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On finiteness of some noncommutative Gröbner bases over finite fields

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Abstract. Eisenbud and al. proved that if $\mathbb{K}$ is a field of characteristic 0 and $\gamma : \mathbb{K}\langle X_1,...,X_n\rangle \longrightarrow \mathbb{K}[x_1,...x_n]$, the map from the noncommutative ploynomial ring to the commutative one which sends X_i to x_i then any noncommutative ideal $\mathcal{J} = \gamma^{-1}(\mathcal{I})$ has a finite Gröbner basis even after a linear change of variables. By an example they prove that if $\mathbb{K}$ is of characteristic $p \neq 0$ then this result does not always hold. In this work, we consider a coefficient finite field $\mathbb{K}$. Then we first give a necessary and sufficient condition for any ideal of the form $\gamma^{-1}(\mathcal{I})$ to have finite Gröbner basis. We secondly prove that this condition is satisfied for any 0-dimensionnal $\mathcal{I}$. We finish by investigating the particular case where $\mathcal{I}$ is a principal ideal.

Key Words: Finite field, Finiteness, Initial ideal, Linear change of variables, Noncommutative Gröbner bases.
2020 MSC: 16Z05, 08A62, 13B25, 13B02.

Dedicated to our Professor David E. Dobbs for his 80th Birthday.

1 Introduction

The concept of Gröbner basis was introduced by Bruno Buchberger in the context of commutative polynomial ring $\mathbb{K}[x_1,...,x_n]$ with coefficients in a field $\mathbb{K}$ during his PhD thesis. See [2]. Naively, a Gröbner basis of an ideal $\mathcal{I} \subset \mathbb{K}[x_1,...,x_n]$ can be defined as a set of generators of $\mathcal{I}$ which solves efficiently the ideal membership problem by using the generalization of the euclidean algorithm to the multivariate polynomials. Furthermore, Buchberger's algorithm guarantees the existence of a finite Gröbner basis for any ideal of $\mathbb{K}[x_1,...,x_n]$.

Except the ideal membership problem solving, Gröbner bases have serveral other interesting applications: ploynomial system solving, determination of a basis of an intersection, a product or a quotient of two ideals, etc. See [3] and [7]

Gröbner bases were then extended to many other algebras. One of the first generalizations was their adaptation to the noncommutative polynomial ring $\mathbb{K}\langle X_1,...,X_n\rangle$ presented in [1] and [8]. An immediate consequence of this extension is the lost of the finiteness. Most of the ideals of $\mathbb{K}\langle X_1,...,X_n\rangle$ do not have a finite Gröbner basis. The classical example is the principal ideal $\mathcal{J} = \langle X_1X_2X_1 - X_2X_1\rangle$. Any generator set of $\mathcal{J}$ which solves the ideal membership problem contains polynomials $f_k = X_1X_2^kX_1 - X_2^kX_1$ for any $k \in \mathbb{N}^*$. So $\mathcal{J}$ does not admit a finite Gröbner basis. Thus characterizing noncommutative ideals that have a finite Gröbner basis became an interesting challenge. In this direction, D. Eisenbud, I. Peeva and B. Sturmfels proved in [6] that if $\mathbb{K}$ is a field of characteristic 0 and if $\gamma : \mathbb{K}\langle X_1,...,X_n\rangle \longrightarrow \mathbb{K}[x_1,...x_n]$ is the map from the noncommutative ploynomial ring to the commutative one which sends X_i to x_i then any noncommutative ideal $\mathcal{J} = \gamma^{-1}(\mathcal{I})$ has a finite Gröbner basis even after a linear change of variables. In [5], Y. Diop and D. Sow treated the opposite problem. They showed that if a noncommutative ideal $\mathcal{J}$ has a finite Gröbner basis and contains all commutators then there exits an ideal $\mathcal{I}$ of $\mathbb{K}[x_1,...x_n]$ such that $\mathcal{J} = \gamma^{-1}(\mathcal{I})$. They also described how to find a generator set of the ideal $\mathcal{I}$.

The combination of these two results yields a characteriszation of noncommutative ideals that have a finite Gröbner basis and contain commutators in the case of a field of characteristic 0.

