper are essential in proving the assertion that $\mathcal{P} \subseteq I$ in the proof of Theorem 5.10 (c). The corresponding assertion in the proof of Corollary 2.6 of [9] was much easier to prove because the ideal that was called I in [9] was a prime ideal of the given quasi-local base domain (as that paper's base domain was a quasi-local going-down domain, hence a quasi-local treed domain, so that the radical ideal I in that paper was an intersection of a chain of prime ideals and, hence, was itself a prime ideal of that base domain). Needing to focus our attention on only the prime ideals of R that compare with a given $P \in \mathrm{Spec}(R)$ led us naturally to work inside the CPI-overring (in the sense of Boisen and Sheldon [5]) $T = R + PR_P$. It became important in the proof to show, with a certain element $u \in PR_P$ in hand, that the extension $R \subseteq S := R[u]$ "looked unibranched insofar as the set of prime ideals of R comparable to P was concerned". The phrase in quotation marks in the preceding sentence indicates how to define a P-unibranched ring extension. While results in [5] ensure that $R \subseteq T$ *is* a P-unibranched extension, the proof of Theorem 5.10 needs that the ring extension $R \subseteq S$ is P-unibranched. Fortunately, that "descent" of P-unibranchedness from $R \subseteq T$ to $R \subseteq S$ is possible in the proof of Theorem 5.10 (c) because an integrality assumption on u allows us to apply the "descent" result in Theorem 5.4 (whose formulation allows us to relax the integrality assumption to the requirement that both $R \subset S$ and $S \subset T$ satisfy the lying-over property). Once $R \subset S$ is known to be P-unibranched, it follows (from integrality) that $R \subset S$ is actually unibranched. However, that is still not, in itself, enough to prove that $\mathcal{P} \subseteq I$. To prove that fact, the proof of Theorem 5.10 (c) needs that the canonical contraction map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is an isomorphism of posets (under inclusion). Unfortunately, a unibranched ring extension need not induce an order isomorphism of prime spectra. Indeed, if $2 \leq n \leq \infty$, Theorem 5.6 produces an example of domains $A \subset B$, each of Krull dimension n, such that $A \subseteq B$ is unibranched (hence, Q-unibranched for each $Q \in \mathrm{Spec}(A)$) but the contraction map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is not an order isomorphism of posets under inclusion. Fortunately, the contraction map $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ in the proof of Theorem 5.10 (c) *is* an order isomorphism of prime spectra because the assumed integrality allows us to apply Proposition 5.7, a result ensuring that certain unibranched extensions *do* induce order isomorphisms of prime spectra. (In fact, the formulation of Proposition 5.7 allows an assumption of integrality to be relaxed to the requirement that the given unibranched extension satisfies the going-up property.)

The following comments complete the summary of Section 5. The notion of seminormality, *per se*, plays no role in Sections 2-5 in this paper until after the proof of Theorem 5.10. Prior to giving five corollaries of that theorem, Remark 5.11 provides an incomplete but serviceable summary of some of the early research on seminormality. Perhaps the most technically sharp upshot of Theorem 5.10 is given in Corollary 5.12 (a), which has a kind of "finite character" application in Corollary 5.13. Also, with the goal of a "one P at a time" approach to seminormality, we define (given a domain R, $P \in \mathrm{Spec}(R)$ and $M \in \mathrm{Max}(R)$ such that $P \subseteq M$) what it means for P to be a "strongly seminormal prime ideal of R with respect to M". This concept (together with its universal quantification over all $M \in \mathrm{Max}(R)$ which contain P) leads to sufficient conditions in parts (b) and (d) of Corollary 5.14 for P to be a locally divided prime ideal of R. These lead directly to the promised sharpening (in Corollary 5.15) of the motivating result that a seminormal domain is a locally divided domain if (and only if) it is a going-down domain. Remark 5.18 generalizes Theorem 5.10 (for base domains that are not necessarily quasi-local) and raises some questions.

About notation: the cardinal number of a set $\mathcal{S}$ will be denoted by $|\mathcal{S}|$; "dim" denotes Krull dimension; and $\subset$ denotes proper inclusion.

We wish to warmly thank Dr. Michael Saum for providing, at our request, the LaTeX keystroke instructions that converted our freehand drawing into parts (a) and (b) of Figure 1, which appears in Section 4.

2 Locally divided prime ideals exhibit GD to P

Consider a ring A, a multiplicatively closed subset S of A and a ring homomorphism $f : A \to B$. Then f induces an A-module structure on B (via $a \cdot b := f(a)b$ for all $a \in A$ and $b \in B$). As a consequence, one can consider the A-module B_S (where B_S can be defined either in the usual way as a set of equivalences classes or as $A_S \otimes_A B$). Since A_S and B are each commutative A-algebras, so is $A_S \otimes_A B$ (in the usual way, induced by $(\lambda \otimes \mu)\cdot(\lambda' \otimes \mu') := \lambda\lambda' \otimes \mu\mu'$ for all $\lambda, \lambda' \in A_S$ and all $\mu, \mu' \in B$). Hence, B_S is also a commutative A-algebra. Moreover, the function $A_S \to B_S$ $(:= A_S \otimes_A B)$, which sends $\xi \in A_S$ to $\xi \otimes 1$, is easily seen to be a ring homomorphism, and so B_S is an A_S-algebra. On the other hand, since $f(S)$ is a multiplicatively closed subset of B, one can also consider the ring of fractions $B_{f(S)}$. Moreover, the assignment $a/z \mapsto f(a)/f(z)$ induces a (well-defined) ring homomorphism $A_S \to B_{f(S)}$, and so $B_{f(S)}$ is also a commutative A_S-algebra. It is natural to ask what relationships may exist between B_S and $B_{f(S)}$.

The answer to the above question may seem obvious in case f is an inclusion map, for in this case, $f(S) = S$ and one checks easily that B_S and $B_{f(S)}$ are isomorphic A_S-algebras. That kind of conclusion is somewhat less obvious when f is not an injection. (For an extreme example of such a situation, let X and Y be commuting algebraically independent indeterminates over a field K, take $A := K[X][Y]$ and $B := K[X]$, take $S := \{Y^n \mid n = 0, 1, 2, \dots\}$ (the multiplicatively closed set generated by Y), and let $f : A \to B$ be the $K[X]$-algebra homomorphism sending Y to 0. One can check that for this example, B_S and $B_{f(S)}$ are isomorphic because each is a/the zero ring.) Regardless of how (un)clear the general situation may seem at first glance, the literature has implicitly realized (at least in important special cases) that B_S and $B_{f(S)}$ can be shown to be isomorphic A_S-algebras in such a transparent way that there is no harm in identifying them. For the sake of completeness, Proposition 2.1 will record the details of that "transparent way" for the general situation. However, whenever f is not an inclusion map, this paper will take care to distinguish between B_S and $B_{f(S)}$ (as, for instance, in Lemma 2.2).

The detailed but elementary proof of Proposition 2.1 is included for the sake of completeness.

Proposition 2.1. *Let $f : A \to B$ be a homomorphism of (commutative) rings and let S be a multiplicatively closed subset of A. Then B_S (as defined above) and $B_{f(S)}$ are isomorphic A_S-algebras. Moreover, the assignment $b/z \mapsto b/f(z)$ (for $b \in B$ and $z \in S$) gives a (well-defined) A_S-isomorphism $B_S \to B_{f(S)}$.*

Proof. The structure of $B_{f(S)}$ as an A_S-algebra is clear, as that structure is due to the ring homomorphism $A_S \to B_{f(S)}$ given by the (well defined) assignment $a/z \mapsto f(a)/f(z)$ (for $a \in A$ and $z \in S$). However, since the structure of B_S as an A_S-algebra was described above by viewing B_S as $A_S \otimes_A B$, it will be helpful to explicate the arithmetic operations in B_S when the elements of B_S are viewed as (equivalence classes of) fractions. Consider $b_1, b_2 \in B$ and $z_1, z_2 \in S$. The sum of fractions, $b_1/z_1 + b_2/z_2$, corresponds to

$$\frac{1}{z_1} \otimes b_1 + \frac{1}{z_2} \otimes b_2 = z_2 \cdot (\frac{1}{z_1 z_2}) \otimes b_1 + z_1 \cdot (\frac{1}{z_1 z_2}) \otimes b_2 = \frac{1}{z_1 z_2} \otimes (z_2 \cdot b_1 + z_1 \cdot b_2)$$

in $A_S \otimes_A B$, which corresponds to the element $(z_2 \cdot b_1 + z_1 \cdot b_2)/(z_1 z_2)$ in B_S (when B_S is constructed via fractions). In short, when the elements of B_S are viewed via fractions, the formula for the addition operation in B_S is as expected. It is even easier to check that the formula (using fractions) for multiplication in B_S is given by $(b_1/z_1)\cdot(b_2/z_2) = (b_1b_2)/(z_1z_2)$, since

$$(\frac{1}{z_1}\otimes b_1)(\frac{1}{z_2}\otimes b_2) = (\frac{1}{z_1})(\frac{1}{z_2})\otimes b_1b_2 = \frac{1}{z_1z_2}\otimes b_1b_2$$

in $A_S\otimes_A B$. It is now apparent that one should consider the assignment $b/z\mapsto b/f(z)$ in attempting to establish an A_S-algebra isomorphism $F: B_S\to B_{f(S)}$. We need only check the following four facts: (1) this assignment is well defined; (2) the resulting function F is an A_S-algebra homomorphism; (3) that function F is a surjection; and (4) that function F is an injection.

(1): We must show that if $b_1/z_1 = b_2/z_2$ in B_S, then $b_1/f(z_1) = b_2/f(z_2)$ in $B_{f(S)}$. By hypothesis, there exists $\zeta\in S$ such that

$$\zeta\cdot(z_2\cdot b_1 - z_1\cdot b_2) = 0$$

in B. It will suffice to show that $\zeta\cdot(f(z_2)b_1 - f(z_1)b_2) = 0$ in B. This, in turn, holds since

$$\zeta\cdot(z_2\cdot b_1 - z_1\cdot b_2) = f(\zeta)[f(z_2)b_1 - f(z_1)b_2)] = \zeta\cdot(f(z_2)b_1 - f(z_1)b_2).$$

(2): By (1), F is a well-defined function. By the first paragraph of this proof (and that fact that f preserves products), we get that F is a homomorphism with respect to both addition and multiplication. It is clear that $F(1) = 1$. It remains only to check that F preserves scalar multiplication by elements of A_S. To that end, we have

$$F(\frac{a}{z_1}\cdot\frac{b}{z_2}) = F(\frac{a\cdot b}{z_1z_2}) = F(\frac{f(a)b}{z_1z_2}) = \frac{f(a)b}{f(z_1z_2)} =$$

$$\frac{f(a)b}{f(z_1)f(z_2)} = (\frac{f(a)}{f(z_1)})(\frac{b}{f(z_2)}) = (\frac{a}{z_1})\cdot F(\frac{b}{z_2}).$$

(3): Clear.

(4): It will suffice to show that if $\xi\in\ker(F)$, then $\xi = 0$. Write $\xi = b/z$ for some $b\in B$ and some $z\in S$. By hypothesis, $F(b/z) = 0$ in B_S; that is, $b/f(z) = 0/1$ in $B_{f(S)}$; that is, there exists $\zeta\in S$ such that $f(\zeta)[1\cdot b - f(z)\cdot 0] = 0$ in B; that is, $f(\zeta)b = 0$ in B. Our task is to show that there exists $\omega\in S$ such that $\omega\cdot[1\cdot b - z\cdot 0] = 0$ in B; that is, such that $f(\omega)b = 0$. Taking $\omega := \zeta$ suffices. The proof is complete. □

Lemma 2.2 focuses attention on a particular behavior of the prime ideals of rings of fractions. This behavior will be used in Proposition 2.3 to show that GD to P is a local property of rings (in the strongest possible sense). That will enable us to prove, in Theorem 2.4, the main/titular result of this section, namely, that if P is a locally divided prime ideal of a domain R and if R is a subring of a domain T, then $R\subseteq T$ satisfies GD to P. As explained in the proof of Corollary 2.5, Theorem 2.4 generalizes the result from [9] that each locally divided domain is a going-down domain.

Lemma 2.2. *Let $f: A\to B$ be a homomorphism of (commutative) rings, let S be a multiplicatively closed subset of A, let $f_S: A_S\to B_{f(S)}$ be the A-algebra homomorphism given by $f_S(a/z) := f(a)/f(z)$ for all $a\in A$ and $z\in S$, and let $Q\in\mathrm{Spec}(B)$ such that $Q\cap f(S) = \emptyset$. Then*

$$(f_S)^{-1}(QB_{f(S)}) = f^{-1}(Q)A_S.$$

Proof. The fact that f_S is an A-algebra homomorphism (indeed, an A_S-algebra homomorphism) was noted in the first paragraph of this section.

We will prove the easier inclusion first, namely, that $f^{-1}(Q)A_S\subseteq(f_S)^{-1}(QB_{f(S)})$. Let $a\in f^{-1}(Q)$, $a^*\in A$ and $z\in S$. Our task is to show that $f_S(a(a^*/z))\in QB_{f(S)}$. We have $f(a)\in Q$, and so $f(aa^*) = f(a)f(a^*)\in QB = Q$. Hence,

$$f_S(a(\frac{a^*}{z})) = f_S(\frac{aa^*}{z}) = \frac{f(aa^*)}{f(z)}\in Q_{f(S)} = QB_{f(S)},$$

as desired.

It remains to prove the reverse inclusion. Our task is to show that if $\xi \in (f_S)^{-1}(QB_{f(S)})$, then $\xi \in f^{-1}(Q)A_S$. We can write $\xi = a/z$ for some $a \in A$ and $z \in S$. It will suffice to prove that $a \in f^{-1}(Q)_S$, for then it would follow that

$$\xi = \frac{a}{z} \in (f^{-1}(Q)_S)_S = f^{-1}(Q)_S = f^{-1}(Q)A_S.$$

We have

$$\frac{f(a)}{f(z)} = (f_S)(\frac{a}{z}) = (f_S)(\xi) \in QB_{f(S)}.$$

Hence, working in $B_{f(S)}$, we have

$$\frac{f(a)}{1} = (\frac{f(a)}{f(z)})(\frac{f(z)}{1}) \in (QB_{f(S)})(B_{f(S)}) = QB_{f(S)}.$$

Let $h : B \to B_{f(S)}$ be the ring homomorphism that is canonically associated with the ring of fractions $B_{f(S)}$. Since $h(f(a)) = f(a)/1 \in QB_{f(S)}$, we get $f(a) \in h^{-1}(QB_{f(S)})$. Recall that $Q \cap f(S) = \emptyset$ by hypothesis. Thus, we also have $f^{-1}(Q) \cap S = \emptyset$. So, by the usual description of the prime ideals of a ring of fractions (cf. Proposition 3.11 (iv) of [2]), we get that $h^{-1}(QB_{f(S)}) = Q$. In particular, $f(a) \in Q$. Consequently, $a \in f^{-1}(Q) \subseteq f^{-1}(Q)_S$. The proof is complete. □

It is worth noting that the special case of Lemma 2.2 when f is an inclusion map, that is when A is a subring of B, admits an especially simple proof. For this case, the result may be stated as follows. If $A \subseteq B$ are rings, S is a multiplicatively closed subset of A and $Q \in \text{Spec}(B)$ satisfies $Q \cap S = \emptyset$, then

$$QB_S \cap A_S = (Q \cap A)_S.$$

The promised "simple proof" is the following sentence. Since $QB_S = Q_S$ and the formation of rings of fractions commutes with finite intersections (cf. Corollary 3.4 (ii) of [2]), we have

$$QB_S \cap A_S = Q_S \cap A_S = (Q \cap A)_S,$$

as required.

In the context of prime ideals of domains, the very definition of the property of being a locally divided prime ideal explains why/how/that this property is a local property. With Lemma 2.2 in hand, it is now only somewhat longer, but not difficult, to show that GD to P is also a local property.

Proposition 2.3. *Let $f : A \to B$ be a ring homomorphism (for instance, an inclusion of rings $A \hookrightarrow B$), and let $P \in \text{Spec}(A)$. Then the following conditions are equivalent:*
(1) f satisfies GD to P;
(2) For each multiplicatively closed subset S of A such that $P \cap S = \emptyset$, the canonically induced ring homomorphism $f_S : A_S \to B_{f(S)}$ satisfies GD to PA_S;
(3) For each $M \in \text{Max}(R)$ such that $P \subseteq M$, the canonically induced ring homomorphism $f_M : A_M \to B_{f(A \setminus M)}$ satisfies GD to PA_M.

Proof. (2) ⇒ (3): Trivial.

(3) ⇒ (1): Assume (3). Let $P \subseteq P_1$ be prime ideals of A and let $Q_1 \in \text{Spec}(B)$ such that $f^{-1}(Q_1) = P_1$. Our task is to find $Q \in \text{Spec}(B)$ such that $Q \subseteq Q_1$ and $f^{-1}(Q) = P$. Pick $M \in \text{Max}(A)$ such that $P_1 \subseteq M$.

Consider $Q_1 B_{f(A \setminus M)} \in \text{Spec}(B_{f(A \setminus M)})$. Since $f^{-1}(Q_1) = P_1$, it follows from Lemma 2.2 (with $S := A \setminus M$) that $(f_M)^{-1}(Q_1 B_{f(A \setminus M)}) = P_1 A_M$. By (3), f_M satisfies GD to PA_M. Hence, there exists $Q \in$

$\mathrm{Spec}(B_{f(A\setminus M)})$ such that $\mathcal{Q} \subseteq Q_1 B_{f(A\setminus M)}$ and $(f_M)^{-1}(\mathcal{Q}) = PA_M$. By the usual description of the prime ideals of a ring of fractions (cf. Proposition 3.11 (iv) of [2]), there exists a unique $Q \in \mathrm{Spec}(B)$ such that $Q \cap f(A \setminus M) = \emptyset$ and $\mathcal{Q} = QB_{f(A\setminus M)}$; and, moreover, if $h : B \to B_{f(A\setminus M)}$ denotes the structure map canonically associated with the ring of fractions $B_{f(A\setminus M)}$, then $Q = h^{-1}(\mathcal{Q})$.

Observe that $Q = h^{-1}(\mathcal{Q}) \subseteq h^{-1}(Q_1 B_{f(A\setminus M)}) = Q_1$. It remains only to check that $f^{-1}(Q) = P$. Note that $f^{-1}(Q) \subseteq f^{-1}(Q_1) = P_1 \subseteq M$. So, since $f^{-1}(Q)$ and P are prime ideals of A that are contained in M, it will suffice to prove that $f^{-1}(Q)A_M = PA_M$. This, in turn, follows from the next displayed chain of equalities, whose first step uses Lemma 2.2:

$$f^{-1}(Q)A_M = (f_M)^{-1}(QB_{f(A\setminus M)}) = (f_M)^{-1}(\mathcal{Q}) = PA_M.$$

.