Recently, L. Mesmoudi and Y. Diop proved in [4] that the map γ can be repleced by any surjective homorphism $\lambda : \mathbb{K}\langle X_1,...,X_n\rangle \longrightarrow \mathbb{K}[x_1,...,x_n]$.

When the coefficient field $\mathbb{K}$ is finite then we can have an ideal $\mathcal{I}$ of $\mathbb{K}[x_1,...x_n]$ such that $\gamma^{-1}(\mathcal{I})$ has no finite Gröbner basis. See *Example 3.* Hence we are interested into finding conditions for which a noncommutative ideal of the form $\mathcal{J} = \gamma^{-1}(\mathcal{I})$ has a finite Gröbner basis when $\mathbb{K}$ is a finite field.

In all the paper:

1. $\mathbb{K}$ is a field;
2. (a) $\mathbb{K}[x_1,...,x_n]$ is the n-variate commutative polynomial ring over $\mathbb{K}$;
   (b) $\mathbb{M} = \{x_1^{\alpha_1}...x_n^{\alpha_n}\}$ is the set of monomials of $\mathbb{K}[x_1,...,x_n]$;
   (c) $\prec$ is a monomial order on $\mathbb{M}$;
   (d) For a non nil polynoial g, $in_\prec(g)$ is the leading monomial g *w.r.t* $\prec$;
   (e) $\mathcal{I} \subset \mathbb{K}[x_1,...x_n]$ is an ideal given by a minimal Gröbner basis G *w.r.t* $\prec$ and $in_\prec(\mathcal{I}) = \langle in_\prec(g),\ g \in \mathcal{I}\rangle$ is the initial ideal of $\mathcal{I}$;
3. (a) $\mathbb{K}\langle X_1,...,X_n\rangle$ is the n-variate noncommutative polynomial ring over $\mathbb{K}$;
   (b) $\mathbb{M}'$ is the set of monomials of $\mathbb{K}\langle X_1,...,X_n\rangle$;
   (c) $<<$ is the lexicographic extension of $\prec$ on $\mathbb{M}'$ (see *Proposition 2.3.* for the definition);
   (d) $in_{<<}(g)$ is the leading monomial of $0 \neq g \in \mathbb{K}\langle X\rangle$ *w.r.t* $<<$;

   (a) $\gamma : \mathbb{K}\langle X_1,...,X_n\rangle \longrightarrow \mathbb{K}[x_1,...,x_n]$ replaces X_i by x_i;
   (b) $\delta : \mathbb{K}[x_1,...,x_n] \longrightarrow \mathbb{K}\langle X_1,...,X_n\rangle$ replaces x_i by X_i by putting variables in the increasing order

2 Preliminaries

In this section, we recall the Gröbner bases computation of an ideal of type $\mathcal{J} = \gamma^{-1}(\mathcal{I})$ designed in [6]. The two elements in the next definition play a fundamental role.

Definition 2.1. Let $\mathcal{L}$ be a monomial ideal of $\mathbb{K}[x_1,...,x_n]$.

1. A monomial $m \in \mathcal{L}$ is called a minimal generator of $\mathcal{L}$ if $\frac{m}{u} \notin \mathcal{L}$ for any monomial $u \neq 1$ which divides m.
   The set of minimal generators of $\mathcal{L}$ will be denoted $\mathcal{M}(\mathcal{L})$.

2. Let $m = x_p^{\alpha_p}\cdots x_q^{\alpha_q}$, $p < \cdots < q$ be a minimal generator of $\mathcal{L}$. We define the set

$$U_{\mathcal{L}}(m) = \{u \in \mathbb{M} \cap \mathbb{K}[x_{p+1},\dots,x_{q-1}] \mid u\frac{m}{x_p} \notin \mathcal{L},\ u\frac{m}{x_q} \notin \mathcal{L}\}.$$

Example 2.2. Let $\mathcal{L} = \langle x_1x_2x_3, x_1^2x_3, x_2x_3^2\rangle$ be an ideal of $\mathbb{R}[x_1,x_2,x_3]$. Then $x_1x_2x_3$ and $x_2x_3^2$ are minimal generators of $\mathcal{L}$.
$U_{\mathcal{L}}(x_1x_2x_3) = \{u \in \mathbb{R}[x_2] \mid ux_2x_3 \notin \mathcal{L},\ ux_1x_2 \notin \mathcal{L}\} = \{x_2^d,\ d \in \mathbb{N}\}$ and $U_{\mathcal{L}}(x_2x_3^2) = \{1\}$