(1) ⇒ (2): Assume (1). Let $\mathcal{P} \subseteq \mathcal{P}_1$ be prime ideals of A_S and let $\mathcal{Q}_1 \in \mathrm{Spec}(B_{f(S)})$ such that $(f_S)^{-1}(\mathcal{Q}_1) = \mathcal{P}_1$. Our task is to find $\mathcal{Q} \in \mathrm{Spec}(B_{f(S)})$ such that $\mathcal{Q} \subseteq \mathcal{Q}_1$ and $(f_S)^{-1}(\mathcal{Q}) = \mathcal{P}$.

Let $h : B \to B_{f(S)}$ and $g : A \to A_S$ denote the structure maps canonically associated with the rings of fractions $B_{f(S)}$ and A_S, respectively. Put $Q_1 := h^{-1}(\mathcal{Q}_1)$ in $\mathrm{Spec}(B)$, as well as $P_1 := g^{-1}(\mathcal{P}_1)$ and $P := g^{-1}(\mathcal{P})$ in $\mathrm{Spec}(A)$. By the usual description of the prime ideals of a ring of fractions (cf. Proposition 3.11 (iv) of [2]), $\mathcal{Q}_1 = Q_1 B_{f(S)}$, $\mathcal{P}_1 = P_1 A_S$ and $\mathcal{P} = PA_S$, with $Q_1 \cap f(S) = \emptyset$ and $P_1 \cap S = \emptyset$ $(= P \cap S)$.

Observe that $P := g^{-1}(\mathcal{P}) \subseteq g^{-1}(\mathcal{P}_1) = P_1$ and, by Lemma 2.2, that

$$f^{-1}(Q_1)A_S = (f_S)^{-1}(Q_1 B_{f(S)}) = (f_S)^{-1}(\mathcal{Q}_1) = \mathcal{P}_1 = P_1 A_S.$$

So, via the oft-cited standard bijection, we get $f^{-1}(Q_1) = P_1$. Therefore, since f satisfies GD to P by hypothesis, there exists $Q \in \mathrm{Spec}(B)$ such that $Q \subseteq Q_1$ and $f^{-1}(Q) = P$. It will suffice to prove the following four facts: (i) $Q \cap f(S) = \emptyset$; (ii) $\mathcal{Q} := QB_{f(S)} \in \mathrm{Spec}(B_{f(S)})$; (iii) $\mathcal{Q} \subseteq \mathcal{Q}_1$; and (iv) $(f_S)^{-1}(\mathcal{Q}) = \mathcal{P}$.

(i): Note that $Q \cap f(S) \subseteq Q_1 \cap f(S) = \emptyset$.

(ii): Since $Q \in \mathrm{Spec}(B)$ satisfies (i), it suffices to apply the oft-cited bijection.

(iii): Note that $\mathcal{Q} = QB_{f(S)} \subseteq Q_1 B_{f(S)} = \mathcal{Q}_1$.

(iv): Using Lemma 2.2, we get $(f_S)^{-1}(\mathcal{Q}) = (f_S)^{-1}(QB_{f(S)}) = f^{-1}(Q)A_S = PA_S = \mathcal{P}$. The proof is complete. □

We next prove the titular result of this section.

Theorem 2.4. Let P be a locally divided prime ideal of a domain R. Then $R \subseteq T$ satisfies GD to P for every domain T that contains R as a subring.

Proof. Suppose that the assertion fails. Then there exists a domain T containing R as a subring such that $R \subseteq T$ does not satisfy GD to P. Hence, by the equivalence (ii) ⇔ (iii) in Exercise 37 on pages 44-45 of [27], there exists $Q \in \mathrm{Spec}(T)$ such that Q is minimal among the prime ideals of T that contain PT and

$$PT \cap (R \setminus P)(T \setminus Q) \neq \emptyset.$$

Thus, $\sum_{i=1}^{n} p_i t_i = rz$ for some elements $p_1, \dots, p_n \in P$; $t_1, \dots, t_n \in T$; $r \in R \setminus P$; and $z \in T \setminus Q$.

Observe that $Q \cap R \in \mathrm{Spec}(R)$. Pick $M \in \mathrm{Max}(R)$ such that $Q \cap R \subseteq M$. Then, since $P \subseteq PT \cap R \subseteq Q \cap R \subseteq M$, we get $P \subseteq M$. As P is a locally divided prime ideal of R, it follows that $PR_P = PR_M$.

If $1 \leq i \leq n$, let $\rho_i := p_i/r$. Note that $\rho_i \in PR_P = PR_M$ for all i and that $\sum_{i=1}^{n} \rho_i t_i = z$. One can "read" this equation in the domain $T_{R\setminus M}$ (as each ρ_i, each t_i and z are elements of $T_{R\setminus M}$). This equation leads to $z \in (PR_M)T = (PT)T_{R\setminus M}$. Thus, $z \in QT_{R\setminus M}$. Consequently, since $Q \cap R \subseteq M$, it follows from the oft-cited bijection (that is, from Proposition 3.11 (iv) of [2]) that

$$z \in QT_{R\setminus M} \cap T = Q,$$

the desired contradiction. The proof is complete. □

By combining two results of [9] (namely, its Proposition 2.1 and Remark 2.7 (b)), one gets the conclusion that every locally divided domain is a going-down domain. We next recover that result, in the "one P at a time" spirit of [30] (and of [14]).

Corollary 2.5. *Every locally divided domain is a going-down domain.*

Proof. Let R be a domain. Essentially by definition, we have the following two conclusions: (1) R is a locally divided domain if and only if each $P \in \text{Spec}(R)$ is a locally divided prime ideal of R; (2) R is a going-down domain if and only if, for each $P \in \text{Spec}(R)$ and each overring T of R, the ring extension $R \subseteq T$ satisfies GD to P. Accordingly, one can obtain the conclusion asserted in the present corollary by applying universal quantification over P (running through the prime spectrum of a given domain) to the assertion in Theorem 2.4. □

Readers of [30], [14] and this paper (to this point) are familiar with the fact that a prime ideal P of a domain R is a locally divided prime ideal of R if and only if P is straight with respect to $R \subseteq T$ for each overring T of R. It is natural to ask whether a locally divided prime ideal P of a domain R is straight with respect to $R \subseteq T$ for each domain extension T of R (regardless of whether T happens to be an overring of R). In Corollary 3.1.3 of [30], Secord answered this question affirmatively. We next offer an alternate proof of that result.

Proposition 2.6. (Secord, Corollary 3.1.3 of *[30]*) *Let R be a domain with quotient field K, and let $P \in$* Spec(R). *Then the following conditions are equivalent:*

(*1*) *P is a locally divided prime ideal of R;*

(*2*) *P is straight with respect to $R \subseteq T$ for each domain T that contains R as a subring.*

Proof. By the above comments, the implication (2) ⇒ (1) can be considered to be well known.

(1) ⇒ (2): Assume (1). Let T be a domain that contains R as a subring. By Proposition 2.3 (b) of [14], it suffices to prove that PR_M is straight with respect to $R_M \subseteq T_{R\setminus M}$ for each $M \in \text{Max}(R)$ that contains P. Fix such an M. Then, by (1), PR_M is a divided prime ideal of (the quasi-local domain) R_M, that is, $PR_P = PR_M$. So, we can replace the quadruple (R, P, M, T) with the quadruple $(R_M, PR_M, MR_M, T_{R\setminus M})$; that is, by *abus de langage*, we can suppose that (R, M) is a quasi-local domain.

Now that we have reduced to the case of a quasi-local base domain, we can simply repeat an argument from the second paragraph of the proof of Proposition 3.7 (a) of [21]. Our task is to prove that if $z \in R \setminus P$ and $\tau \in T$ satisfy $z\tau \in PT$, then $\tau \in PT$. We can write $z\tau = \sum_{i=1}^{n} p_i t_i$ for some finitely many elements $p_1, \ldots, p_n \in P$ and $t_1, \ldots, t_n \in T$. Working in the quotient field of T, put $u_i := p_i/z$ for $i = 1, \ldots, n$. Then for each i, we have $u_i \in PR_P = P$ (with the equality holding because we have reduced to the case where P is a divided prime ideal of the quasi-local domain R). Hence, $\tau = (z\tau)/z = \sum_{i=1}^{n} (p_i/z)t_i = \sum_{i=1}^{n} u_i t_i \in PT$, completing the proof. □

Remark 2.7. By the oft-cited exercise from [27], Proposition 2.6 (that is, Corollary 3.1.3 of [30]) is a stronger result than Theorem 2.4. I owe the reader an explanation for my decision to call the weaker result a "theorem" and the stronger result a "remark". The principal reason is the order in which I discovered the results. I proved Theorem 2.4 several months before noticing Proposition 2.6. No doubt, this historical fact was due, to some extent, to my preference (some may call it an *idée fixe*) which I have expressed plainly in [14], that considerations of the "locally divided" and "straight" concepts should be, whenever possible, conducted in an environment which is attuned to possibly related GD-theoretic concepts, for the simple reason that locally divided domains and straight domains are each examples of going-down domains.

I mentioned my preference/*idée fixe* because I believe that one cannot begin an activity without having a possibly subconscious preconception as to how or where to begin. However, one should take care that a "fixed idea" not become an obsession or a prejudice. I trust that readers will wait until they have read this entire paper before concluding whether my treatment is giving "straight"-related concepts and approaches their just desserts.

To continue the historical account, Theorem 2.4 was typed, hopefully in "final form", before I came upon Proposition 2.6. Only after proving the latter result did I realize that I had just retraced my path of some 20 years earlier (by rediscovering the proof of part of Proposition 3.7 (a) of [21]). Only after that "recherche du temps perdus" (not necessarily in the sense of Proust) did I notice Corollary 3.1.3 of [30]. By way of explanation, I must admit that, until that moment, I had not carefully read the final pages of [30] (where its Corollary 3.1.3 appears). I do not ask to be excused for that human failing, but perhaps it is cosmically balanced by the fact that neither Secord nor I noticed or recalled Proposition 3.7 (a) of [21] soon enough.

I would like to end this section on a humbler and more pedagogic note. Although some dilettantes may think of mathematics in terms of its grand results, many of us who work in the subject's trenches believe that techniques of proof are more important than the specific results of that work which are eventually featured in the headlines. As one who shares that belief, I note that the passage from p_i to $u_i := p_i/z$ in the proof of Proposition 2.6 has the same flavor as the passage from p_i to $\rho_i := p_i/r$ in the proof of Theorem 2.4. In that sense, both proofs make use of a similar technique (and each cites the crucial Exercise 37 from pages 44-45 of [27]). In short, the two results have more in common than one may notice at first glance. In my opinion, my decision to present Theorem 2.4 prior to Proposition 2.6 reflects my preferences and the historical record, hopefully without adversely affecting the assignment of due credit to whomever/whatever may be appropriate.

3 Some characterizations of locally divided prime ideals

The converse of Theorem 2.4 is false. Indeed, there exist a domain R and a prime ideal P of R such that $R \subseteq T$ satisfies GD to P for each domain T that contains R as a subring but P is not a locally divided prime ideal of R. An example illustrating this was given in Example 2.9 of [9]. (The domain R in that example had been constructed for other purposes by Boisen and Sheldon in [4].) That example was, in a certain sense, as "small" as possible, as its base ring R was a quasi-local domain of Krull dimension 2 and its only prime ideal that could play the desired role of "P" was its unique prime ideal of height 1. Later, an infinite class of examples of this phenomenon was constructed using integer valued polynomials in Remark 2.3 of [15].

The question naturally arises whether one can characterize the locally divided prime ideals of a domain by augmenting the "GD to P" conclusion from Theorem 2.4 with a property that seems formally weaker than the "straight prime" property that Secord introduced in [30] (and which we pursued in [14]). Our first (affirmative) answer to that question will be given in Theorem 3.4. Another characterization of locally divided prime ideals of a domain is given in Proposition 3.8.

We begin the section by isolating the essence of a remarkably useful argument that was embedded in Secord's main proof in [30].

Lemma 3.1. *Let $A \subseteq B$ be rings, $J(A)$ the Jacobson radical of A, and $u \in B$ such that u is integral over A. Then $u \in A$ if (and only if) $u \in A + J(A)A[u]$.*

Proof. The "only if" assertion is clear. For the converse, assume $u \in A + J(A)A[u]$. It follows that $A[u] = A + J(A)A[u]$ (since $A + J(A)A[u]$ is a subring of $A[u]$ that contains both A and u). Note that the A-module $E := A[u]/A$ satisfies $J(A)E = E$. Also, to show that $u \in A$, it will suffice to show that $E = 0$. So, by Nakayama's Lemma, we need only prove that E is a finitely generated A-module. Thus, it suffices to show that $A[u]$ is a finitely generated A-module. That, in turn, holds since u is integral over A. The proof is complete. □

The next result begins our process of using Lemma 3.1 to construct the missing ingredient that will play the role of condition (β) in Theorem 3.4. In that regard, recall that a prime ideal P of a domain R is a locally divided prime ideal of R if and only if $PR_P = PR_M$ for each maximal ideal M of R that contains P.

Proposition 3.2. *Let R be a domain, $P \in \mathrm{Spec}(R)$ and $M \in \mathrm{Max}(R)$ such that $P \subseteq M$. Let $u \in PR_P \setminus \{0\}$ such that $R \subseteq R[u^{-1}]$ satisfies* GD *to P. Then $u \in PR_M$ if (and only if) $u \in R_M + MR_M[u]$.*

Proof. The "only if" assertion is clear. For the converse, assume $u \in R_M + MR_M[u]$. By the implication (1) $\Rightarrow$ (3) in Proposition 2.3, $R_M \subseteq R_M[u^{-1}]$ $(= R[u^{-1}]_{R\setminus M})$ satisfies GD to PR_M. Then, by reworking the first paragraph of the proof of Lemma 2.4 (a) in [9], we get that u is integral over R_M. Therefore, by applying Lemma 3.1, with $A := R_M$ and $J(A) = MR_M$, we get $u \in A = R_M$ (as $u \in R_M + MR_M[u] = R_M + (MR_M)R_M[u] = A + J(A)A[u]$), whence $u \in PR_P \cap R_M = PR_M$, to complete the proof. □

A considerable amount of [30] was concerned with a quasi-local base domain (let us call it D here) and (what we later called in [14]) a prime ideal P of D that is "straight with respect to a domain extension" E of D (in the sense that E/PE is a torsion-free module over D/P). For that context, Proposition 3.2 has the following special case.

Corollary 3.3. *Let (R,M) be a quasi-local domain, $P \in \mathrm{Spec}(R)$, and $u \in PR_P \setminus \{0\}$ such that P is straight with respect to $R \subseteq R[u^{-1}]$. Then $u \in P$ if (and only if) $u \in R + MR[u]$.*

Proof. By Exercise 37 on pages 44-45 of [27], our "straight" hypothesis ensures that $R \subseteq R[u^{-1}]$ satisfies GD to P. An application of Proposition 3.2 completes the proof. □

We can now present this paper's first new characterization of a locally divided prime ideal of a domain.

Theorem 3.4. Let P be a domain and $P \in \mathrm{Spec}(R)$. Then P is a locally divided prime ideal of R if and only both of the following conditions, (α) and (β), hold:
(α) $R \subseteq R[u^{-1}]$ satisfies GD to P for each $u \in PR_P \setminus \{0\}$;
(β) $u \in R_M + MR_M[u]$ for all $u \in PR_P$ and all $M \in \mathrm{Max}(R)$ such that $P \subseteq M$.

Proof. Assume first that P is a locally divided prime ideal of R. Then (α) holds by Theorem 2.4. Hence, with (α) in hand, Proposition 3.2 reduces the task of proving (β) to showing that if $M \in \mathrm{Max}(R)$ contains P, then $PR_P \subseteq PR_M$ (equivalently, $PR_P = PR_M$). That, in turn, holds precisely because P is a locally divided prime ideal of R.

For the converse, assume (α) and (β). Then, by Proposition 3.2, it follows that if $M \in \mathrm{Max}(R)$ contains P, then $PR_P \subseteq PR_M$, whence P is a locally divided prime ideal of R. The proof is complete. □

In the spirit of [30], we note that Theorem 3.4 has the following corollary.

Corollary 3.5. *Let P be a domain and $P \in \mathrm{Spec}(R)$. Then P is a locally divided prime ideal of R if and only both of the following conditions, (α^*) and (β), hold:*
(α^*) P is straight with respect to the extension $R \subseteq R[u^{-1}]$ for each $u \in PR_P \setminus \{0\}$;
(β) $u \in R_M + MR_M[u]$ for all $u \in PR_P$ and all $M \in \mathrm{Max}(R)$ such that $P \subseteq M$.

Proof. Once again, we recall the following consequence of Exercise 37 on pages 44-45 of [27]: if $D \subseteq E$ are domains and $P \in \mathrm{Spec}(D)$ is straight with respect to $D \subseteq E$, then $D \subseteq E$ satisfies GD to P. Consequently, (α^*) implies condition (α) in Theorem 3.4. In view of Theorem 3.4, it remains only to show that if P is a locally divided prime ideal of R, then (α^*) holds. That, in turn, holds since P being a locally divided prime ideal of R implies that P is straight with respect to the extension $R \subseteq R[w]$ for each element w of the quotient field of R (cf. the implication (1) $\Rightarrow$ (4) in Proposition 2.4 of [14]). The proof is complete. □

Remark 3.6. (a) Let the data R and P be as in Theorem 3.4 and Corollary 3.5; that is, let R be a domain and let $P \in \mathrm{Spec}(R)$. Consider the following condition:

(γ) PR_M is straight with respect to the extension $R_M \subseteq R_M[u]$ for all $u \in PR_P$ and all $M \in \mathrm{Max}(R)$ that contain P.

For the sake of completeness, we pause to give the elementary proof that (γ) implies the condition (β) (from the statements of Theorem 3.4 and Corollary 3.5). Suppose that $M \in \mathrm{Max}(R)$ contains P and that $u \in PR_P$. Write $u = p/z$ with $p \in P$ and $z \in R \setminus P$. As $zu = p \in P \subseteq PR_M[u] = (PR_M)(R_M[u])$, it follows from ($\gamma$) that $u \in PR_M[u]$. The proof concludes by observing that $PR_M[u] \subseteq MR_M[u] \subseteq R_M + MR_M[u]$.