Remark that the second point of the definition can be extended to the case $p = q$. If $m = x_p^{\alpha_p}$ then $U_{\mathcal{L}}(m) = \{1\}$.
More generally, $1 \in U_{\mathcal{L}}(m)$ for any minimal genarator m of a monomial ideal $\mathcal{L}$.
The following theorem is the main result of [6]. It described a computation of a Gröbner basis of $\gamma^{-1}(\mathcal{I})$ from one of $\mathcal{I}$.

Proposition 2.3. *(See [6]) Let $\prec$ be a monomial order on $\mathbb{M}$. Then the ordering $\ll$ on $k\langle X\rangle$ defined by*

$$u \ll v \Leftrightarrow \begin{cases} \gamma(u) \prec \gamma(v) \\ \text{or} \\ \gamma(u) = \gamma(v) \text{ and } u \prec_{lex} v \end{cases}$$

is a monomial order on $\mathbb{M}'$. It is called the lexicographic extension of $\prec$.

Theorem 2.4 (Eisenbud, Peeva, Sturmfels)**.** If $\mathcal{J} = \gamma^{-1}(\mathcal{I})$ and G, a minimal Gröbner basis of $\mathcal{I}$ *w.r.t.* $\prec$ then the set

$$\{X_iX_j - X_jX_i \mid i > j, x_i, x_j \notin in_{\prec}(\mathcal{I})\} \cup \{\delta(uf) \mid f \in G, u \in U_{in_{\prec}(\mathcal{I})}(in_{\prec}(f))\}$$

is a minimal Gröbner basis of $\mathcal{J}$ *w.r.t.* lexicographic extension << of $\prec$.

This theorem apeared first in [6] (page 2.) and was then revisited and improved in [5] (pages 3. and 4.).
In the rest, when talking about Gröbner bases of $\gamma^{-1}(\mathcal{I})$, it is always *w.r.t.* lexicographic extension << of the considered monomial $\prec$ on $\mathbb{M}$.
It is clear that the Gröbner basis in *Theorem 2.4.* is finite if and only if $U_{in_{\prec}(\mathcal{I})}(in_{\prec}(f))$ is a finite set for any $f \in G$. Some conditions of finiteness are given in the two following results.

Definition 2.5. Let $\mathcal{L}$ be a monomial ideal of $\mathbb{K}[x_1,\dots,x_n]$.

1. $\mathcal{L}$ is said 0-Borel fixed if for any generator m of $\mathcal{L}$ and x_i a divisor of m then $x_j\frac{m}{x_i} \in \mathcal{L}\ \forall x_j \prec x_i$.
2. $\mathcal{L}$ is said p-Borel fixed if the following condition is satisfied: for any generator m of $\mathcal{L}$, if ${x_i}^t$ divides m and ${x_i}^{t+1}$ does not divide m then $(\frac{x_j}{x_i})^s m \in \mathcal{L}\ \forall x_j \prec x_i$ and $s \leq_p t$.

Proposition 2.6. *(Refer to [6] page 3. for the proof)*
If $in_{\prec}(\mathcal{I})$ is 0-Borel fixed or p-Borel fixed for some prime integer p then $U_{in_{\prec}(\mathcal{I})}(m)$ is finite for any minimal generaror m of $in_{\prec}(\mathcal{I})$.

The following theorem is enonciated and used in the *Proof of Corollary 1.1.* in [6]. Moreover a reference for it is given in the same document.

Theorem 2.7 (Bayer-Galligo-Stilmann)**.** If $\mathbb{K}$ is infinite then after a linear change of variables $in_{<}(\mathcal{I})$ is 0-Borel fixed or p-Borel fixed for some prime integer p.

Proposition 2.6. and *Theorem 2.7.* imply that if $\mathbb{K}$ is of characteristic 0 then Gröbner bases computed in *Theorem 2.4.* are finite even after a linear change of variables. In other words, for some well chosen linear change of variables and $\mathcal{I}'$, the ideal corresponding to $\mathcal{I}$, $\gamma^{-1}(\mathcal{I}')$ has a finite Gröbner basis.