I do not know (and, actually, doubt) whether (γ) is equivalent to (β). Nevertheless, I claim that (γ) **can replace** (β) **in the statements of** Theorem 3.4 and Corollary 3.5. This (claimed) fact will have some significant consequences of a motivational nature, so I will devote the rest of (a) to proving the claim.

Consider the conditions in Theorem 3.4 and Corollary 3.5. We know that P is a locally divided prime ideal of $R \Leftrightarrow$ both (α) and (β) hold $\Leftrightarrow$ both (α^*) and (β) hold. We have just noted that (γ) $\Rightarrow$ (β). Also, recall from the proof of Corollary 3.5 that (α^*) $\Rightarrow$ (α). The claim is that P is a locally divided prime ideal of $R \Leftrightarrow$ both (α) and (γ) hold $\Leftrightarrow$ both (α^*) and (γ) hold.

It will suffice to prove that if P is a locally divided prime ideal of R, then (γ) holds. Let $u \in PR_P$ and let $M \in \mathrm{Max}(R)$ such that $P \subseteq M$. Since P is a locally divided prime ideal of R, another appeal to the implication (1) $\Rightarrow$ (4) in Proposition 2.4 of [14] yields that P is straight with respect to the extension $R \subseteq R[u]$. It follows (cf. Corollary 2.3 (b) of [14]) that PR_M is straight with respect to the extension $R_M \subseteq R[u]_{R\setminus M}$. As $R[u]_{R\setminus M} = R_M[u]$, ($\gamma$) holds, so the proof is complete.

(b) While I expect/hope that interest in Theorem 3.4 and Corollary 3.5 will survive the reading of this paper, the result that was established in (a) will be only of motivational interest. Indeed, its main purpose is to draw attention to the fact that a prime ideal P being a locally divided prime ideal of a given domain R can be characterized by the conjunction of two conditions, (α^*) and (γ), **both** of which deal with P (resp., various PR_M) being straight with respect to certain ring extensions of R (resp., R_M). This fact raises the question whether one can prove more. Specifically, can one prove that a prime ideal P being a locally divided prime ideal of a given domain R can be characterized by assuming somewhat less than the conjunction of (α^*) and (γ)? The answer is in the affirmative: see Proposition 3.8. That result will make explicit, through globalization, a result which was essentially proved by Secord in [30] for quasi-local base domains R. The remark is complete.

Let R be a domain and let $P \in \mathrm{Spec}(R)$. In Remark 3.6, we considered the following condition:

(γ) PR_M is straight with respect to the extension $R_M \subseteq R_M[u]$ for all $u \in PR_P$ and all $M \in \mathrm{Max}(R)$ that contain P.

Once again, let R be a domain and $P \in \mathrm{Spec}(R)$. Lemma 2.3 (b) of [14] yields that (γ) is equivalent to the following condition:

(δ) P is straight with respect to $R \subseteq R[u]$ for all $u \in PR_P$.

It will turn out (in Proposition 3.8) that a condition which seems (at first glance) to be slightly formally weaker than (δ) (or (γ)) actually serves to characterize the locally divided prime ideals of a domain R. The “slightly formally weaker" aspect is due to the fact that certain ones of the elements u considered above automatically exhibit the behavior in question. That fact is proved in the following elementary result. While Proposition 3.7 is not strictly needed for the proof of Proposition 3.8, Proposition 3.7 indicates how one should further restrict the elements u that are truly pertinent to the characterization that is stated in condition (2) of Proposition 3.8. To make the statement of that condition meaningful, we will need to remind the reader of a definition that was introduced in [14]. That reminder will be given after the proof of Proposition 3.7.

Proposition 3.7. *Let R be a domain, $P \in \mathrm{Spec}(R)$, $N \in \mathrm{Max}(R)$ such that $P \subseteq N$, and $w \in PR_N$. Then PR_N is straight with respect to $R_N \subseteq R_N[w]$.*

Proof. We must show that $R_N[w]/((PR_N)(R_N[w]))(= R_N[w]/PR_N[w])$ is a torsion-free module over R/P. Since $w \in PR_N$, we have

$$R_N[w]/PR_N[w] \subseteq (R_N + PR_N)/(PR_N[w]) = R_N/PR_N \to R_P/PR_P,$$

where the ring homomorphism $R_N/PR_N \to R_P/PR_P$ (induced by composing the natural maps $R_N \hookrightarrow R_P$ and $R_P \to R_P/PR_P$) is an injection, since $R_N \cap PR_P = PR_N$. This injective ring homomorphism, let us call it f, is easily shown to be an (R/P)-algebra homomorphism. (In detail: the fact that f preserves scalar multiplication by elements of R/P follows from the fact that if $\eta \in R_N$, then $f(\eta + PR_N) = \eta + PR_P$.) Next, note that the canonical (R/P)-algebra homomorphisms $R/P \to (R/P)_{R\setminus N} \to (R/P)_{R\setminus P}$ are each injections and that $(R/P)_{R\setminus P} \cong R_P/PR_P$ is a quotient field of R/P. Thus, R_N/PR_N $(\cong (R/P)_{R\setminus N})$ is (isomorphic as an algebra over R/P to) an overring of R/P. As any overring of a domain D is a torsion-free D-module, R_N/PR_N is torsion-free over R/P. Hence, so is its subring $R_N[w]/PR_N[w]$. The proof is complete. □

The following definition from [14] will be useful here as well. Let R be a domain with quotient field K and let $P \in \mathrm{Spec}(R)$. Put

$$\Omega(R,P) := \{u \in K \mid u \in PR_P \setminus PR_M \text{ for some } M \in \mathrm{Max}(R) \text{ with } P \subseteq M\}.$$

I am glad to acknowledge that my inspiration for developing Proposition 3.8, which characterizes the locally divided prime ideals of a domain, was Secord's innovative approach in [30] which handled the case of that characterization where R is quasi-local. Note that Corollary 3.5 of [14] established the special case of the equivalence (1) ⇔ (2) in Proposition 3.8 where R is assumed to be a (not necessarily quasi-local) integrally closed domain.

Proposition 3.8. *Let R be a domain with quotient field K, and let $P \in \mathrm{Spec}(R)$. Then the following conditions are equivalent:*
(1) P is a locally divided prime ideal of R;
(2) P is straight with respect to $R \subseteq R[u]$ for each $u \in \Omega(R,P)$;
(3) P is straight with respect to $R \subseteq T$ for each overring T of R.

Proof. It was shown in Proposition 2.4 of [14] that (1) ⇔ (3); and (3) ⇒ (2) trivially. It remains to prove that (2) ⇒ (1).

Assume (2). In order to prove (1), it will suffice, by Lemma 2.4 (b) of [14], to prove that, for each $N \in \mathrm{Max}(R)$ such that $P \subseteq N$, PR_N is straight with respect to $R_N \subseteq R_N[u]$ $(= (R[u])_{R\setminus N})$ for each $u \in \Omega(R_N, PR_N)$ (for the just-cited result would then yield that PR_N is a locally divided prime ideal of R_N). Fix $N \in \mathrm{Max}(R)$ such that $P \subseteq N$ and fix $u \in \Omega(R_N, PR_N)$. Then, by the definition of $\Omega(R_N, PR_N)$, $u \in K$ and

$$u \in (PR_N)(R_N)_{PR_N} \setminus PR_N = PR_P \setminus PR_N.$$

Consequently, $u \in \Omega(R,P)$. Hence, by (2), P is straight with respect to $R \subseteq R[u]$. Thus, by Lemma 2.3 (b) of [14], PR_N is straight with respect to $R_N \subseteq R[u]_{R\setminus N}$ $(= R_N[u])$. The proof is complete. □

We have just seen that by finding a formally stronger condition than (β) that is defined in terms of "straight" behavior (namely, (δ)) and then slightly restricting the set of relevant elements u (to lie in $\Omega(R,P)$, as in condition (2) of Proposition 3.8), we have obtained a companion for Theorem 3.4 whose characterization is via a "straight" kind of condition that seems somewhat formally stronger than condition (β) while having no need of a hypothesis that may seem like a variant of (α). This situation raises the question of whether (α) was necessary in the statement of Theorem 3.4. (Of course, something like (β) was necessary in that statement, as (α) alone does not imply that P is a locally divided prime ideal of R: see the initial three sentences of this section.) We will leave open the question whether (α) was necessary in the statement of Theorem 3.4, that is, the question whether (β) implies (α). However, we will devote some of the next section to showing that various hypotheses without the "straight" flavor of (β) *do* imply (α). In that regard, we would point to, for instance, Theorem 4.6 and Corollary 4.7.

4 Additional sufficient conditions for GD to P

We begin this section by raising a question that is somewhat orthogonal to the questions raised in the final paragraph of Section 3. (This "orthogonality" *will* lead to the results that were promised in the final sentence of Section 3.) Is it possible to find various strengthened versions of (β) that would imply (α)? Let us refocus that question as follows: if R is a domain and $P \in \mathrm{Spec}(R)$, are there various strengthened versions of (β) (from the statement of Theorem 3.4) which imply that various certain domain extensions $R \subseteq T$ satisfy GD to P? We will begin to address this question in Remark 4.1. For motivation of that and some of the following material, we recall part of the statement of Proposition 2.1 of [10]: a domain D is a quasi-local going-down domain if and only if each element of QD_Q is integral over D for each $Q \in \mathrm{Spec}(D)$.

Remark 4.1. (a) For the rest of this remark, fix a domain R and $P \in \mathrm{Spec}(R)$. Recall the statement of condition (β) from Theorem 3.4:

(β) $u \in R_M + MR_M[u]$ for all $u \in PR_P$ and all $M \in \mathrm{Max}(R)$ such that $P \subseteq M$.

The remaining parts of this remark and the subsequent results will proceed to find an equivalent of (β) and will then use the result from [10] that was recalled above to motivate the statements of how various strengthenings of that equivalent of (β) will imply that certain domain extensions of R satisfy GD to P.

(b) (β) holds if and only if each $u \in PR_P$ satisfies $MR_M(R_M[u]/R_M) = R_M[u]/R_M$ for each $M \in \mathrm{Max}(R)$ that contains P.

(c) We begin to examine what the condition in (b) that is equivalent to (β) implies if we restrict to a context that is suggested by Proposition 2.1 of [10]. Here in (c), we will do that "one u at a time and one M at a time". Let $u \in PR_P$ and let $M \in \mathrm{Max}(R)$ such that $P \subseteq M$. Suppose that u is integral over R_M (equivalently, that $R_M[u]$ is a finitely generated module over R_M). Then, in regard to M, u behaves as in (β) (that is, by reasoning as in (b), $MR_M(R_M[u]/R_M) = R_M[u]/R_M$). As $R_M[u]/R_M$ inherits the property of being a finitely generated module over R_M, it follows via Nakayama's Lemma that $R_M[u]/R_M = 0$, that is, $u \in R_M$. Hence, $u \in PR_P \cap R_M = PR_M$.

(d) Now, let's continue to fix u as in (c) but universally quantify the result in (c) as M runs over the set of all maximal ideals of R that contain P. This gives the following result. Suppose that $u \in PR_P$ behaves as in (β) and is such that $u \in PR_P$ is integral over R_M for each $M \in \mathrm{Max}(R)$ that contains P. Then $u \in PR_M$ for each $M \in \mathrm{Max}(R)$ that contains P, by (c). So if, in addition, P is contained in the Jacobson radical of R, globalization gives $u \in \cap_M PR_M = P$.

The result that was just noted will yield an important sufficient condition for P to be a locally divided prime ideal of R. That will be isolated in Proposition 4.2 (a). So, this remark is complete.

Proposition 4.2. *Let R be a domain and $P \in \mathrm{Spec}(R)$. Suppose that the condition*

(β) $u \in R_M + MR_M[u]$ *for all* $u \in PR_P$ *and all* $M \in \mathrm{Max}(R)$ *such that* $P \subseteq M$

holds. Suppose also that each $u \in PR_P$ is integral over R_M for each $M \in \mathrm{Max}(R)$ that contains P. Then:

(*a*) *P is a locally divided prime ideal of R.*

(*b*) *$R \subseteq T$ satisfies* GD *to P for every domain T that contains R as a subring.*

Proof. (a) By Remark 4.1 (d), if $u \in PR_P$, then $u \in PR_M$ for each $M \in \text{Max}(R)$ that contains P. Thus, $PR_P = PR_M$ for each such M. In other words, (a) has been proven.

(b) It suffices to combine (a) and Theorem 2.4. □

One important upshot of Proposition 4.2 is the following. Let R be a domain and let $P \in \text{Spec}(R)$ be such that each element of PR_P is integral over R_M for each maximal ideal M of R that contains P. Then, if condition (β) of Theorem 3.4 holds, condition (α) of Theorem 3.4 also holds.

The result in the preceding paragraph raises a pair of questions. After listing these questions, we indicate parenthetically where the reader may look for (possibly partial) answers. First: can one get “close" to the above “GD to P" conclusion (that is, to concluding the condition (α) from Theorem 3.4) *without* assuming (as we did in Proposition 4.2) something that has a “straight" flavor such as condition (β) from Theorem 3.4? In particular, can this sort of endeavor be accomplished by adding some relatively weak hypothesis that is a known property of going-down domains, such as a hypothesis that the base domain satisfies a weak version of being a treed domain? (Answer: Yes, see Corollaries 4.7-4.9 below.)

Grothendieck and Dieudonné referred to the “GD to P" property as “*générisant en P*". That terminology is reminiscent of some terminology from the theory of (partially) ordered sets. We next adapt that terminology and some associated notation. While some of that notation could have been used above, it will be especially useful later in this section and in Section 5.

If P is a prime ideal of a (commutative) ring A, the set of *generalizations of P* (*in* Spec(A)) is

$$P^{\downarrow} :=_A P^{\downarrow} := \{Q \in \text{Spec}(A) \mid Q \subseteq P\}$$

and the set of *specializations of P* (*in* Spec(A)) is

$$P^{\uparrow} :=_A P^{\uparrow} := \{Q \in \text{Spec}(A) \mid P \subseteq Q\}.$$

Consider a maximal ideal M of a (commutative) ring A that contains a given prime ideal P of A. It is trivial that if $M^{\downarrow}$ is linearly ordered (by inclusion), then

$$M^{\downarrow} \cap P^{\downarrow} = \{Q \in \text{Spec}(A) \mid Q \subseteq P\}$$

and

$$M^{\downarrow} \cap P^{\uparrow} = \{Q \in \text{Spec}(A) \mid P \subseteq Q \subseteq M\}$$

are each also linearly ordered by inclusion.

The word “and" in the preceding assertion cannot be changed to “or", even if R is a domain. Indeed, the two parts of Figure 1 each show, for each integer $n \geq 3$, the order-theoretic structure of the prime spectrum of a domain R of (Krull) dimension n, with a prime ideal P of R contained in a maximal ideal M of R, such that exactly one of $M^{\downarrow} \cap P^{\downarrow}$ and $M^{\downarrow} \cap P^{\uparrow}$ is linearly ordered. Note that in part (a) of Figure 1, it is $M^{\downarrow} \cap P^{\downarrow}$ that is linearly ordered; and in part (b) of Figure 1, $M^{\downarrow} \cap P^{\uparrow}$ is linearly ordered.

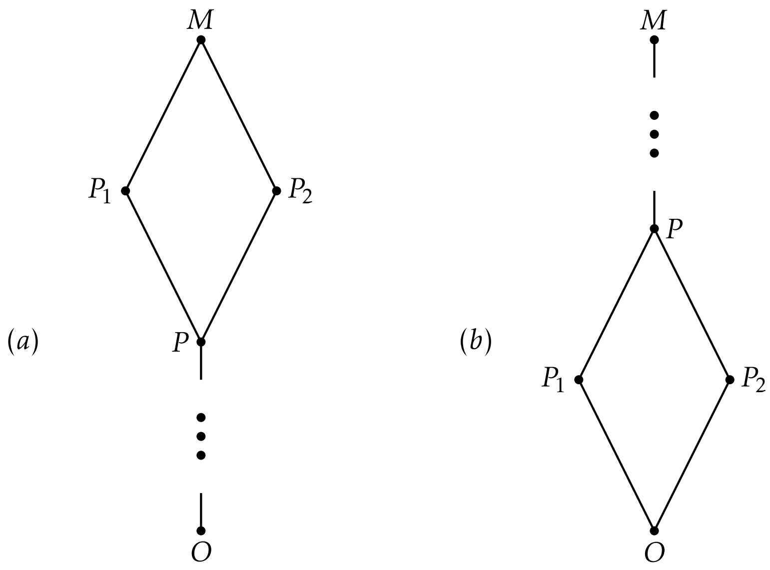

Figure 1

Indeed, it is well known that there exists a commutative ring A such that $\mathrm{Spec}(A)$ is isomorphic as a poset (under inclusion) to the poset that is order-theoretically as depicted in (a) (resp., (b)) of Figure 1, since any finite partially ordered set can be realized in this way (cf. Proposition 10 of [26] or Theorem 2.10 of [28]). Factoring out the "bottom" prime ideal from A gives the desired domain R. Using the just-cited results of Hochster and Lewis, the reader is should have no difficulty in augmenting the data in Figure 1 in order to construct non-quasi-local domains R exhibiting the announced behavior of $M^{\downarrow} \cap P^{\downarrow}$ and $M^{\downarrow} \cap P^{\uparrow}$.

Three paragraphs ago, we noted a certain minor result whose conclusion was an "and" statement. We will now give an example to show that the converse of that minor result is false. In other words, there exist a domain R, $P \in \mathrm{Spec}(R)$, and $M \in \mathrm{Max}(R)$ such that $P \subseteq M$, both $M^{\downarrow} \cap P^{\downarrow}$ and $M^{\downarrow} \cap P^{\uparrow}$ are linearly ordered but $M^{\downarrow}$ is not linearly ordered. For an easy example illustrating this fact (which is available thanks to the above-mentioned results of Hochster and Lewis), consider any quasi-local domain (R, M) of Krull dimension 2 that has exactly two height 1 prime ideals (one of which is P). We will next introduce a "pinched" terminology that explains the just-noted failure of a certain converse.

Once again, consider a (commutative) ring A, with $P \in \mathrm{Spec}(A)$ and $M \in \mathrm{Max}(A)$ such that $P \subseteq M$. One says that $M^{\downarrow}$ is *pinched at* P if each $Q \in M^{\downarrow}$ satisfies either $Q \subseteq P$ or $P \subseteq Q$. Since $M^{\downarrow}$ is pinched at M (and also pinched at 0 if A is a domain), our considerations of $M^{\downarrow}$ possibly being pinched at a given prime ideal P of A will often tacitly ignore the case where P is the maximal ideal M (or 0 if A is a domain). It is clear (and it will be useful to note) that if A, P and M are as above *and if* $M^{\downarrow}$ *is* pinched at P, then $M^{\downarrow}$ is linearly ordered *if* and only if both $M^{\downarrow} \cap P^{\downarrow}$ and $M^{\downarrow} \cap P^{\uparrow}$ are linearly ordered. The "if" assertion fails without the "pinched" hypothesis: that fact is shown by the example that was given at the end of the preceding paragraph.

Our main interest in the "$M^{\downarrow}$ is pinched at P" property comes from the fact that (essentially by definition) a ring A is a treed ring if and only if ${}_{A}M^{\downarrow}$ is linearly ordered (by inclusion) for each $M \in \mathrm{Max}(A)$. Proposition 4.3 next gives a sufficient condition for a maximal ideal M of a domain R to be such that ${}_{R}M^{\downarrow}$ is pinched at some prime ideal P. To motivate that sufficient condition, note the second hypothesis in Proposition 4.2 and the result whose statement was recalled at the end of the first paragraph of this section.

Proposition 4.3. *Let R be a domain, $P \in \mathrm{Spec}(R)$, and $M \in \mathrm{Max}(R)$ such that $P \subseteq M$. Suppose also that each element of PR_P is integral over R_M. Then $M^{\downarrow}$ $(={}_{R} M^{\downarrow})$ is pinched at P.*

Proof. One can prove this result by adapting the first paragraph of the proof of the implication (e) $\Rightarrow$ (a) in Proposition 2.1 of [10]. For the sake of completeness, we provide the details. Suppose that the assertion fails. Pick $P_1 \in M^{\downarrow}$ such that P and P_1 are incomparable. Next, pick $a \in P \setminus P_1$ and $b \in P_1 \setminus P$. Put $v := a/b$. As $v \in PR_P$, a hypothesis ensures that v is integral over R_M. Then v satisfies an n^{th}-degree integrality equation over R_M, for some positive integer n. Multiplying that equation through by b^n leads to $a^n \in R_M b \subseteq P_1 R_M$, whence $a^n \in P_1 R_M \cap R = P_1$. As P_1 is a prime ideal, we get $a \in P_1$, the desired contradiction. The proof is complete. □

The following observations will enable us to add an additional corollary (the next result), additional equivalences as part (b) of several subsequent results, and additional motivation for Theorem 4.10. Let R be a domain, $P \in \text{Spec}(R)$, and $M \in \text{Max}(R)$ such that $P \subseteq M$. By reasoning as in the first paragraph of the proof of Lemma 2.4 (a) of [9], one sees that the hypothesis that "each element of PR_P is integral over R_M" from Proposition 4.3 is implied by the condition that for each nonzero element $v \in PR_P$, $R_M \subseteq R_M[v^{-1}]$ satisfies GD to PR_M. By Proposition 2.3, *that* condition is (assuming $P \subseteq M$), implied by the condition that for each nonzero element $v \in PR_P$, $R \subseteq R[v^{-1}]$ satisfies GD to P.

Corollary 4.4. *Let R be a domain, $P \in \text{Spec}(R)$, and $M \in \text{Max}(R)$ such that $P \subseteq M$. Suppose that for each nonzero element $v \in PR_P$, $R_M \subseteq R_M[v^{-1}]$ satisfies GD to PR_M. Then $M^{\downarrow}$ is pinched at P.*

Proof. It suffices to combine Proposition 4.3 with the first observation from the paragraph preceding the present result. □

Remark 4.5. As noted two (or three) paragraphs before Proposition 4.3, there exist a domain R, a prime ideal P of R and a maximal ideal M of R containing P such that $M^{\downarrow}$ is pinched at P but $M^{\downarrow}$ is not linearly ordered. It seems reasonable to ask if there are natural assumptions that are stronger than in Proposition 4.3 and lead to the sort of "GD to P" conclusion that we are seeking without an overt hypothesis of the "straight" kind. The answer is in the affirmative, and Theorem 4.6 gives our main result along these lines. The remark is complete.

Theorem 4.6. Let R be a domain, $P \in \text{Spec}(R)$, and $M \in \text{Max}(R)$ such that $P \subseteq M$. Suppose also that $M^{\downarrow}$ is linearly ordered by inclusion. Then:

(a) If each element of PR_P is integral over R_M, then $R_M \subseteq T_{R\setminus M}$ satisfies GD to PR_M for each domain T that contains R as a subring.

(b) The following conditions are equivalent:

(1) $R_M \subseteq R_M[v^{-1}]$ satisfies GD to PR_M for each nonzero element $v \in PR_P$;

(2) $R_M \subseteq T_{R\setminus M}$ satisfies GD to PR_M for each domain T that contains R as a subring.

Proof. (a) Since $(PR_M)(R_M)_{PR_M} = PR_P$, we can replace (R, P, M, T) with the quadruple $(R_M, PR_M, MR_M, T_{R\setminus M})$; that is, by *abus de langage*, we can suppose that (R, M) is a quasi-local treed domain ("treed" because we had assumed that $M^{\downarrow}$ is linearly ordered) such that each element of PR_P is integral over R. Having reduced to this context, we can now reason exactly as in the second paragraph of the proof of the implication (e) $\Rightarrow$ (a) of Proposition 2.1 in [10] to complete the proof of (a).

(b) It is trivial that (2) $\Rightarrow$ (1). For the converse, combine (a) with the first observation from the paragraph preceding Corollary 4.4. The proof is complete. □

The next three corollaries deliver the promised triumvirate of answers.

Corollary 4.7. *Let R be a domain and $P \in \text{Spec}(R)$. Suppose that for all $M \in \text{Max}(R)$ such that $P \subseteq M$, the set $M^{\downarrow}$ is linearly ordered by inclusion. Then:*

(a) If each element of PR_P is integral over R_M for each $M \in \text{Max}(R)$ such that $P \subseteq M$, then $R \subseteq T$ satisfies GD to P for each domain T that contains R as a subring.

(*b*) *The following conditions are equivalent:*

(*1*) $R \subseteq R[v^{-1}]$ *satisfies* GD *to* P *for each nonzero element* $v \in PR_P$;

(*2*) $R \subseteq T$ *satisfies* GD *to* P *for each domain* T *that contains* R *as a subring.*

Proof. (a) Let T be a domain that contains R as a subring. Then, by Theorem 4.6 (a), $R_M \subseteq T_{R\setminus M}$ satisfies GD to PR_M for each $M \in \mathrm{Max}(R)$ that contains P. Therefore, by Proposition 2.3, $R \subseteq T$ satisfies GD to P.

(b) It is trivial that (2) $\Rightarrow$ (1). For the converse, combine (a) with both observations from the paragraph preceding Corollary 4.4. The proof is complete. □

Corollary 4.8. *Let* R *be a treed domain and let* $P \in \mathrm{Spec}(R)$. *Then:*

(*a*) *Suppose also that for each* $M \in \mathrm{Max}(R)$ *such that* $P \subseteq M$, *each element of* PR_P *is integral over* R_M. *Then* $R \subseteq T$ *satisfies GD to* P *for each domain* T *that contains* R *as a subring.*

(*b*) *The following conditions are equivalent:*

(*1*) $R \subseteq R[v^{-1}]$ *satisfies* GD *to* P *for each nonzero element* $v \in PR_P$;

(*2*) $R \subseteq T$ *satisfies* GD *to* P *for each domain* T *that contains* R *as a subring.*

Proof. Recall that a domain D is a treed domain if and only if ${}_DN^{\downarrow}$ is linearly ordered for each $N \in \mathrm{Max}(D)$ (regardless of whether or not N contains some specified prime ideal of D). Accordingly, the assertions follow as a special case of Corollary 4.7. □

With obeisance to the motivation that was provided by Proposition 2.1 of [10], we next isolate the "quasi-local base domain" case of Corollary 4.8. Note that the additional hypothesis that the base domain is quasi-local and treed allows Corollary 4.9 (a) to obtain the "GD to P" analogue of the implication (e) $\Rightarrow$ (a) of the motivating result, Proposition 2.1 of [10].

Corollary 4.9. *Let* R *be a quasi-local treed domain and* $P \in \mathrm{Spec}(R)$. *Then:*

(*a*) *If each element of* PR_P *is integral over* R, *then* $R \subseteq T$ *satisfies GD to* P *for each domain* T *that contains* R *as a subring.*

(*b*) *The following conditions are equivalent:*

(*1*) $R \subseteq R[v^{-1}]$ *satisfies* GD *to* P *for each nonzero element* $v \in PR_P$;

(*2*) $R \subseteq T$ *satisfies* GD *to* P *for each domain* T *that contains* R *as a subring.*

The final main result of the second half of this section will characterize the data R, P which satisfy the conclusion of Theorem 2.4, that is, the domains R and their prime ideals P such that $R \subseteq T$ satisfies GD to P for every domain T that contains R as a subring. One way to rephrase the question is to ask which domain extensions T of R suffice as "test domain extensions" for this property. The natural place to look for inspiration is [19], whose Theorem 1 answered the analogous question for GD (instead of for GD to P). Fortunately, the arguments in [19] carry over rather easily, giving the following result. Note that the equivalence (3) $\Leftrightarrow$ (5) in Theorem 4.10 can be viewed as a "GD to P"-theoretic analogue of the "straight"-theoretic equivalence in Corollary 3.1.3 of [30] (which received a different proof in Proposition 2.6 above). Theorem 4.10 is also motivated by our interest in obtaining a result whose conclusion has some of the flavor of the conclusion of Corollary 4.8 (b), but without the "treed" hypothesis in the latter result.

Theorem 4.10. Let R be a domain with quotient field K and let $P \in \mathrm{Spec}(R)$. Then the following conditions are equivalent:

(1) $R \subseteq R[u]$ satisfies GD to P for each $u \in K$;

(2) $R \subseteq V$ satisfies GD to P for each valuation overring V of R (inside K);

(3) $R \subseteq T$ satisfies GD to P for each overring T of R (inside K);

(4) $R \subseteq V$ satisfies GD to P for each valuation domain V that contains R as a subring;

(5) $R \subseteq T$ satisfies GD to P for each domain T that contains R as a subring.

The proof of the above theorem was obtained by adapting the proofs of two results in [19], namely, its Theorem 1 and its Proposition. Frankly, a reader who is familiar with the results and methods in [8] would find only the implication (1) $\Rightarrow$ (5) in Theorem 4.10 difficult to prove. The analogous hardest part to prove of Theorem 1 in [19] is its implication (a) $\Rightarrow$ (c). *That* implication was proved by a slick use of the Proposition in [19]. It turns out that it is not difficult to adapt the statement and proof of the Proposition in [19] so as to obtain a proof of the implication (1) $\Rightarrow$ (5) in Theorem 4.10. The precise statement of that reworded proposition is given next, as Proposition 4.11, for two reasons: we wish to emphasize that it is what was used to prove the hardest part of Theorem 4.10; and its statement may suggest why one should not be surprised that Corollaries 4.8-4.9 used the hypothesis that R is a treed domain. In Remark 4.12, we explain further why Theorem 4.10 did not need to suppose even a weak version of the "treed" property for R.

Proposition 4.11. *Let R be a domain and $P \in \mathrm{Spec}(R)$. Let T be a quasi-local treed domain containing R as a subring. If $R \subseteq R[u]$ satisfies* GD *to P for each $u \in T$, then $R \subseteq T$ satisfies* GD *to P.*

Remark 4.12. Long before embarking upon this project, I had developed a healthy respect for the subtlety inherent in the "treed domain" property. I was able to prove that any going-down domain must be a treed domain in Theorem 2.2 of [8]. (This was done before the question had been settled whether the SGD domains are the same as the GD domains. Later, with Theorem 1 of [19] in hand, we were able to find a faster proof of this fact.) The converse, however, is false. The first example of a treed domain which is not a going-down domain was constructed by W. J. Lewis. A sketch of that example was included as Example 4.4 in the survey article [20] with the kind permission of Dr. Lewis. A different example of a treed domain which is not a going-down domain was given in Example 2.3 of [12]. (Unlike the example of Lewis, the example in [12] had the additional property that each of its overrings is a treed domain). Yet another method of "constructing" a treed domain that is not a going-down domain was given in [18].

In reflecting on how I adapted some of my "old" proofs about going-down domains to the "GD to P" context, I initially found it remarkable that I needed to use the hypothesis that R is a treed domain in order to prove Corollary 4.8, although I did not need to assume even a weak version of R being a treed domain in order to prove Theorem 4.10. Further reflection explained the apparent disparity (to my satisfaction) as follows. Readers who have tried to prove Theorem 4.10 by using Proposition 4.11 and adapting the proof of Theorem 1 in [19] will have already noticed the answer: the proof uses the fact that a suitable domain extension of R (which is denoted by V in [8]) is a valuation overring of R (and, hence, is both quasi-local and treed). On the other hand, in constructing a proof of Corollary 4.8, I could not find a relevant, useful quasi-local treed domain extension of the given domain R. We leave as an open question whether one can remove the hypothesis that R is a treed domain in Corollary 4.8 (or whether one can significantly strengthen Corollary 4.7).

5 P-unibranched extensions and seminormality

While some of the early part of [9] was motivated by applications to the context of integrally closed domains, some later work in that paper was motivated by applications to a subtler context (which I came to understand later as being that of seminormal domains). Anyone who is familiar with seminormal domains and the earlier part of [9] will recognize that Corollary 2.6 of [9] easily implies that a seminormal domain is a locally divided domain if (and only if) it is a going-down domain. More generally, if R is a domain with seminormalization R^+ (in its integral closure), in the sense of [31], then: R is a going-down domain $\Leftrightarrow$ R^+ is a going-down domain $\Leftrightarrow$ R^+ is a locally divided domain. (For a generalization of this result to rings possibly with nontrivial zero-divisors, see Theorem 3.4 of [3].)

The proof of Theorem 5.10 shows how to adapt some reasoning that had concerned the unibranchedness of an overring extension of a given quasi-local going-down domain (see the proofs of Lemma 2.4 (a), Theorem 2.5 and, especially, Corollary 2.6 of [9]) to a more general situation where the given overring extension of a quasi-local domain R satisfies GD to P for some given prime ideal P of R. Insofar as possible, the proof of Theorem 5.10 (c) is organized like the proof of Corollary 2.6 of [9]. To facilitate that organization, the first part of this section introduces the concept of a P-uinbranched (ring) extension. Although we will need properties of this new concept only for ring extensions involving domains, we define it here in a more general setting, in the hope that some later workers will derive some benefit from that ring-theoretic generality.

First, recall that a ring extension $A \subseteq B$ is said to be *unibranched* in case the canonical contraction map $c_A^B : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ (given by $\mathcal{Q} \mapsto \mathcal{Q} \cap A$) is a bijection. (Warning: the literature contains several inequivalent definitions of a "unibranched ring extension"!)

The following generalization of the "unibranched" concept will play a key role in the proof of Theorem 5.10. Let A be a ring and $P \in \mathrm{Spec}(A)$. Let $\mathfrak{P} :=_A^B \mathfrak{P}$ denote the set of $Q \in \mathrm{Spec}(B)$ such that $c_A^B(Q)$ is comparable to P (with respect to inclusion). Then a ring extension $A \subseteq B$ is said to be *P-unibranched* if $c_A^B : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ restricts to a bijection between $\mathfrak{P}$ and the set of prime ideals of A that are comparable to P; that is, if each prime ideal of A that is comparable to P is lain over by exactly one prime ideal of B. For our purposes here, the most important kind of P-unibranched ring extension will be a CPI-extension of a domain. For simplicity of notation, we will confine our summary of results on CPI-extensions in Lemma 5.1 to the context of domains, but Remark 5.2 will review some ring-theoretic generalizations.

Recall from [5] that if A is a ring and $P \in \mathrm{Spec}(A)$, then the associated *CPI-extension* is defined to be the ring $A + PA_P$ (viewed as an A-subalgebra of A_P). Lemma 5.1 states some important properties of CPI-extensions of domains and describes some of the papers where these properties have been established. Remark 5.2 states a ring-theoretic generalization of Lemma 5.1 and describes where (and for which contexts) this generalization has been established.

Lemma 5.1. *Let R be a domain and $P \in \mathrm{Spec}(R)$. Let $T := R + PR_P$ (the associated CPI-extension). Then:*
(*a*) *$R \subseteq T$ is a P-unibranched extension.*
(*b*) *Each prime ideal of T is comparable (with respect to inclusion) to PR_P.*
(*c*) *The only prime ideal of T that meets R in P is PR_P.*
(*d*) *If Q is a prime ideal of T that properly contains PR_P, then $Q = (Q \cap R) + PR_P$.*
(*e*) *If Q is a prime ideal of T that is properly contained in PR_P, then $Q = (Q \cap R)R_P$.*

Proof. (a), (b): These assertions were proven in Proposition 2.7 of [5].

(c): This assertion was proven in Proposition 3.5 of [5].

(d), (e): These assertions can arguably be gleaned from what is explicitly in Proposition 5.1 of [24] and/or [5]. □

Remark 5.2. In [5], Boisen and Sheldon *do* consider CPI-extensions of rings that are not necessarily domains. In doing so, they *do* establish the ring-theoretic generalizations of the assertions in Lemma 5.1. We have declined to state Lemma 5.1 in that generality for two reasons: we simply will not need it here; and the more general ring-theoretic setting would require the more cumbersome description of the contraction map c_A^B as $\mathcal{Q} \mapsto f^{-1}(\mathcal{Q})$ (rather than as $\mathcal{Q} \mapsto \mathcal{Q} \cap A$) where $f : A \to A + PA_P$ ($\subseteq A_P$) is the canonical A-algebra map $A \to A + PA_P$, because f need not be an injection when A is not a domain.

In our opinion, the assertions in parts (b)-(e) of Lemma 5.1 are easiest to verify by direct calculation (somewhat in the spirit of the proofs of parts (b) and (c) of Lemma 2.4 in [9]) once one has the result in Lemma 5.1 (a). We believe that the most elegant and transparent way to obtain *that* result (indeed, its ring-theoretic generalization) is the following. As $R + PR_P$ can be viewed as the pullback of the canonical surjection $R_P \to F := R_P/PR_P$ and the inclusion map $R/P \hookrightarrow F$, consider the order-theoretic impact of the characterization of the Zariski topology of the prime spectrum of a pullback in Theorem 1.4 of [22]. The remark is complete.

The next result collects some (familiar and mostly elementary) examples showing, *i.a.*, that "unibranched extension" and "P-unibranched extension" are inequivalent concepts. I expect that many readers will find it natural to use different kinds of constructions than those given in Example 5.3 to illustrate the various kinds of behavior that are exhibited in its parts (b)-(e).