Example 2.8. Let $\mathcal{I} = \langle x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3\rangle \subset \mathbb{R}[x_1, x_2, x_3]$ be the ideal generated by $x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3$.
For any fixed monomial order $\prec$, $in_{\prec}(\mathcal{I}) = \langle x_1x_2x_3\rangle$.
Then a first computation yields $U_{in_{\prec}(\mathcal{I})}(x_1x_2x_3) = \{x_2^d, d \in \mathbb{N}\}$.
But after the change of variables $\begin{cases} x_1 \longmapsto y_1 + y_3 \\ x_2 \longmapsto y_2 + y_3 \\ x_3 \longmapsto y_3 \end{cases}$ the ideal becomes:

$$\begin{aligned}\mathcal{I}' &= \langle (y_1 + y_3)(y_2 + y_3)y_3 + (y_1 + y_3)(y_2 + y_3) + (y_1 + y_3)y_3 + (y_2 + y_3)y_3\rangle \\ &= \langle y_1y_2y_3 + y_1y_3^2 + y_2y_3^2 + y_3^3 + y_1y_2 + 2y_1y_3 + 2y_2y_3 + 3y_3^2\rangle\end{aligned}$$

Then by the lexicographic order with $y_1 \prec y_2 \prec y_3$ its initial ideal is $\langle y_3^3\rangle$ and then $U_{in_{\prec}(\mathcal{I}')}(y_3^3) = \{1\}$. So by *Theorem 2.4.* the set:

$$\{Y_1Y_2Y_3 + Y_1Y_3^2 + Y_2Y_3^2 + Y_3^3 + Y_1Y_2 + 2Y_1Y_3 + 2Y_2Y_3 + 3Y_3^2,\ Y_2Y_1 - Y_1Y_2, Y_3Y_1 - Y_1Y_3, Y_3Y_2 - Y_2Y_1\}$$

is a Gröbner basis of $\gamma^{-1}(\mathcal{I}')$.

In the case of finite fields there are examples of ideals $\mathcal{I}$ for which there is any finite Gröbner basis for the ideal $\gamma^{-1}(\mathcal{I})$. The following example is discussd in a more general case in [6].

Example 2.9. Let $\mathcal{I} = \langle x_1^4x_2^2x_3 + x_1^4x_2x_3^2 + x_1^2x_2^4x_3 + x_1^2x_2x_3^4 + x_1x_2^4x_3^2 + x_1x_2^2x_3^4\rangle$ be an ideal of $\mathbb{F}_2[x_1, x_2, x_3]$. $\mathcal{I}$ is invariant under any linear change of variables. Then its initial ideal with respect to any monomial order is of the form $in_{\prec}(\mathcal{I}) = \langle x_ix_j^2x_k^4\rangle$ and $U_{in_{\prec}\mathcal{I}}(x_ix_j^2x_k^4) = \{x_j^d, d \in \mathbb{N}\}$.
This implies that $U_{in_{\prec}(\mathcal{I})}(in_{\prec}(f))$ is an infinite set and and then $\gamma^{-1}(\mathcal{I})$ has no finite Gröbner basis.

By *Theorem 2.7.*, we know that if $\mathbb{K}$ is of characteristic 0 then there exists a change of variables that yields a finite Gröbner basis of $\gamma^{-1}(\mathcal{I})$ for any ideal $\mathcal{I}$. Thus in what follows we will work exclusively with finite fields. More precisely, in all the rest $\mathbb{K} = \mathbb{F}_q$ is the finite field of q elements. Under some conditions we can decide whether $\gamma^{-1}(\mathcal{I})$ has a finite Gröbner basis. We discuss them in the next section.

3 On finiteness of Gröbner bases of $\gamma^{-1}(\mathcal{I})$ with a finite coefficient field

For a monomial ideal $\mathcal{L} \subset \mathbb{F}_q[x_1, \ldots, x_n]$ and a minimal generator m of $\mathcal{L}$, the following result yields a sufficients and necessary condition for $U_{\mathcal{L}}(m)$ to be finite.