For possible use in later work on the P-unibranched concept, we will use the technical property of lying-over, which is denoted by LO (as in page 28 of [27]), in stating (and proving) Example 5.3 and Theorem 5.4. A familiar sufficient condition for a ring extension $A \subseteq B$ to satisfy LO is that B be integral over A (cf. Theorem 44 of [27]). Corollary 5.5 gives the special case of Theorem 5.4 where the hypothesis of LO (from Theorem 5.4) is replaced the hypothesis of an integral ring extension. Corollary 5.5 will be used in the proof of Theorem 5.10.

Example 5.3. (a) If $A \subseteq B$ is a unibranched (ring) extension and $P \in \text{Spec}(A)$, then $A \subseteq B$ is a P-unibranched extension.

(b) There exist domains $A \subseteq B$ and a prime ideal P of A such that $A \subseteq B$ is a P-unibranched extension but is not a unibranched extension.

(c) There exist domains $A \subseteq B$ and a prime ideal P of A such that $c_A^B : \text{Spec}(B) \to \text{Spec}(A)$ is an injection but $A \subseteq B$ is not a P-unibranched extension (and so $A \subseteq B$ is not a unibranched extension).

(d) There exist rings $A \subseteq B$ and a prime ideal P of A such that c_A^B is surjective (that is, $A \subseteq B$ satisfies LO) but $A \subseteq B$ is not a P-unibranched extension (and so $A \subseteq B$ is not a unibranched extension).

(e) There exist domains $A \subseteq B$ such that c_A^B is neither injective nor surjective. In any such example, $A \subseteq B$ is not unibranched and there exists $P \in \text{Spec}(A)$ such that $A \subseteq B$ is not P-unibranched.

Proof. (a) The assertion is obvious. (However, its converse is false: see (b).)

(b) Let A be a domain of Krull dimension 2 with exactly two maximal ideals, say M and N, such that M has height 2, N has height 1, and M contains exactly one prime ideal of height 1. Let P denote that height 1 prime ideal. (The existence of such a domain A is guaranteed by applying the results from [26] or [28] that were mentioned in our earlier discussion involving Figure 1 in Section 4. Another way to construct such a domain A, is the following. Take A to be the intersection of two incomparable valuation domains, one of which is of Krull dimension 2 and the other of which is of Krull dimension 1, such that these two valuation domains have the same quotient field and are incomparable; then apply Theorem 22.8 of [23].) Note that $P \not\subseteq N$. Put $B := A_M$. Using familiar properties of localizations, one sees that c_A^B is injective and that the image of c_A^B is $\{0, P, M\} = \text{Spec}(A) \setminus \{N\} \subset \text{Spec}(A)$. The assertions now follow from the definitions of a P-unibranched extension and a unibranched extension.

c) Let A, M, N and P be as in the proof of (b). Put $B := A_P$. (Note: **this B is not the same as the ring that was denoted by B in (b)**.) One sees that c_A^B is injective and that the image of c_A^B is $\{0, P\}$. Then $A \subseteq B$ is not P-unibranched (because M is not in that image). The failure of c_A^B to be surjective explains why $A \subseteq B$ is not unibranched, but that conclusion could also be inferred now by applying the contrapositive of (a).

(d) Perhaps the easiest example of this phenomenon is given by taking A to be any nonzero ring and $B := A \times A$. (It is conventional to view A as a subring of B via the diagonal injection $\delta : A \to B$ given by $\delta(a) := (a, a)$ for all $a \in A$.) Since any $Q \in \text{Spec}(A)$ is lain over by the distinct prime ideals $Q \times A$ and $A \times Q$ of B, the assertions follow. If one wishes an example involving domains, it suffices to take $A := \mathbb{Z}$ and to take B to be the ring of (algebraic) integers of any quadratic algebraic number field such that some prime ideal P of A "splits in B" (in the sense of classical algebraic number theory that $PB = Q_1Q_2$ for some distinct prime ideals Q_1 and Q_2 of B.)

(e) Take A to be any domain which is not a field. Let L be any field extension of the quotient field of A. Let X be a (commuting) indeterminate over L. Let B denote the polynomial ring $L[X]$. Then the image of c_A^B is the singleton set $\{0\}$ (since any $\mathcal{Q} \in \mathrm{Spec}(B)$ satisfies $\mathcal{Q} \cap A = (\mathcal{Q} \cap L) \cap A = 0 \cap A = 0$). Hence, c_A^B is not injective since it sends each of the distinct prime ideals 0 and XB to 0; and c_A^B is not surjective since each maximal ideal of A fails to be in the image of c_A^B.

We turn to the final assertion. If $P \in \mathrm{Spec}(A)$ is not in the image of c_A^B, then $A \subseteq B$ is not P-unibranched (since P is comparable to P). Then, by the contrapositive of (a), $A \subseteq B$ is not unibranched. The proof is complete. □

The next result gives a sufficient condition for a subextension of a P-unibranched extension to be P-unibranched itself. For possible use in later work on the P-unibranched concept, we state (and prove) Theorem 5.4 using the technical property of lying-over, which is denoted by LO, as in page 28 of [27]. We *will* use its consequence, Corollary 5.5, where the hypothesis of LO from Theorem 5.4 is replaced by *its* sufficient condition of integrality, in the proof of Theorem 5.10.

Theorem 5.4. Let R be a domain, let $P \in \mathrm{Spec}(R)$, and let T be a domain such that $R \subseteq T$ is a P-unibranched extension. Let S be a ring (in fact, a domain) such that $R \subset S \subset T$. Suppose also that both $R \subset S$ and $S \subset T$ satisfy LO. Then $R \subseteq S$ is a P-unibranched extension.

Proof. Since $R \subseteq S$ satisfies LO, it remains only to prove that if $\mathcal{P} \in \mathrm{Spec}(R)$ is comparable to P, then there exists at most one prime ideal of S that contracts to $\mathcal{P}$. Let us first consider the case where $\mathcal{P} = P$. As $R \subseteq T$ is P-unibranched, there exists a unique $Q^* \in \mathrm{Spec}(T)$ such that $Q^* \cap R = P$. Put $Q := Q^* \cap S$. Then $Q \in \mathrm{Spec}(S)$ contracts to P, since

$$Q \cap R = (Q^* \cap S) \cap R = Q^* \cap (S \cap R) = Q^* \cap R = P.$$

It remains to show that if $\mathcal{Q} \in \mathrm{Spec}(S)$ contracts to P, then $\mathcal{Q} = Q$. Since $S \subseteq T$ satisfies LO, there exists $\mathcal{Q}^* \in \mathrm{Spec}(T)$ such that $\mathcal{Q}^* \cap S = \mathcal{Q}$. Then, arguing as in the last display, we get $\mathcal{Q}^* \cap R = (\mathcal{Q}^* \cap S) \cap R = \mathcal{Q} \cap R = P$. As $R \subseteq T$ is P-unibranched, it follows that $\mathcal{Q}^* = Q^*$ (as both of these prime ideals of T contract to P). Thus, $\mathcal{Q} = \mathcal{Q}^* \cap S = Q^* \cap S = Q$. This completes the proof in case $\mathcal{P} = P$.

Two cases remain, but we can treat them together. Suppose, then, that $\mathcal{P} \in \mathrm{Spec}(R)$ is comparable to P but $\mathcal{P} \neq P$. Assume that $\mathcal{P} \subset P$ (resp., $P \subset \mathcal{P}$). As $R \subseteq T$ is P-unibranched, there exists a unique $\mathfrak{Q}^* \in \mathrm{Spec}(T)$ such that $\mathfrak{Q}^* \cap R = \mathcal{P}$. Put $\mathfrak{Q} := \mathfrak{Q}^* \cap S$. Arguing as above, one shows that $\mathfrak{Q} \in \mathrm{Spec}(S)$ contracts to $\mathcal{P}$. It will suffice to show that if $\mathcal{Q} \in \mathrm{Spec}(S)$ satisfies $\mathcal{Q} \cap R = \mathcal{P}$, then $\mathcal{Q} = \mathfrak{Q}$. Since $S \subseteq T$ satisfies LO, there exists $(\mathfrak{Q}^*)^* \in \mathrm{Spec}(T)$ such that $(\mathfrak{Q}^*)^* \cap S = \mathcal{Q}$. It follows that $(\mathfrak{Q}^*)^* \cap R = \mathcal{P}$. Thus, as $R \subseteq T$ is P-unibranched, we get $(\mathfrak{Q}^*)^* = \mathfrak{Q}^*$. Therefore,

$$\mathcal{Q} = (\mathfrak{Q}^*)^* \cap S = \mathfrak{Q}^* \cap S = \mathfrak{Q},$$

completing the proof. □

Corollary 5.5. *Let R be a domain, let $P \in \mathrm{Spec}(R)$, and let T be a domain such that $R \subseteq T$ is an integral P-unibranched extension. Let S be a ring (in fact, a domain) such that $R \subset S \subset T$. Then $R \subseteq S$ is a P-unibranched extension.*

Proof. Since both of the extensions $R \subseteq S$ and $S \subseteq T$ inherit integrality from $R \subseteq T$, each of these extensions satisfies LO (cf. Theorem 44 of [27]), and so an application of Theorem 5.4 completes the proof. □

Recall that if $A \subseteq B$ are rings, then the contraction map $c_A^B : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an *isomorphism of posets* (*under inclusion*) [sometimes this is called an "order isomorphism of prime spectra"] if and only if c_A^B is a bijection and the following holds: if $Q_1, Q_2 \in \text{Spec}(B)$ with $P_i := Q_i \cap A$ for all $i \in \{1,2\}$, then $P_1 \subseteq P_2$ if and only if $Q_1 \subseteq Q_2$.

Let $A \subseteq B$ be rings. If c_A^B is an order isomorphism of prime spectra, then $A \subseteq B$ is unibranched and (one can check easily that) A and B have the same Krull dimension. Let us examine what can be said about a possible converse. Suppose, then, that $A \subseteq B$ is a unibranched ring extension such that A and B have the same Krull dimension n, for some $n \in \{0,1,2,3,\dots,\infty\}$. If $n = 0$, then it is evident that c_A^B is an order isomorphism of prime spectra. For $n > 0$, let us also assume that A and B each have a unique minimal prime ideal. (As our main interests here are domain-theoretic, this additional assumption is not inappropriate. Indeed, every nonzero ring does have at least one minimal prime ideal (cf. Theorem 10 of [27]); and a well known application of Zorn's Lemma (in Exercise 1, page 41 of [27]) shows that (for *any* ring extension $A \subseteq B$) every minimal prime ideal of A is the contraction of some minimal prime ideal of B.) Let us next consider the case $n = 1$. We are assuming that $A \subseteq B$ is a unibranched extension involving one-dimensional rings, both of which have a unique minimal prime ideal. Then c_A^B restricts to a bijection from the set of nonminimal prime ideals of B onto the set of nonminimal prime ideals of A. It then follows easily that c_A^B is an order isomorphism. On the basis of the amassed evidence, would it be wise to conjecture that a unibranched extension $A \subseteq B$ of n-dimensional domains must imply that c_A^B is an order isomorphism? No! In fact, possibly contrary to some readers' intuition, the next result will show that such a conjecture fails for *every* $n > 1$.

Theorem 5.6. If $2 \leq n \leq \infty$, there exist domains $R \subset T$, each of Krull dimension n, such that $R \subseteq T$ is unibranched (hence, P-unibranched for each $P \in \text{Spec}(R)$) but $c_R^T : \text{Spec}(T) \to \text{Spec}(R)$ is not an order-isomorphism of posets under inclusion.

Proof. We begin by addressing the case where $n = 2$. This will be done by finding a suitable way to modify the reasoning in the proofs of a couple of results from [18] and [13]. In Remark 5.9, we will explain why it is simply impossible to directly apply those results at this point.

We begin by defining a poset $(Y, \leq)$, with $Y := \{y_0, y_1, y_2, y_3\}$, by imposing the requirements that $y_0 < y_1 < y_2$ and $y_0 < y_3$ (and $|Y| = 4$), with no other occurrences of "<", apart from $y_0 < y_2$, which is required in view of the transitivity axiom for a partial order. (As usual, a statement of the form "$a < b$" means "$a \leq b$ and $a \neq b$".) We claim that Y is a spectral set. This claim means that there is a ring D such that $(Y, \leq)$ is order isomorphic to the poset structure $(\text{Spec}(D), \leq_D)$ that is induced by the Zariski topology on $\text{Spec}(D)$ (where, as in [26, page 53, lines 13-14], if P and Q are prime ideals of D, $P \leq_D Q$ means that Q is in the Zariski-topology closure of $\{P\}$). Moreover, we claim that $(Y, \leq)$ is an L-spectral set, in the sense of page 229 of [16]. This second (and stronger) claim means that Y^L, the topological space obtained by imposing the left topology on the poset Y, is a spectral space. (Recall from Exercice 1, page 89 of [6] that an open basis for Y^L consists of the sets of the form $v^{\downarrow}$ as v runs through the elements of Y, where $v^{\downarrow} := \{u \in Y \mid u \leq v\}$; and recall from the second paragraph on page 43 of [26] that a *spectral space* is a topological space that is homeomorphic to $\text{Spec}(E)$ with the Zariski topology for some ring E.) It is straightforward to check that the poset structure induced on Y by the left topology on Y is precisely $(Y, \leq)$. (Because Y is finite, the preceding assertion is also an immediate consequence of either the Main Theorem of [11] or Corollary 2.6 of [11].) Consequently, every L-spectral set is a spectral set. Thus, we need only prove the second claim (as the first claim will then follow.) To that end, one need only verify the four order-theoretic conditions (α) - (δ) in the characterization of L-spectral sets in Theorem 2.4 of [16]. Since Y is finite, it is evident that (α), (γ) and (δ) hold in Y; moreover, since Y is finite, we can now conclude that (β) also holds in Y for reasons of universal algebra, as explained in Remark 2.5 (a) of [16]. This completes the proof of the above claims about Y and Y^L.

Next, define a four-element poset $X := \{x_0, x_1, x_2, x_3\}$ by imposing the requirements that $x_0 < x_1 < x_3$, $x_0 < x_2$ and $x_1 < x_2$ (and $|X| = 4$), with no other occurrences of "<" (apart from $x_0 < x_3$, which is required in view of the transitivity axiom for a partial order). Since X is a finite poset, one can adapt the reasoning in the preceding paragraph (especially, the citations from [11] and [16]) to conclude that X is an L-spectral set and also a spectral set.

Next, define the function $\varphi : Y \to X$ by $\varphi(y_0) = x_0$, $\varphi(y_1) = x_1$, $\varphi(y_2) = x_3$ and $\varphi(y_3) = x_2$. Observe that φ is surjective and order-preserving. (We have also arranged that φ is *not* an order-isomorphism: note that $y_1 \mapsto x_1$, $y_3 \mapsto x_2$ and $x_1 \leq x_2$, although $y_1 \not\leq y_3$. That data-driven explanation will play an important role in our proof that the eventual domain extension does not induce an order isomorphism of posets; indeed, the only elements $\xi, \eta \in Y$ such that $\varphi(\xi) = x_1$ and $\varphi(\eta) = x_2$ are $\xi = y_1$ and $\eta = y_3$, and $x_1 \leq x_2$ although $y_1 \not\leq y_3$. This single fact captures most of the novelty that distinguishes the constructions here of Y, X and φ from the constructions of the similarly denoted quantities in the proof on pages 3-5 of [18] and the proof of Theorem 2.1 of [13]. Also, note that φ is continuous when viewed as a map $Y^L \to X^L$, since Lemma 2.6 (a) of [16] states that any order-preserving map of posets is continuous when these posets are each equipped with the left topology. Next, recall from page 43 of [26] that a map h of spectral spaces is said to be a *spectral map* if h is continuous and the inverse image under h of any quasi-compact open subset of the codomain of h is quasi-compact (and open). Since φ is a continuous function between finite spectral spaces, we can also conclude that φ is a spectral map (the point being that the finitude of X and Y ensures that every subset of X (resp., Y) is quasi-compact). In short, $\varphi : Y^L \to X^L$ is both spectral and surjective.

Readers who are familiar with the proof of Main Theorem in [18] or the proof of Theorem 2.1 in [13] (for its case $n = 3$) will probably now be able to complete the proof here for the case $n = 2$. For the sake of completeness, we next provide those delicate details.

The above data are made to order for the realization assertion in Theorem 6 (b) of [26]. This result states that when Spec is viewed as a contravariant functor from the category of commutative rings (and ring homomorphisms) to the category of spectral spaces (and spectral maps), then Spec is invertible on the (nonfull) subcategory of all spectral spaces and surjective spectral maps. In particular, one infers the existence of a ring homomorphism $f : A \to B$ and homeomorphisms $\alpha : \mathrm{Spec}(A) \to X$, $\beta : \mathrm{Spec}(B) \to Y$ (where Spec(A) and Spec(B) are each endowed with the Zariski topology), such that $\alpha \circ \mathrm{Spec}(f) = \varphi \circ \beta$. It follows that Spec($f$) inherits the "surjective" property of φ. Moreover, since the homeomorphisms α, β are necessarily order isomorphisms, it also follows that Spec(f) has all the order-theoretic properties of φ.

We next reduce to the case of injective f. Indeed, the First Isomorphism Theorem gives the factorization $f = j \circ \pi$, where $\pi : A \to A/\ker(f)$ is the canonical projection and $j : A/\ker(f) \hookrightarrow B$ is the canonical injection. Note that Spec(π) is a homeomorphism, the key point being that $P \supseteq \ker(f)$ for each prime ideal P of A. (In greater detail: if $P \in \mathrm{Spec}(A)$, then the surjectivity of Spec(f) provides $Q \in \mathrm{Spec}(B)$ such that $f^{-1}(Q) = P$; hence, $\ker(f) = f^{-1}(0) \subseteq f^{-1}(Q) = P$, as asserted. In yet greater detail: we have now shown that Spec(π) is a continuous bijection; to conclude that Spec(π) is a homeomorphism, and hence an order isomorphism, it is enough to note that a standard homomorphism theorem ensures that Spec(π) is a closed map.) As $\mathrm{Spec}(j) = (\mathrm{Spec}(\pi))^{-1} \circ \mathrm{Spec}(f)$, we see that Spec($j$) has all the order-theoretic properties of Spec(f) and, hence, all the order-theoretic properties of φ. By *abus de langage*, we henceforth replace f with j, viewed as an inclusion (and thus replace A with $A/\ker(f)$). Later in this proof, it will be important to have noted the following consequence of Spec(π) being a homeomorphism: in replacing A with $A/\ker(f)$ at *this* point of the proof, we have *not* changed Spec(A) (up to homeomorphism or order isomorphism). In particular, in view of the construction of the poset X, it is *still* true that Spec(A) "looks like" X order-theoretically and A has Krull dimension 2.

Recall that if E is any ring, then the *associated reduced ring* of E is $E_{\mathrm{red}} := E/\sqrt{E}$, where $\sqrt{E}$ denotes the set of all nilpotent elements of E. It is well known that for any ring E, applying the Spec functor to the canonical projection $E \to E_{\mathrm{red}}$ produces a homeomorphism. As we have canonical surjective ring homomorphisms $p_1 : A \to A_{\mathrm{red}}$ and $p_2 : B \to B_{\mathrm{red}}$, we get that both Spec($p_1$) and Spec($p_2$) are homeomorphisms (hence, order isomorphisms). Moreover, the inclusion map $f : A \hookrightarrow B$ induces an injective ring homomorphism $f_{\text{red}} : A_{\text{red}} \to B_{\text{red}}$ (defined by $a + \sqrt{A} \mapsto f(a) + \sqrt{B} = a + \sqrt{B}$). Since $p_2 \circ f = f_{\text{red}} \circ p_1$, it follows from the (contravariant) functoriality of Spec that

$$\text{Spec}(f) \circ \text{Spec}(p_2) = \text{Spec}(p_1) \circ \text{Spec}(f_{\text{red}}).$$

Since $\text{Spec}(p_2)$ and $\text{Spec}(p_1)$ are homeomorphisms (hence order isomorphisms), it follows that $\text{Spec}(f)$ has the same order-theoretic properties as $\text{Spec}(f_{\text{red}})$. By more *abus de langage*, we replace the inclusion map f with the injective ring homomorphism f_{red}, which is now viewed as an inclusion. Note A and B have been replaced by A_{red} and B_{red}, respectively. Thus, we have reduced to the context in which both A and B are reduced rings (that is, rings with no nonzero nilpotents) each having a unique minimal prime ideal, that is, domains. In the spirit of the comment at the end of the last paragraph, we wish to point out that after this latest *abus de langage*, it is *still* the case that $\text{Spec}(A)$ "looks like" X order-theoretically and A has Krull dimension 2. It is even easier to see that what is now being called B *still* "looks like" Y and also has Krull dimension 2.

The above instances of *abus de langage* have served to given us domains $A \subseteq B$. It remains only to show that these domains and this domain extension have the properties that were asserted for some domains $R \subseteq T$. (In short, we will show that it suffices to take $R := A$ and $T := B$.) First, $A \subseteq B$ is unibranched. (Here is a summary of the main observations that lead to this conclusion: it was shown that $\text{Spec}(j)$ has all the order-theoretic properties of $\text{Spec}(f)$ and, hence, all the order-theoretic properties of φ; that allowed us to replace f with j; next, we were able to replace (the new) $f : A \hookrightarrow B$ with $f_{\text{red}} : A_{\text{red}} \hookrightarrow B_{\text{red}}$, because of the canonical homeomorhism (hence, order isomorphism) from $\text{Spec}(E_{\text{red}})$ onto $\text{Spec}(E)$ for any ring E. In particular, the newest f, which is the domain extension $A \hookrightarrow B$, is such that $\text{Spec}(f)$ has all the order-theoretic properties of φ. Therefore, $A \subseteq B$ is unibranched (that is, $\text{Spec}(f)$ is a bijection) because φ is a bijection. Next, $c_A^B = \text{Spec}(f) : \text{Spec}(B) \to \text{Spec}(A)$ is not an order isomorphism. (For a proof of this fact, recall that $\text{Spec}(f)$ has all the order-theoretic properties of φ and reread the description of the order-theoretic behavior of φ that was given in the second and third sentences of the fourth paragraph of this proof.) Finally, as mentioned at the end of the preceding paragraph, A and B are each of Krull dimension 2. That is the sort of thing which we considered to be obvious at the end of each of the last two paragraphs, but we think it prudent to provide more detail here about the proof of this final conclusion. Note that none of the changes of notation (that is, the instances of *abus de langage*) changed the Krull dimension of A or the Krull dimension of B; and that the homeomorhisms $\alpha : \text{Spec}(A) \to X$ and $\beta : \text{Spec}(B) \to Y$ are also order isomorphisms. So, it will be enough to show that the posets Y and X are each two-dimensional (in the obvious sense). That, in turn, holds for the following reasons: the maximal elements of Y are y_2 and y_3, these elements have respective heights 2 and 1 (in the obvious sense, by taking the supremum of the lengths of chains emanating downward from the element in question) and $\sup(2,1) = 2$; the maximal elements of X are x_2 and x_3, these elements each have height 2 and $\sup(2,2) = 2$. This completes the proof for the case $n = 2$.

It remains to address the case where $3 \leq n \leq \infty$. During the proof for this case, it will be convenient to let $\mathcal{R}$ and $\mathcal{T}$, respectively, denote the rings that had been denoted by R and T in the treatment of the earlier case (where n had been 2). Pick any field L that contains $\mathcal{T}$ (and hence $\mathcal{R}$) as a subring. Using, for instance, the proof of Corollary 18.5 of [23], we can construct a valuation domain of the form $V = L+M$ (which $= L \oplus M$ additively) such that V has Krull dimension $n-2$ (where $\infty \pm z := \infty =: z+\infty$ for any integer z, as usual) and M is the maximal ideal of V. It is interesting to observe that the domains $R := \mathcal{R} + M$ and $T := \mathcal{T} + M$ have the same quotient field, since they share M as a common nonzero ideal. The standard lore of the classical $(D+M)$-construction, as in Exercise 12, p. 202 of [23], yields that

$$\text{Spec}(R) = \text{Spec}(V) \cup \{\mathcal{P} + M \mid \mathcal{P} \in \text{Spec}(\mathcal{R})\},$$

with, of course, a similar description of $\text{Spec}(T)$. (The same conclusions are also available via Theorem 1.4 of [22].) We have that

$$\dim(R) = \dim(\mathcal{R}+M) = \dim(\mathcal{R}) + \dim(V) = 2 + (n-2) = n,$$

and, similarly, $\dim(T) = n$.

Next, the above description of prime spectra implies that the ring extension $R := \mathcal{R}+M \subseteq \mathcal{T}+M = T$ inherits from $\mathcal{R} \subseteq \mathcal{T}$ the property of being unibranched. (This assertion is an easy consequence of the following observations: if $Q \in \mathrm{Spec}(V)$ is viewed in $\mathrm{Spec}(\mathcal{T}+M)$, then $Q \cap R = Q \cap \mathcal{R}$ (since $L \cap M = 0$); and if $\mathcal{Q} \in \mathrm{Spec}(\mathcal{T})$, then $(\mathcal{Q}+M) \cap (\mathcal{R}+M) = (\mathcal{Q} \cap \mathcal{R}) + M$.) Finally, the ring extension $R \subseteq T$ also inherits from $\mathcal{R} \subseteq \mathcal{T}$ the property of not inducing an order-isomorphism of prime spectra. Indeed, if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are prime ideals of $\mathcal{T}$ such that $\mathcal{Q}_1 \cap \mathcal{R} \subseteq \mathcal{Q}_2 \cap \mathcal{R}$ but $\mathcal{Q}_1 \not\subseteq \mathcal{Q}_2$, then $\mathcal{Q}_1 + M \not\subseteq \mathcal{Q}_2 + M$ as prime ideals of T, while

$$(\mathcal{Q}_1 + M) \cap R = (\mathcal{Q}_1 \cap \mathcal{R}) + M \subseteq (\mathcal{Q}_2 \cap \mathcal{R}) + M = (\mathcal{Q}_2 + M) \cap R.$$

Hence, the domain extension $\mathcal{R}+M \subseteq \mathcal{T}+M$ has all the asserted properties for the case $3 \leq n \leq \infty$. The proof is complete. □

Despite Theorem 5.6, the next result shows that, by adding the hypothesis that a unibranched ring extension $A \subseteq B$ satisfies the technical property of going-up (which is denoted by GU, as in page 28 of [27]), one can guarantee that $A \subseteq B$ does not exhibit the somewhat pathological behavior of the extensions $R \subseteq T$ in Theorem 5.6. This result will be particularly useful, since integrality of a ring extension guarantees that the ring extension satisfies GU (cf. Theorem 44 of [27]). It is of technical interest that GU $\Rightarrow$ LO (cf. Theorem 42 of [27]) but the converse is false (cf. Exercise 3, page 41 of [27]).

Proposition 5.7. *Let $A \subseteq B$ be a unibranched ring extension that satisfies* GU. *Then the contraction map $c_A^B : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an order-isomorphism of posets under inclusion.*

Proof. Since $A \subseteq B$ is unibranched, c_A^B is a bijection. It remains to prove that if $Q_1, Q_2 \in \mathrm{Spec}(B)$ with $P_i := Q_i \cap A$ for all $i \in \{1,2\}$, then $P_1 \subseteq P_2$ if and only if $Q_1 \subseteq Q_2$. By intersecting with A, we see that the "if" assertion is clear. Lastly, suppose that $P_1 \subseteq P_2$; that is, $Q_1 \cap A \subseteq Q_2 \cap A$. Since $A \subseteq B$ satisfies GU, there exists $Q_3 \in \mathrm{Spec}(B)$ such that $Q_1 \subseteq Q_3$ and $Q_3 \cap A = P_2$. In particular, $c_A^B(Q_3) = Q_3 \cap A = P_2 = Q_2 \cap A = c_A^B(Q_2)$. As c_A^B is an injection, it follows that $Q_3 = Q_2$. So, since $Q_1 \subseteq Q_3$, we get $Q_1 \subseteq Q_2$, completing the proof. □

We next isolate the special case of Proposition 5.7 where the unibranched ring extension $A \subseteq B$ is also assumed to be integral. Corollary 5.8 will play a crucial role in the proof of Theorem 5.10, as the use of Corollary 5.8 in that proof will enable us to avoid having to assume that the domain R in Theorem 5.10 is treed.

Corollary 5.8. *Let $A \subseteq B$ be a ring extension that is both integral and unibranched. Then the contraction map $c_A^B : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an order-isomorphism of posets under inclusion.*

Remark 5.9. As promised in the proof of Theorem 5.6, we will now explain why neither the construction in the proof of the Main Theorem in [18] nor any the constructions in the proof of Theorem 2.1 in [13] could be used to give the assertion for $n = 2$ in Theorem 5.6. The former construction would be inappropriate for a proof of Theorem 5.6 because the posets that were called X and Y in the proof in [18] had unequal cardinalities (namely, 4 and 3, respectively), so that the resulting domain extension in [18] was, perforce, not unibranched. As for the constructions in the proof of Theorem 2.1 in [13]: each of those constructions would be inappropriate for a proof of Theorem 5.6 because each of those constructions was designed to produce a base ring of Krull dimension at least 3.

We freely admit the following two things: the proof of the Main Theorem in [18] provided a hint as to how to proceed in proving the case of finite $n \geq 3$ in Theorem 2.1 of [13] and the case of $n = 2$ in Theorem 5.6; and all three proofs made use of the same realization theorem of Hochster [26]. However, the fact of the matter is that the statements of those three results were sufficiently different from one another that we needed to devise different posets X and Y in each of those proofs (and different functions $\varphi : Y \to X$ in each of those proofs). That is why the proof that was given for the case of $n = 2$ in Theorem 5.6 carefully defined X, Y and φ, while expecting the reader to be (or to become) somewhat familiar with the details from [18] or [13] explaining how that kind of poset-theoretic data could be used to get suitable ring extensions and then to get suitable domain extensions. The remark is complete.

We come now to the main result of this section. From the “one P at a time" point of view, Theorem 5.10 can be considered a generalization of Corollary 2.6 of [9]. So, as an adjunct to the first sentence of the Introduction, I would like to add that this paper is also a sequel to [9].

Theorem 5.10. Let R be a quasi-local domain with maximal ideal M and quotient field K. Let P be a non-divided prime ideal of R (that is, $P \in \text{Spec}(R)$ and $PR_P \neq P$). Suppose that each element of PR_P is integral over R. Consider the CPI−extension $T := R + PR_P$. Then:

(a) $M^{\downarrow}$ is pinched at P and $R \subseteq T$ is a unibranched integral extension (and, hence, a P-unibranched extension).

(b) T is quasi-local.

(c) Let $u \in PR_P \setminus P$. Put $S := R[u]$. Then $R \subseteq S$ is an integral extension that is both unibranched and P-unibranched (and so S is quasi-local) and there exists an integer $n \geq 2$ such that $u^n S \subseteq P$.

(d) There exists $w \in PR_P \setminus P$ such that $w^n \in P$ for all $n \geq 2$.

Proof. (a) The hypothesis that each element of PR_P is integral over R gives two dividends at once: T is an integral ring extension of R (cf. Theorem 13 of [27]) and $M^{\downarrow}$ is pinched at P (by Proposition 4.3). Next, recall from Lemma 5.1 (a) that the canonical map $c_R^T : \text{Spec}(T) \to \text{Spec}(R)$ (given by $\mathcal{Q} \mapsto \mathcal{Q} \cap R$) is an injection that maps onto the set of prime ideals of R that are comparable to P (under inclusion). *That* set is all of Spec(R) since $M^{\downarrow}$ is pinched at P, and so $R \subseteq T$ is a unibranched extension. (Since T is integral over R, here is an alternate proof that c_R^T is surjective: simply apply the classical Lying-over Theorem (cf. Theorem 11.5 of [23]).) Finally, since c_R^T is bijective and $M^{\downarrow}$ is pinched at P, we get the parenthetical assertion that the extension $R \subseteq T$ is P-unibranched.

(b) As R is quasi-local with maximal ideal M and $R \subseteq T$ is integral, a well-known property of integral ring extensions (cf. Corollary 5.8 of [2]) ensures that Max(T) is the set of prime ideals of T that meet R in M. Since c_R^T is an injection (because $R \subseteq T$ is unibranched), it follows that Max(T) is a singleton set; that is, T is quasi-local.

(c) By (a), $R \subseteq T$ is an integral extension, and so $S \subseteq T$ inherits integrality from $R \subseteq T$. Therefore, since it was shown in (b) that T is quasi-local, the parenthetical assertion that S is quasi-local follows from the above-mentioned Corollary 5.8 of [2]. Also, $R \subseteq S$ inherits P-unibranchedness from $R \subseteq T$ (thanks to its integrality and Corollary 5.4, with the latter applying since we saw in (a) that $R \subseteq T$ is P-unibranched). Moreover, the extension $R \subseteq S$ is unibranched, since it is P-unibranched and $M^{\downarrow}$ (= Spec(R)) is pinched at P. It remains to prove the existence of a suitable n. That endeavor will occupy most of the rest of the proof of this theorem. As mentioned above, it will, insofar as possible, be organized like the proof of Corollary 2.6 of [9].

Since $R \subseteq S$ is P-unibranched, it follows from Lemma 5.1 (c) that the unique prime ideal of S that contracts to P is $\mathcal{P} := PR_P \cap S$.

This proof will not need the fact that Spec(S) is pinched at $\mathcal{P}$. Readers who join me in finding this fact to be interesting are invited to prove it. (Here is an essentially complete hint. Combine the following three facts: $R \subseteq S$ is unibranched; $M^{\downarrow}$ is pinched at P; and $R \subseteq S$ satisfies GU, thanks to integrality).

We return in earnest to the proof. Consider the conductor

$$J := (R :_T S) := (R : S) = \{\rho \in T \mid \rho S \subseteq R\}.$$

Let I denote $\text{rad}_S(J)$, the radical in S of J. In other words,

$$I := \{\sigma \in S \mid \text{there exists a positive integer } k \text{ such that } \sigma^k \in J\};$$

equivalently, I is the intersection of the set of prime ideals of S that contain J (cf. Proposition 1.14 of [2]).

We can write $u = a/b$ for some elements $a \in P$ and $b \in R \setminus P$. By hypothesis, u is integral over R, and so u is a root of some monic polynomial, say f, in $R[X]$ (where X is a commuting indeterminate over R). Let k be the degree of f. Then $k \geq 1$ and

$$S = R[u] = \sum_{i=0}^{k-1} Ru^i = \sum_{i=0}^{k-1} R(\frac{a^i}{b^i}).$$

Hence, $b^k S = \sum_{i=0}^{k-1} Ra^i b^{k-i} \subseteq R$. Thus, $b^k \in (R :_T S) = J$. Therefore, $b \in I$. As $\mathcal{P} \cap R = P$, while $b \in R \setminus P$, we get that $b \notin \mathcal{P}$.

We claim that $\mathcal{P} \subseteq I$. An equivalent claim is that if $\mathcal{Q} \in \text{Spec}(S)$ satisfies $J \subseteq \mathcal{Q}$, then $\mathcal{P} \subseteq \mathcal{Q}$. Pick $\mathcal{Q} \in \text{Spec}(S)$ such that $J \subseteq \mathcal{Q}$. As Corollary 5.8 ensures that $c_R^S : \text{Spec}(S) \to \text{Spec}(R)$ is an order-isomorphism of posets under inclusion, the claim will be established if we show that $\mathcal{P} \cap R \subseteq \mathcal{Q} \cap R$ (that is, if we show that $P \subseteq \mathcal{Q} \cap R$). As M is the unique maximal ideal of R and $M^{\downarrow}$ is pinched at P, the claim will have been established if we show that $\mathcal{Q} \cap R \not\subseteq P$. As $b^k \in J \subseteq \mathcal{Q}$, the primeness of $\mathcal{Q}$ gives $b \in \mathcal{Q}$, whence $b \in \mathcal{Q} \cap R$. As $b \in R \setminus P$, it follows that $\mathcal{Q} \cap R \not\subseteq P$, thus completing the proof of the above claim.

Since $u \in PR_P \cap S = \mathcal{P}$, the above (proven) claim gives that $u \in I$ $(= \text{rad}_S(J))$. Hence, there exists a positive integer d such that $u^d \in J$, that is, $u^d R[u] \subseteq R$. In fact, as $u \in PR_P$ and $d \geq 1$, we have $u^d R[u] \subseteq PR_P \cap R = P$. Moreover, $d \geq 2$ (since $d = 1$ would lead to the absurdity that $u \in R$). The proof of (c) is complete.