Proposition 3.1. *Let $\mathcal{L} \subset \mathbb{F}_q[x_1, \ldots, x_n]$ be a monomial ideal and $m = x_p^{\alpha_p} \ldots x_q^{\alpha_q}$ a minimal generator of $\mathcal{L}$. The two following statements are equivalent.*

1. *For any integer i in $[p+1, q-1]$ there exists $d_i \in \mathbb{N}$ such that $x_i^{d_i} \notin U_{\mathcal{L}}(m)$.*
2. *$U_{\mathcal{L}}(m)$ is a finite set.*

Proof. Let $\mathcal{L}$ be a monomial ideal, $m = x_p^{\alpha_p} ... x_q^{\alpha_q}$, a minimal generator of $\mathcal{L}$ and $u \in U_{\mathcal{L}}(m)$. Then we can write $u = x_{p+1}^{\beta_{p+1}} ... x_{q-1}^{\beta_{q-1}}$, $\beta_{p+1}, ..., \beta_{q-1} \in \mathbb{N}$.
$1) \Longrightarrow 2)$
Suppose that for any integer i in $[p+1, q-1]$ there is $d_i \in \mathbb{N}$ such that $x_i^{d_i} \notin U_{\mathcal{L}}(m)$.

$$\begin{aligned} x_i^{d_i} \notin U_{\mathcal{L}}(m) &\Longrightarrow x_i^{d} \notin U_{\mathcal{L}}(m)\ \forall d > d_i \\ &\Longrightarrow U_{\mathcal{L}}(m) \subseteq \{x_{p+1}^{\beta_{p+1}} ... x_{q-1}^{\beta_{q-1}},\ \beta_i \leq d_{i_0}\}\ where\ d_{i_0} = \max\{d_i : x_i^{d_i} \in U_{\mathcal{L}}(m)\} \\ &\Longrightarrow Card(U_{\mathcal{L}}(m)) \leq \prod_{p+1}^{q-1} d_{i_0} \end{aligned}$$

$2) \Longrightarrow 1)$
Conversely if there exists i such that $x_i^{d_i} \in U_{\mathcal{L}}(m)\ \forall\ d_i$ then $U_{\mathcal{L}}(m)$ is an infinite set. So if $U_{\mathcal{L}}(m)$ is a finite set then for any i there exists $d_i \in \mathbb{N}$ such that $x_i^{d_i} \notin U_{\mathcal{L}}(m)$. ☐

Corollary 3.2. *Let $\mathcal{I} \subset \mathbb{F}_q[x_1, ..., x_n]$ be an ideal. Then $\gamma^{-1}(\mathcal{I})$ has a finite Gröbner basis if and only if the first condition of Proposition 3.1. is satisfied for any minimal generator of the initial ideal of $\mathcal{I}$.*

For so well-called 0-dimensionnal ideals, we prove that this condition is always satisfied. Those ideals have interesting properties and applications.

Definition 3.3. Let $\mathbb{K}$ be a field. An ideal $\mathcal{I} \subset \mathbb{K}[x_1, ..., x_n]$ is called a 0-dimensionnal ideal if the quotient ring $\mathbb{K}[x_1, ..., x_n]/\mathcal{I}$ has a finite dimension as a vector space.

Theorem 3.4. Let $\mathbb{K}$ be a field, $\mathcal{I} \subset \mathbb{K}[x_1, ..., x_n]$, an ideal and G, a Gröbner basis of $\mathcal{I}$ *w.r.t.* $\prec$. Then $\mathcal{I}$ is 0-dimensionnal if and only if for any x_i, there exists $d_i \in \mathbb{N}^*$ and $g \in G$ such that $in_{\prec}(g) = x_i^{d_i}$.

From *Theorem 3.4.* and *Corollary 3.2.* we have the following result.