(d) By (c) and the hypothesis that P is not a divided prime ideal of R, there exists a least integer $n \geq 2$ such that $v^n R[v] \subseteq P$ for some element $v \in PR_P \setminus P$. Pick such an element v. Put $w := v^{n-1}$. For each integer $k \geq 2$, $w^k = v^{kn-k} \in P$ since $kn - k \geq n$ (since $n \geq 2 \geq k/(k-1)$). By the minimality of n, we get $n - 1 = 1$, whence $n = 2$, and so v can serve as the required w. □

Theorem 5.10 will have several corollaries. Before giving those, we will devote the next remark to some history concerning a concept that was mentioned in the title of this section of the paper (and the so-called "(2,3)-closed" property which characterizes that concept within the universe of domains).

Remark 5.11. We pause to give a brief (and incomplete) account of the work published on seminormality during (for the most part) 1969-1980. (A more historically accurate account would also discuss, at least, a 1962 paper of Bass in the Trans. Amer. Math. Soc., a 1963 paper of Endo in J. Math. Soc. Japan, and a 1967 paper of Bass and Murthy in Ann. Math.) Our account begins with a paper, [32], that was published in 1970. In it, Traverso was mostly concerned with ring extensions $A \subseteq B$ where A is Noetherian (possibly a domain, definitely reduced) and B is a finitely generated integral extension of A. As one may have surmised from the title of [32], Traverso was especially interested in domains A such that $\text{Pic}(A) = \text{Pic}(A[X_1, \dots, X_n])$, where $X_1, \dots, X_n$ are finitely many commuting algebraically independent indeterminates over A. (For rings $C \subseteq D$ with inclusion map $i : C \to D$, one conventionally writes $\text{Pic}(C) = \text{Pic}(D)$ if the monomorphism $i : C \hookrightarrow D$ has a left inverse in the category of C-modules and $\mathrm{Pic}(i) : \mathrm{Pic}(C) \to \mathrm{Pic}(D)$ is surjective.) Possible applications to algebraic K-theory were clearly relevant. The work on/in [32] became of quick, widespread interest. Andreotti and Bombieri published a paper (published in 1969, whose bibliography listed [32] as being "*in preparazione*") that introduced the concept of "weak seminormalization", motivated by a question in algebraic geometry concerning a possible analogue of Zariski's Main Theorem that would go beyond the context of a "normalization". During the next decade (and subsequently), there has been intense work done on seminormality. Near the beginning of their 1979 paper [7], Brewer and Costa did a good job of summarizing some of that activity prior to proving their Theorem 1. That proof cleverly exploited and extended that work by showing, *i.a.*, that a domain D, with quotient field K, is seminormal (in the sense that $\mathrm{Pic}(D) = \mathrm{Pic}(D[X_1, \dots, X_n])$) if and only if each $u \in K$ such that $u^2 \in D$ and $u^3 \in D$ must actually satisfy $u \in D$. The "only half" of this characterization of seminormal domains was essentially due to an observation of Schanuel that had been widely publicized by Bass and others. (In the Spring of 1967, while discussing Schanuel's Lemma in the first course that I took on homological algebra, the instructor mentioned that Schanuel was "the kind of person whose most interesting results ended up in other people's papers". During that same semester, Schanuel was teaching the first course that I took on algebraic number theory.) Since seminormal rings are now a rather well understood class of reduced rings, perhaps the most efficient "modern" way to reprove the above-mentioned part of Theorem 1 of [7] would be to apply the construction/description of the seminormalization in Theorem 2.8 of [31]. The remark is complete.

We next give five corollaries of Theorem 5.10. The first of these moves beyond the "quasi-local base domain" context of Theorem 5.10, and its part (b) begins this paper's use of seminormality.

Corollary 5.12. *Let R be a domain, $P \in \mathrm{Spec}(R)$, and $M \in \mathrm{Max}(R)$ be such that $P \subseteq M$ and also such that each $u \in PR_P$ is integral over R_M. Then:*

(a) For each $u \in PR_P$, there exists a positive integer n_M (possibly depending on u) such that $u^{n_M} R_M[u] \subseteq PR_M$.

(b) Suppose, in addition, that R_M is seminormal (for instance, suppose, in addition, that R is seminormal). Then $PR_P = PR_M$.

(c) Suppose that for each $N \in \mathrm{Max}(R)$ such that $P \subseteq N$, R_N is seminormal and each $u \in PR_P$ is integral over R_N. Then P is a locally divided prime ideal of R.

Proof. (a) The assertion is clear if $u \in PR_M$. So, without loss of generality, $u \in PR_P \setminus PR_M$. As $(PR_M)(R_M)_{PR_M} = PR_P$, the proof of (a) ends by applying the final assertion in Theorem 5.10 (c) to the base domain R_M.

(b) It is well known (and easy to prove) that a domain D is seminormal if and only if D_N is seminormal for all $N \in \mathrm{Max}(D)$. It is also well known (and almost as easy to prove) that a domain D (say, with quotient field L) is semnormal if and only if each $w \in L$ such that there exists a positive integer k satisfying $\{w^j \mid j \geq k\} \subseteq D$ must actually satisfy $w \in D$. The first of these well known facts explains the parenthetical assertion in (b). The second of these well known facts explains how (b) is a consequence of (a), the point being that we need only show that if $u \in PR_P$, then $u \in PR_M$.

(c) When the assertion in (b) is universally quantified over the set of all $N \in \mathrm{Max}(R)$ such that $P \subseteq N$, we get that $PR_P = PR_N$ for all such N. *That* conclusion is logically equivalent to the asserted conclusion. The proof is complete. □

The next corollary features one of our few uses of a hypothesis having the flavor of the "finite character" condition.

Corollary 5.13. *Let R be a domain and $P \in \mathrm{Spec}(R)$. Suppose that each $u \in PR_P$ is integral over R_M for each $M \in \mathrm{Max}(R)$ such that $P \subseteq M$. Suppose also there are only finitely many maximal ideals of R that contain P. Then for each $u \in PR_P$, there exists a positive integer n (possibly depending on u) such that $u^n R_M[u] \subseteq PR_M$ for each $M \in \mathrm{Max}(R)$ such that $P \subseteq M$.*

Proof. Let $\{M_1, \dots, M_e\}$ be the set of maximal ideals of R that contain P. Consider any element $u \in PR_P$. For each i such that $1 \leq i \leq e$, Corollary 5.12 (a) provides a positive integer n_i such that $u^{n_i} R_{M_i}[u] \subseteq PR_{M_i}$. Put $n := \max(n_1, \dots, n_e)$. Then, if $1 \leq i \leq e$, we have

$$u^n R_{M_i}[u] \subseteq u^{n_i} R_{M_i}[u] \subseteq PR_{M_i}.$$

The proof is complete. □

We will develop a "one P at a time" kind of approach that will recover (and sharpen) the result (cf. Corollary 2.6 of [9]) that a seminormal domain is a locally divided domain if and only if it is a going-down domain (see Corollary 5.15). This approach will begin by giving two definitions. Then we will indicate how/where those definitions play a role in the next several results on the way to the promised sharpening.

Let R be a domain and $P \in \text{Spec}(R)$. We say that P is a *strongly seminormal prime ideal of R with respect to M* if the three conditions $u \in PR_P$, $u^2 \in PR_M$ (equivalently, $u^2 \in R_M$, as $PR_P \cap R_M = PR_M$), and $u^3 \in PR_M$ (equivalently, $u^3 \in R_M$) jointly imply that $u \in PR_M$. (Of course, this definition is motivated by the "(2,3)-closed" characterization of seminormal domains that was recalled above.) Next, we say that P is a *strongly seminormal prime ideal of R* if P is a strongly seminormal prime ideal of R with respect to M for each $M \in \text{Max}(R)$ such that $P \subseteq M$. (Note that this definition does not necessarily apply to all $M \in \text{Max}(R)$.) Of course, if R is a seminormal domain, then each $P \in \text{Spec}(R)$ is a strongly seminormal prime ideal of R.

Parts (b) and (d) of Corollary 5.14 will show how, by suitably augmenting the hypothesis that P is a strongly seminormal prime ideal of a domain R, we can conclude that P is a locally divided prime ideal of R. Part (a) of Corollary 5.15 will then infer that for a seminormal domain R and $P \in \text{Spec}(R)$, P is a locally divided prime ideal of R if and only if $R \subseteq R[v^{-1}]$ satisfies GD to P for all nonzero elements $v \in PR_P$. By universally quantifying over R, this result leads, in Corollary 5.15 (b), to our new proof that a seminormal domain is a locally divided domain if (and only if) it is a going-down domain.

Corollary 5.14. *Let R be a domain and let $P \in \text{Spec}(R)$. Then:*

(*a*) *Let $M \in \text{Max}(R)$ such that $P \subseteq M$. Suppose also that P is a strongly seminormal prime ideal of R with respect to M and that each $u \in PR_P$ is integral over R_M. Then $PR_P = PR_M$.*

(*b*) *Suppose that P is a strongly seminormal prime ideal of R and that each $u \in PR_P$ is integral over R_M for each $M \in \text{Max}(R)$ such that $P \subseteq M$. Then P is a locally divided prime ideal of R.*

(*c*) *Let $M \in \text{Max}(R)$ such that $P \subseteq M$. Suppose also that P is a strongly seminormal prime ideal of R with respect to M and that $R_M \subseteq R_M[v^{-1}]$ satisfies GD to PR_M for each nonzero element $v \in PR_P$. Then $PR_P = PR_M$.*

(*d*) *Suppose that P is a strongly seminormal prime ideal of R and that $R \subseteq R[v^{-1}]$ satisfies GD to P for each nonzero element $v \in PR_P$. Then P is a locally divided prime ideal of R.*

Proof. (a) As $PR_M \subseteq PR_P$ in general, it remains only to prove the reverse inclusion. Suppose that the assertion fails; that is, suppose that $PR_M \subset PR_P$. Pick $u \in PR_P \setminus PR_M$. Since u is integral over R_M, it follows from Theorem 5.10 (c), as applied to the base domain R_M, that there exists

$$w \in (PR_M)(R_M)_{PR_M} \setminus PR_M = PR_P \setminus PR_M$$

such that $w^n \in PR_M$ for all $n \geq 2$. Since P is a strongly seminormal prime ideal of R with respect to M, the facts that $w \in PR_P$, $w^2 \in PR_M$ and $w^3 \in PR_M$ entail that $w \in PR_M$, the desired contradiction.

(b) The hypotheses of (b) entail that for each $M \in \text{Max}(R)$ such that $P \subseteq M$, P is a strongly seminormal prime ideal of R with respect to M and that each $u \in PR_P$ is integral over R_M. Consequently, by (a), $PR_P = PR_M$ for each $M \in \text{Max}(R)$ such that $P \subseteq M$. *That* conclusion is precisely what it means for P to be a locally divided prime ideal of R.

(c) Suppose that v is a nonzero element of PR_P and that $R_M \subseteq R_M[v^{-1}]$ satisfies GD to PR_M. Then, by the first paragraph of the proof of Lemma 2.4 (a) of [9], it follows that v is integral over R_M. Therefore, we see that (c) is a special case of (a).

(d) Let $M \in \mathrm{Max}(R)$ such that $P \subseteq M$. The first hypothesis ensures that P is a strongly seminormal prime ideal of R with respect to M. The second hypothesis, when coupled with the implication (1) $\Rightarrow$ (3) in Proposition 2.3, ensures that $R_M \subseteq R_M[v^{-1}]$ satisfies GD to PR_M. Hence, by (c), $PR_P = PR_M$. Universally quantifying over all the maximal ideals of R that contain P, we obtain the desired conclusion. The proof is complete. □

The next two corollaries give results for seminormal domains. They are motivated by Corollary 2.6 of [9]. For all intents and purposes, that result established that a seminormal domain is a locally divided domain if (and only if) it is a going-down domain. Corollary 5.15 (b) will give a new proof of that motivating result, by applying Corollary 5.15 (a). The latter result, which is in the "one P at a time" spirit, is a new characterization of the locally divided prime ideals of a seminormal domain. Thus, Corollary 5.15 (a) can be viewed as a "seminormal" companion for some of the results in [30], [14], parts of Sections 3 and 4 of this paper and, of course, Corollary 5.12 (c).

Corollary 5.15. *Let R be a seminormal domain. Then:*

(*a*) *Let* $P \in \mathrm{Spec}(R)$. *Then the following two conditions are equivalent:*

(*1*) $R \subseteq R[v^{-1}]$ *satisfies* GD *to P for all nonzero elements* $v \in PR_P$;

(*2*) *P is a locally divided prime ideal of R.*

(*b*) *The following three conditions are equivalent:*

(*i*) *R is a going-down domain;*

(*ii*) $R \subseteq R[v^{-1}]$ *satisfies* GD *to P for all* $P \in \mathrm{Spec}(R)$ *and for all nonzero elements* $v \in PR_P$;

(*iii*) *R is a locally divided domain.*

Proof. (a): (2) $\Rightarrow$ (1): The proof of this implication will not use the "seminormal" hypothesis. Assume (2). Then (1) follows by combining Corollary 3.1.3 of [30] with the oft-cited Exercise 37 on pages 44-45 of [27]. (For a faster proof of this implication, simply apply Theorem 2.4.)

(1) $\Rightarrow$ (2): We will offer two proofs of this implication. For the first proof, let us begin by assuming (1). Let v be a nonzero element of PR_P. Let M be any maximal ideal of R that contains P. Our task is to prove that $PR_P \subseteq PR_M$ (as the reverse inclusion holds in general). It will suffice to show that $v \in PR_M$. By combining (1) and the implication (1) $\Rightarrow$ (3) in Proposition 2.3, we get that $R_M \subseteq R_M[v^{-1}]$ satisfies GD to PR_M. Therefore, it follows from the first paragraph of the proof of Lemma 2.4 (a) of [9] that v is integral over R_M. (Of course, the same conclusion would hold if $v = 0$.) Hence, since R is assumed seminormal, we can apply Corollary 5.12 (b) to get $v \in PR_M$. This completes the first proof that (1) $\Rightarrow$ (2).

We next give a second proof that (1) $\Rightarrow$ (2). It follows from Corollary 5.14 (d) that it suffices to prove that P is a strongly seminormal prime ideal of R. We must show the following: if $M \in \mathrm{Max}(R)$ such that $P \subseteq M$ and $u \in PR_P$ satisfies $u^2 \in PR_M$ and $u^3 \in PR_M$, then $u \in PR_M$. Since R is seminormal, so is R_M. Therefore, as the element u in the quotient field of R_M satisfies $u^2 \in R_M$ and $u^3 \in R_M$, the seminormality of R_M yields that $u \in R_M$. Hence, $u \in PR_P \cap R_M = PR_M$, completing the second proof that (1) $\Rightarrow$ (2).

(b): (iii) $\Rightarrow$ (i): The proof of this implication will not use the "seminormal" hypothesis. It suffices to apply Remark 2.7 (b) of [9].

(i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): A domain D is a locally divided domain if and only if each prime ideal of D is a locally divided prime ideal of D. Therefore, it suffices to apply universal quantification on P to the implication (1) $\Rightarrow$ (2) in (a). The proof is complete. □

Let D be a domain with quotient field L. Consider an integer $n \geq 2$. Recall that D is said to be (an) *n-root closed* (domain) if each $\xi \in L$ such that $\xi^n \in D$ actually satisfies $\xi \in D$. It is clear (from the (2,3)-closed criterion for seminormal domains) that if D is an n-root closed domain, then D must be a seminormal domain. (The converse is false.) It is well known (and easy to prove) that D is n-root closed if and only if D_N is n-root closed for each $N \in \mathrm{Max}(D)$. Since n-root closure has been studied in its own right (that is, outside the context of studies of seminormality), it seems appropriate to include the following corollary which is in somewhat the same spirit as Corollary 5.15 (b). Notice, as well, that Corollary 5.16 harkens back to the "root closed" formulation of Corollary 2.8 of [9].

Corollary 5.16. *Let R be an n-root closed domain, for some integer $n \geq 2$. Then the following three conditions are equivalent:*
(1) *R is a going-down domain;*
(2) *For all $P \in \mathrm{Spec}(R)$ and for all nonzero elements $v \in PR_P$, $R \subseteq R[v^{-1}]$ satisfies* GD *to P;*
(3) *R is a locally divided domain.*

Proof. While readers should have no trouble in finding rather convoluted proofs of this corollary, here is the easiest proof of this corollary: since every n-root closed domain is a seminormal domain (assuming, of course, that the integer $n \geq 2$), the assertion is a special case of Corollary 5.15 (b). □

I do not know whether the next result has been noticed before. Proposition 5.17 establishes that the seminormalization of the base domain R in Theorem 5.10 contains the CPI-extension T figuring in that result. With Proposition 5.17 in hand, one could bypass the first (and easier) half of the above proof of Theorem 5.10 (c) and resume the proof of Theorem 5.10 (c) at the point where I had written that "We return in earnest to the proof.". It would be interesting to know if Proposition 5.17 could be used to complete the proof of Theorem 5.10 (c) from that point onward in a way that is simpler than what was given above. Note, however, that the proof of Proposition 5.17 will begin by using more explicit information about seminormality (and seminormalization) than was needed in the above proof of Theorem 5.10.

Proposition 5.17. *Let R be a domain with quotient field K. Let $P \in \mathrm{Spec}(R)$ such that each element of PR_P is integral over R. Put $T := R + PR_P$. Let R^+ be the seminormalization of R (in K). Then:*
(*a*) *$T \subseteq R^+$.*
(*b*) *Suppose, in addition, that R is a seminormal domain. Then P is a divided prime ideal of R (that is, $PR_P = P$).*

Proof. (a) It is known that R^+ is the largest subintegral extension of R inside K and, in fact, R^+ contains each subintegral extension of R inside K. Hence, it suffices to prove that the ring extension $R \subseteq T$ is subintegral. Of course, $R \subseteq T$ is integral because each element of PR_P is assumed to be integral over R. Moreover, the contraction map $c_R^T : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is a injection by parts (b)-(e) of Lemma 5.1 and a surjection by the Lying-over Theorem. Thus, it remains only to prove that if $\mathfrak{Q} \in \mathrm{Spec}(T)$ and $\mathfrak{P} := \mathfrak{Q} \cap R$ (in $\mathrm{Spec}(R)$), then the canonical injective R-algebra homomorphism $R/\mathfrak{P} \to T/\mathfrak{Q}$ (which we will regard as an inclusion map) induces an isomorphism of quotient fields. Of course, it induces an injection. There are three cases to consider, corresponding to parts (c), (d) and (e) of Lemma 5.1.

In the first case, $\mathfrak{Q} = PR_P$ (and $\mathfrak{P} = P$). Then a standard isomorphism theorem allows the canonical injection of domains, $R/\mathfrak{P} \hookrightarrow T/\mathfrak{Q}$, to be viewed as an identity map. Of course, that identity map induces an isomorphism of quotient fields.

In the next case, $PR_P \subset \mathfrak{Q} = \mathfrak{P} + PR_P$ (and $P \subset \mathfrak{P}$). It is easy to check that in this case, the canonical inclusion $R/\mathfrak{P} \hookrightarrow T/\mathfrak{Q}$ is an identity map, and so the argument concludes as it did for the first case.

In the last case, $\mathfrak{Q} = \mathfrak{P}R_P \subset PR_P$ (and $\mathfrak{P} \subset P$). (This, the hardest case, is not hard.) Consider elements $\lambda \in P \setminus \mathfrak{P}$ and $v \in R \setminus P$. It will suffice to prove that all cosets of $\mathfrak{P}R_P$ in T that are represented by elements of the form λ/ν are in the image of the induced map of quotient fields. (Indeed, it will then follow that the induced map of quotient fields is surjective, since its image will be a field containing $T/\mathfrak{Q}$.) Since multiplicative homomorphisms send (multiplicative) inverses to inverses and the coset represented by λ *is* in the image of the induced map of quotient fields, it will suffice to show that the image of $\nu + \mathfrak{P}$ in the quotient field of $T/\mathfrak{Q}$ is nonzero. Thus, it will suffice to show that the canonical image of $\nu + \mathfrak{P}$ in $T/\mathfrak{Q}$ is nonzero. That, in turn, holds since $\nu \notin \mathfrak{P} = \mathfrak{Q} \cap R$. The proof of (a) is complete.

(b) The hypothesis that R is a seminormal domain is equivalent to $R = R^+$. By (a), this condition implies that $PR_P \subseteq T \subseteq R^+ = R$, whence $PR_P = PR_P \cap R = P$. The proof is complete. □

It will be convenient to let J(R) denote the Jacobson radical of an ambient ring R.

Notice that Proposition 5.17 did not assume that R is quasi-local. However, in view of Lemma 5.1 (b) and the Lying-over Theorem, one can show that the assumptions in Proposition 5.17 do imply that $P \subseteq$ J(R). We leave this as an exercise here, as parts of Remark 5.18 will dwell on some related matters. For the moment, we note that this observation provides additional motivation for the study of the "$P \subseteq$ J(R)" condition, which will figure prominently in Remark 5.18.

We close with a ten-part remark. Readers are forewarned that some Propositions and a Theorem are stated and proved *within* the various parts of Remark 5.18. Motivated in part by the statement of Proposition 5.17, the initial parts of Remark 5.18 study the condition that a prime ideal P of a domain R is such that each element of PR_P is integral over R. One upshot, in Remark 5.18 (f), is a generalization of Theorem 5.10 for base domains that need not be quasi-local. The final three parts of Remark 5.18 raise some questions and suggest a deeper study of some properties.

Remark 5.18. (a) We begin our path to a generalization of Theorem 5.10 with the following result. It may come as a surprise to the reader (it did to the author) that Proposition 1 turns out to be the most important "new" agent in proving that generalization.

Proposition 1. Let R be a domain and let $P \in$ Spec(R) such that each element of PR_P is integral over R. Then each prime ideal of R is comparable to P (under inclusion), and so $P \subseteq$ J(R).

To prove the first assertion in Proposition 1, one need only adapt the proof of Proposition 4.3 (or the first paragraph of the proof of the implication (e) ⇒ (a) in Proposition 2.1 of [10]). It then follows that $P \subseteq M$ for each $M \in$ Max(R). As J(R) is the intersection of all such M, the proof of Proposition 1 is complete.

(b) Now that we have seen in (a) that the property "each element of PR_P is integral over R" can be a sufficient condition for some potentially useful properties, it makes sense to ask if the property "each element of PR_P is integral over R" can serve as a necessary condition for some clearly relevant property. Proposition 2 answers this question in the affirmative.

Proposition 2. Let R be a domain and let $P \in$ Spec(R) such that $P \subseteq$ J(R) and $R \subseteq R[v^{-1}]$ satisfies GD to P for each nonzero element $v \in PR_P$. Then each element of PR_P is integral over R.

To prove Proposition 2, one need only adapt the first paragraph of the proof of Lemma 2.4 of [9]. As the just-cited result assumed a quasi-local base domain, the proposed adaptation deserves serious scrutiny. The adaptation succeeds if a certain expression, "$(1-p_0)^{-1}$", in that proof describes an element of R. It will suffice to show that $p_0 \in$ J(R). That, in turn, holds by virtue of our hypotheses, since the element p_0 in the cited proof is an element of P. This completes the proof of Proposition 2.

(c) One cannot delete the hypothesis that $P \subseteq$ J(R) from the statement of Proposition 2 in (b). To see this, consider, any going-down domain R which is not quasi-local (for instance, $\mathbb{Z}$ or $\mathbb{Z}_{2\mathbb{Z}} \cap \mathbb{Z}_{3\mathbb{Z}}$). Let P be any nonzero prime ideal of R. Since R is a going-down domain, $R \subseteq R[v^{-1}]$ satisfies GD (and hence satisfies GD to P) for each nonzero element $v \in PR_P$. If the conclusion of Proposition 2 did not need the hypothesis that $P \subseteq \mathrm{J}(R)$, it would follow that each element of PR_P is integral over R, and then (by Proposition 1 in (a)) it would follow that $P \subseteq \mathrm{J}(R)$, and then (since P was an arbitrary nonzero prime ideal of R) it would follow that any two maximal ideals of R would be comparable under inclusion, and then it would follow that R has only one maximal ideal, contradicting the supposition that R is not quasi-local.

(d) I believe that it may be instructive for some readers to compare the reasoning in (b) and (c) with the proof of the equivalence (e) $\Leftrightarrow$ (a) in Proposition 2.1 of [10]. That equivalence states that a domain R is such that each element of PR_P is integral over R for each $P \in \mathrm{Spec}(R)$ if and only if R is a quasi-local going-down domain.

(e) It seems timely to pause here for an example illustrating Proposition 2 for a going-down domain R that is not quasi-local. We will do so with an example where such a domain R has the smallest interesting Krull dimension and the fewest possible prime ideals. Note that the assertion in Proposition 2 is trivial if R is not quasi-local and has Krull dimension less than 2. So, we will produce a non-quasi-local Pruefer domain R of Krull dimension 2, such that $\mathrm{Max}(R) = \{M, N\}$ (necessarily, $M \neq N$), both M and N have height 2, and R has only one prime ideal of height 1 (say, P, which is necessarily contained in both M and N).

To that end, define a 4-element poset $\mathcal{S} := (\{a, b, c, d\}, \leq)$ by imposing the requirements that $a < b < c$ and $b < d$, with no other occurrences of "<", apart from $a < c$ and $a < d$, which are required in view of the transitivity axiom for a partial order. Note that $\mathcal{S}$ is a finite tree and has a unique minimal element (namely, a). Therefore, a celebrated realization result of W. J. Lewis (see Theorem 3.1 of [28]) produces a Pruefer domain (in fact, a Bézout domain) R such that $\mathrm{Spec}(R)$ is order isomorphic (as a poset) to $\mathcal{S}$. Choose such a domain R and an order isomorphism $f : \mathcal{S} \to \mathrm{Spec}(R)$. Put $0 := f(a)$, $P := f(b)$, $M := f(c)$ and $N := f(d)$. It is now clear that R is a non-quasi-local Pruefer domain R of Krull dimension 2, such that $\mathrm{Max}(R) = \{M, N\}$, both M and N have height 2, and R has only one prime ideal of height 1 (namely, P) which is necessarily contained in both M and N. Thus, $P \subseteq M \cap N = \mathrm{J}(R)$.

To complete the verification that R illustrates Proposition 2 (and its conclusion that each element of PR_P is integral over R), we need only explain why we know that $R \subseteq R[v^{-1}]$ satisfies GD to P for each nonzero element $v \in PR_P$. (The reason is that *any* Pruefer domain is a going-down domain. Ideal theorists could see this because each localization of R at a maximal ideal is a valuation domain, hence a divided domain, hence a going-down domain. On the other hand, homological algebraists could see this by observing that Pruefer domains are the semi-hereditary domains, that is, the domains D all of whose torsion-free modules are flat, whence $D \subseteq T$ satisfies GD for all domains T containing D as a subring (since flat ring homomorphisms satisfy GD). The homological explanation, while longer, is in fact how I motivated the first proposed definition of going-down domains in [8].) I expect that any reader who is interested in seeing a more "concrete" example illustrating Proposition 2 would be able to fashion one by applying Theorem 22.8 of [23].

(f) The example in (e) featured, in particular, a non-quasi-local domain R and a nonzero, nonmaximal prime ideal P of R such that ($P \subseteq \mathrm{J}(R)$ and) each element of PR_P is integral over R. With such an example in hand, we believe that it is time to move past the motivating context from Corollary 2.6 of [9]. In that spirit, we present the following generalization of Theorem 5.10 to the context of base domains that need not be quasi-local. In short, we seek to determine what can be said about such data R and P. While the proof given below assumes that the reader is somewhat familiar with the proof of Theorem 5.10, it is intended to make clear how Proposition 1 from (b) will enable us to avoid Theorem 5.10's recourse to the existence of a unique maximal ideal M such that $M^{\downarrow}$ is pinched at P. The statement of the following Theorem is organized to make clear that all of Theorem 5.10

is being generalized here. As was the case for the proof of Theorem 5.10, our results (from earlier in this section) on (P)-unibranchedness and on order isomorphisms of spectral posets will play key roles in the proof of the following Theorem.

Theorem. Let R be a domain and let $P \in \mathrm{Spec}(R)$ such that each element of PR_P is integral over R. Put $T := R + PR_P$. Then:

(α) Each prime ideal of R is comparable to P (under inclusion) and $R \subseteq T$ is a unibranched integral extension (and, hence, a P-unibranched extension).

(β) $|\mathrm{Max}(T)| = |\mathrm{Max}(R)|$.

(γ) Let $u \in PR_P$. Put $S := R[u]$. Then $R \subseteq S$ is an integral extension that is both unibranched and P-unibranched (and so $|\mathrm{Max}(S)| = |\mathrm{Max}(R)|$) and there exists an integer $n \geq 1$ such that $u^n S \subseteq P$.

(δ) There exists a nonzero element $w \in PR_P$ such that $w^n \in P$ for all $n \geq 2$.

Proof. (α), (β): The first assertion in (α) follows from Proposition 1 in (a). Also, as in the proof of Theorem 5.10 (a), the contraction map $c_R^T : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is an injection (by Lemma 5.1 (a)) and a surjection (by the Lying-over Theorem, which applies here because the ring extension $R \subseteq T$ is integral). Thus, $R \subseteq T$ is unibranched (hence P-unibranched) and Corollary 5.8 ensures that c_R^T is an order-isomorphism of posets under inclusion. Moreover, since $R \subseteq T$ is unibranched and integral, it follows, as in the proof of Theorem 5.10 (b), from Corollary 5.8 of [2] that c_R^T induces a bijection $\mathrm{Max}(T) \to \mathrm{Max}(R)$, whence $|\mathrm{Max}(T)| = |\mathrm{Max}(R)|$.

(γ) We proved above that $R \subseteq T$ is an integral unibranched ring extension. Note that if $Q \in \mathrm{Spec}(T)$, then $Q \cap R = (Q \cap S) \cap R$, with $Q \cap S \in \mathrm{Spec}(S)$. Since the (integral) ring extension $S \subseteq T$ satisfies the lying-over property, it follows easily that $R \subseteq S$ is an unibranched (hence, P-unibranched) ring extension. (As $D \subseteq D^+$ is known to be an integral unibranched ring extension for any domain D, we could have gotten the same conclusion by replacing T with R^+ throughout the preceding three sentences and then invoking Proposition 5.17.) Therefore, by Corollary 5.8, the contraction map $c_R^S : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is an order-isomorphism of posets under inclusion. It remains to prove the existence of a suitable n. Insofar as possible, that will be done as in the proof of Theorem 5.10 (c).

We have the following information, as in the proof of Theorem 5.10 (c). The unique prime ideal of S that contracts to P is $\mathcal{P} := PR_P \cap S$; the ideals J and I are defined as in that proof; $u = a/b$ for some elements $a \in P$ and $b \in R \setminus P$; $b^k \in (R :_T S) = J$ for some integer $k \geq 1$, whence $b \in I$, whence $b \notin \mathcal{P}$; and it will suffice to prove the claim that if $\mathcal{Q} \in \mathrm{Spec}(S)$ satisfies $J \subseteq \mathcal{Q}$, then $\mathcal{P} \subseteq \mathcal{Q}$. (Indeed, if we establish the above claim, we would get, equivalently, that $\mathcal{P} \subseteq \cap \mathcal{Q} = \mathrm{rad}_S(J) = I$, whence $u \in PR_P \cap S = \mathcal{P} \subseteq I = \mathrm{rad}_S((R :_T S))$, whence some positive integer n satisfies $u^n S \subseteq R$, as desired.)

As explained in the preceding paragraph, the proof of (γ) will be complete if we show the following: if $\mathcal{Q} \in \mathrm{Spec}(S)$ satisfies $J \subseteq \mathcal{Q}$, then $\mathcal{P} \subseteq \mathcal{Q}$. Observe that $b^k \in \mathcal{Q} \setminus P$. Next: since c_R^S is an order-isomorphism of posets, it will suffice to prove that $\mathcal{P} \cap R \subseteq \mathcal{Q} \cap R$ (that is, equivalently, that $P \subseteq \mathcal{Q} \cap R$). At this point, we make the most important (perhaps the only) significant deviation from the methodology that is being adapted here from the proof of Theorem 5.10: invoke Proposition 1 from part (a) of this Remark. As Proposition 1 yields that P is comparable to $\mathcal{Q} \cap R$ and we already know that $b^k \in (\mathcal{Q} \cap R) \setminus P$, it must be the case that $P \subseteq \mathcal{Q} \cap R$. This completes the proof of (γ).

(δ) Since it is possible (but not necessary) that P is a divided prime ideal of R, one cannot simply repeat the proof of Theorem 5.10 (b) *verbatim* here. However, only a few small changes to that earlier argument will suffice. For the sake of completeness, we will indicate the details. As the assertion is clear in case $P = 0$, we may assume, without loss of generality, that $P \neq 0$. By (γ), there exists a least integer $n \geq 1$ such that $v^n R[v] \subseteq P$ for some nonzero element $v \in PR_P$. If $n \leq 2$, the proof of (δ) is complete. So, without loss of generality, $n \geq 3$. It will suffice to infer a contradiction. With $v \in PR_P$ chosen such that $v^n R[v] \subseteq P$, put $w := v^{n-1}$. For each integer $k \geq 2$, $w^k = v^{kn-k} \in P$ since $kn - k \geq n$ (since $n \geq 2 \geq k/(k-1)$). Thus, $w^2 R[v] \subseteq P$. As $w \in PR_P \setminus \{0\}$ and $2 < n$, this contradicts the minimality of n. The proof is complete.

(g) In view of the Theorem in (f), some readers may wish to rework the proofs of some of the corollaries of Theorem 5.10 in ways that minimize (or possibly eliminate) the need to pass from a given domain R to various (quasi-local) localizations R_M. That style of proof for Theorem 5.10 was not adopted above because of the following three reasons: my wish to adhere, as much as possible, to the style of the formulation of the result (Corollary 2.6 of [9]) that was being generalized; my sense that the condition "each element of PR_P is integral over R" is less general than "each element of PR_P is integral over R_M for each $M \in \mathrm{Max}(R)$ that contains P" (because it is not known, *a priori*, whether $P \subseteq \mathrm{J}(R)$); and my wish to make the proof of Theorem 5.10 more accessible by eliminating appeals to information concerning seminormality. I do not mean to denigrate the usefulness of such information. In fact, it was the rereading of Corollary 4.6 of [31] that reminded me that any domain D satisfies

$$D^+ = \cap_{M \in \mathrm{Max}(D)} (D_M)^+.$$

That recollection motivated me to consider if one could eliminate the "quasi-local" hypothesis on R while proving at least some of the conclusions in Theorem 5.10. Without access to [31] and all the work cited there, I may never have found the Theorem in (f).

(h) The description of D^+ that was displayed in (g) has the following application. If R is a domain and $P \in \mathrm{Spec}(R)$ such that $P \subseteq \mathrm{J}(R)$ and $R_M + PR_P \subseteq (R_M)^+$ for each $M \in \mathrm{Max}(R)$, then $R + PR_P \subseteq R^+$. I would like to express the hope that the experience of verifying this application will stimulate some readers to discover further applications of the above description of D^+.

(i) [19] was the third and final paper in a series of papers that led, *i.a.*, to a unambiguous definition of going-down domains. As the titles of those papers indicated, I had become interested in the role of "simple overrings" (that is, singly generated overrings) in characterizing going-down domains. That role was illustrated in various classes of domains of long-standing interest in the first paper in that series; an initial definition of going-down domains was proposed in the second paper in that series ([8]); and Theorem 1 in the third paper in that series ([19]) showed that the definition of going-down domains in [8] was equivalent to the condition that $R \subseteq R[u]$ satisfies GD for all elements u in the quotient field of the domain R. The "one P at a time" analogue of that result appeared as Theorem 4.10 above. Among its list of equivalent conditions was condition (1): $R \subseteq R[u]$ satisfies GD to P for each $u \in K$. My continuing interest in simple overrings led to a frequent use in the present paper of the condition that $R \subseteq R[v^{-1}]$ satisfies GD to P (most often, in situations where R was quasi-local). In several results in this paper, this condition was the only hypothesis that seemed to involve a "going-down kind of condition". This circumstance raises the following (vague, but I think, tempting) questions about a non-field domain R with quotient field K. If $u \in K \setminus R$, must there exist $P \in \mathrm{Spec}(R)$ and a nonzero element $v \in PR_P$ such that $R[u] = R[v^{-1}]$? (This question seems particularly apt in case R is seminormal, in view of Corollary 5.15.) If the answer is in the negative in general, what are some interesting sufficient conditions for the answer to be positive? More generally, are there any deep necessary and sufficient conditions for elements u and v of the quotient field K of a given domain R to satisfy $R[u] = R[v]$? What if we restrict attention to certain (hopefully tractable) classes of domains of long-standing interest in multiplicative theory and/or homological algebra?

(j) I would like to, once again, draw attention to the following two topics/concepts: P-unibranched extensions and contraction maps $c_A^B : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ that give order isomorphisms of prime spectra. These were the two essential ingredients that allowed me to complete the proof of Theorem 5.10, which achieved my goal of generalizing Corollaries 2.6 and 2.8 of [9] (and their proofs) to results (and proofs) that would fit into the "one P at a time" theme. I would like to express the hope that other workers will be stimulated to find reasons to study these two topics more deeply.

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