Corollary 3.5. *Let $\mathcal{I} \subset \mathbb{F}_q[x_1, ..., x_n]$ be a 0-dimensionnal ideal. Then $\gamma^{-1}(\mathcal{I})$ has a finite Gröbner basis.*

Proof. Let $\mathcal{I}$ be a 0-dimensionnal and G, a minimal Gröbner basis of $\mathcal{I}$. Then for any $i \in [1, n]$ there exists $g \in G$ $s.t.$ $in_{\prec}(g) = x_i^{d_i}$ $for\ some$ $d_i \in \mathbb{N}^*$.
Thus for any $f \in G$, $x_i^{d_i} \notin U_{\mathcal{L}}(in_{\prec}(f))$. ☐

Because of *Corollary 3.5.*, the ideals we consider in the rest of the paper are non 0-dimensionnal. Then for any ideal $\mathcal{I} \subset \mathbb{F}_q[x_1, ..., x_n]$ in the following there exists $i \in [1, n]$ such that $x_i^{d_i} \neq in_{\prec}(g)$ for any $d_i \in \mathbb{N}^*$ and any $g \in G$. It is equivalent to say $x_i^{d_i} \notin in_{\prec}(\mathcal{I})$ for $d_i \in \mathbb{N}^*$. So the set $A = \{i \in [1, n] \mid x_i^{d_i} \notin in_{\prec}(\mathcal{I})\ \forall\ d_i \in \mathbb{N}^*\}$ is non empty.
In the following, we improve the result of *Proposition 3.1.*.
For a monomial $m = x_1^{\beta_1} ... x_n^{\beta_n}$, we set $\deg_{(m)}(x_i) = \beta_i$.

Proposition 3.6. *Let $\mathcal{I} \subset \mathbb{F}_q[x_1, ..., x_n]$ be a non 0-dimensinonal ideal and A defined as before. Let $i \in A$, $d = \max\{\deg_{m'}(x_i),\ m' \in \mathcal{M}(in_{\prec}(\mathcal{I}))\}$ and $m = x_p^{\alpha_p} ... x_i^{\alpha_i} ... x_q^{\alpha_q}$ a minimal generator of $in_{\prec}(\mathcal{I})$ such that $\alpha_i \neq d$.*
If $x_i^{d-\alpha_i} \in U_{in_{\prec}(\mathcal{I})}(m)$ then $x_i^{d_i} \in U_{in_{\prec}(\mathcal{I})}(m)\ \forall\ d_i \in \mathbb{N}$.

Proof. Let m and d choosen as declared and suppose that $x_i^{d-\alpha_i} \in U_{in_{<}(\mathcal{I})}(m)$. Then $u \in U_{in_{<}(\mathcal{I})}(m)$ for any monomial u which divides $x_i^{d-\alpha_i}$ It follows that $x_i^{d_i} \in U_{in_{<}(\mathcal{I})}(m)$ for any $d_i < d - \alpha_i$ (1$*$).
In the other side, from $x_i^{d-\alpha_i} \in U_{in_{<}(\mathcal{I})}(m)$ we obtain $m_p = x_i^{d-\alpha_i}\dfrac{m}{x_p} \notin in_{<}(\mathcal{I})$ and $m_q = x_i^{d-\alpha_i}\dfrac{m}{x_q} \notin in_{<}(\mathcal{I})$.
We can remark that $\deg_{m_p}(x_i) = \deg_{m_q}(x_i) = d$.
Let m' be a minimal generator.

$$\begin{aligned} m_p,\ m_q \notin in_{<}(\mathcal{I}) &\Longrightarrow m' \nmid m_p \wedge m' \nmid m_q \\ &\Longrightarrow \forall \beta_i \in \mathbb{N}^*,\ m' \nmid x_i^{\beta_i} m_p \ \wedge\ m' \nmid x_i^{\beta_i} m_q \\ &\Longrightarrow x_i^{d-\alpha_i} x_i^{\beta_i} \in U_{in_{<}(\mathcal{I})}(m) \\ &\Longrightarrow x_i^{d_i} \in U_{in_{<}(\mathcal{I})}(m)\ \forall\ d_i > d - \alpha_i\ (2*) \end{aligned}$$

From (1$*$) and (2$*$) we have the result. ☐

The *Proposition 3.6.* yields a tool to check whether the noncommutative ideal $\gamma^{-1}(\mathcal{I})$, for a commutative ideal $\mathcal{I}$ given by a minimal Gröbner basis, has a finite Gröbner basis. It requires less computation than *Proposition 3.1.*.

Now we investigate the special case of principal ideals. For any principal ideal $\mathcal{I} = \langle g \rangle \subset \mathbb{F}_q[x_1, \ldots, x_n]$ and its initial ideal $\mathcal{L} = \langle in_{<}(g) \rangle$, we have $U_{\mathcal{L}}(in_{<}(g)) = \{1\}$ or $U_{\mathcal{L}}(in_{<}(g))$ is an infinite set. A necessary and sufficient condition to obtain $U_{\mathcal{L}}(in_{<}(g)) = \{1\}$ is given in the following result.

Proposition 3.7. *Let $\mathcal{I} = \langle g \rangle \subset \mathbb{F}_q[x_1, \ldots, x_n]$ be a principal ideal. Then $\gamma^{-1}(\mathcal{I})$ has a finite Gröbner basis if and only if $in_{<}(g)$ is a product of powers of two consecutive variables even after a linear change of variables.*

Proof. After a linear change of variables, if $in_{<}(g) = y_i^{\alpha_i} y_{i+1}^{\alpha_{i+1}}$ for some $1 \le i \le n-1$ and $\alpha_i, \alpha_{i+1} \in \mathbb{N}$ then $U_{\mathcal{L}}(in_{<}(g)) = \{1\}$.
If $in_{<}(g) \neq y_i^{\alpha_i} y_{i+1}^{\alpha_{i+1}}$ for any $1 \le i \le n-1$ and $\alpha_i, \alpha_{i+1} \in \mathbb{N}$ then $in_{<}(g) = y_p^{\alpha_p} \ldots y_q^{\alpha_q}$ with $p+1 < q$ and $\alpha_p \neq 0$, $\alpha_q \neq 0$. Thus $y_t^{\alpha_t} \in u_{\mathcal{L}}(in_{<}(g))\ \forall (t, \alpha_t) \in [p, q] \times \mathbb{N}$. And then $U_{\mathcal{L}}(in_{<}(g))$ is an infinite set. ☐

The previous result means that in the case of a principal ideal $\mathcal{I} = \langle g \rangle$, to decide whether $\gamma^{-1}(\mathcal{I})$ has a finite Gröbner basis or not, on has only to look for the existence of a linear change variables for which $in_{<}(g) \in \mathbb{F}_q[y_i, y_{i+1}]$.

Example 3.8. Let $\mathcal{I} = \langle x_1^4 x_2^2 x_3 \rangle \subset \mathbb{F}_2[x_1, x_2, x_3]$. By the change of variables in *Example 2.8.*,
$\{Y_1^4 Y_2^2 Y_3 + Y_1^4 Y_3^3 + Y_2^2 Y_3^5 + Y_3^7, Y_2 Y_1 - Y_1 Y_2, Y_3 Y_1 - Y_1 Y_3, Y_3 Y_2 - Y_2 Y_3\}$ is a Gröbner basis of $\gamma^{-1}(\mathcal{I})$.

More generally, for a principal monomial ideal, such a change of variables always works.

Corollary 3.9. *$\gamma^{-1}(\mathcal{I})$ has a finite Gröbner basis for any monomial principal ideal $\mathcal{I} \subset \mathbb{F}_q[x_1, \ldots, x_n]$.*

Proof. Let m be a monomial and $\mathcal{I} = \langle m \rangle$. We order variables such that those of m are the last q ones. Namely $m = x_{n-q+1}^{\alpha_{n-q+1}} \ldots x_n^{\alpha_n}$ where $\alpha_i \neq 0\ \forall\ 1 \le i \le q$. Consider the change of variables $\begin{cases} x_n \longmapsto y_n \\ x_i \longmapsto y_n + y_i\ \forall\ i < n \end{cases}$

Then m becomes $m' = y_n^{\beta_n} + \sum\limits_j \lambda_j m_j$, *where m_j is a monomial $\forall j$ and*

$\deg_{m_j}(y_n) < \beta_n$. So by the lexicographic order with $y_1 \prec \ldots \prec y_n$ we have $in_{<}(\mathcal{I}') = \langle y_n^{\beta_n} \rangle$ and then $U_{in_{<}(\mathcal{I}')}(y_n^{\beta_n}) = \{1\}$. ☐

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